Monday, December 5, 2022

1 Conservative and Irrotational Vector Fields

Definition 1 *Given two intervals $I, J \subseteq \mathbb{R}$, and two functions $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and $\boldsymbol{\psi} : J \to \mathbb{R}^N$ of class C^k, $k \in \mathbb{N}_0$, we say that they are* equivalent *if there exists a bijective function $h : I \to J$ with h and h^{-1} of class C^k such that*

$$\boldsymbol{\varphi}(t) = \boldsymbol{\psi}(h(t))$$

for all $t \in I$. We write $\boldsymbol{\varphi} \sim \boldsymbol{\psi}$ and we call $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ parametric representations of class C^k *and the function h a* parameter change of class C^k*. A* curve $\boldsymbol{\gamma}$ *of class C^k is an equivalence class of parametric representations of class C^k, that is, $[\boldsymbol{\varphi}] := \{\boldsymbol{\psi} : \boldsymbol{\psi} \sim \boldsymbol{f}\}$. The set $\Sigma = \boldsymbol{\varphi}(I)$ is called the* range of the curve*.*

Definition 2 *A curve $\boldsymbol{\gamma}$ of class C^k is* closed *if it has a parametric representation $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ with $\boldsymbol{\varphi}(a) = \boldsymbol{\varphi}(b)$.*

Similarly we can define C^∞ curves, Lipschitz curves, analytic curves, and so on.

Remark 3 *Note that given a curve $\boldsymbol{\gamma}$ of class C^k with parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$, the function $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is not in general a local chart for a one-dimensional manifold, since we are not assuming that $\boldsymbol{\varphi}$ is injective or that $\boldsymbol{\varphi}'(t) \neq \mathbf{0}$ for every $t \in I$. In particular, a curve could self intersects but a one-dimensional manifold cannot.*

Next we introduce the notion of an oriented curve.

Definition 4 *Given a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^N$ of class C^k, $k \in \mathbb{N}_0$, with parametric representations $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and $\boldsymbol{\psi} : J \to \mathbb{R}^N$, we say that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the* same orientation *if the parameter change $h : I \to J$ is increasing and* opposite orientation *if the parameter change $h : I \to J$ is decreasing. If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the same orientation, we write $\boldsymbol{\varphi} \overset{*}{\sim} \boldsymbol{\psi}$.*

Exercise 5 *Prove that $\overset{*}{\sim}$ is an equivalence relation.*

Definition 6 *An* oriented curve $\boldsymbol{\gamma}$ *in $\mathbb{R}^N$ of class C^k, $k \in \mathbb{N}_0$, is an equivalence class of parametric representations with the same orientation.*

Note that any curve $\boldsymbol{\gamma}$ in $\mathbb{R}^N$ gives rise to two oriented curves. Indeed, it is enough to fix a parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and considering the equivalence class $\boldsymbol{\gamma}^+$ of parametric representations with the same orientation of $\boldsymbol{\varphi}$ and the equivalence class $\boldsymbol{\gamma}^-$ of parametric representations with the opposite orientation of φ.

Definition 7 *Given a Lipschitz continuous oriented curve $\boldsymbol{\gamma}$ in $\mathbb{R}^N$ and a function $\boldsymbol{g} : E \to \mathbb{R}^N$, where E contains the range of $\boldsymbol{\gamma}$, we define the* curve *(or* line*)* integral *of $\boldsymbol{g}$ along the curve $\boldsymbol{\gamma}$ as the number*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} := \int_I \boldsymbol{g}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{\varphi}'(t)\, dt.$$

provided the function $t \in I \mapsto \boldsymbol{g}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{\varphi}'(t)$ is Lebesgue integrable for every parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ of $\boldsymbol{\gamma}$ and the value of the integral does not change with the representation.

Exercise 8 *Let $\boldsymbol{\gamma}$ be an oriented Lipschitz continuous curve in $\mathbb{R}^N$ with parametric representations $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ and $\boldsymbol{\psi} : [c,d] \to \mathbb{R}^N$. Given a continuous function $\boldsymbol{g} : E \to \mathbb{R}^N$, where E contains the range of $\boldsymbol{\gamma}$, prove that*

$$\int_a^b \boldsymbol{g}(\boldsymbol{\varphi}(t)) \cdot \boldsymbol{\varphi}'(t)\, dt = \int_c^d \boldsymbol{g}(\boldsymbol{\psi}(\tau)) \cdot \boldsymbol{\psi}'(\tau)\, d\tau.$$

Proposition 9 *Let $\boldsymbol{\gamma}$ be an oriented Lipschitz continuous curve and $\boldsymbol{f}, \boldsymbol{g} : E \to \mathbb{R}^N$, where E contains the range of $\boldsymbol{\gamma}$. Then*

(i) if $\int_{\boldsymbol{\gamma}} \boldsymbol{f}$ and $\int_{\boldsymbol{\gamma}} \boldsymbol{g}$ are well defined, then for all $a, b \in \mathbb{R}$,

$$\int_{\boldsymbol{\gamma}} (a\boldsymbol{f} + b\boldsymbol{g}) = a \int_{\boldsymbol{\gamma}} \boldsymbol{f} + b \int_{\boldsymbol{\gamma}} \boldsymbol{g},$$

(ii) If $\int_{\boldsymbol{\gamma}} \boldsymbol{f}$ is well defined and $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is a parametric representation of $\boldsymbol{\gamma}$, then $\left| \int_{\boldsymbol{\gamma}} \boldsymbol{f} \right| \le \operatorname{Var}_I \boldsymbol{\varphi} \sup_\Sigma \|\boldsymbol{f}\|$, where Σ is the range of $\boldsymbol{\gamma}$,

(iii) If $\int_{\boldsymbol{\gamma}} \boldsymbol{f}$ is well defined, $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is a parametric representation of $\boldsymbol{\gamma}$, $c \in I^\circ$, and $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are the oriented curves of parametric representations $\boldsymbol{\varphi}_1 : I \cap (-\infty, c] \to \mathbb{R}^N$ and $\boldsymbol{\varphi}_2 : I \cap [c, \infty) \to \mathbb{R}^N$, then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{f} = \int_{\boldsymbol{\gamma}_1} \boldsymbol{f} + \int_{\boldsymbol{\gamma}_2} \boldsymbol{f}.$$

Definition 10 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\boldsymbol{g} : U \to \mathbb{R}^N$. We say that $\boldsymbol{g}$ is* conservative vector field *if there exists a differentiable function $f : U \to \mathbb{R}$ such that*

$$\nabla f(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in U$. The function f is called a scalar potential *for $\boldsymbol{g}$.*

Wednesday, December 7, 2022

Theorem 11 (Fundamental Theorem of Calculus for Curves) *Let $U \subseteq \mathbb{R}^N$ be an open set, let $f \in C^1(U)$, let $\boldsymbol{x}$, $\boldsymbol{y} \in U$ and let $\boldsymbol{\gamma}$ a Lipschitz oriented curve with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}(b) = \boldsymbol{x}$, $\boldsymbol{\varphi}(a) = \boldsymbol{y}$, and $\boldsymbol{\varphi}([a,b]) \subset U$. Then*

$$\int_{\boldsymbol{\gamma}} \nabla f = f(\boldsymbol{x}) - f(\boldsymbol{y}).$$

Proof. Define $p(t) := f(\boldsymbol{\varphi}(t))$ and observe that p is Lipschitz with

$$p'(t) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\boldsymbol{\varphi}(t))\, \varphi_i'(t)$$

for $\mathcal{L}^1$ a.e. $t \in [a,b]$. Hence,

$$\int_{\boldsymbol{\gamma}} \nabla f = \int_a^b \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\boldsymbol{\varphi}(t))\, \varphi_i'(t)\, dt = \int_a^b p'(t)\, dt = p(b) - p(a) = f(\boldsymbol{x}) - f(\boldsymbol{y}),$$

where we have used the fundamental theorem of calculus for Lebesgue integral. ■

The previous theorem shows that if a conservative vector field is continuous, then its integral along a curve joining two points depends only on the value at the two points and not on the particular curve. If U is pathwise connected, then this condition turns out to be equivalent to the vector field being conservative.

Definition 12 *A set $E \subseteq \mathbb{R}^N$ is* pathwise connected *if for every $\boldsymbol{x}, \boldsymbol{y} \in E$ there exists a continuous curve with range in E joining $\boldsymbol{x}$ with $\boldsymbol{y}$, that is, $\boldsymbol{\gamma} = [\boldsymbol{\varphi}]$, and $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ is such that $\boldsymbol{\varphi}(b) = \boldsymbol{x}$, $\boldsymbol{\varphi}(a) = \boldsymbol{y}$.*

Exercise 13 *Prove that if $U \subseteq \mathbb{R}^N$ is open and pathwise connected, then for every $\boldsymbol{x}, \boldsymbol{y} \in E$ there exists a polygonal path with range in U joining $\boldsymbol{x}$ with $\boldsymbol{y}$.*

Theorem 14 *Let $U \subseteq \mathbb{R}^N$ be an open pathwise connected set and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be a continuous function. Then the following conditions are equivalent.*

(i) $\boldsymbol{g}$ is a conservative vector field,

(ii) for every $\boldsymbol{x}$, $\boldsymbol{y} \in U$ and for every two Lipschitz oriented curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ with parametric representations $\boldsymbol{\varphi}_1 : [a,b] \to \mathbb{R}^N$ and $\boldsymbol{\varphi}_2 : [c,d] \to \mathbb{R}^N$, respectively, such that $\boldsymbol{\varphi}_1(b) = \boldsymbol{\varphi}_2(d) = \boldsymbol{x}$, $\boldsymbol{\varphi}_1(a) = \boldsymbol{\varphi}_2(c) = \boldsymbol{y}$, and $\boldsymbol{\varphi}_1([a,b]), \boldsymbol{\varphi}_2([c,d]) \subset U$,

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{g} = \int_{\boldsymbol{\gamma}_2} \boldsymbol{g}.$$

(iii) for every Lipschitz closed oriented curve $\boldsymbol{\gamma}$ with range contained in U,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = \boldsymbol{0}.$$

Proof. We prove that (i) implies (ii). Assume that $\boldsymbol{g}$ is a conservative vector field with scalar potential $f : U \to \mathbb{R}$, let $\boldsymbol{x}$, $\boldsymbol{y} \in U$ and let $\boldsymbol{\varphi}_1 : [a, b] \to \mathbb{R}^N$ and $\boldsymbol{\varphi}_2 : [c, d] \to \mathbb{R}^N$ be as in (ii). Then by the previous theorem

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{g} = \int_{\boldsymbol{\gamma}_1} \nabla f = f(\boldsymbol{x}) - f(\boldsymbol{y}) = \int_{\boldsymbol{\gamma}_2} \nabla f = \int_{\boldsymbol{\gamma}_2} \boldsymbol{g}.$$

Conversely assume that (ii) holds. We need to find a scalar potential for $\boldsymbol{g}$. Fix a point $\boldsymbol{x}_0 \in U$ and for every $\boldsymbol{x} \in U$ define

$$f(\boldsymbol{x}) := \int_{\boldsymbol{\gamma}} \boldsymbol{g},$$

where $\boldsymbol{\gamma}$ a Lipschitz continuous oriented curve with parametric representation $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}(b) = \boldsymbol{x}$, $\boldsymbol{\varphi}(a) = \boldsymbol{x}_0$, and $\boldsymbol{\varphi}([a, b]) \subset U$. We claim that there exist

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}) = g_i(\boldsymbol{x}).$$

Since U is open and $\boldsymbol{x} \in U$, there exists $B(\boldsymbol{x}, r) \subseteq U$. Fix $|h| < r$, then the segment joining the point $\boldsymbol{x} + h\boldsymbol{e}_i$ with $\boldsymbol{x}$ is contained in $B(\boldsymbol{x}, r)$. Define the curve $\boldsymbol{\psi} : [a, b+1] \to \mathbb{R}^N$ as follows

$$\boldsymbol{\psi}(t) := \begin{cases} \boldsymbol{\varphi}(t) & \text{if } t \in [a, b], \\ \boldsymbol{x} + (t - b)\, h\boldsymbol{e}_i & \text{if } t \in [b, b+1]. \end{cases}$$

Using (ii), we have that

$$\begin{aligned} f(\boldsymbol{x} + h\boldsymbol{e}_i) = \int_{\boldsymbol{\psi}} \boldsymbol{g} &= f(\boldsymbol{x}) + \int_b^{b+1} \sum_{j=1}^N g_j(\boldsymbol{x} + (t-b)h\boldsymbol{e}_i)\, h\delta_{ij}\, dt \\ &= f(\boldsymbol{x}) + \int_b^{b+1} g_i(\boldsymbol{x} + (t-b)\, h\boldsymbol{e}_i)\, h\, dt = \\ &= f(\boldsymbol{x}) + \int_0^h g_i(\boldsymbol{x} + s\boldsymbol{e}_i)\, ds, \end{aligned}$$

where in the last equality we have used the change of variable $s = (t - b)\, h$. It follows by the mean value theorem that

$$\frac{f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})}{h} = \frac{1}{h} \int_0^h g_i(\boldsymbol{x} + s\boldsymbol{e}_i)\, ds = g_i(\boldsymbol{x} + s_h \boldsymbol{e}_i),$$

where s_h is between 0 and h. As $h \to 0$, we have that $s_h \to 0$ and so $\boldsymbol{x} + s_h \boldsymbol{e}_i \to \boldsymbol{x}$. Using the continuity of g_i, we have that there exists

$$\lim_{h \to 0} \frac{f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})}{h} = \lim_{h \to 0} g_i(\boldsymbol{x} + s_h \boldsymbol{e}_i) = g_i(\boldsymbol{x}),$$

which proves the claim.

The equivalence between (ii) and (iii) is left as an exercise. ∎

Remark 15 *The previous theorem is used to prove that a vector field is not conservative. Indeed, if $U \subseteq \mathbb{R}^N$ is an open pathwise connected set and $\boldsymbol{g} : U \to \mathbb{R}^N$ is a continuous function, if you can construct a Lipschitz closed oriented curve $\boldsymbol{\gamma}$ with range contained in U such that*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} \neq \mathbf{0},$$

then $\boldsymbol{g}$ cannot be conservative.

Given a curve $\boldsymbol{\gamma}$ in $\mathbb{R}^N$ of class C^k, $k \in \mathbb{N}_0$, with parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is a proper interval, the *multiplicity* of a point $\boldsymbol{x} \in \mathbb{R}^N$ is the (possibly infinite) number of points $t \in I$ such that $\boldsymbol{\varphi}(t) = \boldsymbol{x}$. Since every parameter change $h : I \to J$ is bijective, the multiplicity of a point does not depend on the particular parametric representation. The *range* of $\boldsymbol{\gamma}$ is the set of points of $\mathbb{R}^N$ with positive multiplicity, that is, $\boldsymbol{\varphi}(I)$. If one of the endpoints of I belongs to I, its image through $\boldsymbol{\varphi}$ is called an *endpoint* of the curve.

A point in the range of $\boldsymbol{\gamma}$ with multiplicity one is called a *simple point.* If every point of the range is simple, then $\boldsymbol{\gamma}$ is called a *simple arc*. A closed curve is called *simple* if every point of the range of $\boldsymbol{\gamma}$ is simple, with the exception of $\boldsymbol{\varphi}(a)$, which has multiplicity two.

Remark 16 *Note that in view of Exercise 13, we can replace item (iii) with the weaker requirement that*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = \mathbf{0}$$

for every simple closed polygonal path $\boldsymbol{\gamma}$ with range contained in U, where $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ with $\boldsymbol{\varphi}(a) = \boldsymbol{\varphi}(b)$

Friday, December 9, 2022

Next we give a simple necessary condition for a field $\boldsymbol{g}$ to be conservative.

Definition 17 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be differentiable. We say that $\boldsymbol{g}$ is an* irrotational vector field *or* a curl-free vector field *if*

$$\frac{\partial g_i}{\partial x_j}(\boldsymbol{x}) = \frac{\partial g_j}{\partial x_i}(\boldsymbol{x})$$

for all $i, j = 1, \ldots, N$ and all $\boldsymbol{x} \in U$.

Theorem 18 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be a conservative vector field of class C^1. Then $\boldsymbol{g}$ is irrotational.*

Proof. Since $\boldsymbol{g}$ is a conservative vector field, there exists a a scalar potential $f : U \to \mathbb{R}$ with $\nabla f = \boldsymbol{g}$ in U. But since $\boldsymbol{g}$ is of class C^1, we have that f is of class C^2. Hence, we are in a position to apply the Schwartz theorem to conclude that

$$\frac{\partial g_i}{\partial x_j}(\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{x}) = \frac{\partial g_j}{\partial x_i}(\boldsymbol{x})$$

for all $i, j = 1, \ldots, N$ and all $\boldsymbol{x} \in U$. ■

The next example shows that there exist irrotational vector fields that are not conservative.

Example 19 *Let $U := \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the function*

$$\boldsymbol{g}(x,y) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right).$$

Then $\boldsymbol{g}$ is irrotational but not conservative. Indeed,

$$\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2} \right) = -\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) = -\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

but, taking the oriented curve $\boldsymbol{\gamma}$ parametrized by $\boldsymbol{\varphi}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we get

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \boldsymbol{g} &= \int_0^{2\pi} \boldsymbol{g}(\cos t, \sin t) \cdot (-\sin t, \cos t)\, dt \\ &= \int_0^{2\pi} \left(-\frac{\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right) \cdot (-\sin t, \cos t)\, dt = 2\pi \neq 0.\end{aligned}$$

Hence, by Theorem 14(iii), $\boldsymbol{g}$ cannot be conservative.

The problem here is the fact that the domain has a hole.

Definition 20 *A set $E \subseteq \mathbb{R}^N$ is* starshaped *with respect to a point $\boldsymbol{x}_0 \in \mathbb{R}^N$ if for every $\boldsymbol{x} \in E$, the segment joining $\boldsymbol{x}$ and $\boldsymbol{x}_0$ is contained in E.*

Theorem 21 (Poincaré's Lemma) *Let $U \subseteq \mathbb{R}^N$ be an open set starshaped with respect to a point $\boldsymbol{x}_0$ and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be an irrotational vector field of class C^1. Then $\boldsymbol{g}$ is a conservative vector field.*

Proof. For every $\boldsymbol{x} \in U$ define

$$f(\boldsymbol{x}) := \int_{\boldsymbol{\gamma}} \boldsymbol{g},$$

where $\boldsymbol{\gamma}$ is the curve given by the parametric representation $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^N$ is defined by

$$\boldsymbol{\varphi}(t) := \boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0).$$

Note that

$$f(\boldsymbol{x}) = \int_0^1 \sum_{j=1}^N g_j(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))(x_j - x_{0j})\, dt.$$

Since $\boldsymbol{g}$ is of class C^1 we can differentiate under the integral sign to get

$$\begin{aligned}
\frac{\partial f}{\partial x_i}(\boldsymbol{x}) &= \int_0^1 \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N g_j(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))(x_j - x_{0j})\right) dt \\
&= \int_0^1 \left(\sum_{j=1}^N \frac{\partial g_j}{\partial x_i}(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))\, t\,(x_j - x_{0j}) + g_i(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))\, 1\right) dt \\
&= \int_0^1 \left(\sum_{j=1}^N \frac{\partial g_i}{\partial x_j}(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))\, t\,(x_j - x_{0j}) + g_i(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))\, 1\right) dt,
\end{aligned}$$

where we have used the fact that $\boldsymbol{g}$ is an irrotational vector field. Define

$$h(t) := t g_i(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0)).$$

By the chain rule,

$$h'(t) = \sum_{j=1}^N \frac{\partial g_i}{\partial x_j}(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0))\, t\,(x_j - x_{0j}) + g_i(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0)).$$

Hence, by the fundamental theorem of calculus,

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}) = \int_0^1 h'(t)\, dt = h(1) - h(0) = 1 g_i(\boldsymbol{x}) - 0,$$

which completes the proof. ■

Definition 22 *Given a set $E \subseteq \mathbb{R}^N$, $\boldsymbol{x}$, $\boldsymbol{y} \in E$, and two continuous oriented curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ with range in E and parametric representations $\boldsymbol{\varphi}_1 : [a, b] \to \mathbb{R}^N$ and $\boldsymbol{\varphi}_2 : [a, b] \to \mathbb{R}^N$, respectively, such that $\boldsymbol{\varphi}_1(a) = \boldsymbol{\varphi}_2(a) = \boldsymbol{x}$ and $\boldsymbol{\varphi}_1(b) = \boldsymbol{\varphi}_2(b) = \boldsymbol{y}$, we say that $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are* path homotopic *in E if there exists a continuous function $\boldsymbol{h} : [0, 1] \times [a, b] \to \mathbb{R}^N$ such that $\boldsymbol{h}([0, 1] \times [a, b]) \subseteq E$,*

$$\begin{aligned}
&\boldsymbol{h}(0, t) = \boldsymbol{\varphi}_1(t) \text{ for all } t \in [a, b], \quad \boldsymbol{h}(1, t) = \boldsymbol{\varphi}_2(t) \text{ for all } t \in [a, b], \\
&\boldsymbol{h}(s, a) = \boldsymbol{x},\ \boldsymbol{h}(s, b) = \boldsymbol{y} \text{ for all } s \in [0, 1].
\end{aligned}$$

The function $\boldsymbol{h}$ is called a path-homotopy in E *or* fixed endpoint homotopy *between the two curves.*

Roughly speaking, two curves are path homotopic in E if it is possible to deform the first continuously until it becomes the second *without leaving* the set E.

Definition 23 *Given a set $E \subseteq \mathbb{R}^N$ and $\boldsymbol{x} \in E$, a continuous oriented closed curve $\boldsymbol{\gamma}_1$ with range in E and parametric representation $\boldsymbol{\varphi}_1 : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}_1(a) = \boldsymbol{\varphi}_1(b) = \boldsymbol{x}$, we say that $\boldsymbol{\gamma}_1$ is* null homotopic *in E if it is path homotopic in E to the continuous oriented curves $\boldsymbol{\gamma}_2$ parametrized by the constant function $\boldsymbol{\varphi}_2(t) := \boldsymbol{x}$.*

Definition 24 *A set $E \subseteq \mathbb{R}^N$ is* simply connected *if it is pathwise connected and if every continuous closed curve with range in E is null homotopic in E.*

Example 25 *A star-shaped set is simply connected. Indeed, let $E \subseteq \mathbb{R}^N$ be star-shaped with respect to some point $\boldsymbol{x}_0 \in E$ and consider a continuous closed curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}([a,b]) \subseteq E$. Then the function*

$$\boldsymbol{h}(s,t) := s\boldsymbol{\varphi}(t) + (1-s)\,\boldsymbol{x}_0$$

is an homotopy between $\boldsymbol{\gamma}$ and the point $\boldsymbol{x}_0$.

Theorem 26 *Let $U \subseteq \mathbb{R}^N$ be an open set, let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be two oriented closed Lipschitz continuous curves which are path homotopic in U and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be of class C^1 and irrotational. Then*

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{g} = \int_{\boldsymbol{\gamma}_2} \boldsymbol{g}.$$

In particular, if U is simply connected, then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = 0$$

for every Lipschitz continuous closed oriented curve $\boldsymbol{\gamma}$ with range in U.

Wednesday, January 18, 2023

In what follows, given the unit square $Q = [0,1] \times [0,1]$, we consider the oriented closed simple curve obtained by moving along ∂Q counterclockwise starting from $(0,0)$. Denote by $\boldsymbol{\varphi}_0 : [0,4] \to \partial Q$ the parametric representation obtained by using arclength.

Theorem 27 *Let $U \subseteq \mathbb{R}^N$ be an open set, let $\boldsymbol{h} : Q \to U$ be Lipschitz continuous, let $\boldsymbol{\gamma}$ be the Lipschitz continuous oriented closed curve parametrized by $\boldsymbol{h} \circ \boldsymbol{\varphi}_0 : [0,4] \to U$, and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be of class C^1 and irrotational. Then*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = 0.$$

Proof. Assume by contradiction that

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = c \neq 0.$$

By replacing $\boldsymbol{g}$ with $\boldsymbol{g}/c$, without loss of generality, we may assume that $c = 1$. Divide Q into four squares $Q_{1,1}$, $Q_{1,2}$, $Q_{1,3}$, $Q_{1,4}$ of side-length $\frac{1}{2}$ and parametrize their boundaries as we did for ∂Q. Let $\boldsymbol{\varphi}_{1,1}$, $\boldsymbol{\varphi}_{1,2}$, $\boldsymbol{\varphi}_{1,3}$, $\boldsymbol{\varphi}_{1,4}$ be the corresponding parametric representations and let $\boldsymbol{\gamma}_{1,1}$, $\boldsymbol{\gamma}_{1,2}$, $\boldsymbol{\gamma}_{1,3}$, $\boldsymbol{\gamma}_{1,4}$ be the oriented closed curve parametrized by $\boldsymbol{h} \circ \boldsymbol{\varphi}_{1,k} : [0, 4/2^1] \to U$, $k = 1, \ldots, 4$, respectively. Since integrals over opposite curves cancel out, we have that

$$1 = \int_{\boldsymbol{\gamma}_{1,1}} \boldsymbol{g} + \int_{\boldsymbol{\gamma}_{1,2}} \boldsymbol{g} + \int_{\boldsymbol{\gamma}_{1,3}} \boldsymbol{g} + \int_{\boldsymbol{\gamma}_{1,4}} \boldsymbol{g}$$

and thus there exists $k_1 \in \{1, \ldots, 4\}$ such that

$$\left| \int_{\boldsymbol{\gamma}_{1,k}} \boldsymbol{g} \right| \geq \frac{1}{4}.$$

Let $Q_1 := Q_{1,k_1}$ and $\boldsymbol{\gamma}_1 := \boldsymbol{\gamma}_{1,k_1}$. We now divide Q_1 into four squares $Q_{2,1}$, $Q_{2,2}$, $Q_{2,3}$, $Q_{2,4}$ of side-length $\frac{1}{16}$. Proceeding as before we find $k_2 \in \{1, \ldots, 4\}$ such that

$$\left| \int_{\boldsymbol{\gamma}_{2,k_2}} \boldsymbol{g} \right| \geq \frac{1}{16}.$$

Inductively we obtain a decreasing sequence of closed squares Q_n of side-length $\frac{1}{2^n}$ such that

$$\left| \int_{\boldsymbol{\gamma}_n} \boldsymbol{g} \right| \geq \frac{1}{4^n}. \tag{1}$$

where $\boldsymbol{\gamma}_n$ is the oriented closed curve parametrized by $\boldsymbol{h} \circ \boldsymbol{\varphi}_n : [0, \frac{4}{2^n}] \to U$ and $\boldsymbol{\varphi}_n : [0, \frac{4}{2^n}] \to \partial Q_n$. By Cantor's theorem there exists $(r_0, t_0) \in Q_n$ for all n. Let $\boldsymbol{x}_0 = \boldsymbol{h}((r_0, t_0))$. Since $\boldsymbol{g}$ is differentiable, we can write

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}_0) + J_{\boldsymbol{g}}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + \boldsymbol{R}(\boldsymbol{x}),$$

where

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \frac{\boldsymbol{R}(\boldsymbol{x})}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} = 0. \tag{2}$$

Since $\boldsymbol{g}$ is irrotational, the Jacobian matrix $J_{\boldsymbol{g}}(\boldsymbol{x}_0)$ is symmetric. Hence, the affine function $\boldsymbol{g}(\boldsymbol{x}_0) + J_{\boldsymbol{g}}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)$ is conservative, since a scalar potential is given by

$$f(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}_0) \cdot \boldsymbol{x} + \frac{1}{2}(J_{\boldsymbol{g}}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0))^T \cdot (\boldsymbol{x} - \boldsymbol{x}_0).$$

It follows by the fundamental theorem of calculus,

$$\int_{\boldsymbol{\gamma}_n} \boldsymbol{g} = \int_{\boldsymbol{\gamma}_n} \nabla f + \int_{\boldsymbol{\gamma}_n} \boldsymbol{R} = 0 + \int_{\boldsymbol{\gamma}_n} \boldsymbol{R}.$$

Let Γ_n be the range of $\boldsymbol{\gamma}_n$. If $\boldsymbol{x} \in \Gamma_n = \boldsymbol{h}(\boldsymbol{\varphi}_n([0, \frac{4}{2^n}]))$, we can find $(r,t) \in \partial Q_n$ such that $\boldsymbol{x} = \boldsymbol{h}(r,t)$. Hence, if $L > 0$ is the Lipschitz constant of $\boldsymbol{h}$, we have that

$$\|\boldsymbol{x}-\boldsymbol{x}_0\| = \|\boldsymbol{h}(r,t)-\boldsymbol{h}(r_0,t_0)\| \leq L\sqrt{(r-r_0)^2+(t-t_0)^2} \leq L \operatorname{diam} Q_n = L\frac{\sqrt{2}}{2^n}.$$

In turn, by (2),

$$\|\boldsymbol{R}(\boldsymbol{x})\| = o\left(\|\boldsymbol{x}-\boldsymbol{x}_0\|\right) = o\left(\frac{1}{2^n}\right),$$

where $\varepsilon_n \to 0^+$. Hence,

$$\begin{aligned}\left|\int_{\boldsymbol{\gamma}_n} \boldsymbol{g}\right| = \left|\int_{\boldsymbol{\gamma}_n} \boldsymbol{R}\right| &= \left|\int_0^{\frac{4}{2^n}} \boldsymbol{R}((\boldsymbol{h}\circ\boldsymbol{\varphi}_n)(s)) \cdot (\boldsymbol{h}\circ\boldsymbol{\varphi}_n)'(s)\, ds\right| \\ &\leq \int_0^{\frac{4}{2^n}} \|\boldsymbol{R}((\boldsymbol{h}\circ\boldsymbol{\varphi}_n)(s))\| \|(\boldsymbol{h}\circ\boldsymbol{\varphi}_n)'(s)\|\, ds \\ &\leq o\left(\frac{1}{2^n}\right) \int_0^{\frac{4}{2^n}} \|(\boldsymbol{h}\circ\boldsymbol{\varphi}_n)'(s)\|\, ds \\ &\leq o\left(\frac{1}{2^n}\right) L \int_0^{\frac{4}{2^n}} \|\boldsymbol{\varphi}_n'(s)\|\, ds = o\left(\frac{1}{2^n}\right) L \int_0^{\frac{4}{2^n}} 1\, ds = o\left(\frac{1}{2^n}\right)\frac{4L}{2^n}.\end{aligned}$$

Using (2) we get

$$\frac{1}{4^n} \leq \left|\int_{\boldsymbol{\gamma}_n} \boldsymbol{g}\right| \leq o\left(\frac{1}{4^n}\right)$$

as $n \to \infty$, which is a contradiction. ∎

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Next we consider the case in which $\boldsymbol{h}$ is only continuous.

Theorem 28 *Let $U \subseteq \mathbb{R}^N$ be an open set, let $\boldsymbol{h} : Q \to U$ be continuous, let $\boldsymbol{\gamma}$ be the oriented closed curve parametrized by $\boldsymbol{h}\circ\boldsymbol{\varphi}_0 : [0,4] \to U$, and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be of class C^1 and irrotational. If $\boldsymbol{h}\circ\boldsymbol{\varphi}_0 : [0,4] \to U$ is Lipschitz continuous, then*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{g} = 0.$$

Proof. Subdivide Q into small subsquares of side-length $\frac{1}{n}$, define $\boldsymbol{h}_n = \boldsymbol{h}$ at the vertex of each subsquare and interpolate linearly in each subsquare. The corresponding function $\boldsymbol{h}_n$ will be Lipschitz continuous. Homework. ∎

Corollary 29 *Let $U \subseteq \mathbb{R}^N$ be an open set, let $\boldsymbol{h} : Q \to U$ be continuous and such that $\boldsymbol{h}(s,0) = \boldsymbol{h}(s,1)$ for all $s \in [0,1]$, let $\boldsymbol{\gamma}$ be the oriented closed curve parametrized by $\boldsymbol{h}\circ\boldsymbol{\varphi}_0 : [0,4] \to U$, and let $\boldsymbol{g} : U \to \mathbb{R}^N$ be of class C^1 and irrotational. Assume that the curves $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ parametrized by $\boldsymbol{h}\circ\boldsymbol{\varphi}_0 : [1,2] \to U$ and $\boldsymbol{h}\circ\boldsymbol{\varphi}_0 : [3,4] \to U$ are Lipschitz continuous, then*

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{g} + \int_{\boldsymbol{\gamma}_2} \boldsymbol{g} = 0.$$

Proof. Since $\boldsymbol{h}(s,0) = \boldsymbol{h}(s,1)$ for all $s \in [0,1]$, by your homework we will have $\boldsymbol{h}_n(s,0) = \boldsymbol{h}_n(s,1)$ for all $s \in [0,1]$. Hence, the Lipschitz curves parametrized by $\boldsymbol{h} \circ \boldsymbol{\varphi}_0 : [0,1] \to U$ and $\boldsymbol{h} \circ \boldsymbol{\varphi}_0 : [2,3] \to U$ are one the opposite of the other and so their corresponding integrals will cancel each other. In turn,

$$\int_{\boldsymbol{\gamma}_{1,n}} \boldsymbol{g} + \int_{\boldsymbol{\gamma}_{2,n}} \boldsymbol{g} = 0.$$

Letting $n \to \infty$ will give the desired result. ■

We turn to the proof of Theorem 26

Proof of Theorem 26. Let $\boldsymbol{\varphi}_1 : [0,1] \to U$ and $\boldsymbol{\varphi}_2 : [0,1] \to U$ be parametric representations of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$, respectively, and let $\boldsymbol{h} : [0,1] \times [0,1]$ be a corresponding homotopy. Then $\boldsymbol{h} \circ \boldsymbol{\varphi}_0$ is composed of four curves: first $s \in [0,1] \to \boldsymbol{h}(s,0)$ followed by $\boldsymbol{\gamma}_1$, then the opposite of $s \in [0,1] \to \boldsymbol{h}(s,1)$ and finally the opposite of $\boldsymbol{\gamma}_2$. Since the first and the third of these four curves are the opposite to each other, the corresponding integrals will cancel out. Hence, in view of Corollary 29,

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{g} + \int_{-\boldsymbol{\gamma}_2} \boldsymbol{g} = 0.$$

■

Part I

Fixed Point Theorems and Applications

2 Brouwer's Fixed Point Theorem

Theorem 30 (Brouwer's fixed point theorem) *Let $K \subset \mathbb{R}^N$ be a nonempty compact convex set and let $\boldsymbol{g} : K \to K$ be a continuous function. Then there exists $\boldsymbol{x} \in K$ such that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$.*

We begin with a preliminary lemma.

Lemma 31 *There is no function $\boldsymbol{f} : \overline{B(\mathbf{0},1)} \to \partial B(\mathbf{0},1)$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial B(\mathbf{0},1)$ and which is continuous together with all its partial derivatives.*

Proof. Assume by contradiction that $\boldsymbol{f}$ exists and for $t \in [0,1]$ define

$$\boldsymbol{f}_t(\boldsymbol{x}) := t\boldsymbol{f}(\boldsymbol{x}) + (1-t)\boldsymbol{x}.$$

Then for every $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$,

$$\|\boldsymbol{f}_t(\boldsymbol{x})\| \le t\|\boldsymbol{f}(\boldsymbol{x})\| + (1-t)\|\boldsymbol{x}\| \le 1,$$

thus $\boldsymbol{f}_t : \overline{B(\mathbf{0},1)} \to \overline{B(\mathbf{0},1)}$. Moreover, $\boldsymbol{f}_t(\boldsymbol{x}) = \boldsymbol{x}$ for every $\boldsymbol{x} \in \partial B(\mathbf{0},1)$. ■

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Proof. Define $\boldsymbol{h}(\boldsymbol{x}) := \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}$. Since the derivatives of $\boldsymbol{h}$ are bounded, by the mean value theorem applied to each component, we obtain that $\boldsymbol{h}$ is Lipschitz continuous with Lipschitz constant $L \geq 1$. We claim that $\boldsymbol{f}_t$ is injective for every $0 < t < 1/L$. Indeed, assume by contradiction that there exist $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \overline{B(\mathbf{0},1)}$ such that $\boldsymbol{f}_t(\boldsymbol{x}_1) = \boldsymbol{f}_t(\boldsymbol{x}_2)$ and $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Since $\boldsymbol{f}_t(\boldsymbol{x}) = \boldsymbol{x} + t\boldsymbol{h}(\boldsymbol{x})$ it follows that

$$\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| = \|t(\boldsymbol{h}(\boldsymbol{x}_2) - \boldsymbol{h}(\boldsymbol{x}_1))\| \leq tL\|\boldsymbol{x}_2 - \boldsymbol{x}_1\| < \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|,$$

which is a contradiction. Hence, the claim holds.

Since $D\boldsymbol{f}_t = I_N + tD\boldsymbol{h}$ and $D\boldsymbol{h}$ is bounded in $\overline{B(\mathbf{0},1)}$, by taking t_0 smaller, if necessary, we can assume that $\det D\boldsymbol{f}_t(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$. To see this, note that the function

$$\xi \in \mathbb{R}^{N\times N} \mapsto \det \xi$$

is continuous and so, taking $\varepsilon = \frac{1}{2} > 0$ we can find $0 < \delta < 1$ such that

$$|\det \xi - \det I_N| \leq \frac{1}{2}$$

for all $\xi \in \mathbb{R}^{N\times N}$, with $\|\xi - I_N\|_{N\times N} < \delta$. Then $\|D\boldsymbol{f}_t(\boldsymbol{x}) - I_N\|_{N\times N} = |t|\|D\boldsymbol{h}(\boldsymbol{x})\|_{N\times N} \leq |t|L < \delta$, for $|t| < \delta/L$.

Since $\det D\boldsymbol{f}_t(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$ and all $0 \leq t < \delta/L =: t_0$. It follows by the inverse function theorem that the set $U_t := \boldsymbol{f}_t(B(\mathbf{0},1))$ is open for all $0 < t < t_0$.

We claim that $U_t = B(\mathbf{0},1)$ for every $0 < t < t_0$. Indeed, assume that this is not the case. Since $U_t \subseteq \overline{B(\mathbf{0},1)}$ by what we proved above, then $\partial U_t \subseteq \overline{B(\mathbf{0},1)}$, and so if $U_t \neq B(\mathbf{0},1)$, then there must exist $\boldsymbol{y}_0 \in \partial U_t$ such that $\boldsymbol{y}_0 \in B(\mathbf{0},1)$. Let $\boldsymbol{y}_n \in U_t$ be such that $\boldsymbol{y}_n \to \boldsymbol{y}_0$ and find $\boldsymbol{x}_n \in B(\mathbf{0},1)$ such that $\boldsymbol{y}_n = \boldsymbol{f}_t(\boldsymbol{x}_n)$. By compactness, up to a subsequence, we may assume that $\boldsymbol{x}_n \to \boldsymbol{x}_0 \in \overline{B(\mathbf{0},1)}$ with $\boldsymbol{f}_t(\boldsymbol{x}_0) = \boldsymbol{y}_0$, by the continuity of $\boldsymbol{f}_t$. Note that $\boldsymbol{x}_0$ cannot belong to $B(\mathbf{0},1)$ as otherwise $\boldsymbol{y}_0 = \boldsymbol{f}_t(\boldsymbol{x}_0)$ would belong to $\boldsymbol{f}_t(B(\mathbf{0},1)) = U_t$, so, necessarily $\boldsymbol{x}_0 \in \partial B(\mathbf{0},1)$. But then $\boldsymbol{f}_t(\boldsymbol{x}_0) = \boldsymbol{x}_0$ and so $\boldsymbol{x}_0 = \boldsymbol{y}_0$, which is again a contradiction since $\boldsymbol{y}_0 \in B(\mathbf{0},1)$. This proves that $U_t = B(\mathbf{0},1)$ for every $0 < t < t_0$.

For $t \in [0,1]$ define the function

$$g(t) := \int_{B(\mathbf{0},1)} \det D\boldsymbol{f}_t(\boldsymbol{x})\,d\boldsymbol{x} = \int_{B(\mathbf{0},1)} \det(I_N + tD\boldsymbol{h}(\boldsymbol{x}))\,d\boldsymbol{x}.$$

Since for every $0 < t < t_0$ the function $\boldsymbol{f}_t : B(\mathbf{0},1) \to B(\mathbf{0},1)$ is a bijection and $\det J_{\boldsymbol{f}_t} > 0$, it follows by the theorem on change of variables that $g(t) = \operatorname{meas}(B(\mathbf{0},1))$ for all $0 < t < t_0$. Since g is a polynomial of degree N, we have that $g(t) = \operatorname{meas}(B(\mathbf{0},1))$ for all $0 \leq t \leq 1$. In particular,

$$0 < \operatorname{meas}(B(\mathbf{0},1)) = g(1) = \int_{B(\mathbf{0},1)} \det D\boldsymbol{f}(\boldsymbol{x})\,d\boldsymbol{x}. \tag{3}$$

On the other hand, since by hypothesis $\|\boldsymbol{f}(\boldsymbol{x})\|^2 = \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 1$ for every $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$, if $\boldsymbol{x} \in B(\mathbf{0},1)$ and $\boldsymbol{v} \in \mathbb{R}^N$, then by replacing $\boldsymbol{x}$ with $\boldsymbol{x} + s\boldsymbol{v}$ differentiating with respect to s we get

$$(D\boldsymbol{f}(\boldsymbol{x})\boldsymbol{v}) \cdot \boldsymbol{f}(\boldsymbol{x}) = 0,$$

which shows that the range of $D\boldsymbol{f}(\boldsymbol{x})$ is orthogonal to the vector $\boldsymbol{f}(\boldsymbol{x})$. In turn, $D\boldsymbol{f}(\boldsymbol{x})$ has rank less than or equal to $N-1$ and so $\det D\boldsymbol{f}(\boldsymbol{x}) = 0$, which contradicts (3). This concludes the proof. ■

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Remark 32 *The previous lemma continues to hold for any ball. There is no function* $\boldsymbol{f} : \overline{B(\mathbf{0},r)} \to \partial B(\mathbf{0},r)$ *such that* $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \partial B(\mathbf{0},r)$ *and which is continuous together with all its partial derivatives. Just consider the function* $\boldsymbol{f}_r(\boldsymbol{x}) := \boldsymbol{f}(r\boldsymbol{x})$, $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$.

Lemma 33 *There is no continuous function* $\boldsymbol{f} : \overline{B(\mathbf{0},1)} \to \partial B(\mathbf{0},1)$ *such that* $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \partial B(\mathbf{0},1)$.

Proof. Assume that $\boldsymbol{f}$ exists. Extend $\boldsymbol{f}$ to be the identity outside $\overline{B(\mathbf{0},1)}$. Then $\boldsymbol{f}$ is continuous. Define $\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{x}$ and consider the mollification $\boldsymbol{g}_\varepsilon$ of $\boldsymbol{g}$. Since $\boldsymbol{g}(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x}$ outside $\overline{B(\mathbf{0},1)}$, we have that $\boldsymbol{g}_\varepsilon(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x}$ outside $\overline{B(\mathbf{0},1+\varepsilon)}$. Moreover, $\boldsymbol{g}_\varepsilon \to \boldsymbol{g}$ uniformly on compact sets. Take $\varepsilon = \frac{1}{n}$ and define $\boldsymbol{f}_n(\boldsymbol{x}) := \boldsymbol{g}_{1/n}(\boldsymbol{x}) + \boldsymbol{x}$. Then $\boldsymbol{f}_n$ is in $C^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $\boldsymbol{f}_n$ is the identity outside $\overline{B(\mathbf{0},1+\frac{1}{n})}$ and $\boldsymbol{f}_n \to \boldsymbol{f}$ uniformly in $\mathbb{R}^N$. Hence, for all n large,

$$\sup_{\boldsymbol{x}\in\mathbb{R}^N} \|\boldsymbol{f}_n(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})\| < \frac{1}{2}$$

Since $\|\boldsymbol{f}(\boldsymbol{x})\| \geq 1$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0},1+\frac{1}{n})}$, it follows that

$$\|\boldsymbol{f}_n(\boldsymbol{x})\| \geq \|\boldsymbol{f}(\boldsymbol{x})\| - \|\boldsymbol{f}_n(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})\| \geq 1 - \frac{1}{2}.$$

Define

$$\boldsymbol{h}_n(\boldsymbol{x}) := (1 + \frac{1}{n})\frac{\boldsymbol{f}_n(\boldsymbol{x})}{\|\boldsymbol{f}_n(\boldsymbol{x})\|}.$$

Then $\boldsymbol{h}_n$ is C^∞, $\|\boldsymbol{h}_n(\boldsymbol{x})\| = 1 + \frac{1}{n}$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0},1+\frac{1}{n})}$, and if $\|\boldsymbol{x}\| = 1 + \frac{1}{n}$, then

$$\boldsymbol{h}_n(\boldsymbol{x}) = (1 + \frac{1}{n})\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \boldsymbol{x}.$$

This is a contradiction in view of the previous lemma. ■

We now turn to the proof of Brouwer's fixed point theorem.

Proof of Theorem ??. Step 1: Assume first that $K = \overline{B(\mathbf{0},1)}$ and that $\boldsymbol{f} : \overline{B(\mathbf{0},1)} \to \overline{B(\mathbf{0},1)}$ is continuous. We claim that $\boldsymbol{f}$ has a fixed point. Indeed, if not then $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{x}$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$. For each $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$ let $\boldsymbol{g}(\boldsymbol{x})$ be the point where the ray from $\boldsymbol{f}(\boldsymbol{x})$ to $\boldsymbol{x}$ meets $\partial B(\mathbf{0},1)$. To be precise, we consider the ray

$$\boldsymbol{f}(\boldsymbol{x})+s(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})), \quad s \geq 0,$$

through $\boldsymbol{f}(\boldsymbol{x})$ in the direction $\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x}) \neq \mathbf{0}$ and then find $s \geq 0$ such that

$$\begin{aligned} 1 &= \|\boldsymbol{f}(\boldsymbol{x})+s(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x}))\|^2 \\ &= s^2\|\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})\|^2+2s\boldsymbol{f}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x}))+\|\boldsymbol{f}(\boldsymbol{x})\|^2. \end{aligned}$$

Note if we let $s \in \mathbb{R}$ then the previous equation must have two distinct roots since the line intersects the boundary of the ball in two distinct points. Hence, the discriminant of the quadratic equation must be strictly positive. Solving for s we find

$$\begin{aligned} s(\boldsymbol{x}) := &\frac{-\boldsymbol{f}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x}))}{\|\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})\|^2} \\ &+ \frac{\sqrt{(\boldsymbol{f}(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})))^2+\|\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})\|^2(1-\|\boldsymbol{f}(\boldsymbol{x})\|^2)}}{\|\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})\|^2}. \end{aligned}$$

Since the discriminant is strictly positive the function s is continuous and $s(\boldsymbol{x}) = 1$ if $\|\boldsymbol{x}\| = 1$. Hence, the function $\boldsymbol{g} : \overline{B(\mathbf{0},1)} \to \partial B(\mathbf{0},1)$, defined by

$$\boldsymbol{g}(\boldsymbol{x}) := \boldsymbol{f}(\boldsymbol{x})+s(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{f}(\boldsymbol{x})),$$

is continuous and is the identity on the unit sphere since $s(\boldsymbol{x}) = 1$ if $\|\boldsymbol{x}\| = 1$. This contradicts the previous lemma.

Step 2: Let $K = \overline{B(\mathbf{0},R)}$ for some $R > 0$ and let $\boldsymbol{f} : \overline{B(\mathbf{0},R)} \to \overline{B(\mathbf{0},R)}$ be a continuous transformation. To obtain a fixed point, it suffices to apply the previous step to the rescaled function $\boldsymbol{f}_R(\boldsymbol{x}) := R^{-1}\boldsymbol{f}(R\boldsymbol{x})$, $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$.

Step 3: Let $K \subset \mathbb{R}^N$ be a nonempty compact convex set and let $\boldsymbol{f} : K \to K$ be a continuous transformation. Find $R > 0$ such that $K \subseteq \overline{B(\mathbf{0},R)}$ and for each $\boldsymbol{x} \in \overline{B(\mathbf{0},R)}$ consider the continuous transformation $\boldsymbol{h}(\boldsymbol{x}) := \boldsymbol{f}(\Pi(\boldsymbol{x}))$, $\boldsymbol{x} \in \overline{B(\mathbf{0},R)}$, where $\Pi : \mathbb{R}^N \to K$ is the projection onto the convex set K. Note that $\boldsymbol{h}(K) \subseteq K \subseteq \overline{B(\mathbf{0},R)}$, and so by the continuity of Π we have that $\boldsymbol{h} : \overline{B(\mathbf{0},R)} \to \overline{B(\mathbf{0},R)}$ is continuous. By the previous step there exists $\boldsymbol{x} \in \overline{B(\mathbf{0},R)}$ such that $\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{f}(\Pi(\boldsymbol{x}))$. On the other hand, since $\boldsymbol{h}(K) \subseteq K$, we have that $\boldsymbol{x} \in K$, and so $\Pi(\boldsymbol{x}) = \boldsymbol{x}$. Thus, the previous identity reduces to $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$ and the proof is completed. ■

Remark 34 *If a set $K \subset \mathbb{R}^N$ is homeomorphic to a closed ball, then a continuous function $\boldsymbol{f} : K \to K$ has a fixed point. To see this, let $\boldsymbol{g} : K \to \overline{B(\mathbf{0},1)}$ be a homeomorphism. Then the function $\boldsymbol{g}\circ\boldsymbol{f}\circ\boldsymbol{g}^{-1} : \overline{B(\mathbf{0},1)} \to \overline{B(\mathbf{0},1)}$ has a fixed point $\boldsymbol{x} \in \overline{B(\mathbf{0},1)}$, so that,*

$$\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{g}^{-1}(\boldsymbol{x}))) = \boldsymbol{x}.$$

By applying $\boldsymbol{g}^{-1}$ to both sides we get that $\boldsymbol{y} := \boldsymbol{g}^{-1}(\boldsymbol{x}) \in K$ is a fixed point for $\boldsymbol{f}$.

We present some examples that show the importance of the hypotheses in the Brouwer fixed point theorem.

Example 35 *In* $\mathbb{R}^2$ *consider the annulus*

$$K := \{\boldsymbol{x} \in \mathbb{R}^2 : \varepsilon \leq \|\boldsymbol{x}\| \leq 1\}.$$

This set is compact, path-connected, but not convex. Consider the function given in polar coordinates by

$$\boldsymbol{f}(r, \theta) = (r, \theta + \pi), \quad r \in [\varepsilon, 1], \quad \theta \in [0, 2\pi].$$

It is continuous, maps K *into itself but has no fixed point.*

Example 36 *If* $E = [0, 1)$ *then* $f(x) = (x + 1)/2$ *has no fixed point. The set* E *is bounded, convex, but not closed.*

Example 37 *If* $E = \mathbb{R}$ *then* $f(x) = x + 1$ *has no fixed points. The set* E *is closed, convex, but not bounded.*

In the proof of the Brouwer fixed point theorem we used the following result.

Theorem 38 *Let* $C \subseteq \mathbb{R}^N$ *be a nonempty closed convex set. Then for every* $\boldsymbol{x} \in \mathbb{R}^N$ *there exists a unique point* $\Pi(\boldsymbol{x}) \in C$ *such that*

$$\|\boldsymbol{x} - \Pi(\boldsymbol{x})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\| \quad \text{for all } \boldsymbol{y} \in C. \tag{4}$$

Moreover, the mapping $\Pi : \mathbb{R}^N \to C$ *is Lipschitz continuous with Lipschitz constant less than or equal one, that is*

$$\|\Pi(\boldsymbol{x}_1) - \Pi(\boldsymbol{x}_2)\| \leq \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$$

for all $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in \mathbb{R}^N$.

Proof. Fix $\boldsymbol{x}_0 \in \mathbb{R}^N$. For $r > 0$ sufficiently large, the set $\overline{B(\boldsymbol{x}_0, r)} \cap C$ is compact and nonempty. Hence the continuous function

$$\boldsymbol{x} \in \mathbb{R}^N \mapsto \|\boldsymbol{x}_0 - \boldsymbol{x}\|$$

attains a minimum on this set, say at $\boldsymbol{y}_0 \in \overline{B(\boldsymbol{x}_0, r)} \cap C$. Hence

$$\|\boldsymbol{x}_0 - \boldsymbol{y}_0\| \leq \|\boldsymbol{x}_0 - \boldsymbol{x}\| \quad \text{for all } \boldsymbol{x} \in \overline{B(\boldsymbol{x}_0, r)} \cap C.$$

If $\boldsymbol{x} \in C \setminus \overline{B(\boldsymbol{x}_0, r)}$, then $\|\boldsymbol{x}_0 - \boldsymbol{x}\| > r \geq \|\boldsymbol{x}_0 - \boldsymbol{y}_0\|$, and so we have shown (4).

The remaining of the proof will be in your homework. ■

Friday, January 27, 2023

3 Application I of BrFTT: Invariance of Domain

Theorem 39 *Let $U \subseteq \mathbb{R}^N$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^N$ be continuous and injective. Then $\boldsymbol{f}(U)$ is open.*

Lemma 40 *Let $\boldsymbol{f} : \overline{B(\boldsymbol{x}_0, r)} \to \mathbb{R}^N$ be continuous and injective. Then $\boldsymbol{f}(\boldsymbol{x}_0)$ belongs to the interior of $\boldsymbol{f}(\overline{B(\boldsymbol{x}_0, r)})$.*

Proof. By replacing $\boldsymbol{f}$ with $\boldsymbol{f}_1(\boldsymbol{x}) := \boldsymbol{f}(r\boldsymbol{x} + \boldsymbol{x}_0)$, without loss of generality we may assume that $\boldsymbol{x}_0 = \boldsymbol{0}$ and $r = 1$.

Since $\boldsymbol{f}$ is continuous and $\overline{B(\boldsymbol{0}, 1)}$ is compact, $\boldsymbol{f}^{-1} : \boldsymbol{f}(\overline{B(\boldsymbol{0}, r)}) \to \overline{B(\boldsymbol{0}, 1)}$ is continuous. By your homework, there exists a continuous function $\boldsymbol{g} : \mathbb{R}^N \to \mathbb{R}^N$ which extends $\boldsymbol{f}^{-1}$. Note that $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{0})) = \boldsymbol{0}$. We begin by showing a stability result.

Step 1: Let $\boldsymbol{h} : \boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)}) \to \mathbb{R}^N$ be a continuous function such that

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{h}(\boldsymbol{y})\| \leq 1$$

for all $\boldsymbol{y} \in \boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$. We claim that there exists $\boldsymbol{y} \in \boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$ such that $\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{0}$. To see this, we apply Brouwer's fixed point theorem to the function

$$\boldsymbol{F}(\boldsymbol{x}) := \boldsymbol{x} - \boldsymbol{h}(\boldsymbol{f}(\boldsymbol{x})) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{h}(\boldsymbol{f}(\boldsymbol{x})), \quad \boldsymbol{x} \in \overline{B(\boldsymbol{0}, 1)}.$$

Note that $\boldsymbol{F}$ maps $\overline{B(\boldsymbol{0}, 1)}$ into $\overline{B(\boldsymbol{0}, 1)}$ since

$$\|\boldsymbol{F}(\boldsymbol{x})\| = \|\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{h}(\boldsymbol{f}(\boldsymbol{x}))\| \leq 1 \quad \text{for all } \boldsymbol{x} \in \overline{B(\boldsymbol{0}, 1)}.$$

It follows that there is $\boldsymbol{x}_1 \in \overline{B(\boldsymbol{0}, 1)}$ such that $\boldsymbol{x}_1 = \boldsymbol{F}(\boldsymbol{x}_1) = \boldsymbol{x}_1 - \boldsymbol{h}(\boldsymbol{f}(\boldsymbol{x}_1))$ and so $\boldsymbol{h}(\boldsymbol{f}(\boldsymbol{x}_1)) = \boldsymbol{0}$.

Step 2: Assume by contradiction that $\boldsymbol{f}(\boldsymbol{0})$ does not belong to the interior of $\boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$. We are going to construct a perturbation $\boldsymbol{h}$ of $\boldsymbol{g}$ which has no zeros, thus contradicting Step 1. Since $\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{0})) = \boldsymbol{0}$, by continuity we can find $\delta > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{0}\| = \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{0}))\| \leq \frac{1}{10}$$

for all $\boldsymbol{y} \in \mathbb{R}^N$ with $\|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{0})\| \leq 2\delta$. Since $\boldsymbol{f}(\boldsymbol{0})$ is not an interior point of $\boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$, there exists $\boldsymbol{c} \in \mathbb{R}^N$ with $\|\boldsymbol{f}(\boldsymbol{0}) - \boldsymbol{c}\| < \delta$ such that $\boldsymbol{c}$ does not belong to $\boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$. Note that

$$\|\boldsymbol{g}(\boldsymbol{y})\| \leq \frac{1}{10} \tag{5}$$

for all $\boldsymbol{y} \in \mathbb{R}^N$ with $\|\boldsymbol{y} - \boldsymbol{c}\| \leq \delta$ (since $\|\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{0})\| \leq \|\boldsymbol{y} - \boldsymbol{c}\| + \|\boldsymbol{f}(\boldsymbol{0}) - \boldsymbol{c}\| < 2\delta$). Consider the set $K := K_1 \cup K_2$, where

$$K_1 := \{\boldsymbol{y} \in \boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)}) : \|\boldsymbol{y} - \boldsymbol{c}\| \geq \delta\}, \quad K_2 := \{\boldsymbol{y} \in \mathbb{R}^N : \|\boldsymbol{y} - \boldsymbol{c}\| = \delta\}.$$

Then K is compact and $\boldsymbol{f}(\boldsymbol{0}) \notin K$ since $\|\boldsymbol{f}(\boldsymbol{0}) - \boldsymbol{c}\| < \delta$. Since $\boldsymbol{g} = \boldsymbol{f}^{-1}$ on $\boldsymbol{f}(\overline{B(\boldsymbol{0}, 1)})$, we have that $\boldsymbol{g} \neq \boldsymbol{0}$ in K_1. Since K_1 is compact there exists $0 < 2\eta < \frac{1}{10}$ such that

$$\|\boldsymbol{g}(\boldsymbol{y})\| \geq 2\eta \quad \text{for all } \boldsymbol{y} \in K_1.$$

By mollification we can find $\varepsilon > 0$ so small that

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}_\varepsilon(\boldsymbol{y})\| < \eta \quad \text{for all } \boldsymbol{y} \in K.$$

In particular, $\boldsymbol{g}_\varepsilon$ does not vanish on K_1, since

$$\|\boldsymbol{g}_\varepsilon(\boldsymbol{y})\| \geq \|\boldsymbol{g}(\boldsymbol{y})\| - \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}_\varepsilon(\boldsymbol{y})\| > 2\eta - \eta.$$

However, $\boldsymbol{g}_\varepsilon$ could vanish on K_2. To fix this, observe that K_2 has Lebesgue measure zero. Since $\boldsymbol{g}_\varepsilon$ is smooth, it follows that $\mathcal{L}_o^N(\boldsymbol{g}_\varepsilon(K_2)) \leq C\mathcal{L}_o^N(K_2) = 0$ (exercise) and thus there exists $\boldsymbol{d} \in B(\boldsymbol{0}, \eta) \setminus \boldsymbol{g}_\varepsilon(K_2)$. Consider the function $\boldsymbol{p} := \boldsymbol{g}_\varepsilon - \boldsymbol{d}$. Then $\boldsymbol{p}$ does not vanish on K_2 by construction and

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{p}(\boldsymbol{y})\| \leq \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}_\varepsilon(\boldsymbol{y})\| + \|\boldsymbol{d}\| < \eta + \eta \quad \text{for all } \boldsymbol{y} \in K, \tag{6}$$

so $\|\boldsymbol{p}(\boldsymbol{y})\| \geq \|\boldsymbol{g}(\boldsymbol{y})\| - \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{p}(\boldsymbol{y})\| > 2\eta - 2\eta$. Thus $\boldsymbol{p}$ does not vanish on K. ■

Monday, January 30, 2023

Proof. Consider the function

$$\boldsymbol{q} : \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)}) \to \mathbb{R}^N$$

given by

$$\boldsymbol{q}(\boldsymbol{y}) := \boldsymbol{c} + \max\left\{\frac{\delta}{\|\boldsymbol{y} - \boldsymbol{c}\|}, 1\right\}(\boldsymbol{y} - \boldsymbol{c}). \tag{7}$$

Note that if $\boldsymbol{y} \in \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)})$ and $\|\boldsymbol{y} - \boldsymbol{c}\| \geq \delta$, then $\frac{\delta}{\|\boldsymbol{y}-\boldsymbol{c}\|} \leq 1$, so $\boldsymbol{q}(\boldsymbol{y}) = \boldsymbol{c} + \boldsymbol{y} - \boldsymbol{c} = \boldsymbol{y} \in K_1$, while if $\boldsymbol{y} \in \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)})$ and $\|\boldsymbol{y} - \boldsymbol{c}\| < \delta$, then $\frac{\delta}{\|\boldsymbol{y}-\boldsymbol{c}\|} > 1$, so $\boldsymbol{q}(\boldsymbol{y}) = \boldsymbol{c} + \delta\frac{\boldsymbol{y}-\boldsymbol{c}}{\|\boldsymbol{y}-\boldsymbol{c}\|}$, and

$$\|\boldsymbol{q}(\boldsymbol{y}) - \boldsymbol{c}\| = \delta,$$

so $\boldsymbol{q}(\boldsymbol{y}) \in K_2$. Thus, $\boldsymbol{q} : \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)}) \to K$. Moroever, the function $\boldsymbol{q}$ is continuous, since $\boldsymbol{c} \notin \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)})$. Define $\boldsymbol{h} : \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)}) \to \mathbb{R}^N$ as follows

$$\boldsymbol{h}(\boldsymbol{y}) := \boldsymbol{p}(\boldsymbol{q}(\boldsymbol{y})), \quad \boldsymbol{y} \in \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)}).$$

By construction $\boldsymbol{h}$ does not vanish since $\boldsymbol{p}$ does not vanish on K and $\boldsymbol{q} : \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)}) \to K$.

It remains to show that is close to $\boldsymbol{g}$. If $\boldsymbol{y} \in \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)})$ is such that $\|\boldsymbol{y} - \boldsymbol{c}\| > \delta$, then $\boldsymbol{q}(\boldsymbol{y}) = \boldsymbol{y} \in K_1$ (see (7)) and so by (6),

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{h}(\boldsymbol{y})\| = \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{p}(\boldsymbol{y})\| < 2\eta < \frac{1}{10},$$

while if $\boldsymbol{y} \in \boldsymbol{f}\,(\overline{B(\boldsymbol{0}, 1)})$ is such that $\|\boldsymbol{y} - \boldsymbol{c}\| \leq \delta$, then $\|\boldsymbol{q}(\boldsymbol{y}) - \boldsymbol{c}\| = \delta$ (see (7)) and so by (5) we have that

$$\|\boldsymbol{g}(\boldsymbol{y})\| \leq \frac{1}{10}, \quad \|\boldsymbol{g}(\boldsymbol{q}(\boldsymbol{y}))\| \leq \frac{1}{10}.$$

In turn, by (6),

$$\begin{aligned}\|\boldsymbol{g}(\boldsymbol{y})-\boldsymbol{h}(\boldsymbol{y})\| &\leq \|\boldsymbol{g}(\boldsymbol{y})\|+\|\boldsymbol{g}(\boldsymbol{q}(\boldsymbol{y}))\|+\|\boldsymbol{g}(\boldsymbol{q}(\boldsymbol{y}))-\boldsymbol{h}(\boldsymbol{y})\| \\ &= \|\boldsymbol{g}(\boldsymbol{y})\|+\|\boldsymbol{g}(\boldsymbol{q}(\boldsymbol{y}))\|+\|\boldsymbol{g}(\boldsymbol{q}(\boldsymbol{y}))-\boldsymbol{p}(\boldsymbol{q}(\boldsymbol{y}))\| \\ &\leq \frac{1}{10}+\frac{1}{10}+2\varepsilon<\frac{3}{10}.\end{aligned}$$

Thus we have contradicted Step 1. ■

We now turn to the proof of Theorem 39.

Proof. Let $\boldsymbol{y}_0 \in \boldsymbol{f}(U)$. Then there is $\boldsymbol{x}_0 \in U$ such that $\boldsymbol{f}(\boldsymbol{x}_0)=\boldsymbol{y}_0$. Since U is open, we can find a ball $B(\boldsymbol{x}_0,\delta) \subseteq U$. In turn, $\boldsymbol{f}:\overline{B(\boldsymbol{x}_0,\delta)} \to \mathbb{R}^N$ is continuous and injective. Thus by the previous lemma $\boldsymbol{f}(\boldsymbol{x}_0)=\boldsymbol{y}_0$ belongs to the interior of $\boldsymbol{f}(\overline{B(\boldsymbol{x}_0,\delta)})$, that is, there is $B(\boldsymbol{y}_0,r) \subseteq \boldsymbol{f}(\overline{B(\boldsymbol{x}_0,\delta)}) \subseteq \boldsymbol{f}(U)$. ■

Wednesday, February 1, 2023

Remark 41 *The previous theorem fails if the dimensions of the space and codomain are different. For example the function* $\boldsymbol{f}:(0,1)\to\mathbb{R}^2$*, given by* $\boldsymbol{f}(t)=(t,0)$*, is continuous and injective but the image is not open in* $\mathbb{R}^2$*. It also fails for infinitely dimensional spaces. Indeed if we consider the space* ℓ^∞ *of all bounded sequences, endowed with the sup norm, then the shift function* $f:\ell^\infty\to\ell^\infty$ *given by* $f((x_1,x_n,\ldots)):=(0,x_1,x_2,\ldots)$ *is continuous and injective but the image is not open.*

An important consequence of the previous theorem is the invariance of the domain.

Theorem 42 (Dimension Invariance Theorem) *If* $N>M$ *and* $U\subseteq\mathbb{R}^N$ *is an open set, then there is no continuous injective function* $\boldsymbol{g}:U\to\mathbb{R}^M$*. In particular,* $\mathbb{R}^M$ *and* $\mathbb{R}^N$ *are not homeomorphic.*

Proof. Assume that such function exists. Define $\boldsymbol{\Pi}:\mathbb{R}^M\to\mathbb{R}^N$ as

$$\boldsymbol{\Pi}(\boldsymbol{y})=(y_1,\ldots,y_M,0,\ldots,0)$$

and $\boldsymbol{f}:U\to\mathbb{R}^N$ as $\boldsymbol{f}:=\boldsymbol{\Pi}\circ\boldsymbol{g}$. Then $\boldsymbol{f}$ is continuous and injective. However, $\boldsymbol{f}(U)=\boldsymbol{\Pi}(\boldsymbol{g}(U))\subseteq\boldsymbol{\Pi}(\mathbb{R}^M)$ and $\boldsymbol{\Pi}(\mathbb{R}^M)$ has empty interior. This contradicts the previous theorem. ■

Another important application of Brouwer fixed point theorem is Jordan's curve theorem. Before we discuss it, we need to introduce connected sets.

4 Connectedness and Pathwise Connectedness

Definition 43 *Let* (X,τ) *be a topological space.*

(i) A set $E\subseteq X$ *is* disconnected *if it can be written as union of two disjoint nonempty relatively open sets, that is, if there exist two open sets* $U_1,U_2\subseteq X$ *such that* $E\cap U_1\neq\emptyset$*,* $E\cap U_2\neq\emptyset$*,*

$$E=(E\cap U_1)\cup(E\cap U_2),\quad E\cap U_1\cap U_2=\emptyset.$$

(ii) A set $E \subseteq X$ is connected *if it is not disconnected.*

Next we show that continuous functions preserve connectedness.

Proposition 44 *Consider two topological spaces (X, τ_X) and (Y, τ_Y) and a continuous function $f : E \to Y$, where E is connected. Then $f(E)$ is connected.*

Proof. Assume by contradiction that $f(E)$ is disconnected. Then there exist two open sets $V_1, V_2 \subseteq Y$ such that $f(E) \cap V_1 \neq \emptyset$, $f(E) \cap V_2 \neq \emptyset$,

$$f(E) = (f(E) \cap V_1) \cup (f(E) \cap V_2), \quad (f(E) \cap V_1) \cap (f(E) \cap V_2) = \emptyset.$$

Since f is continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are relatively open in E, that is, there exist open sets U_1 and $U_2 \subseteq X$ such that $f^{-1}(V_1) = E \cap U_1$, $f^{-1}(V_2) = E \cap U_2$. Since $f(E) \cap V_1 \neq \emptyset$, $f(E) \cap V_2 \neq \emptyset$, it follows that $E \cap U_1$ and $E \cap U_2$ are nonempty. If $x \in E$, then $f(x) \in f(E)$ and so either $f(x) \in f(E) \cap V_1$ or $f(E) \cap V_2$, and so either $x \in f^{-1}(V_1) = E \cap U_1$ or $x \in f^{-1}(V_2) = E \cap U_2$. Hence,

$$E = (E \cap U_1) \cup (E \cap U_2).$$

Finally, if there existed $x \in E \cap U_1 \cap U_2$, then $f(x) \in f(E) \cap V_1 \cap V_2 = \emptyset$, which is a contradiction. This shows that E is disconnected, which is a contradiction and completes the proof. ■

We recall that $I \subseteq \mathbb{R}$ is an *interval* if for every $x, y \in I$ with $x \leq y$, we have that $[x, y] \subseteq I$.

Theorem 45 *A set $E \subseteq \mathbb{R}$ is connected if and only if it is an interval.*

Proof. Recitation. ■

We now introduce another notion of connectedness, which is simpler to verify.

Definition 46 *Given a topological space (X, τ), a set $E \subseteq X$ is called* pathwise connected *if for all $x, y \in E$ there exists a continuous function $f : [0, 1] \to E$ such that $f(0) = x$ and $f(1) = y$.*

Proposition 47 *Let (X, τ) be a topological space and let $E \subseteq X$ be pathwise connected. Then E is connected.*

Proof. We claim that E is connected. If not, then there exist two open sets $U_1, U_2 \subseteq X$ such that $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$,

$$E = (E \cap U_1) \cup (E \cap U_2), \quad E \cap U_1 \cap U_2 = \emptyset.$$

Let $x \in E \cap U_1$ and $y \in E \cap U_2$. By hypothesis we can find a continuous function $f : [0, 1] \to E$ such that $f(0) = x$ and $f(1) = y$. By Proposition 44 and Theorem 45, we have that $f([0, 1])$ is connected. On the other hand,

$$f([0, 1]) \subseteq E \subseteq U_1 \cup U_2, \quad x \in f([0, 1]) \cap U_1, \quad y \in f([0, 1]) \cap U_2,$$

which is a contradiction. ■

Proposition 48 *Let (X, τ) be a topological space and let $E \subseteq X$ be a connected set. Then $\overline{E}$ is connected.*

Proof. Recitation ■

The next example shows that in $\mathbb{R}^N$ a connected set may fail to be pathwise connected, unless the set is open.

Example 49 *Let $E \subset \mathbb{R}^2$ be the set given by*

$$E_1 = \left\{(x, y) \in \mathbb{R}^2 : x = 0, -1 \leq y \leq 1\right\},$$
$$E_2 = \left\{(x, y) \in \mathbb{R}^2 : 0 < x \leq \frac{1}{\pi}, y = \sin\frac{1}{x}\right\},$$
$$E = E_1 \cup E_2.$$

The set E is connected but not pathwise connected.

Definition 50 *Given a normed space $(X, \|\cdot\|)$, a* polygonal path *is a continuous curve represented by a continuous function $f : [a, b] \to X$ for which there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ with the property that $f : [t_{i-1}, t_i] \to X$ is affine for all $i = 1, \ldots, n$, that is,*

$$f(t) = c_i + td_i \quad \text{for } t \in [x_{i-1}, x_i],$$

for some $c_i, d_i \in X$.

Theorem 51 *Given a normed space $(X, \|\cdot\|)$, let $O \subseteq X$ be open and connected. Then O is pathwise connected.*

Proof. Recitation. ■

Exercise 52 *Prove that the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.*

Next we show that if a set is not connected, we can decompose it uniquely into a disjoint union of maximal connected subsets.

Proposition 53 *Let (X, τ) be a topological space and let $E \subseteq X$. Assume that*

$$E = \bigcup_{\alpha \in \Lambda} E_\alpha,$$

where each E_α is a connected set. If $\bigcap_{\alpha \in \Lambda} E_\alpha$ is nonempty, then E is connected.

Proof. We claim that E is connected. If not, then there exist two open sets $U_1, U_2 \subseteq X$ such that $E \cap U_1 \neq \emptyset$, $E \cap U_2 \neq \emptyset$,

$$E = (E \cap U_1) \cup (E \cap U_2), \quad E \cap U_1 \cap U_2 = \emptyset.$$

Since each E_α is connected, we must have that either $E_\alpha \subseteq U_1$ or $E_\alpha \subseteq U_2$. On the other hand, if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta$ is nonempty, while $E \cap U_1 \cap U_2 = \emptyset$. Thus, all E_α either belong to U_1 or to U_2. This contradicts the fact that $E \cap U_1 \neq \emptyset$ and that $E \cap U_2 \neq \emptyset$. ■

Let (X, τ) be a topological space and let $E \subseteq X$. For every $x \in E$, let E_x be the union of all the connected subsets of E that contain x. Note that E_x is nonempty, since $\{x\}$ is a connected subset of E. In view of the previous proposition, the set E_x is connected. Moreover, if $x, y \in E$ and $x \neq y$, then either $E_x \cap E_y = \emptyset$ or $E_x = E_y$. Indeed, if not, then again by the previous proposition the set $E_x \cup E_y$ would be connected, contained in E, and would contain x and y, which would contradict the definition of E_x and of E_y. Thus, we can partition E into a disjoint union of maximal connected subsets, called the *connected components* of E.

Friday, February 3, 2023

Proposition 54 *Let (X, τ) be a topological space and let $C \subseteq X$ be a closed set. Then the connected components of C are closed.*

Proof. Let C_α be a connected component of C. Then $C_\alpha \subseteq \overline{C_\alpha} \subseteq \overline{C} = C$. By Proposition 48, $\overline{C_\alpha}$ is connected, and so by the maximality of C_α, $\overline{C_\alpha} = C_\alpha$, i.e., C_α is closed. ■

Proposition 55 *Let $(X, \|\cdot\|)$ be a normed space and let $U \subseteq X$ be an open set. Then the connected components of U are open.*

Proof. Let U_α be a connected component of U. If U_α is not open, then there exists $x \in U_\alpha \cap \partial U_\alpha$. Since U is open, we can find $B(x, r) \subseteq U$. But then $U_\alpha \cup B(x, r)$ is still connected by Proposition 53, which contradicts the maximality of U_α. ■

Example 56 *Consider the metric space $X = \mathbb{Q}$ with the metric induced by the one on the real line. Then $\mathbb{Q}$ is open (since it is the entire space) but the connected components of $\mathbb{Q}$ are singletons (why?) which are not open.*

Proposition 57 *Let (X, d) be a metric space and let $E_1, E_2 \subseteq X$ be two connected sets. If $E_1 \cap \overline{E_2}$ is nonempty, then $E_1 \cup E_2$ is connected.*

Proof. Let $E := E_1 \cup E_2$. If E is disconnected, then there exist two nonempty open sets $U_1, U_2 \subseteq X$ such that

$$E \subseteq U_1 \cup U_2, \quad E \cap U_1 \cap U_2 = \emptyset, \quad E \cap U_1 \neq \emptyset, \quad E \cap U_2 \neq \emptyset.$$

Since E_1 is connected, we must have that either $E_1 \subseteq U_1$ or $E_1 \subseteq U_2$, say, $E_1 \subseteq U_1$. But then, $E_2 \cap U_2 \neq \emptyset$, and since E_2 is connected, it follows that $E_2 \subseteq U_2$. But since there exists $x \in E_1 \cap \overline{E_2}$ and $E_1 \subseteq U_1$, we have that $x \in U_1 \cap \overline{E_2}$, which implies that there exists $y \in E_2$ such that $y \in U_1$, this contradicts the fact that $E \cap U_1 \cap U_2$ is empty. ■

Exercise 58 *Let (X, τ) be a topological space and let $E_1, E_2 \subseteq X$ be two connected sets. If $E_1 \cap \overline{E_2}$ is nonempty, then $E_1 \cup E_2$ is connected.*

5 Application II of BrFTT: Jordan's Curve Theorem

Another important application of Brouwer fixed point theorem is Jordan's curve theorem.

Theorem 59 (Jordan's curve theorem) *Given a continuous closed simple curve* $\boldsymbol{\gamma}$ *in* $\mathbb{R}^2$ *with range* Γ*, the set* $\mathbb{R}^2 \setminus \Gamma$ *consists of two connected components.*

Lemma 60 *Let* $a < b$*,* $c < d$ *and let* $\boldsymbol{f} : [-1,1] \to [a,b] \times [c,d]$ *and* $\boldsymbol{g} : [-1,1] \to [a,b] \times [c,d]$ *be two continuous functions such that* $f_1(-1) = a$*,* $f_1(1) = b$*, and* $g_2(-1) = c$*,* $g_2(1) = d$*. Then there exists* $t_0, s_0 \in [0,1]$ *such that* $\boldsymbol{f}(s_0) = \boldsymbol{g}(t_0)$*.*

Proof. Assume by contradiction that $\boldsymbol{f}(s) \neq \boldsymbol{g}(t)$ for all $s, t \in [-1,1]$. Let $Q := [-1,1] \times [-1,1]$ and consider the continuous function $\boldsymbol{h} : Q \to Q$ defined by

$$\boldsymbol{h}(s,t) := \left(\frac{g_1(t) - f_1(s)}{\|\boldsymbol{f}(s) - \boldsymbol{g}(t)\|_\infty}, \frac{f_2(s) - g_2(t)}{\|\boldsymbol{f}(s) - \boldsymbol{g}(t)\|_\infty} \right),$$

where $\|\boldsymbol{x}\|_\infty = \max\{|x|, |y|\}$. Note that $\boldsymbol{h}(s,t) \in \partial Q$. By the Brouwer fixed point theorem there exists $(s_0, t_0) \in Q$ such that $\boldsymbol{h}(s_0, t_0) = (s_0, t_0)$. But since $\boldsymbol{h}(Q) \subseteq \partial Q$, necessarily, $(s_0, t_0) \in \partial Q$. Hence, $s_0 = \pm 1$ or $t_0 = \pm 1$. If $s_0 = 1$, then

$$1 = \frac{g_1(t_0) - f_1(1)}{\|\boldsymbol{f}(1) - \boldsymbol{g}(t_0)\|_\infty} = \frac{g_1(t_0) - b}{\|\boldsymbol{f}(1) - \boldsymbol{g}(t_0)\|_\infty} \leq 0,$$

while if $s_0 = -1$, then

$$-1 = \frac{g_1(t_0) - f_1(-1)}{\|\boldsymbol{f}(-1) - \boldsymbol{g}(t_0)\|_\infty} = \frac{g_1(t_0) - a}{\|\boldsymbol{f}(-1) - \boldsymbol{g}(t_0)\|_\infty} \geq 0,$$

which give a contradiction.

On the other hand, if $t_0 = 1$, then

$$1 = \frac{f_2(s_0) - g_2(1)}{\|\boldsymbol{f}(s_0) - \boldsymbol{g}(1)\|_\infty} = \frac{f_2(s_0) - d}{\|\boldsymbol{f}(s_0) - \boldsymbol{g}(1)\|_\infty} \leq 0,$$

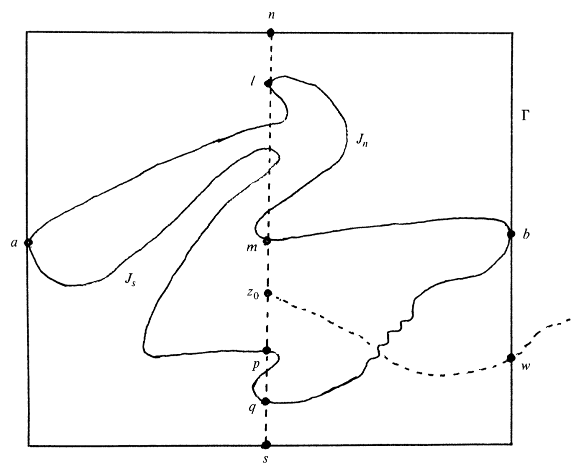
while if $t_0 = -1$, then

$$-1 = \frac{f_2(s_0) - g_2(-1)}{\|\boldsymbol{f}(s_0) - \boldsymbol{g}(-1)\|_\infty} = \frac{f_2(s_0) - c}{\|\boldsymbol{f}(s_0) - \boldsymbol{g}(1)\|_\infty} \geq 0,$$

which give again a contradiction. This completes the proof. ■

Remark 61 *Since* Γ *is bounded, the set* $\mathbb{R}^2 \setminus \Gamma$ *has only one unbounded component. Indeed, let* $\Gamma \subset B(\mathbf{0}, r)$*. The set* $\mathbb{R}^2 \setminus \overline{B(\mathbf{0}, r)}$ *is open and pathwise connected and so it is connected. Moreover,* $\mathbb{R}^2 \setminus \overline{B(\mathbf{0}, r)} \subseteq \mathbb{R}^2 \setminus \Gamma$*, and so there is a connected component that contains* $\mathbb{R}^2 \setminus \overline{B(\mathbf{0}, r)}$*.*

Monday, February 6, 2023



Lemma 62 *Given a continuous closed simple curve $\boldsymbol{\gamma}$ in $\mathbb{R}^2$ with range Γ, if $\mathbb{R}^2 \setminus \Gamma$ is not connected and U is a bounded connected component of $\mathbb{R}^2 \setminus \Gamma$, then $\partial U = \Gamma$.*

Taking for granted Lemma 62, let's prove Jordan's curve theorem.
Proof. Since Γ is compact, there exist $\boldsymbol{a}, \boldsymbol{b} \in \Gamma$ such that $\|\boldsymbol{a} - \boldsymbol{b}\| = \operatorname{diam} \Gamma$. By changing coordinates, without loss of generality we may assume that $\boldsymbol{a} = (1, 0)$ and $\boldsymbol{b} = (-1, 0)$. Then the rectangle $R = [-1, 1] \times [-2, 2]$ contains Γ and has only $\boldsymbol{a}$ and $\boldsymbol{b}$ on its boundary. Let $\boldsymbol{n} = (0, 2)$ and $\boldsymbol{s} = (0, -2)$. By Lemma 60 the segment $\overrightarrow{\boldsymbol{ns}}$ intersects Γ. Let $\boldsymbol{l}$ be the point in $\Gamma \cap \overrightarrow{\boldsymbol{ns}}$ with maximal y-component.

The points $\boldsymbol{a}$ and $\boldsymbol{b}$ divide Γ in two arcs, let Γ_n be the one containing $\boldsymbol{l}$ and let Γ_s be the other one. Let $\boldsymbol{m}$ be the point in $\Gamma_n \cap \overrightarrow{\boldsymbol{ns}}$ with minimal y-component.

Then the segment $\overrightarrow{\boldsymbol{ms}}$ intersects Γ_s, since otherwise, denoting by $\widehat{\boldsymbol{lm}}$ the subarc contained in Γ_n with endpoints $\boldsymbol{l}$ and $\boldsymbol{m}$, the curve given by $\overrightarrow{\boldsymbol{nl}} + \widehat{\boldsymbol{lm}} + \overrightarrow{\boldsymbol{ms}}$ would not intersect the curve Γ_s, contradicting Lemma 60. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the points in $\Gamma_s \cap \overrightarrow{\boldsymbol{ms}}$ be the points with maximal and minimal y-component, respectively. Finally let $\boldsymbol{z}_0$ be the middle point of the segment $\overrightarrow{\boldsymbol{mp}}$. Note that $\boldsymbol{m} \neq \boldsymbol{p}$ since γ is simple. Hence, $\boldsymbol{z}_0$ does not belong to Γ. Let U be the connected component of $\mathbb{R}^2 \setminus \Gamma$ which contains $\boldsymbol{z}_0$. We claim that U is bounded.

Assume by contradiction that U is unbounded. Since U is open and connected, it is path-connected and so we can find a polygonal path in U joining $\boldsymbol{z}_0$ to a point outside R. Let $\boldsymbol{w}$ be the point at which this polygonal path first intersects ∂R and denote with γ_1 the portion of this polygonal arc joining $\boldsymbol{z}_0$ and $\boldsymbol{w}$. If $\boldsymbol{w}$ is on the lower half of R, let $\widehat{\boldsymbol{ws}}$ the subarc contained in ∂R with endpoints $\boldsymbol{w}$ and $\boldsymbol{s}$ and not intersecting $\boldsymbol{a}, \boldsymbol{b}$. Then the curve given by $\overrightarrow{\boldsymbol{nl}} + \widehat{\boldsymbol{lm}} + \overrightarrow{\boldsymbol{mz}_0} + \gamma_1 + \widehat{\boldsymbol{ws}}$ would not intersect the curve Γ_s, contradicting Lemma 60. On the other hand, if $\boldsymbol{w}$ is on the upper half of R, let $\widehat{\boldsymbol{wn}}$ the subarc contained in ∂R with endpoints $\boldsymbol{w}$ and $\boldsymbol{n}$ and not intersecting $\boldsymbol{a}, \boldsymbol{b}$. Then the curve $\overrightarrow{\boldsymbol{s}\boldsymbol{z}_0} + \gamma_1 + \widehat{\boldsymbol{w}\boldsymbol{n}}$ would not intersect the curve Γ_n, contradicting again Lemma 60.

This shows that U is bounded. It remains to show that U is the only bounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Assume by contradiction that there is another one, say, V. Then $V \subset R$, since $\mathbb{R}^2 \setminus R$ is pathwise connected and thus contained in the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Since the segments $\overrightarrow{\boldsymbol{n}\boldsymbol{l}} \setminus \{\boldsymbol{l}\}$ and $\overrightarrow{\boldsymbol{q}\boldsymbol{s}} \setminus \{\boldsymbol{q}\}$ are contained in the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$, it does not intersect V. Similarly, since $\boldsymbol{z}_0 \in U$, the segment $\overrightarrow{\boldsymbol{m}\boldsymbol{p}} \setminus \{\boldsymbol{m}, \boldsymbol{p}\}$ is contained in U, and thus it does not intersect V. It follows that the curve γ_2 given by $\overrightarrow{\boldsymbol{n}\boldsymbol{l}} + \widehat{\boldsymbol{l}\boldsymbol{m}} + \overrightarrow{\boldsymbol{m}\boldsymbol{p}} + \widehat{\boldsymbol{p}\boldsymbol{q}} + \overrightarrow{\boldsymbol{q}\boldsymbol{s}}$, where $\widehat{\boldsymbol{p}\boldsymbol{q}}$ the subarc contained in Γ_s with endpoints $\boldsymbol{p}$ and $\boldsymbol{q}$, does not intersect V. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not in γ_2, there are balls $B(\boldsymbol{a}, r)$ and $B(\boldsymbol{b}, r)$ which do not intersect γ_2. By Lemma 62, $\partial V = \Gamma$ and so $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to ∂V. Hence, there exist $\boldsymbol{a}_1 \in V \cap B(\boldsymbol{a}, r)$ and $\boldsymbol{b}_1 \in V \cap B(\boldsymbol{b}, r)$. Let $\widehat{\boldsymbol{a}_1\boldsymbol{b}_1}$ be a polygonal path in V joining $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$. Then the curve $\overrightarrow{\boldsymbol{a}\boldsymbol{a}_1} + \widehat{\boldsymbol{a}_1\boldsymbol{b}_1} + \overrightarrow{\boldsymbol{b}_1\boldsymbol{b}}$ does not intersect γ_2, contradicting again Lemma 60. This concludes the proof. ■

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We prove Lemma 62.

Proof of Lemma 62. Step 1: We claim that $\partial U \subseteq \Gamma$. Indeed, if not then there would exist $\boldsymbol{x}_0 \in \partial U \cap (\mathbb{R}^2 \setminus \Gamma)$. Since U is open, $\boldsymbol{x}_0$ does not belong to U, so there is another connected component V of $\mathbb{R}^2 \setminus \Gamma$ with $\boldsymbol{x}_0 \in V$. But then $\overline{U} \cap V \neq \emptyset$ and so by Proposition 57, $U \cup V$ is connected, which contradicts the maximality of U. Thus $\partial U \subseteq \Gamma$.

Step 2: Assume by contradiction that $\partial U \subset \Gamma$. Let $\boldsymbol{x}_0 \in \Gamma \setminus \partial U$. Since $\mathbb{R}^2 \setminus \overline{U}$ is open, there exists $B(\boldsymbol{x}_0, \delta) \subset \mathbb{R}^2 \setminus \overline{U}$. Consider a simple arc whose range $C \subseteq \Gamma$ is closed and contains ∂U but does not intersect $B(\boldsymbol{x}_0, \delta)$. Let V be the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Take $\boldsymbol{p} \in U$. Let $r > 0$ be so large that $B(\boldsymbol{p}, r) \supset \Gamma$. Then $\partial B(\boldsymbol{p}, r)$ is contained in V. Define $\boldsymbol{f}(\boldsymbol{x}) := \boldsymbol{x}$ for all $\boldsymbol{x} \in C$. Since C is homeomorphic to $[0, 1]$, by Tietze's extension theorem we can extend $\boldsymbol{f}$ to a continuous function $\boldsymbol{f} : \mathbb{R}^2 \to C$. Define

$$\boldsymbol{g}(\boldsymbol{x}) := \begin{cases} \boldsymbol{f}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \overline{U}, \\ \boldsymbol{x} & \text{if } \boldsymbol{x} \in \mathbb{R}^2 \setminus U, \end{cases}$$

Then $\boldsymbol{g} : \overline{B(\boldsymbol{p}, r)} \to \overline{B(\boldsymbol{p}, r)} \setminus \{\boldsymbol{p}\}$. Indeed, $\boldsymbol{p}$ belongs to U and $\boldsymbol{g}$ is mapped into $C \subseteq \Gamma$. Moreover, since $\partial U \subseteq C$ and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in C$, we have that $\boldsymbol{g}$ is continuous. Note that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial B(\boldsymbol{p}, r)$. It follows that the map

$$\boldsymbol{h}(\boldsymbol{x}) = \frac{\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{p}}{\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{p}\|} r + \boldsymbol{p}$$

maps $\overline{B(\boldsymbol{p}, r)}$ into $\partial B(\boldsymbol{p}, r)$ and is the identity on $\partial B(\boldsymbol{p}, r)$. However, this contradicts Lemma 33. ■

6 Banach Fixed Point Theorem

Definition 63 *Given a metric space* (X, d) *and a sequence* $\{x_n\}_n$ *in* X*, we say that* $\{x_n\}_n$ *is a* Cauchy sequence *if for every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \mathbb{N}$ *such that*

$$d(x_n, x_m) < \varepsilon$$

for all $n, m \geq N_\varepsilon$*,*

Proposition 64 *Given a metric space* (X, d) *and a sequence* $\{x_n\}_n$ *in* X*, if* $\{x_n\}_n$ *converges to some* $x \in X$*, then* $\{x_n\}_n$ *is a Cauchy sequence.*

Proof. Since $\{x_n\}$ converges to $x \in X$, given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$ in the definition of convergence. Then there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x) < \frac{\varepsilon}{2}$$

for all $n \geq N_\varepsilon$. Hence, by the triangle inequality and symmetry of d, if $n, m \geq N_\varepsilon$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

■

The opposite is not true, that is, there are Cauchy sequences that do not have a limit.

Example 65 *Consider* $X = (0, 1)$ *with the metric* $d(x, y) = |x - y|$ *and consider the sequence* $x_n = \frac{1}{n}$*. Then* $x_n \to 0$ *which does not belong to* $X = (0, 1)$*, but* $\{x_n\}_n$ *is a Cauchy (just applied the previous proposition in the metric space* $\mathbb{R}$*).*

Exercise 66 *Let* $\{x_n\}_n$ *be a sequence in a metric space* (X, d)*.*

(i) Prove that if $\{x_n\}_n$ *is a Cauchy sequence and if a subsequence* $\{x_{n_k}\}_k$ *of* $\{x_n\}_n$ *converges to some* $x \in X$*, then* $\{x_n\}_n$ *converges to* x*.*

(ii) Prove that if there exists $x \in X$ *such that for* every *subsequence* $\{x_{n_k}\}_k$ *of* $\{x_n\}_n$ *there exists a further subsequence* $\{x_{n_{k_j}}\}_j$ *that converges to* x*, then* $\{x_n\}_n$ *converges to* x*.*

Definition 67 *A metric space* (X, d) *is said to be* complete *if every Cauchy sequence has a limit in* X*.*

Example 68 *Let* $X = (0, 1)$ *with the metric* $d(x, y) = |x - y|$*. The sequence* $\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}}$ *converges to* 0 *in* $\mathbb{R}$*, and so it is a Cauchy sequence in* $\mathbb{R}$*. In particular, it is a Cauchy sequence in* X*. However, it does not converge to an element of* X*, since* $0 \notin X$*.*

Theorem 69 $\mathbb{R}^N$ *is a complete metric space.*

Proof. Step 1: Let $\{\boldsymbol{x}_n\}_n$ be a Cauchy sequence. We claim that $\{\boldsymbol{x}_n\}_n$ in $\mathbb{R}^N$ is bounded. Fix $\varepsilon = 1$. By the definition of Cauchy sequence, there is exists $N_1 \in \mathbb{N}$ such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| < 1$$

for all $n, m \geq N_1$. In particular, taking $m = N_1$, we have that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_{N_1}\| < 1,$$

for all $n \geq N_1$. Taking

$$R := \max\{1, \|\boldsymbol{x}_1 - \boldsymbol{x}_{N_1}\| + 1, \ldots, \|\boldsymbol{x}_{N_1-1} - \boldsymbol{x}_{N_1}\| + 1\},$$

we have that $\boldsymbol{x}_n \in B(\boldsymbol{x}_{N_1}, R)$ for all $n \in \mathbb{N}$.

Step 2: We claim that $\{\boldsymbol{x}_n\}_n$ in $\mathbb{R}^N$ admits a convergent subsequence. Consider the set $E := \{\boldsymbol{x}_n : n \in \mathbb{N}\}$.

Case 1: There exists $\boldsymbol{\ell} \in \mathbb{R}^N$ such that $\boldsymbol{x}_n = \boldsymbol{\ell}$ for infinitely many n. In this case we can find a subsequence $\{\boldsymbol{x}_{n_k}\}_k$ such that $\boldsymbol{x}_{n_k} = \boldsymbol{\ell}$ for all $k \in \mathbb{N}$ and so $\boldsymbol{x}_{n_k} = \boldsymbol{\ell} \to \boldsymbol{\ell}$ as $k \to \infty$. ■

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Proof. Case 2: Since no element of E is repeated infinitely many times, the set E has infinitely many distinct elements. Since E is bounded and has infinitely many distinct elements, the set E has an accumulation point $\boldsymbol{\ell} \in \mathbb{R}^N$.

Take $\varepsilon_1 = 1$. Since $\boldsymbol{\ell}$ is an accumulation point of E there exists $\boldsymbol{x}_{n_1} \in B(\boldsymbol{\ell}, \varepsilon_1)$ with $\boldsymbol{x}_{n_1} \neq \boldsymbol{\ell}$. Let

$$\varepsilon_2 := \min\left\{\frac{1}{2}, \min\{\|\boldsymbol{x}_n - \boldsymbol{\ell}\| : n = 1, \ldots, n_1, \ \boldsymbol{x}_n \neq \boldsymbol{\ell}\}\right\} > 0.$$

Since $\boldsymbol{\ell}$ is an accumulation point of E there exists $\boldsymbol{x}_{n_2} \in B(\boldsymbol{\ell}, \varepsilon_2)$ with $\boldsymbol{x}_{n_2} \neq \boldsymbol{\ell}$. It follows from the definition of ε_2 that necessarily $n_2 > n_1$.

Inductively, assume that $n_1 < n_2 < \cdots < n_{k-1}$ have been chosen so that $0 < \|\boldsymbol{x}_{n_i} - \boldsymbol{\ell}\| \leq \frac{1}{i-1}$ for all $i = 1, \ldots, k-1$. Let

$$\varepsilon_k := \min\left\{\frac{1}{k}, \min\{\|\boldsymbol{x}_n - \boldsymbol{\ell}\| : n = 1, \ldots, n_{k-1}, \ \boldsymbol{x}_n \neq \boldsymbol{\ell}\}\right\} > 0.$$

Since $\boldsymbol{\ell}$ is an accumulation point of E there exists $\boldsymbol{x}_{n_k} \in B(\boldsymbol{\ell}, \varepsilon_k)$ with $\boldsymbol{x}_{n_k} \neq \boldsymbol{\ell}$. It follows from the definition of ε_k that necessarily $n_k > n_{k-1}$.

Thus, by induction we have constructed a subsequence $\{\boldsymbol{x}_{n_k}\}_k$ such that

$$0 < \|\boldsymbol{x}_{n_k} - \boldsymbol{\ell}\| \leq \frac{1}{k} \quad \text{for all } k \in \mathbb{N}.$$

By the squeeze theorem $\boldsymbol{x}_{n_k} \to \boldsymbol{\ell}$ as $k \to \infty$.

Step 3: We claim that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{\ell}$. Let $\varepsilon > 0$. Since $\{\boldsymbol{x}_n\}_n$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. On the other hand, since $x_{n_k} \to \ell$ as $k \to \infty$, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\|\boldsymbol{x}_{n_k} - \boldsymbol{\ell}\| \leq \varepsilon$$

for all $k \geq k_\varepsilon$. Let $k_* \geq k_\varepsilon$ be so large that $n_{k_*} \geq n_\varepsilon$. Then for all $n \geq n_{k_*}$, we have

$$\|\boldsymbol{x}_n - \boldsymbol{\ell}\| \leq \|\boldsymbol{x}_n - \boldsymbol{x}_{n_k}\| + \|x_{n_{k_*}} - \boldsymbol{\ell}\| \leq \varepsilon + \varepsilon.$$

This implies that $\{\boldsymbol{x}_n\}_n$ converges to $\boldsymbol{\ell}$. ■

Definition 70 *A normed space* $(X, \|\cdot\|)$ *is a* Banach space *if it is a complete metric space.*

Theorem 71 *Given a nonempty set* X*, consider the space*

$$\ell^\infty(X) := \{f : X \to \mathbb{R} : f \text{ is bounded}\}$$

with the norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Then $\ell^\infty(X)$ *is a Banach space.*

Proof. Let $\{f_n\}_n$ in $\ell^\infty(X)$ be a Cauchy sequence. Let $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ so large that

$$\sup_{x \in X} |f_n(x) - f_m(x)| = d_\infty(f_n, f_m) \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. This implies that for every fixed $x \in E$, the sequence of real numbers $\{f_n(x)\}_n$ is a Cauchy sequence in $\mathbb{R}$ and so there exists

$$\lim_{n \to \infty} f_n(x) = f(x) \in \mathbb{R}.$$

Since

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$, letting $n \to \infty$ gives (why?)

$$|f(x) - f_m(x)| \leq \varepsilon$$

for all $m \geq n_\varepsilon$. This holds for every $x \in X$. Hence, taking the supremum over all $x \in \ell^\infty(X)$ gives

$$\sup_{x \in X} |f(x) - f_m(x)| = d_\infty(f, f_m) \leq \varepsilon$$

for all $m \geq n_\varepsilon$; that is $d_\infty(f, f_m) \to 0$ as $m \to \infty$. Moreover, for every $x \in X$,

$$|f(x)| \leq |f(x) + f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x)| \leq \sup_{x \in X} |f(x) - f_{n_\varepsilon}(x)| + \sup_{x \in X} |f_{n_\varepsilon}(x)| \leq \varepsilon + \sup_{x \in X} |f_{n_\varepsilon}(x)|,$$

which implies that $f \in \ell^\infty(X)$. ■

The next theorem shows that uniform convergence preserves continuity.

Corollary 72 *Let (X, d) be a metric space and let*

$$C_b(X) := \{f : X \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

with then norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)| .$$

Then $C_b(X)$ is a Banach space.

Proof. Let $\{f_n\}_n$ be a Cauchy sequence in $C_b(X)$. By the previous theorem there exists a bounded function $f \in \ell^\infty(X)$ such that $f_n \to f$ in $\ell^\infty(X)$. Let's prove that f is continuous. Let $x_0 \in X$ and let $\varepsilon > 0$. Since $f_n \to f$ in $\ell^\infty(X)$, there exists $n_\varepsilon \in \mathbb{N}$ so large that

$$\sup_{x \in X} |f_n(x) - f(x)| \le \varepsilon$$

for all $n \ge n_\varepsilon$. Since the function f_{n_ε} is continuous at x_0, there exists $\delta > 0$ such that

$$|f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)| \le \varepsilon$$

for all $x \in X$ with $d(x, x_0) \le \delta$. In turn,

$$|f(x) - f(x_0)| \le |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)| + |f_{n_\varepsilon}(x_0) - f(x_0)| \le 3\varepsilon$$

for all $x \in X$ with $d(x, x_0) \le \delta$. This completes the proof. ■

Theorem 73 (Banach's contraction principle) *Let (X, d) be a nonempty complete metric space and let $f : X \to X$ be a contraction, that is f is Lipschitz with Lipschitz constant less than one. Then f has a unique fixed point; that is, there is a unique $x \in X$ such that $f(x) = x$.*

Proof. Step 1: Let's first prove uniqueness. Assume that x_1 and x_2 are fixed points of f. Then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \le L d(x_1, x_2),$$

which implies that

$$(1 - L) d(x_1, x_2) \le 0.$$

Since $L < 1$, we have that $d(x_1, x_2) = 0$, and so $x_1 = x_2$.
Step 2: To prove existence, fix $x_0 \in X$ and define inductively

$$x_1 := f(x_0), \quad x_{n+1} := f(x_n).$$

We claim that $\{x_n\}_n$ is a Cauchy sequence. Indeed, note that

$$d(x_1, x_2) = d(f(x_0), f(x_1)) \le L d(x_0, x_1)$$

and by induction

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \le L^n d(x_0, x_1).$$

Hence, for every $m, n \in \mathbb{N}$, by the triangle inequality

$$\begin{aligned} d\left(x_n, x_{n+m}\right) &\leq \sum_{i=n}^{n+m-1} d\left(x_i, x_{i+1}\right) \leq d\left(x_0, x_1\right) \sum_{i=n}^{n+m-1} L^i \\ &\leq d\left(x_0, x_1\right) \sum_{i=n}^{\infty} L^i = d\left(x_0, x_1\right) \frac{L^n}{1-L}. \end{aligned}$$

Letting $n \to \infty$, we have that $\{x_n\}_n$ is a Cauchy sequence. Since the space is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. But by the continuity of f,

$$x \leftarrow x_{n+1} = f\left(x_n\right) \to f\left(x\right),$$

which shows that $f(x) = x$. ■

Example 74 *The function $f(x) = x + 1$, $x \in \mathbb{R}$, has Lipschitz constant 1 but no fixed points.*

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7 Application I of BaFTT: The Inverse Function Theorem

Theorem 75 (Inverse Function) *Let $U \subseteq \mathbb{R}^N$ be open, let $\boldsymbol{f} : U \to \mathbb{R}^N$, and let $\boldsymbol{x}_0 \in U$. Assume that there exists $B\left(\boldsymbol{x}_0, r_0\right) \subseteq U$ such that $\boldsymbol{f}$ is continuous in $B\left(\boldsymbol{x}_0, r_0\right)$ and that for all $\boldsymbol{x} \in B\left(\boldsymbol{x}_0, r_0\right)$ there exist $\frac{\partial f_j}{\partial x_i}$, $i, j = 1, \ldots, N$, and that they are continuous at $\boldsymbol{x}_0$. If*

$$\det J_{\boldsymbol{f}}\left(\boldsymbol{x}_0\right) \neq 0,$$

then there exists $0 < r_1 < r_0$ such that the function

$$\boldsymbol{f} : B\left(\boldsymbol{x}_0, r_1\right) \to \boldsymbol{f}\left(B\left(\boldsymbol{x}_0, r_1\right)\right)$$

is invertible, $\boldsymbol{f}\left(B\left(\boldsymbol{x}_0, r_1\right)\right)$ is open, and $\boldsymbol{f}^{-1} : \boldsymbol{f}\left(B\left(\boldsymbol{x}_0, r_1\right)\right) \to B\left(\boldsymbol{x}_0, r_1\right)$ is Lipschitz continuous in $\boldsymbol{f}\left(B\left(\boldsymbol{x}_0, r_1\right)\right)$ and differentiable at $\boldsymbol{f}\left(\boldsymbol{x}_0\right)$, with

$$J_{\boldsymbol{f}^{-1}}\left(\boldsymbol{f}\left(\boldsymbol{x}_0\right)\right) = \left(J_{\boldsymbol{f}}\left(\boldsymbol{x}_0\right)\right)^{-1}.$$

Proof. Step 1: Assume that $\boldsymbol{x}_0$=$\mathbf{0}$, that $f(\mathbf{0})$ =0, and that $J_{\boldsymbol{f}}\left(\mathbf{0}\right) = I_N$, the identity matrix. Write

$$\boldsymbol{f}\left(\boldsymbol{x}\right) = \boldsymbol{x} + \boldsymbol{h}\left(\boldsymbol{x}\right).$$

Then $h(\mathbf{0})$ =0 and $J_{\boldsymbol{h}}\left(\mathbf{0}\right) = 0_N$. Since $\frac{\partial h_j}{\partial x_i}$ are continuous at $\mathbf{0}$, there exists $r > 0$ such that $\overline{B\left(\mathbf{0}, r\right)} \subset U$ and

$$\|\nabla h_j(\boldsymbol{x})\| \leq \frac{1}{2\sqrt{N}} \quad \text{for all } \boldsymbol{x} \in \overline{B\left(\mathbf{0}, r\right)}.$$

By the mean value theorem for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \overline{B(\mathbf{0}, r)}$ and $j = 1, \ldots, N$,

$$|h_j(\boldsymbol{x}_1) - h_j(\boldsymbol{x}_2)| \leq \|\nabla h_j(\boldsymbol{z}_{i,j})\| \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq \frac{1}{2\sqrt{N}} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$

where $\boldsymbol{z}_{i,j} = \theta_{i,j}\boldsymbol{x}_1 + (1 - \theta_{i,j})\boldsymbol{x}_2$, and so

$$\|\boldsymbol{h}(\boldsymbol{x}_1) - \boldsymbol{h}(\boldsymbol{x}_2)\| = \sqrt{\sum_{j=1}^{N} |h_j(\boldsymbol{x}_1) - h_j(\boldsymbol{x}_2)|^2} \leq \frac{1}{2} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|.$$

This shows that $\boldsymbol{h} : \overline{B(\mathbf{0}, r)} \to \mathbb{R}^N$ is a contraction. Moreover, since $\boldsymbol{h}(\mathbf{0}) = \mathbf{0}$, $\|\boldsymbol{h}(\boldsymbol{x})\| \leq \frac{1}{2}\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \overline{B(\mathbf{0}, r)}$.

Fix $\boldsymbol{y} \in \overline{B\left(\mathbf{0}, \frac{1}{2}r\right)}$ and consider the function

$$\boldsymbol{h}_{\boldsymbol{y}}(\boldsymbol{x}) := \boldsymbol{y} - \boldsymbol{h}(\boldsymbol{x}).$$

Then $\boldsymbol{h}_{\boldsymbol{y}}$ is a contraction, and for all $\boldsymbol{x} \in \overline{B(\mathbf{0}, r)}$,

$$\|\boldsymbol{h}_{\boldsymbol{y}}(\boldsymbol{x})\| \leq \|\boldsymbol{h}(\boldsymbol{x})\| + \|\boldsymbol{y}\| \leq \frac{1}{2}\|\boldsymbol{x}\| + \|\boldsymbol{y}\| \leq \frac{1}{2}r + \frac{1}{2}r.$$

Thus, we can apply the Banach fixed point theorem to $\boldsymbol{h}_{\boldsymbol{y}} : \overline{B(\mathbf{0}, r)} \to \overline{B(\mathbf{0}, r)}$ to conclude that $\boldsymbol{h}_{\boldsymbol{y}}$ has a unique fixed point $\boldsymbol{x} \in \overline{B(\mathbf{0}, r)}$, that is,

$$\boldsymbol{y} - \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{h}_{\boldsymbol{y}}(\boldsymbol{x}) = \boldsymbol{x}.$$

In turn,

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{y}.$$

Hence, we proved that for every $\boldsymbol{y} \in \overline{B\left(\mathbf{0}, \frac{1}{2}r\right)}$ there exists a unique $\boldsymbol{x} \in \overline{B(\mathbf{0}, r)}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$. This means that we can define $\boldsymbol{f}^{-1} : \overline{B\left(\mathbf{0}, \frac{1}{2}r\right)} \to \overline{B(\mathbf{0}, r)}$.

To prove that $\boldsymbol{f}^{-1}$ is continuous, let $\boldsymbol{y}_1, \boldsymbol{y}_1 \in \overline{B\left(\mathbf{0}, \frac{1}{2}r\right)}$ and define $\boldsymbol{x}_1 := \boldsymbol{f}^{-1}(\boldsymbol{y}_1)$ and $\boldsymbol{x}_2 := \boldsymbol{f}^{-1}(\boldsymbol{y}_1)$. Then

$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{h}(\boldsymbol{x}_1) &= \boldsymbol{y}_1, \\ \boldsymbol{x}_2 + \boldsymbol{h}(\boldsymbol{x}_2) &= \boldsymbol{y}_2, \end{aligned}$$

and since $\|\boldsymbol{h}(\boldsymbol{x}_1) - \boldsymbol{h}(\boldsymbol{x}_2)\| \leq \frac{1}{2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$, we have

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| + \|\boldsymbol{h}(\boldsymbol{x}_1) - \boldsymbol{h}(\boldsymbol{x}_2)\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| + \frac{1}{2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|,$$

and so

$$\frac{1}{2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \leq \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|,$$

which shows that

$$\left\|\boldsymbol{f}^{-1}(\boldsymbol{y}_1) - \boldsymbol{f}^{-1}(\boldsymbol{y}_2)\right\| \leq 2\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|.$$

Thus, $\boldsymbol{f}^{-1}$ is Lipschitz continuous in $\overline{B\left(\mathbf{0}, \frac{1}{2}r\right)}$ with Lipschitz constant at most 2. In particular, since $f^{-1}(\mathbf{0})$ =0, we have that $\boldsymbol{f}^{-1}$ maps the open ball $B\left(\mathbf{0}, \frac{1}{2}r\right)$ into a subset of $B(\mathbf{0}, r)$. To prove that $\boldsymbol{f}^{-1}$ is differentiable at $\mathbf{0}$, let $\boldsymbol{y} \in \overline{B\left(\mathbf{0}, \frac{1}{2}r\right)}$ and define $\boldsymbol{x} := \boldsymbol{f}^{-1}(\boldsymbol{y})$. Then

$$\boldsymbol{x}+\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{y},$$

and so

$$\boldsymbol{f}^{-1}(\boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{h}\left(\boldsymbol{f}^{-1}(\boldsymbol{y})\right). \tag{8}$$

Since $\boldsymbol{h}$ is differentiable at $\mathbf{0}$, $h(\mathbf{0})$ =0 and $J_{\boldsymbol{h}}(\mathbf{0}) = 0_N$, we have that

$$\mathbf{0} = \lim_{\boldsymbol{x}\to\mathbf{0}} \frac{\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{h}(\mathbf{0}) - J_{\boldsymbol{h}}(\mathbf{0})(\boldsymbol{x}-\mathbf{0})}{\|\boldsymbol{x}-\mathbf{0}\|} = \lim_{\boldsymbol{x}\to\mathbf{0}} \frac{\boldsymbol{h}(\boldsymbol{x})}{\|\boldsymbol{x}\|}.$$

Thus given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|\boldsymbol{h}(\boldsymbol{x})\|}{\|\boldsymbol{x}\|} \le \varepsilon$$

for all $\boldsymbol{x}$ with $0 < \|\boldsymbol{x}\| \le \delta$. But since $\boldsymbol{f}^{-1}$ is Lipschitz continuous, it follows that $0 < \left\|\boldsymbol{f}^{-1}(\boldsymbol{y})\right\| \le 2\|\boldsymbol{y}\| \le \delta$ for all $0 < \|\boldsymbol{y}\| \le \delta/2$. Hence,

$$\frac{\left\|\boldsymbol{h}(\boldsymbol{f}^{-1}(\boldsymbol{y}))\right\|}{\|\boldsymbol{y}\|} = \frac{\left\|\boldsymbol{h}(\boldsymbol{f}^{-1}(\boldsymbol{y}))\right\|}{\left\|\boldsymbol{f}^{-1}(\boldsymbol{y})\right\|} \frac{\left\|\boldsymbol{f}^{-1}(\boldsymbol{y})\right\|}{\|\boldsymbol{y}\|} \le 2\varepsilon$$

for all $0 < \|\boldsymbol{y}\| \le \delta/2$. In turn, by (8),

$$\begin{aligned} &\frac{\left\|\boldsymbol{f}^{-1}(\boldsymbol{y}) - \boldsymbol{f}^{-1}(\mathbf{0}) - I_N(\boldsymbol{y}-\mathbf{0})\right\|}{\|\boldsymbol{y}-\mathbf{0}\|} \\ &= \frac{\left\|\boldsymbol{f}^{-1}(\boldsymbol{y}) - \mathbf{0} - \boldsymbol{y}\right\|}{\|\boldsymbol{y}-\mathbf{0}\|} = \frac{\left\|\boldsymbol{y} - \boldsymbol{h}(\boldsymbol{f}^{-1}(\boldsymbol{y})) - \boldsymbol{y}\right\|}{\|\boldsymbol{y}-\mathbf{0}\|} \\ &= \frac{\left\|\boldsymbol{h}(\boldsymbol{f}^{-1}(\boldsymbol{y}))\right\|}{\|\boldsymbol{y}\|} \le 2\varepsilon, \end{aligned}$$

which implies that $\boldsymbol{f}^{-1}$ is differentiable at $\mathbf{0}$ with $J_{\boldsymbol{f}^{-1}}(\mathbf{0}) = I_N$. ■

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Proof. Next we claim that $\boldsymbol{f}(B(\mathbf{0}, r/3))$ is open. Take $\boldsymbol{y}_1 \in \boldsymbol{f}(B(\mathbf{0}, r/3))$. Then there is $\boldsymbol{x}_1 \in B(\mathbf{0}, r/3)$ such that $\boldsymbol{y}_1 = \boldsymbol{f}(\boldsymbol{x}_1)$. Since $\boldsymbol{f}$ is Lipschitz continuous with Lipschitz constant $\frac{3}{2}$ and $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ we have that $\|\boldsymbol{y}_1\| = \|\boldsymbol{f}(\boldsymbol{x}_1)\| = \|\boldsymbol{f}(\boldsymbol{x}_1) - \boldsymbol{f}(\mathbf{0})\| \le \frac{3}{2}\|\boldsymbol{x}_1\| < \frac{1}{2}r$ and so $\boldsymbol{y}_1 \in B(\mathbf{0}, r/2)$ and $\boldsymbol{f}^{-1}$ is defined at $\boldsymbol{y}_1$. Since $\boldsymbol{f}^{-1}$ is Lipschitz continuous with Lipschitz constant 2, we have that

$$\left\|\boldsymbol{f}^{-1}(\boldsymbol{y}) - \boldsymbol{f}^{-1}(\boldsymbol{y}_1)\right\| \le 2\|\boldsymbol{y}-\boldsymbol{y}_1\| < \frac{r}{3} - \left\|\boldsymbol{f}^{-1}(\boldsymbol{y}_1)\right\|,$$

provided $\|\boldsymbol{y}-\boldsymbol{y}_1\| < \min\left\{\frac{r}{6}-\frac{1}{2}\left\|\boldsymbol{f}^{-1}(\boldsymbol{y}_1)\right\|, \frac{1}{2}r\right\} =: \delta$. In particular,

$$\left\|\boldsymbol{f}^{-1}(\boldsymbol{y})\right\| \leq \left\|\boldsymbol{f}^{-1}(\boldsymbol{y})-\boldsymbol{f}^{-1}(\boldsymbol{y}_1)\right\| + \left\|\boldsymbol{f}^{-1}(\boldsymbol{y}_1)\right\| < \frac{r}{3}$$

for all $\boldsymbol{y} \in B(\boldsymbol{y}_1, \delta)$. This implies $B(\boldsymbol{y}_1, \delta) \subseteq \boldsymbol{f}(B(\mathbf{0}, r/3))$, and so $\boldsymbol{f}(B(\mathbf{0}, r/3))$ is open. Hence we take $r_1 := \frac{r}{3}$.
Step 2: In the general case, since $\det J_{\boldsymbol{f}}(\boldsymbol{x}_0) \neq 0$, we have that the matrix $J_{\boldsymbol{f}}(\mathbf{0})$ is invertible. Hence, we can apply Step 1 to the function $g(\boldsymbol{x}) = (J_{\boldsymbol{f}}(\boldsymbol{x}_0))^{-1}(\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{x}_0)-\boldsymbol{f}(\boldsymbol{x}_0))$. We obtain that $g : \boldsymbol{g}^{-1}\left(B\left(\mathbf{0}, \frac{1}{2}r\right)\right) \to B\left(\mathbf{0}, \frac{1}{2}r\right)$ is a homeomorphism. Consider the invertible linear mapping $\boldsymbol{T} : \mathbb{R}^N \to \mathbb{R}^N$ given by $\boldsymbol{T}(\boldsymbol{y}) = (J_{\boldsymbol{f}}(\boldsymbol{x}_0))(\boldsymbol{y}+\boldsymbol{f}(\boldsymbol{x}_0))$. Since $\boldsymbol{T}$ is invertible, it follows that $\boldsymbol{T}\circ\boldsymbol{g} : \boldsymbol{g}^{-1}\left(B\left(\mathbf{0}, \frac{1}{2}r\right)\right) \to \boldsymbol{T}\left(B\left(\mathbf{0}, \frac{1}{2}r\right)\right)$ is a homeomorphism, but

$$(\boldsymbol{T}\circ\boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{x}_0),$$

so $\boldsymbol{f}$ is locally invertible. ■

Corollary 76 *Let $U \subseteq \mathbb{R}^N$ be open, let $\boldsymbol{f} : U \to \mathbb{R}^N$, and let $\boldsymbol{x}_0 \in U$. Assume that $\boldsymbol{f} \in C^m(U)$ for some $m \in \mathbb{N}$ and that*

$$\det J_{\boldsymbol{f}}(\boldsymbol{x}_0) \neq 0.$$

Then there exists $B(\boldsymbol{x}_0, r) \subseteq U$ such that $\boldsymbol{f}(B(\boldsymbol{x}_0, r))$ is open, the function

$$\boldsymbol{f} : B(\boldsymbol{x}_0, r) \to \boldsymbol{f}(B(\boldsymbol{x}_0, r))$$

is invertible and $\boldsymbol{f}^{-1} \in C^m(\boldsymbol{f}(B(\boldsymbol{x}_0, r)))$.

Proof. By the previous theorem there exists $0 < r_1 < r_0$ such that the function

$$\boldsymbol{f} : B(\boldsymbol{x}_0, r_1) \to \boldsymbol{f}(B(\boldsymbol{x}_0, r_1))$$

is invertible, $\boldsymbol{f}(B(\boldsymbol{x}_0, r_1))$ is open, and $\boldsymbol{f}^{-1} : \boldsymbol{f}(B(\boldsymbol{x}_0, r_1)) \to B(\boldsymbol{x}_0, r_1)$ is Lipschitz continuous in $\boldsymbol{f}(B(\boldsymbol{x}_0, r_1))$ and differentiable at $\boldsymbol{f}(\boldsymbol{x}_0)$, with

$$J_{\boldsymbol{f}^{-1}}(\boldsymbol{f}(\boldsymbol{x}_0)) = (J_{\boldsymbol{f}}(\boldsymbol{x}_0))^{-1}.$$

Assume that $\det J_{\boldsymbol{f}}(\boldsymbol{x}_0) > 0$. Since $J_{\boldsymbol{f}}$ is continuous at $\boldsymbol{x}_0$ there exists $r_2 > 0$ such that $\det J_{\boldsymbol{f}}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r_2)$. Take $r := \min\{r_1, r_2\}$ and consider $\boldsymbol{f}^{-1} : \boldsymbol{f}(B(\boldsymbol{x}_0, r)) \to B(\boldsymbol{x}_0, r)$. Since $\det J_{\boldsymbol{f}}(\boldsymbol{x}) > 0$, for every $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$ we can apply the inverse function theorem at the point $\boldsymbol{x}$ to that the function

$$\boldsymbol{f} : B(\boldsymbol{x}, r_{\boldsymbol{x}}) \to \boldsymbol{f}(B(\boldsymbol{x}, r_{\boldsymbol{x}}))$$

is invertible and the inverse is differentiable at $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. Taking $0 < r_{\boldsymbol{x}} < r_1 - \|\boldsymbol{x}-\boldsymbol{x}_0\|$, we have that the two inverse functions coincide (this is an important point, this theorem only provides a local inverse function so the inverse functions could be different) and so we have shown that $\boldsymbol{f}^{-1}: \boldsymbol{f}(B(\boldsymbol{x}_0, r)) \to B(\boldsymbol{x}_0, r)$ is differentiable. To prove that $\boldsymbol{f}^{-1}$ is of class C^m, we use the fact that

$$J_{\boldsymbol{f}^{-1}}(\boldsymbol{y}) = \left(J_{\boldsymbol{f}}\left(\boldsymbol{f}^{-1}(\boldsymbol{y})\right)\right)^{-1}$$

for every $\boldsymbol{y} \in \boldsymbol{f}(B(\boldsymbol{x}_0, r))$. If $m \geq 2$, then since $\boldsymbol{f}^{-1}$ and $J_{\boldsymbol{f}}$ are differentiable, it follows that $J_{\boldsymbol{f}^{-1}}$ is differentiable and we can apply the chain rule to compute the second order order derivatives of $\boldsymbol{f}^{-1}$. Note that on the right-hand side they will appear second order derivatives of $\boldsymbol{f}$ computed at $\boldsymbol{f}^{-1}(\boldsymbol{y})$ and ONLY first order derivatives of $\boldsymbol{f}^{-1}$. Hence, if $m \geq 3$ we can apply the chain rule one more time to obtain that the second order derivatives of $\boldsymbol{f}^{-1}$ are differentiable. We will continue in this way. ■

Given a function f of two variables $(x, y) \in \mathbb{R}^2$, consider the equation

$$f(x, y) = 0.$$

We want to solve for y, that is, we are interested in finding a function $y = g(x)$ such that

$$f(x, g(x)) = 0.$$

We will see under which conditions we can do this. The result is going to be local.

In what follows given $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{y} \in \mathbb{R}^M$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$, we write

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}, \boldsymbol{y}) & \cdots & \frac{\partial f_1}{\partial x_N}(\boldsymbol{x}, \boldsymbol{y}) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial x_1}(\boldsymbol{x}, \boldsymbol{y}) & \cdots & \frac{\partial f_M}{\partial x_N}(\boldsymbol{x}, \boldsymbol{y}) \end{pmatrix}$$

and

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\boldsymbol{x}, \boldsymbol{y}) & \cdots & \frac{\partial f_1}{\partial y_M}(\boldsymbol{x}, \boldsymbol{y}) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial y_1}(\boldsymbol{x}, \boldsymbol{y}) & \cdots & \frac{\partial f_M}{\partial y_M}(\boldsymbol{x}, \boldsymbol{y}) \end{pmatrix}.$$

Theorem 77 (Implicit Function) *Let $U \subseteq \mathbb{R}^N \times \mathbb{R}^M$ be open, let $\boldsymbol{f}: U \to \mathbb{R}^M$, and let $(\boldsymbol{a}, \boldsymbol{b}) \in U$. Assume that $\boldsymbol{f} \in C^m(U)$ for some $m \in \mathbb{N}$, that*

$$\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0} \quad \text{and} \quad \det \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{b}) \neq 0.$$

Then there exist $B_N(\boldsymbol{a}, r_0) \subset \mathbb{R}^N$ and $B_M(\boldsymbol{b}, r_1) \subset \mathbb{R}^M$, with $B_N(\boldsymbol{a}, r_0) \times B_M(\boldsymbol{b}, r_1) \subseteq U$, and a unique function

$$\boldsymbol{g}: B_N(\boldsymbol{a}, r_0) \to B_M(\boldsymbol{b}, r_1)$$

of class C^m such that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) = \boldsymbol{0}$ for all $\boldsymbol{x} \in B_N(\boldsymbol{a}, r_0)$ and $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$.

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In 21-269 we have seen how the implicit function theorem follows from the inverse function theorem. Now we give a second proof of the implicit function theorem which does not use the inverse function theorem.

Proof. We present a proof in the case $M = 1$.

Step 1: Existence of g. Since $\frac{\partial f}{\partial y}(\boldsymbol{a}, b) \neq 0$, without loss of generality, we can assume that $\frac{\partial f}{\partial y}(\boldsymbol{a}, b) > 0$ (the case $\frac{\partial f}{\partial y}(\boldsymbol{a}, b) < 0$ is similar). Using the fact that $\frac{\partial f}{\partial y}$ is continuous at $(\boldsymbol{a}, b)$, we can find $r > 0$ such that

$$R := \overline{Q_N(\boldsymbol{a}, r)} \times [b - r, b + r] \subseteq U,$$

where $Q_N(\boldsymbol{a}, r) = (a_1 - r, a_1 + r) \times \cdots \times (a_N - r, a_N + r)$,

$$\frac{\partial f}{\partial y}(\boldsymbol{x}, y) > 0 \quad \text{for all } (\boldsymbol{x}, y) \in R.$$

Consider the function $h(y) := f(\boldsymbol{a}, y)$, $y \in [b - r, b + r]$. Since

$$h'(y) = \frac{\partial f}{\partial y}(\boldsymbol{a}, y) > 0 \quad \text{for all } y \in [b - r, b + r],$$

we have that h is strictly increasing. Using the fact that $h(b) = f(\boldsymbol{a}, b) = 0$, it follows that

$$0 > h(b - r) = f(\boldsymbol{a}, b - r), \quad 0 < h(b + r) = f(\boldsymbol{a}, b + r).$$

Consider the function $k_1(\boldsymbol{x}) := f(\boldsymbol{x}, b - r)$, $\boldsymbol{x} \in Q_N(\boldsymbol{a}, r)$. Since $k_1(\boldsymbol{a}) < 0$ and k_1 is continuous at $\boldsymbol{a}$, there exists $0 < \delta_1 < r$ such that

$$0 > k_1(\boldsymbol{x}) = f(\boldsymbol{x}, b - r) \quad \text{for all } \boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta_1).$$

Similarly, consider the function $k_2(\boldsymbol{x}) := f(\boldsymbol{x}, b + r)$, $\boldsymbol{x} \in Q_N(\boldsymbol{a}, r)$. Since $k_2(\boldsymbol{a}) > 0$ and k_2 is continuous at $\boldsymbol{a}$, there exists $0 < \delta_2 < r$ such that

$$0 < k_2(\boldsymbol{x}) = f(\boldsymbol{x}, b + r) \quad \text{for all } \boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta_2).$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then for all $\boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta)$,

$$f(\boldsymbol{x}, b - r) < 0, \quad f(\boldsymbol{x}, b + r) > 0.$$

Fix $\boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta)$ and consider the function $k(y) := f(\boldsymbol{x}, y)$, $y \in [b - r, b + r]$. Since

$$k'(y) = \frac{\partial f}{\partial y}(\boldsymbol{x}, y) > 0 \quad \text{for all } y \in [b - r, b + r],$$

we have that k is strictly increasing. Using the fact that $k(b - r) = f(\boldsymbol{x}, b - r) < 0$ and $k(b + r) = f(\boldsymbol{x}, b + r) > 0$, it follows that there exists a unique $y \in (b - r, b + r)$ (depending on $\boldsymbol{x}$) such that $0 = k(y) = f(\boldsymbol{x}, y)$.

Thus, we have shown that for every $\boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta)$ there exists a unique $y \in (b - r, b + r)$ depending on $\boldsymbol{x}$ such that $f(\boldsymbol{x}, y) = 0$. We define $g(\boldsymbol{x}) := y$.

Step 2: Continuity of g. Fix $\boldsymbol{x}_0 \in Q_N(\boldsymbol{a}, \delta)$. Note that $b-r < g(\boldsymbol{x}_0) < b+r$. Let $\varepsilon > 0$ be so small that

$$b - r < g(\boldsymbol{x}_0) - \varepsilon < g(\boldsymbol{x}_0) < g(\boldsymbol{x}_0) + \varepsilon < b + r.$$

Consider the function $j(y) := f(\boldsymbol{x}_0, y)$, $y \in [b-r, b+r]$. Since

$$j'(y) = \frac{\partial f}{\partial y}(\boldsymbol{x}_0, y) > 0 \quad \text{for all } y \in [b-r, b+r],$$

we have that j is strictly increasing. Using the fact that $j(g(\boldsymbol{x}_0)) = f(\boldsymbol{x}_0, g(\boldsymbol{x}_0)) = 0$, it follows that

$$f(\boldsymbol{x}_0, g(\boldsymbol{x}_0) - \varepsilon) < 0, \quad f(\boldsymbol{x}_0, g(\boldsymbol{x}_0) + \varepsilon) > 0.$$

Consider the function $j_1(\boldsymbol{x}) := f(\boldsymbol{x}, g(\boldsymbol{x}_0) - \varepsilon)$, $\boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta)$. Since $j_1(\boldsymbol{x}_0) < 0$ and j_1 is continuous at $\boldsymbol{x}_0$, there exists $0 < \eta_1 < \delta$ such that

$$0 > j_1(\boldsymbol{x}) = f(\boldsymbol{x}, g(\boldsymbol{x}_0) - \varepsilon) \quad \text{for all } \boldsymbol{x} \in Q_N(\boldsymbol{x}_0, \eta_1).$$

Similarly, consider the function $j_2(\boldsymbol{x}) := f(\boldsymbol{x}, g(\boldsymbol{x}_0) + \varepsilon)$, $\boldsymbol{x} \in Q_N(\boldsymbol{a}, \delta)$. Since $j_2(\boldsymbol{x}_0) > 0$ and j_2 is continuous at $\boldsymbol{x}_0$, there exists $0 < \eta_2 < \delta$ such that

$$0 < j_2(\boldsymbol{x}) = f(\boldsymbol{x}, g(\boldsymbol{x}_0) + \varepsilon) \quad \text{for all } \boldsymbol{x} \in Q_N(\boldsymbol{x}_0, \eta_2).$$

Let $\eta := \min\{\eta_1, \eta_2\}$. Then for all $\boldsymbol{x} \in Q_N(\boldsymbol{x}_0, \eta)$,

$$f(\boldsymbol{x}, g(\boldsymbol{x}_0) - \varepsilon) < 0, \quad f(\boldsymbol{x}, g(\boldsymbol{x}_0) + \varepsilon) > 0.$$

But $f(\boldsymbol{x}, g(\boldsymbol{x})) = 0$ and $y \in [b-r, b+r] \mapsto f(\boldsymbol{x}, y)$ is strictly increasing. It follows that

$$g(\boldsymbol{x}_0) - \varepsilon < g(\boldsymbol{x}) < g(\boldsymbol{x}_0) + \varepsilon$$

and so g is continuous at $\boldsymbol{x}_0$.

Step 3: Differentiability of g. Fix $\boldsymbol{x}_0 \in Q_N(\boldsymbol{a}, \delta)$ and $i = 1, \ldots, N$. Consider the open segment S joining $(\boldsymbol{x}_0 + t\boldsymbol{e}_i, g(\boldsymbol{x}_0 + t\boldsymbol{e}_i))$ and $(\boldsymbol{x}_0, g(\boldsymbol{x}_0))$. By the mean value theorem there exists $(\bar{\boldsymbol{x}}, \bar{y}) \in S$ such that

$$0 = f(\boldsymbol{x}_0 + t\boldsymbol{e}_i, g(\boldsymbol{x}_0 + t\boldsymbol{e}_i)) - f(\boldsymbol{x}_0, g(\boldsymbol{x}_0)) = \frac{\partial f}{\partial x_i}(\bar{\boldsymbol{x}}, \bar{y})(\boldsymbol{x} - \boldsymbol{x}_0)$$
$$+ \frac{\partial f}{\partial y}(\bar{\boldsymbol{x}}, \bar{y})(g(\boldsymbol{x}_0 + t\boldsymbol{e}_i) - g(\boldsymbol{x}_0)).$$

Hence,

$$\frac{g(\boldsymbol{x}_0 + t\boldsymbol{e}_i) - g(\boldsymbol{x}_0)}{t} = -\frac{\frac{\partial f}{\partial x_i}(\bar{\boldsymbol{x}}, \bar{y})}{\frac{\partial f}{\partial y}(\bar{\boldsymbol{x}}, \bar{y})}$$

letting $\boldsymbol{x} \to \boldsymbol{x}_0$ and using the continuity of g and of $\frac{\partial f}{\partial \boldsymbol{x}}$ and of $\frac{\partial f}{\partial y}$, we get that $(\bar{\boldsymbol{x}}, \bar{y}) \to (\boldsymbol{x}_0, g(\boldsymbol{x}_0))$ as $t \to 0^+$ and so

$$\lim_{t \to 0} \frac{g(\boldsymbol{x}_0 + t\boldsymbol{e}_i) - g(\boldsymbol{x}_0)}{t} = -\frac{\frac{\partial f}{\partial \boldsymbol{x}}(\boldsymbol{x}_0, g(\boldsymbol{x}_0))}{\frac{\partial f}{\partial y}(\boldsymbol{x}_0, g(\boldsymbol{x}_0))}.$$

This shows that

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}_0) = -\frac{\frac{\partial f}{\partial x_i}(\boldsymbol{x}_0, g(\boldsymbol{x}_0))}{\frac{\partial f}{\partial y}(\boldsymbol{x}_0, g(\boldsymbol{x}_0))}.$$

Since the right-hand side is continuous, it follows that $\frac{\partial g}{\partial x_i}$ is continuous. Thus g is of class C^1.

Step 4: The case $M \geq 2$ is done using induction on M. ■

Assuming the implicit function theorem, we can give an alternative proof of the inverse fuction theorem.

Corollary 78 *Let $U \subseteq \mathbb{R}^N$ be open, let $\boldsymbol{f} : U \to \mathbb{R}^N$, and let $\boldsymbol{x}_0 \in U$. Assume that $\boldsymbol{f} \in C^m(U)$ for some $m \in \mathbb{N}$ and that*

$$\det J_{\boldsymbol{f}}(\boldsymbol{x}_0) \neq 0.$$

Then there exists $B(\boldsymbol{x}_0, r) \subseteq U$ such that $\boldsymbol{f}(B(\boldsymbol{x}_0, r))$ is open, the function

$$\boldsymbol{f} : B(\boldsymbol{x}_0, r) \to \boldsymbol{f}(B(\boldsymbol{x}_0, r))$$

is invertible and $\boldsymbol{f}^{-1} \in C^m(\boldsymbol{f}(B(\boldsymbol{x}_0, r)))$.

Second proof. Step 1: We apply the implicit function theorem to the function $\boldsymbol{h} : U \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}.$$

Let $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$. Then $\boldsymbol{h}(\boldsymbol{x}_0, \boldsymbol{y}_0) = 0$ and

$$\det \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \det J_{\boldsymbol{f}}(\boldsymbol{x}_0) \neq 0.$$

Hence, by the implicit function theorem there exists $B(\boldsymbol{x}_0, r_0) \subset \mathbb{R}^N$ and $B(\boldsymbol{y}_0, r_1) \subset \mathbb{R}^N$ such that $B(\boldsymbol{x}_0, r_0) \times B(\boldsymbol{y}_0, r_1) \subseteq U \times \mathbb{R}^N$ and a function $\boldsymbol{g} : B(\boldsymbol{y}_0, r_1) \to B(\boldsymbol{x}_0, r_0)$ of class C^m such that $\boldsymbol{h}(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in B(\boldsymbol{y}_0, r_1)$, that is,

$$\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y})) = \boldsymbol{y}$$

for all $\boldsymbol{y} \in B(\boldsymbol{y}_0, r_1)$. This implies that $\boldsymbol{g} = \boldsymbol{f}^{-1}$. Moreover,

$$\frac{\partial \boldsymbol{g}}{\partial y_k}(\boldsymbol{y}) = -\left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y})\right)^{-1} \frac{\partial \boldsymbol{h}}{\partial y_k}(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}),$$

that is,

$$\frac{\partial \boldsymbol{f}^{-1}}{\partial y_k}(\boldsymbol{y}) = \left(J_{\boldsymbol{f}}\left(\boldsymbol{f}^{-1}(\boldsymbol{y})\right)\right)^{-1} e_k.$$

■

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Proof. Assume that $\det J_{\boldsymbol{f}}(\boldsymbol{x}_0) > 0$. Since $J_{\boldsymbol{f}}$ is continuous at $\boldsymbol{x}_0$ there exists $r_2 > 0$ such that $\det J_{\boldsymbol{f}}(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r_2)$. Take $r := \min\{r_1, r_2\}$

and consider $\boldsymbol{f}^{-1} : \boldsymbol{f}(B(\boldsymbol{x}_0, r)) \to B(\boldsymbol{x}_0, r)$. Since $\det J_{\boldsymbol{f}}(\boldsymbol{x}) > 0$, for every $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$ we can apply the inverse function theorem at the point $\boldsymbol{x}$ to that the function

$$\boldsymbol{f} : B(\boldsymbol{x}, r_{\boldsymbol{x}}) \to \boldsymbol{f}(B(\boldsymbol{x}, r_{\boldsymbol{x}}))$$

is invertible and the inverse is differentiable at $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. Taking $0 < r_{\boldsymbol{x}} < r_1 - \|\boldsymbol{x} - \boldsymbol{x}_0\|$, we have that the two inverse functions coincide (this is an important point, this theorem only provides a local inverse function so the inverse functions could be different) and so we have shown that $\boldsymbol{f}^{-1} : \boldsymbol{f}(B(\boldsymbol{x}_0, r)) \to B(\boldsymbol{x}_0, r)$ is differentiable. To prove that $\boldsymbol{f}^{-1}$ is of class C^m, we use the fact that

$$J_{\boldsymbol{f}^{-1}}(\boldsymbol{y}) = \left(J_{\boldsymbol{f}}\left(\boldsymbol{f}^{-1}(\boldsymbol{y})\right)\right)^{-1}$$

for every $\boldsymbol{y} \in \boldsymbol{f}(B(\boldsymbol{x}_0, r))$. If $m \geq 2$, then since $\boldsymbol{f}^{-1}$ and $J_{\boldsymbol{f}}$ are differentiable, it follows that $J_{\boldsymbol{f}^{-1}}$ is differentiable and we can apply the chain rule to compute the second order order derivatives of $\boldsymbol{f}^{-1}$. Note that on the right-hand side they will appear second order derivatives of $\boldsymbol{f}$ computed at $\boldsymbol{f}^{-1}(\boldsymbol{y})$ and ONLY first order derivatives of $\boldsymbol{f}^{-1}$. Hence, if $m \geq 3$ we can apply the chain rule one more time to obtain that the second order derivatives of $\boldsymbol{f}^{-1}$ are differentiable. We will continue in this way. ■

Exercise 79 *Under the hypotheses of Corollary 78, prove that if* $0 < r < r_0$ *is sufficiently small, then*

$$\boldsymbol{f} : B(\boldsymbol{x}_0, r) \to \mathbb{R}^N$$

is injective and $\boldsymbol{f}(B(\boldsymbol{x}_0, r))$ *is open.*

Theorem 80 (Lagrange Multipliers) *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $f : U \to \mathbb{R}$ *be a function of class* C^1 *and let* $\boldsymbol{g} : U \to \mathbb{R}^M$ *be a class of function* C^1*, where* $M < N$*, and let*

$$F := \{\boldsymbol{x} \in U : \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}\}.$$

Let $\boldsymbol{x}_0 \in F$ *and assume that* f *attains a constrained local minimum (or maximum) at* $\boldsymbol{x}_0$*. If* $J_{\boldsymbol{g}}(\boldsymbol{x}_0)$ *has maximum rank* M*, then there exist* $\lambda_1, \ldots, \lambda_M \in \mathbb{R}$ *such that*

$$\nabla f(\boldsymbol{x}_0) = \lambda_1 \nabla g_1(\boldsymbol{x}_0) + \cdots + \lambda_M \nabla g_M(\boldsymbol{x}_0).$$

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Proof. Assume that f attains a constrained local minimum at $\boldsymbol{x}_0$ (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0)$ for all $\boldsymbol{x} \in U \cap B(\boldsymbol{x}_0, r)$ such that $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}$. By taking $r > 0$ smaller, and since U is open, we can assume that $B(\boldsymbol{x}_0, r) \subseteq U$ so that

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_0) \text{ for all } \boldsymbol{x} \in B(\boldsymbol{x}_0, r) \text{ with } \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0}. \tag{9}$$

Since $J_{\boldsymbol{g}}(\boldsymbol{x}_0)$ has maximum rank M and $\boldsymbol{g}$ is of class C^1, without loss of generality, we can assume that $J_{\boldsymbol{g}}$ has maximum rank M in $B(\boldsymbol{x}_0, r)$, which implies that $M \cap B(\boldsymbol{x}_0, r)$ is an $(N-M)$th dimensional manifold of class C^1. Let $\boldsymbol{\varphi} : B_{N-M}(\boldsymbol{z}_0, r_0) \to \mathbb{R}^N$ be a local chart, with $\boldsymbol{\varphi}(\boldsymbol{z}_0) = \boldsymbol{x}_0$. Then by (9),

$$p(\boldsymbol{z}) = f(\boldsymbol{\varphi}(\boldsymbol{z})) \geq f(\boldsymbol{x}_0) = f(\boldsymbol{\varphi}(\boldsymbol{z}_0)) = p(\boldsymbol{z}_0)$$

for all $\boldsymbol{z} \in B_{N-M}(\boldsymbol{z}_0, r_0)$. Hence, the function p attains a local minimum at $\boldsymbol{z}_0$. It follows that

$$\mathbf{0} = J_p(\boldsymbol{z}_0) = \nabla f(\boldsymbol{x}_0) J_{\boldsymbol{\varphi}}(\boldsymbol{z}_0).$$

Considering the transpose of this expression, we get

$$(J_{\boldsymbol{\varphi}}(\boldsymbol{z}_0))^T (\nabla f(\boldsymbol{x}_0))^T = \mathbf{0},$$

which implies that the vector $\nabla f(\boldsymbol{x}_0)$, $i = 1, \ldots, M$ belong to the kernel of kernel of the linear transformation $\boldsymbol{T} : \mathbb{R}^N \to \mathbb{R}^M$ defined by

$$\boldsymbol{T}(\boldsymbol{x}) := (J_{\boldsymbol{\varphi}}(\boldsymbol{z}_0))^T \boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^N.$$

On the other hand, since

$$\boldsymbol{g}(\boldsymbol{\varphi}(\boldsymbol{z})) = \mathbf{0} \quad \text{for all } \boldsymbol{z} \in B_{N-M}(\boldsymbol{a}, r_0).$$

by the chain rule,

$$\mathbf{0} = J_{\boldsymbol{g}}(\boldsymbol{x}_0) J_{\boldsymbol{\varphi}}(\boldsymbol{z}_0),$$

Considering the transpose of this expression, we get

$$\mathbf{0} = (J_{\boldsymbol{\varphi}}(\boldsymbol{z}_0))^T (J_{\boldsymbol{g}}(\boldsymbol{x}_0))^T,$$

which implies that the vectors $\nabla g_i(\boldsymbol{x}_0)$, $i = 1, \ldots, M$, belong to the kernel of $\boldsymbol{T}$. Hence,

$$V := \operatorname{span}\{\nabla g_1(\boldsymbol{x}_0), \ldots, \nabla g_M(\boldsymbol{x}_0)\} \subseteq \ker \boldsymbol{T}.$$

But $\dim V = \operatorname{rank} J_{\boldsymbol{g}}(\boldsymbol{x}_0) = M = N - \operatorname{rank}(J_{\boldsymbol{k}}(\boldsymbol{a}))^T = \dim \ker \boldsymbol{T}$. Hence,

$$V = \ker \boldsymbol{T}.$$

Since $\nabla f(\boldsymbol{x}_0) \in V$, it follows from the definition of V that there exist $\lambda_1, \ldots,$ $\lambda_M \in \mathbb{R}$ such that

$$\nabla f(\boldsymbol{x}_0) = \lambda_1 \nabla g_1(\boldsymbol{x}_0) + \cdots + \lambda_m \nabla g_M(\boldsymbol{x}_0).$$

■

8 Application II of BaFPT: Differential Equations

An important application of Banach's contraction principle is the existence and uniqueness of solutions of ODE.

Definition 81 *Given a set* $E \subseteq \mathbb{R} \times \mathbb{R}^d$*, and interval* $I \subseteq \mathbb{R}$*, and a function* $\boldsymbol{f} : E \to \mathbb{R}^d$*, we say that a differentiable function* $\boldsymbol{u} : I \to \mathbb{R}^d$ *is a* solution of the differential equation

$$\boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u}(t)) \tag{10}$$

in I *if* $(t, \boldsymbol{u}(t)) \in E$ *and (10) holds for all* $t \in I$*.*

Definition 82 *Given a set* $E \subseteq \mathbb{R} \times \mathbb{R}^d$*, an interval* $I \subseteq \mathbb{R}$*, a function* $\boldsymbol{f} : E \to \mathbb{R}^d$*, and* $(t_0, \boldsymbol{u}_0) \in E$ *we say that a differentiable function* $\boldsymbol{u} : I \to \mathbb{R}^d$ *is a* solution of the initial value problem *or* (Cauchy problem)

$$\begin{cases} \boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u}(t)), \\ \boldsymbol{u}(t_0) = \boldsymbol{u}_0 \end{cases} \tag{11}$$

in I *if* $\boldsymbol{u}$ *is a solution of (10) in* I*,* $t_0 \in I$ *and* $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$*.*

We say that the Cauchy problem (11) admits a *local solution* if there exists an interval I containing t_0 and a solution $\boldsymbol{u} : I \to \mathbb{R}^d$ of (11) in I.

We say that the Cauchy problem (11) admits a *global solution* if for every interval I containing t_0 and with the property that for every $t \in I$ there exists $\boldsymbol{z} \in \mathbb{R}^d$ with $(t, \boldsymbol{z}) \in E$, there exists a solution $\boldsymbol{u} : I \to \mathbb{R}^d$ of (11) in I.

Example 83 *Consider the Cauchy problem*

$$\begin{cases} u'(t) = u^2(t), \\ u(0) = 1. \end{cases}$$

Since $f(s) = s^2$ *is continuous, any solution of the Cauchy problem in some interval* I *will be of class* $C^1(I)$*. Hence, near* $t = 0$ *we have that* $u(t) > 0$ *and so as long as this happens*

$$\frac{u'(t)}{u^2(t)} = 1.$$

Integrating both sides we get

$$\int_0^t \frac{u'(s)}{u^2(s)}\, ds = \int_0^t 1\, ds = t - 0.$$

Using the change of variable $y = u(s)$ *gives*

$$\int_{u(0)}^{u(t)} \frac{1}{y^2}\, dy = t.$$

Hence,

$$\left[-\frac{1}{y}\right]_1^{u(t)} = \int_{u(0)}^{u(t)} \frac{1}{y^2}\, dy = t,$$

that is, $-\frac{1}{u(t)} + 1 = t$*. Thus, we have shown that as long at* $u(t)$ *stays positive it is given by*

$$u(t) = \frac{1}{1 - t}$$

which exists in the interval $(-\infty, 1)$*. Thus,* u *is a local solution but not a global solution.*

Theorem 84 *Let* $I = [t_0, t_0 + T_0]$*, where* $t_0 \in \mathbb{R}$ *and* $T_0 > 0$*, let* $\boldsymbol{u}_0 \in \mathbb{R}^d$ *and let* $\boldsymbol{f} : I \times \overline{B_d(\boldsymbol{u}_0, r)} \to \mathbb{R}^d$ *be a continuous function such that*

$$\|\boldsymbol{f}(t, \boldsymbol{z}_1) - \boldsymbol{f}(t, \boldsymbol{z}_2)\| \leq L \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|$$

for all $t \in I$*,* $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \overline{B_d(\boldsymbol{u}_0, r)}$*, and for some* $L, r > 0$*. Then there exists* $0 < T \leq T_0$ *such that the initial value problem*

$$\begin{aligned} \boldsymbol{u}'(t) &= \boldsymbol{f}(t, \boldsymbol{u}(t)), \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0 \end{aligned}$$

admits a unique solution in some interval $[t_0, t_0 + T]$*.*

Proof. Consider the space $X = \{\boldsymbol{g} : [t_0, t_0 + T] \to \mathbb{R}^d$ continuous such that $\|\boldsymbol{g} - \boldsymbol{u}_0\|_\infty \leq r\}$, where $0 < T \leq T_1$ will be chosen later and consider the operator

$$F : X \to X$$

given by

$$F(\boldsymbol{g})(t) = \boldsymbol{u}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{g}(s))\, ds$$

for $g \in X$ and $t \in [t_0, t_0 + T]$. By the Weierstrass theorem there exists $M := \max\{\|\boldsymbol{f}(t, \boldsymbol{z})\| : (t, \boldsymbol{z}) \in I \times \overline{B_d(\boldsymbol{u}_0, r)}\} \in [0, \infty)$. If $\boldsymbol{g} \in X$, then

$$\|F(\boldsymbol{g})(t) - \boldsymbol{u}_0\| = \left\| \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{g}(s))\, ds \right\| \leq \int_{t_0}^{t} \|\boldsymbol{f}(s, \boldsymbol{g}(s))\|\, ds \leq M(t - t_0) \leq MT \leq r,$$

provided $T \leq r/M$. Thus, F is well-defined. Let's prove that F is a contraction. Take $\boldsymbol{g}_1, \boldsymbol{g}_2 \in X$. Then

$$\begin{aligned} \|F(\boldsymbol{g}_1)(t) - F(\boldsymbol{g}_2)(t)\| &= \left\| \int_{t_0}^{t} [\boldsymbol{f}(s, \boldsymbol{g}_1(s)) - \boldsymbol{f}(s, \boldsymbol{g}_2(s))]\, ds \right\| \\ &\leq \int_{t_0}^{t} \|\boldsymbol{f}(s, \boldsymbol{g}_1(s)) - \boldsymbol{f}(s, \boldsymbol{g}_2(s))\|\, ds \\ &\leq L \int_{t_0}^{t} \|\boldsymbol{g}_1(s) - \boldsymbol{g}_2(s)\|\, ds \leq L(t - t_0) \max_{y \in [t_0, t_0 + T]} \|\boldsymbol{g}_1(y) - \boldsymbol{g}_2(y)\| T \\ &\leq LT \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_\infty, \end{aligned}$$

and so taking the maximum over all $t \in [t_0, t_0 + T]$, we get

$$\|F(\boldsymbol{g}_1) - F(\boldsymbol{g}_2)\|_\infty \leq LT \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_\infty.$$

If we take T so small that $LT < 1$ and $t_0 \pm T \in I$, then F is a contraction. ■

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Proof. By Banach's contraction principle there exists a unique fixed point $\boldsymbol{u} \in X$, that is,

$$\boldsymbol{u}(t) = F(\boldsymbol{u})(t) = \boldsymbol{u}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{u}(s))\, ds$$

for all $t \in [t_0, t_0 + T]$. Since $\boldsymbol{u}$ is continuous, the right-hand side is of class C^1, and so $\boldsymbol{u}$ is actually of class C^1. By differentiating both sides, we get that $\boldsymbol{u}$ is a solution of the ODE. Moreover, $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. Since any other solution of the initial value problem is a fixed point of F, we have uniqueness. ■

Remark 85 *If $\boldsymbol{u}(t_0 + T) \in B_d(\boldsymbol{u}_0, r)$, then we can prove that the solution exists for some time after $t_0 + T$. Indeed, set $\boldsymbol{u}_1 := \boldsymbol{u}(t_0 + T)$ and consider*

$$F_1(\boldsymbol{g})(t) = \boldsymbol{u}_1 + \int_{t_0+T}^{t} \boldsymbol{f}(s, \boldsymbol{g}(s))\, ds, \quad t \in [t_0 + T, t_0 + T_1],$$

where $T_1 \leq T_0$ and F is defined on the set

$$X_1 = \{\boldsymbol{g} : [t_0 + T, t_0 + T_1] \to \mathbb{R}^d \text{ continuous and such that } \|\boldsymbol{g} - \boldsymbol{u}_1\|_\infty \leq r_1\},$$

where $r_1 = r - \|\boldsymbol{u}_1 - \boldsymbol{u}_0\| > 0$.

Definition 86 *Given a set $E \subseteq \mathbb{R} \times \mathbb{R}^d$, a function $\boldsymbol{f} : E \to \mathbb{R}^d$, and a solution $\boldsymbol{u} : I \to \mathbb{R}^d$ of the differential equation (10), we say that $\boldsymbol{u}$ is a* maximal solution of *(10) if there does not exists an interval $J \supset I$ (in the strict sense) and a solution $\boldsymbol{v} : J \to \mathbb{R}^d$ of (10) which coincides with $\boldsymbol{u}$ in I. A similar definition can be given for the Cauchy problem (11).*

To prove the existence of maximal solutions we will need to use Zorn's lemma.

Given two nonempty sets X, Y, a (binary) *relation* is a subset $\mathcal{R} \subseteq X \times Y$. Usually, we associate a symbol to it, say $*$, so that $x * y$ means that $(x, y) \in \mathcal{R}$.

A *partial ordering* on a nonempty set X is a relation $\mathcal{R} \subseteq X \times X$, denoted $\leq$, such that

(i) $x \leq x$ for every $x \in X$; that is $(x, x) \in \mathcal{R}$ (reflexivity).

(ii) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$; that is, if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$ (antisymmetry).

(iii) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; that is, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$ (transitivity).

The word "partial" means that given $x, y \in X$, in general we cannot always say that $x \leq y$ or $y \leq x$.

Example 87 *Let $X = \mathcal{P}(\mathbb{R}) = \{$all subsets of $\mathbb{R}\}$. Given $E, F \in X$, we say that $E \leq F$ if $E \subseteq F$. Then $\leq$ is a partial ordering, but given the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$, one is not contained into the other.*

Given a partially ordered set $(X, \leq)$, a *totally ordered set*, or *chain*, $E \subseteq X$ is a set with the property that for all $x, y \in X$, either $x \leq y$ or $y \leq x$ (or both).

In the previous example $E = \{\{1, 2, 3\}, \{1, 2\}, \{2\}\}$ is a chain.

Given a partially ordered set $(X, \leq)$, and a set $E \subseteq X$, an *upper bound of* E is an element $x \in X$ such that $y \leq x$ for all $y \in E$. A set E may not have any upper bounds. A *maximal element* of E is an element $x \in E$ such that if $x \leq y$ for some $y \in E$, then $x = y$. A set E may not have maximal elements or it may have maximal elements that are not upper bounds (it can happen that a maximal element cannot be compared with all the elements of E).

Proposition 88 (Zorn's lemma) *Given a partially ordered set $(X, \leq)$, if every totally ordered subset of X has an upper bound, then X has a maximal element.*

Theorem 89 (Maximal Solutions) *Let $E \subseteq \mathbb{R} \times \mathbb{R}^d$ and let $\boldsymbol{f} : E \to \mathbb{R}^d$ be a function. Assume that there exists a solution function $\boldsymbol{u} : I \to \mathbb{R}^d$ of the differential equation (10). Then $\boldsymbol{u}$ can be extended to a maximal solution of (10).*

Proof. Consider the set X of all functions $\boldsymbol{v} : J \to \mathbb{R}^d$ such that J is an interval, $I \subseteq J$ and $\boldsymbol{v}$ is a solution of (10) which coincides with $\boldsymbol{u}$ in I. The set X is nonempty since $\boldsymbol{u}$ belongs to X. We define a partial order in X. Given two functions $\boldsymbol{v}_1 : J_1 \to \mathbb{R}^d$ and $\boldsymbol{v}_1 : J_1 \to \mathbb{R}^d$ in X, we say thatr $\boldsymbol{v}_1 \leq \boldsymbol{v}_2$ if $J_1 \subseteq J_2$ and $\boldsymbol{v}_2$ coincides with $\boldsymbol{v}_1$ in J_1.

Given a chain $Y \subseteq X$, write $Y = \{\boldsymbol{v}_\alpha : J_\alpha \to \mathbb{R}^d\}_{\alpha \in F}$, for some set F. Define

$$J_{\max} := \bigcup_{\alpha \in F} J_\alpha.$$

We claim that $J_{\max}$ is an interval. Indeed, if $s, t \in J_{\max}$, with $s < t$, let J_α and J_β be such that $s \in J_\alpha$ and $t \in J_\beta$. Since Y is a chain, $J_\alpha \subseteq J_\beta$ or $J_\beta \subseteq J_\alpha$. In both cases s and t belong to the larger interval and so does the segment $[s, t]$. In turn, $[s, t] \subseteq J_{\max}$.

For every $t \in J_{\max}$ let $\alpha \in F$ be such that $t \in J_\alpha$ and define $\boldsymbol{v}_{\max}(t) := \boldsymbol{v}_\alpha(t)$. Note that the function $\boldsymbol{v}_{\max}$ is well-defined, since if $\beta \in F$ is such that $t \in J_\beta$, then v_β and v_α coincide in $J_\alpha \cap J_\beta$. Let's prove that $\boldsymbol{v}_{\max}$ is a solution of (10) in $J_{\max}$. Let $t \in (J_{\max})^\circ$. Then there exists $(t - \varepsilon, t + \varepsilon) \subseteq J_{\max}$. Reasoning as before (when we proved that $J_{\max}$ is an interval), we can find $\alpha \in F$ such that $(t - \varepsilon, t + \varepsilon) \subseteq J_\alpha$. In turn, since $\boldsymbol{v}_{\max} = \boldsymbol{v}_\alpha$ in $(t - \varepsilon, t + \varepsilon)$, it follows that $\boldsymbol{v}_{\max}$ is differentiable in $(t - \varepsilon, t + \varepsilon)$ and is a solution of (10) in $(t - \varepsilon, t + \varepsilon)$. On the other hand, if $J_{\max}$ contains one or both of its endpoints, let $t \in J_{\max} \setminus (J_{\max})^\circ$, then there exists $\alpha \in F$ such that $t \in J_\alpha$ and t is an endpoint of J_α. But then the function $\boldsymbol{v}_{\max}$ coincides with $\boldsymbol{v}_\alpha$ in J_α. In particular, it is differentiable at t with $\boldsymbol{v}'_{\max}(t) = \boldsymbol{v}'_\alpha(t) = \boldsymbol{f}\,(t, \boldsymbol{v}_\alpha\,(t)) = \boldsymbol{f}\,(t, \boldsymbol{v}_{\max}\,(t))$. This shows that $\boldsymbol{v}_{\max}$ is a solution of (10) in $J_{\max}$. By construction $\boldsymbol{v}_{\max} \in X$ and $\boldsymbol{v}_{\max} \geq \boldsymbol{v}_\alpha$ for all $\alpha \in Y$. Thus, Y has an upper bound.

It now follows from Zorn's lemma that X admits a maximal element. ■

Theorem 90 (Gronwall's Inequality) *Let $I \subseteq \mathbb{R}$ be an interval, let $t_0 \in I$, let $u : I \to \mathbb{R}$ and $\beta : I \to [0, \infty)$ be continuous functions and let $\alpha : I \to \mathbb{R}$ locally integrable. Assume that*

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s) u(s)\, ds$$

for all $t \in I$ *with* $t \geq t_0$. *Then*

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(\int_r^t \beta(s)\ ds\right) dr$$

for all $t \in I$ *with* $t \geq t_0$. *Moreover, if* α *is increasing,*

$$u(t) \leq \alpha(t) \exp \int_{t_0}^{t} \beta(s)\ ds$$

for all $t \in I$ *with* $t \geq t_0$.

Proof. Consider the function

$$v(t) := \int_{t_0}^{t} \beta(s)u(s)\ ds \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right).$$

Then

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s)u(s)\ ds = \alpha(t) + \exp\left(\int_{t_0}^{t} \beta(s)\ ds\right) v(t).$$

On the other hand,

$$\begin{aligned} v'(t) &= \beta(t)\left(u(t) - \int_{t_0}^{t} \beta(s)u(s)\ ds\right) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) \\ &\leq \alpha(t)\beta(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right). \end{aligned}$$

Since $v(t_0) = 0$ integrating the previous inequality gives

$$v(t) \leq \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(-\int_{t_0}^{r} \beta(s)\ ds\right) dr,$$

and so

$$\begin{aligned} u(t) &\leq \alpha(t) + \exp\left(\int_{t_0}^{t} \beta(s)\ ds\right) \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(-\int_{t_0}^{r} \beta(s)\ ds\right) dr \\ &= \alpha(t) + \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(\int_r^t \beta(s)\ ds\right) dr. \end{aligned}$$

If α is increasing then we can bound the right-hand side with

$$\begin{aligned} &\alpha(t) + \alpha(t) \int_{t_0}^{t} \beta(r) \exp\left(\int_r^t \beta(s)\ ds\right) dr. \\ &= \alpha(t) + \alpha(t) \left[-\exp\left(\int_r^t \beta(s)\ ds\right)\right]_{r=t_0}^{r=t} \\ &= \alpha(t) \exp\left(\int_{t_0}^{t} \beta(s)\ ds\right). \end{aligned}$$

This concludes the proof. ■

Remark 91 *To understand Gronwall's inequality, assume that equality holds, that is, that*

$$u(t) = \alpha(t) + \int_{t_0}^{t} \beta(s) u(s)\ ds$$

for all $t \in I$ *with* $t \geq t_0$*. If* α *is differentiable, then by differentiating both sides we get a linear differential equation*

$$u'(t) = \alpha'(t) + \beta(t) u(t).$$

To solve it, multiply both sides by $\exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right)$*. Then*

$$\exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) u'(t) - \beta(t) u(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) = \alpha'(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right)$$

which we can write as

$$\frac{d}{dt}\left(u(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right)\right) = \alpha'(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right).$$

Integrating we get

$$\begin{aligned} u(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) &= u(t_0) + \int_{t_0}^{t} \alpha'(r) \exp\left(-\int_{t_0}^{r} \beta(s)\ ds\right) dr \\ &= u(t_0) + \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) dr + \alpha(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) - \alpha(t_0) \\ &= \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) dr + \alpha(t) \exp\left(-\int_{t_0}^{t} \beta(s)\ ds\right) \end{aligned}$$

where we integrated by parts and used the fact that $u(t_0) = \alpha(t_0)$*. In turn,*

$$\begin{aligned} u(t) &= \exp\left(\int_{t_0}^{t} \beta(s)\ ds\right) \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(-\int_{t_0}^{r} \beta(s)\ ds\right) dr + \alpha(t) \\ &= \alpha(t) + \int_{t_0}^{t} \alpha(r)\beta(r) \exp\left(\int_{r}^{t} \beta(s)\ ds\right) dr. \end{aligned}$$

Theorem 92 (Global Existence) *Let* $I \subseteq \mathbb{R}$ *be an interval,let* $t_0 \in I$*,* $\boldsymbol{u}_0 \in \mathbb{R}^d$*, and let* $\boldsymbol{f} : I \times \mathbb{R}^d \to \mathbb{R}^d$ *be a* C^1 *function such that*

$$\|\boldsymbol{f}\ (t, \boldsymbol{z})\| \leq a(t) + \beta(t) \|\boldsymbol{z}\|$$

for all $t \in I$*,* $\boldsymbol{z} \in \mathbb{R}^d$*, and where* $a : I \to [0, \infty)$ *is locally integrable integrable and* $\beta : I \to [0, \infty)$ *is continuous. Then the Cauchy problem (11) admits a global solution* $\boldsymbol{u} : I \to \mathbb{R}^d$*.*

Exercise 93 *Consider the Cauchy problem*

$$\begin{cases} u' = u(u-1)(u-2) \\ u(0) = u_0, \end{cases}$$

where $u_0 \in \mathbb{R}$. Study local existence, uniqueness, global existence, and the asymptotic behavior of solutions.

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Proof. By Theorem 84 the Cauchy problem (11) admits a solution $\boldsymbol{u} : I_0 \to \mathbb{R}^d$ in some interval $I_0 \subseteq I$. By Theorem 89 the function $\boldsymbol{u}$ can be extended to a maximal solution $\boldsymbol{u} : J \to \mathbb{R}^d$, where J is an interval contained in I. We claim that $J = I$. Assume by contradiction that $J \neq I$. Then there exists $T_1 \in I \setminus J$. Assume $T_1 > t_0$ (the case $T_1 < t_0$ is similar). For every $t \in J$ with $t \geq t_0$ we have

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{u}(s))\, ds$$

and so

$$\|\boldsymbol{u}(t)\| \leq \|\boldsymbol{u}_0\| + \int_{t_0}^{t} \|\boldsymbol{f}(t, \boldsymbol{u}(s))\|\, ds \leq \|\boldsymbol{u}_0\| + \int_{t_0}^{t} a(s)\, ds + \int_{t_0}^{t} \beta(s)\, \|\boldsymbol{u}(s)\|)\, ds.$$

Thus we can apply Gronwall's inequality with $\alpha(t) = \|\boldsymbol{u}_0\| + \int_{t_0}^{t} a(s)\, ds$ to conclude that

$$\begin{aligned} \|\boldsymbol{u}(t)\| &\leq (\|\boldsymbol{u}_0\| + \int_{t_0}^{t} a(s)\, ds) \exp\left(\int_{t_0}^{t} \beta(s)\, ds\right) \\ &\leq \left(\|\boldsymbol{u}_0\| + \int_{t_0}^{T_1} a(s)\, ds\right) \exp\left(\int_{t_0}^{T_1} \beta(s)\, ds\right) =: R. \end{aligned}$$

Let

$$M := \max\{\|\boldsymbol{f}(t, \boldsymbol{z})\| : \ (t, \boldsymbol{z}) \in [t_0, T_1] \times \overline{B_d(\boldsymbol{u}_0, R)}\}.$$

Then

$$\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\| = \left\| \int_{t_1}^{t_2} \boldsymbol{f}(t, \boldsymbol{u}(s))\, ds \right\| \leq \int_{t_0}^{t_2} \|\boldsymbol{f}(t, \boldsymbol{u}(s))\|\, ds \leq M(t_2 - t_1)$$

for all $t_1, t_2 \in J \cap [t_0, T_1]$ with $t_1 < t_2$. Let $b := \sup J$. Since $\boldsymbol{u}$ is Lipschitz continuous, there exists

$$\lim_{t \to b^-} \boldsymbol{u}(t) = \boldsymbol{\ell} \in \mathbb{R}^d.$$

If $b < T_1$ consider the Cauchy problem

$$\begin{cases} \boldsymbol{v}'(t) = \boldsymbol{f}(t, \boldsymbol{v}(t)), \\ \boldsymbol{v}(b) = \boldsymbol{\ell}. \end{cases}$$

By Theorem 84 there exists a solution $\boldsymbol{v} : [b, b+T] \to \mathbb{R}^d$ for some $T > 0$. Consider the function

$$\boldsymbol{w}(t) := \begin{cases} \boldsymbol{u}(t) & \text{if } t < b, \\ \boldsymbol{v}(t) & \text{if } b \leq t \leq b+T, \end{cases}$$

By L'Hôpital's rule'

$$\lim_{t \to b^-} \frac{\boldsymbol{u}(t) - \boldsymbol{\ell}}{t-b} = \lim_{t \to b^-} \boldsymbol{u}'(t) = \lim_{t \to b^-} \boldsymbol{f}(t, \boldsymbol{u}(t)) = \boldsymbol{f}(b, \boldsymbol{\ell}),$$

by the continuity of $\boldsymbol{f}$, which shows that $\boldsymbol{w}'_-(b) = \boldsymbol{f}(b, \boldsymbol{\ell})$. Similarly, $\boldsymbol{w}'_+(b) = \boldsymbol{f}(b, \boldsymbol{\ell})$, and so $\boldsymbol{w}$ is a solution of the Cauchy problem (11) in $J \cup [b, b+T]$, which contradicts the fact that $\boldsymbol{u}$ is a maximal solution.

On the other hand if $b = T_1 \notin J$, then the function

$$\boldsymbol{w}(t) := \begin{cases} \boldsymbol{u}(t) & \text{if } t < b, \\ \boldsymbol{\ell} & \text{if } t = b, \end{cases}$$

satisfies $\boldsymbol{w}'_-(b) = \boldsymbol{f}(b, \boldsymbol{\ell})$ (again by y L'Hôpital's rule') and so is a solution of the Cauchy problem in $J \cup \{b\}$, which contradicts the fact that $\boldsymbol{u}$ is a maximal solution. ■

Remark 94 *In view of Theorem 120 below, this corollary continues to hold if we assume that* $\boldsymbol{f}$ *is continuous instead of* C^1.

9 Schauder Fixed Point Theorem

Definition 95 *Let* (X, d) *be a metric space. A set* $K \subseteq X$

(i) is sequentially compact *if for every sequence* $\{x_n\} \subseteq K$*, there exist a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *and* $x \in K$ *such that* $x_{n_k} \to x$ *as* $k \to \infty$*,*

(ii) is totally bounded *if for every* $\varepsilon > 0$ *there exist* $x_1, \ldots, x_m \in K$ *such that*

$$K \subseteq \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

The following theorem is one of the main results of this subsection.

Theorem 96 *Let* (X, d) *be a metric space. A set* $K \subseteq X$ *is sequentially compact if and only if* K *is compact.*

Proof. Step 1: Assume that K is sequentially compact. We claim that K is totally bounded. Assume by contradiction that K is not totally bounded. Then there exists $\varepsilon_0 > 0$ such that K cannot be covered by a finite number of balls of radius ε_0. Fix $x_1 \in K$. Then there exists $x_2 \in K$ such that $d(x_1, x_2) \geq \varepsilon_0$ (otherwise $B(x_1, \varepsilon_0)$ would cover K). Similarly, we can find $x_3 \in K$ such that

$d(x_1, x_3) \geq \varepsilon_0$ and $d(x_2, x_3) \geq \varepsilon_0$ (otherwise $B(x_1, \varepsilon_0)$ and $B(x_2, \varepsilon_0)$ would cover K). Inductively, construct a sequence $\{x_n\} \subseteq K$ such that $d(x_n, x_m) \geq \varepsilon_0$ for all $n, m \in \mathbb{N}$ with $n \neq m$. The sequence $\{x_n\}$ cannot have a convergent subsequence, which contradicts the fact that K is sequentially compact.

Next we prove that K is compact. Let $\{U_\alpha\}_\alpha$ be an open cover of K. Since K is totally bounded, for every $n \in \mathbb{N}$ let $\mathcal{B}_n$ be a finite cover of K with balls of radius $\frac{1}{n}$ and centers in K. We want to prove that there exists $\overline{n} \in \mathbb{N}$ such that every ball in $\mathcal{B}_{\overline{n}}$ is contained in some U_α. Note that this would conclude the proof. Indeed, for every $B \in \mathcal{B}_{\overline{n}}$ fix one U_α containing B. Since $\mathcal{B}_{\overline{n}}$ is a finite family and covers K, the subcover of $\{U_\alpha\}$ just constructed has the same properties.

To find $\overline{n}$, assume by contradiction that for every $n \in \mathbb{N}$ there exists a ball $B\left(x_n, \frac{1}{n}\right) \in \mathcal{B}_n$ that is not contained in any U_α. By sequential compactness, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Since $x \in K$, there exists β such that $x \in U_\beta$. But U_β is open, and so there exists $r > 0$ such that $B(x, r) \subseteq U_\beta$. Since, $x_{n_k} \to x$, we have that $\|x_{n_k} - x\| < \frac{r}{2}$ for all k sufficiently large. In turn, if $\frac{1}{n_k} < \frac{r}{2}$, by the triangle inequality, $B\left(x_{n_k}, \frac{1}{n_k}\right) \subseteq B(x, r) \subseteq U_\beta$, which contradicts the fact that ball $B\left(x_{n_k}, \frac{1}{n_k}\right)$ is not contained in any U_α. This shows that K is compact. ∎

Monday, February 27, 2023

Proof. Step 2: Assume that K is compact. By a theorem proved last semester, K is closed, and so sequentially closed by Proposition **??**. We claim that K is sequentially compact. To see this, assume by contradiction that there exists a sequence $\{x_n\} \subseteq K$ that has no subsequence converging in K. Then for every $m \in \mathbb{N}$ the number of $n \in \mathbb{N}$ such that $x_n = x_m$ is finite (otherwise, if $x_n = x_m$ for infinitely many $n \in \mathbb{N}$, then this would be a convergent subsequence). Moreover, the set $C := \{x_n : n \in \mathbb{N}\}$ has no accumulation points. Indeed, if C had an accumulation point, then since K is sequentially closed, there would a subsequence of $\{x_n\}$ converging to K. Since C has no accumulation point, it follows, in particular, that C is closed. Similarly, for every $m \in \mathbb{N}$ the sets $C_m := \{x_n : n \in \mathbb{N}, n \geq m\}$ are closed. Moreover, $C_{m+1} \subseteq C_m$ and by what we said before,

$$\bigcap_{m=1}^{\infty} C_m = \emptyset. \tag{12}$$

For every $m \in \mathbb{N}$ the set $U_m := X \setminus C_m$ is open, $U_{m+1} \supseteq U_m$ and by (12) and De Morgan's laws

$$\bigcup_{m=1}^{\infty} U_m = \bigcup_{m=1}^{\infty} (X \setminus C_m) = X \setminus \left(\bigcap_{m=1}^{\infty} C_m \right) = X.$$

In particular, $\{U_m\}_m$ is an open cover of K. By compactness, it follows that there $\overline{m} \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{m=1}^{\overline{m}} U_m = U_{\overline{m}} = X \setminus C_{\overline{m}},$$

which implies that $K \cap C_{\overline{m}} = \emptyset$. This is a contradiction, since $C_{\overline{m}}$ is nonempty and contained in K. ■

Remark 97 *Neither direction holds for topological spaces.*

Definition 98 *Given a topological space X, A set $E \subseteq X$ is* relatively compact *(or* precompact*) if its closure $\overline{E}$ is compact.*

Example 99 *A finite set $K \subseteq \mathbb{R}^N$ is compact, sequentially compact, and totally bounded.*

Exercise 100 *Let (X, d) be a metric space. Prove that a set $K \subseteq X$ is compact if and only if K is complete and totally bounded.*

Definition 101 *Given a vector space X and a set $E \subseteq X$, the* convex hull *of E is the intersection of all convex sets that contain E. It is denoted* $\operatorname{co} E$

Remark 102 *The convex hull is the smallest convex set that contains E. It consists of all convex combinations of elements of E, that is*

$$\operatorname{co} E = \left\{ \sum_{i=1}^{n} \theta_i y_i : \theta_i \geq 0, \sum_{i=1}^{n} \theta_i = 1, \, y_i \in E, \, n \in \mathbb{N} \right\}.$$

The proof is left as an exercise.

Theorem 103 (Schauder Fixed Point Theorem) *Let X be a Banach space, let $K \subset X$ be a compact, convex set and let $g : K \to K$ be a continuous function. Then g has a fixed point.*

Proof. Fix $n \in \mathbb{N}$ then by compactness we may find $x_1, \ldots, x_{\ell_n} \in K$ such that the balls $B_i := B(x_i, \frac{1}{n})$, $i = 1, \ldots, \ell_n$, cover K. Let $K_n \subseteq K$ be the convex hull of $\{x_1, \ldots, x_{\ell_n}\}$ and consider the function $f_n : K \to K_n$ given by

$$f_n(x) := \sum_{i=1}^{\ell_n} \frac{\operatorname{dist}(x, K \setminus B_i)}{\sum_{j=1}^{\ell_n} \operatorname{dist}(x, K \setminus B_j)} x_i.$$

Note that if $x \in K$, then $x \in B_i$ for some i and so $\operatorname{dist}(x, K \setminus B_i) > 0$ since $K \setminus B_i$ is closed. The function f_n is continuous since the distance function is Lipschitz continuous. Moreover, if $x \in K$, then

$$\|f_n(x) - x\| \leq \sum_{i=1}^{\ell_n} \frac{\operatorname{dist}(x, K \setminus B_i)}{\sum_{j=1}^{\ell_n} \operatorname{dist}(x, K \setminus B_j)} \|x_i - x\| < \frac{1}{n}. \tag{13}$$

Define

$$(f_n \circ g)(x) = \sum_{i=1}^{\ell_n} \frac{\operatorname{dist}(g(x), K \setminus B_i)}{\sum_{j=1}^{\ell_n} \operatorname{dist}(g(x), K \setminus B_j)} x_i, \quad x \in K_n.$$

Since $(f_n \circ g)(x)$ is a convex combination of $x_1, \ldots, x_{\ell_n}$, it belongs to K_n. Hence, $f_n \circ g : K_n \to K_n$ is a continuous function. But K_n is homeomorphic to a compact convex set of a finite dimensional vector space. Hence, we are in a position to apply the Brouwer xed point theorem to find $y_n \in K_n$ such that $f_n(g(y_n)) = y_n$. Since K is compact, there exist a subsequence of $\{y_n\}_n$, not relabeled, and $y \in K$ such that $y_n \to y$. By applying (13) to $g(y_n)$ we get

$$\|y_n - g(y_n)\| = \|f_n(g(y_n)) - g(y_n)\| < \frac{1}{n}.$$

Letting $n \to \infty$ and using the fact that g is continuous gives $y = g(y)$. ■

Wednesday, March 1, 2023

Given two normed spaces X and Y and a set $E \subseteq X$ and a function $f : E \to Y$, we say that f is *compact* if for every bounded sequence $\{x_n\}_n$ there exist a subsequence $\{x_{n_k}\}_k$ and $y \in Y$ such that $g(x_{n_k}) \to y$ as $k \to \infty$.

Corollary 104 *Let X be a Banach space, let $K \subset X$ be a bounded, closed, convex set and let $g : K \to K$ be a continuous and compact function. Then g has a fixed point.*

Proof. The set $g(K)$ is relatively compact, that is, its closure is compact. Let C be the closure of the convex hull $\overline{g(K)}$, that is,

$$C = \overline{\mathrm{co}(\overline{g(K)})}.$$

Note that $g(K) \subseteq K$, and K is closed, so $\overline{g(K)} \subseteq K$. Since K is convex, $\mathrm{co}(\overline{g(K)}) \subseteq K$, and since K is closed, $C \subseteq K$. Also, $g(C) \subseteq g(K) \subseteq C$, so $g : C \to C$.

Assuming that C is compact (homework), by the previous theorem, g has a fixed point, so there is $x \in C \subseteq K$ such that $g(x) = x$. ■

Carathéodory's theorem improves Remark 102 in that it limits the number of terms in the convex combination to at most $N + 1$.

Theorem 105 (Carathéodory) *Let $E \subseteq \mathbb{R}^N$. Then*

$$\mathrm{co}E = \left\{ \sum_{i=1}^{N+1} t_i \boldsymbol{x}_i : \sum_{i=1}^{N+1} t_i = 1,\, t_i \geq 0,\, \boldsymbol{x}_i \in E,\, i = 1, \ldots, N+1 \right\}.$$

Proof. Fix $\boldsymbol{x} \in \mathrm{co}E$ and let

$$S := \{\ell \in \mathbb{N} : \boldsymbol{x} \text{ is a convex combination of } \ell \text{ vectors of } E\}.$$

Note that by Remark 102, S is nonempty. Let $k := \min S$. We claim that $k \leq N + 1$. Assume by contradiction that $k > N + 1$ and let

$$\boldsymbol{x} = \sum_{i=1}^{k} t_i \boldsymbol{x}_i,$$

where $\sum_{i=1}^{k} t_i = 1$, $t_i \in (0,1)$, $\boldsymbol{x}_i \in E$, $i = 1, \ldots, k$. Since $k - 1 > N$, the $k - 1$ vectors $\boldsymbol{x}_2 - \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k - \boldsymbol{x}_1$ are linearly dependent, and so we may find $s_2, \ldots, s_k \in \mathbb{R}$ not all zero such that

$$\sum_{i=2}^{k} s_i (\boldsymbol{x}_i - \boldsymbol{x}_1) = 0.$$

Let $s_1 := -\sum_{i=2}^{k} s_i$. Then $\sum_{i=1}^{k} s_i \boldsymbol{x}_i = 0$ and $\sum_{i=1}^{k} s_i = 0$. Since not all the s_i are zero, there must be positive ones. Define

$$c := \min \left\{ \frac{t_i}{s_i} : s_i > 0, \ i = 1, \ldots, k \right\}$$

and let m be such that $c = \frac{t_m}{s_m}$. Then $t_i - c s_i \geq 0$ for all $i = 1, \ldots, k$ (if $s_i > 0$, then this follows from the definition of c, while if $s_i \leq 0$, then $-c s_i \geq 0$), $t_m - c s_m = 0$, and

$$\sum_{i=1}^{k} (t_i - c s_i) = \sum_{i=1}^{k} t_i - c \sum_{i=1}^{k} s_i = 1 - 0.$$

Since

$$\boldsymbol{x} = \sum_{i=1}^{k} t_i \boldsymbol{x}_i = \sum_{i=1}^{k} t_i \boldsymbol{x}_i - 0 = \sum_{i=1}^{k} (t_i - c s_i) \boldsymbol{x}_i,$$

we have written $\boldsymbol{x}$ as a convex combination of less than k elements ($t_m - c s_m = 0$), which contradicts the definition of k. ■

Exercise 106 *Let $K \subset \mathbb{R}^N$ be compact. Prove that* $\operatorname{co} K$ *is compact.*

Exercise 107 *Let $(X, \|\cdot\|)$ be a normed space and let $K \subset X$ be compact. Prove that* $\operatorname{co} K$ *is pre-compact.*

10 Application of SFPT: ODE

An important application of Schauder's fixed point theorem is local existence of solutions to the Cauchy problem in the case that $\boldsymbol{f}$ is only continuous. To prove it, we need first to understand compactness in the space of continuous functions.

Definition 108 *A metric space (X, d) is* separable *if there exists a sequence $\{x_n\}_n$ in X that is dense in X.*

Example 109 *We discuss separability of some of the examples introduced before.*

(i) $\mathbb{R}^N$ is separable, since $\mathbb{Q}^N$ is dense in $\mathbb{R}^N$.

(ii) *Given a nonempty set X with discrete metric*

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

X is separable if and only if X is countable. Why?

(iii) *Using uniform continuity, one can show that piecewise affine functions are dense in $C([a,b])$. By approximating a piecewise affine function with one with rational slopes and endpoints, it follow that $C([a,b])$ is separable.*

(iv) *$\ell^\infty = \ell^\infty(\mathbb{N})$ is not separable (exercise).*

(v) *The space $C_b(\mathbb{R})$ of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ is not separable (exercise).*

Exercise 110 *Let (X,d) be a compact metric space. Prove that X is separable and complete.*

Exercise 111 *Let (X,d) be a separable metric space and let $E \subseteq X$. Prove that (E,d) is separable.*

The previous exercise fails for topological spaces.

Example 112 *Let $X := C([0,1])$. The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0,1]$$

is bounded in $C([0,1])$, but no subsequence converges uniformly to a continuous function. This shows that $\overline{B_X(0,1)}$ is closed and bounded but not compact. Hence, Bolzano–Weierstrass theorem fails for infinite dimensional metric spaces.

Definition 113 *Let (X,d_X) and (Y,d_Y) be metric spaces and let $E \subseteq X$. A family $\mathcal{F}$ of functions $f : E \to Y$ is said to be* equicontinuous at a point $x_0 \in E$ *if for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that*

$$d_Y(f(x), f(x_0)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x \in E$ with $d(x,x_0) \leq \delta$. The family $\mathcal{F}$ of functions $f : E \to Y$ is said to be uniformly equicontinuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in E$ with $d(x,y) \leq \delta$.

Remark 114 *To negate equicontinuity at one point x_0 it is enough to show that there exist a sequence $\{x_n\}_n$ in E and a sequence $\{f_n\}_n$ in $\mathcal{F}$ such that $x_n \to x_0$ but $d_Y(f_n(x_n), f_n(x_0)) \nrightarrow 0$.*

Example 115 *The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0,1],$$

is not equicontinuous at $x = 1$*. To see this, take* $x_n = 1 - \frac{1}{n} \to 1$*. Then*

$$f_n(1) - f_n(x_n) = 1 - \left(1 - \frac{1}{n}\right)^n \to 1 - \frac{1}{e} \neq 0,$$

and so by the previous remark, $\{f_n\}_n$ *is not equicontinuous at* $x = 1$*.*

Example 116 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a family* $\mathcal{F}$ *of functions from* X *into* Y*. If there exist* $\alpha \in (0,1]$ *if there exists* $L > 0$ *such that*

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in X$ *and for all* $f \in \mathcal{F}$*, then the family* $\mathcal{F}$ *is uniformly equicontinuous. The sequence of functions*

$$f_n(x) = \frac{x^n}{n}, \quad x \in [0,1],$$

is pointwise bounded and equicontinuous at $x = 1$*. Indeed,*

$$f_n'(x) = x^{n-1}, \quad x \in [0,1],$$

so that $\max_{x \in [0,1]} \left|x^{n-1}\right| = 1$*, which shows that the sequence* $\{f_n\}$ *is equi-Lipschitz (take* $L = 1$*). Hence, it is (uniformly) equicontinuous.*

Friday, March 3, 2023

Theorem 117 (Ascoli–Arzelà) *Let* (X, d) *be a separable metric space and let* $\mathcal{F} \subseteq C_b(X)$ *be a family of functions. Assume that* $\mathcal{F}$ *is bounded and equicontinuous at every point* $x \in X$*. Then every sequence* $\{f_n\}_n$ *in* $\mathcal{F}$ *has a subsequence* $\{f_{n_j}\}_j$ *that converges pointwise to a function* $g \in C_b(X)$ *and uniformly on every compact subset of* X*.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ has infinite many elements, otherwise there is nothing to prove. Since X is separable, there exists a countable set $E \subseteq X$ such that $X = \overline{E}$. Since $\mathcal{F}$ is bounded, there exists $M > 0$ such that

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \leq M \tag{14}$$

for all $f \in \mathcal{F}$.

Step 1: Let $\{f_n\}_n$ be a sequence in $\mathcal{F}$. We claim that there exists a subsequence $\left\{f_{n_j}\right\}_j$ such that the limit $\lim_{j\to\infty} f_{n_j}(x)$ exists in $\mathbb{R}$ for all $x \in E$. The proof makes use of the *Cantor diagonal argument*. Write $E = \{x_k : k \in G \subseteq \mathbb{N}\}$. Since the set

$$\{f_n(x_1) : n \in \mathbb{N}\}$$

is bounded in $\mathbb{R}$ by (14), by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,1}\}_n$ of $\{f_n\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,1}(x_1) = \ell_1 \in \mathbb{R}.$$

Since the set

$$\{f_{n,1}(x_2) : n \in \mathbb{N}\}$$

is bounded in $\mathbb{R}$ by (14), again by the Bolzano–Weierstrass theorem we can find a subsequence sequence $\{f_{n,2}\}_n$ of $\{f_{n,1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,2}(x_2) = \ell_2 \in \mathbb{R}.$$

By induction for every $k \in G$, $k > 1$, we can find a subsequence $\{f_{n,k}\}_n$ of $\{f_{n,k-1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,k}(x_k) = \ell_k \in \mathbb{R}.$$

We now consider the diagonal elements of the infinite matrix, that is, the sequence $\{f_{n,n}\}_n$. For every fixed $x_k \in E$ we have that the sequence $\{f_{n,n}(x_k)\}_{n=k}^{\infty}$ is a subsequence of $\{f_{n,k}(x_k)\}_n$, and thus it converges to ℓ_k as $n \to \infty$. This completes the proof of the claim. Set $g_n := f_{n,n}$ and define $g : E \to \mathbb{R}$ by

$$g(x) := \lim_{n\to\infty} g_n(x) \in \mathbb{R}, \quad x \in E. \tag{15}$$

Step 2: Fix $\varepsilon > 0$ and $x \in X$. By equicontinuity, there exists $\delta_{x,\varepsilon} > 0$ (depending on x and ε) such that

$$|f_n(x) - f_n(y)| < \varepsilon \tag{16}$$

for all $n \in \mathbb{N}$ and for all $y \in X$ with $d(x,y) < \delta_{x,\varepsilon}$. Since E is dense in X there exists $y \in E$ with $d(x,y) < \delta_{x,\varepsilon}$. Using (15), we have that there exists an integer $n_{\varepsilon,y} \in \mathbb{N}$ (depending on ε and y) such that

$$|g_n(y) - g(y)| < \varepsilon \tag{17}$$

for all $n \in \mathbb{N}$ with $n \geq n_{\varepsilon,y}$. Using (16) and (17), we have that

$$\begin{aligned}|g_n(x) - g_m(x)| &\leq |g_n(x) - g_n(y)| + |g_n(y) - g(y)| \\ &\quad + |g_m(y) - g(y)| + |g_m(y) - g_m(x)| < 4\varepsilon\end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n, m \geq n_{\varepsilon,y}$, which shows that the sequence $\{g_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, there exists

$$g(x) := \lim_{n\to\infty} g_n(x) \in \mathbb{R}. \tag{18}$$

Moreover, since

$$|g_n(x) - g_n(y)| < \varepsilon$$

for all $y \in X$ with $d(x,y) < \delta_{x,\varepsilon}$ and for all $n \in \mathbb{N}$, letting $n \to \infty$, we conclude that

$$|g(x) - g(y)| \leq \varepsilon$$

for all $y \in X$ with $d(x,y) < \delta_{x,\varepsilon}$, which shows that g is continuous at x (with the same $\delta_{x,\varepsilon}$)

Since this is true for every $x \in X$, we have proved that $\{g_n\}$ converges pointwise to a continuous function g. By (14), we have that $|g_n(x)| \leq M$ for all $x \in X$. Letting $n \to \infty$, we conclude that

$$|g(x)| \leq M$$

for all $x \in X$. This proves that g belongs to the space $C_b(X)$.

Step 3: It remains to show that $\{g_n\}$ converges to g uniformly on compact sets. Let $K \subseteq X$ be compact. Fix $\varepsilon > 0$ and let $\delta_{x,\varepsilon} > 0$ be the number given in (16). Since

$$K \subseteq \bigcup_{x \in K} B(x, \delta_{x,\varepsilon}),$$

by compactness there exist $x_1, \ldots, x_M \in K$ such that

$$K \subseteq \bigcup_{i=1}^{M} B(x_i, \delta_{x_i,\varepsilon}).$$

Using (18), for all $i = 1, \ldots, M$ we have that there exists an integer $n_{\varepsilon,x_i} \in \mathbb{N}$ such that

$$|g_n(x_i) - g(x_i)| \leq \varepsilon \tag{19}$$

and for all $n \in \mathbb{N}$ with $n \geq n_{\varepsilon,x_i}$. Let $n_\varepsilon = \max\{n_{\varepsilon,x_1}, \ldots, n_{\varepsilon,x_M}\}$. Let $n \geq n_\varepsilon$ and $x \in K$. Then x belongs to $B(x_i, \delta_{x_i,\varepsilon})$ for some i. Using (16) and (19), we have that

$$|g_n(x) - g(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g(x_i)| + |g(x_i) - g(x)| \leq 3\varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$. Thusm, for all $x \in K$ and all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, we have

$$\sup_{x \in K} |g_n(x) - g(x)| \leq 3\varepsilon,$$

which shows that $\{g_n\}$ converges to g uniformly on K. ■

Remark 118 *If we assume the stronger hypothesis that $\mathcal{F}$ is uniformly equicontinuous, then the function g turns out to be uniformly continuous.*

Monday, March 13, 2023

Corollary 119 *Let (X, d) be a compact metric space. Then $\mathcal{F} \subseteq C(X)$ is compact if and only if is closed, bounded, and uniformly equicontinuous.*

Proof. If $\mathcal{F}$ is closed, bounded, and uniformly equicontinuous, then by the previous theorem it follows that $\mathcal{F}$ is sequentially compact, and so by Theorem 96, $\mathcal{F}$ is compact.

Conversely, assume that $\mathcal{F} \subseteq C(X)$ is compact. Then by a theorem proved last semester, $\mathcal{F}$ is bounded. It remains to show that $\mathcal{F}$ is uniformly equicontinuous. Assume, by contradiction, that this is not the case. Then there exist $\varepsilon > 0$, $\{f_n\}_n$ in $\mathcal{F}$, and $\{x_n\}, \{y_n\}$ in X such that

$$|f_n(x_n) - f_n(y_n)| > \varepsilon$$

and $d(x_n, y_n) \le \frac{1}{n}$. Since X is compact (and so sequentially compact), there exist a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $x_0 \in X$ such that $d(x_{n_k}, x_0) \to 0$ as $k \to \infty$. In turn, since $\{f_{n_k}\}_k$ is in $\mathcal{F}$, again by Theorem 96, there exist a subsequence $\left\{f_{n_{k_j}}\right\}_j$ of $\{f_{n_k}\}_k$ and $f_0 \in C(X)$ such that $d_{C(X)}(f_{n_{k_j}}, f_0) \to 0$ as $j \to \infty$. In particular, for all j sufficiently large, say, $j \ge j_0$,

$$\max_{x \in X} \left| f_{n_{k_j}}(x) - f_0(x) \right| < \frac{\varepsilon}{4}. \tag{20}$$

Using the continuity of f_0 at x_0, we may find $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|f_0(x) - f_0(x_0)| < \frac{\varepsilon}{4} \tag{21}$$

for all $x \in X$ with $d(x, x_0) \le \delta$. Since $d(x_{n_{k_j}}, x_0) \to 0$ and $d(x_{n_{k_j}}, y_{n_{k_j}}) \to 0$, by taking j_0 larger, if necessary, we may assume that $d(x_{n_{k_j}}, x_0) \le \delta$ and $d(y_{n_{k_j}}, x_0) \le \delta$ for all $j \ge j_0$. Hence, by (20) and (21), for all $j \ge j_0$,

$$\begin{aligned}\varepsilon < \left|f_{n_{k_j}}(x_{n_{k_j}}) - f_{n_{k_j}}(y_{n_{k_j}})\right| &\le \left|f_{n_{k_j}}(x_{n_{k_j}}) - f_0(x_{n_{k_j}})\right| + \left|f_0(x_{n_{k_j}}) - f_0(x_0)\right| \\ + \left|f_0(x_0) - f_0(y_{n_{k_j}})\right| &+ \left|f_0(y_{n_{k_j}}) - f_{n_{k_j}}(y_{n_{k_j}})\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,\end{aligned}$$

which is a contradiction. ■

As an application of the previous theorem and of Schauder's fixed point theorem, we can prove local existence of the Cauchy problem only assuming $\boldsymbol{f}$ continuous.

Theorem 120 (Local Existence) *Let $I = [t_0, t_0 + T_0]$, where $t_0 \in \mathbb{R}$ and $T_0 > 0$, let $\boldsymbol{u}_0 \in \mathbb{R}^d$, let $r > 0$, and let $\boldsymbol{f} : I \times \overline{B_d(\boldsymbol{u}_0, r)} \to \mathbb{R}^d$ be a continuous function. Then there exists $0 < T \le T_0$ such that the Cauchy problem*

$$\begin{cases} \boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u}(t)), \\ \boldsymbol{u}(t_0) = \boldsymbol{u}_0 \end{cases}$$

admits a solution in some interval $[t_0, t_0 + T]$.

Proof. By the Weierstrass theorem there exists

$$M := \max\{\|\boldsymbol{f}(t, \boldsymbol{z})\| : (t, \boldsymbol{z}) \in I \times \overline{B_d(\boldsymbol{u}_0, r)}\}.$$

Consider $0 < T \leq \min\{T_0, r/M\}$ and let $X = C([t_0, t_0 + T]; \mathbb{R}^d)$, with the supremum norm $\|\cdot\|_\infty$. Let $K = \{\boldsymbol{g} \in C([t_0, t_0 + T]\,; \mathbb{R}^d) : \|\boldsymbol{g} - \boldsymbol{u}_0\|_\infty \leq r\}$. Note that K is closed and convex but not compact. Consider the the function

$$F : K \to X$$

given by

$$F(\boldsymbol{g})(t) = \boldsymbol{u}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{g}(s))\, ds$$

for $g \in X$ and $t \in [t_0, t_0 + T]$. We claim that $F(K) \subseteq K$. If $\boldsymbol{g} \in K$, then

$$\|F(\boldsymbol{g})(t) - \boldsymbol{u}_0\| = \left\| \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{g}(s))\, ds \right\| \leq \int_{t_0}^{t} \|\boldsymbol{f}(s, \boldsymbol{g}(s))\|\, ds \leq M(t - t_0) \leq MT \leq r,$$

provided $T \leq r/M$.

Let's prove that F is a sequentially continuous. Let $\boldsymbol{g}_n \in K$ be such that $\boldsymbol{g}_n \to \boldsymbol{g}$ uniformly in $[a, b]$. Then

$$\begin{aligned} \|F(\boldsymbol{g}_n)(t) - F(\boldsymbol{g})(t)\| &= \left\| \int_{t_0}^{t} [\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}(s))]\, ds \right\| \\ &\leq \int_{t_0}^{t_0+T} \|\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}(s))\|\, ds \end{aligned}$$

and so taking the maximum over all $t \in [t_0, t_0 + T]$, we get

$$\|F(\boldsymbol{g}_n) - F(\boldsymbol{g})\|_\infty \leq \int_{t_0}^{t_0+T} \|\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}(s))\|\, ds.$$

Since $\|\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}(s))\| \leq 2M$ and $\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}(s)) \to 0$, it follows by the Lebesgue dominated convergence theorem the righ-hand side goes to zero. This shows that F is sequentially continuous.

Next we shows that F is compact. Let $\boldsymbol{g}_n \in K$. We claim that the set $\{F(\boldsymbol{g}_n) : n \in \mathbb{N}\}$ is bounded and uniformly equicontinuous in X. We have

$$\begin{aligned} \|F(\boldsymbol{g}_n)(t_2) - F(\boldsymbol{g}_n)(t_1)\| &= \left\| \int_{t_1}^{t_2} [\boldsymbol{f}(s, \boldsymbol{g}_n(s)) - \boldsymbol{f}(s, \boldsymbol{g}_n(s))]\, ds \right\| \\ &\leq \int_{t_1}^{t_2} \|\boldsymbol{f}(s, \boldsymbol{g}_n(s))\|\, ds \leq M(t_2 - t_1), \end{aligned}$$

which proves that $\{F(\boldsymbol{g}_n) : n \in \mathbb{N}\}$ uniformly equicontinuous in X. We have already seen that $F(K) \subseteq K$. Hence, we can apply the Ascoli-Arzelá's theorem to show that $\{F(\boldsymbol{g}_n)\}_n$ admits a convergence subsequence. This proves that F is compact.

By Schauder's theorem there exists a fixed point $\boldsymbol{u} \in X$, that is,

$$\boldsymbol{u}(t) = F(\boldsymbol{u})(t) = \boldsymbol{u}_0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{u}(s))\, ds$$

for all $t \in [t_0, t_0 + T]$. Since $\boldsymbol{u}$ is continuous, the right-hand side is of class C^1, and so $\boldsymbol{u}$ is actually of class C^1. By differentiating both sides, we get that $\boldsymbol{u}$ is a solution of the ODE. Moreover, $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. Since any other solution of the initial value problem is a fixed point of F, we have uniqueness. ■

In general the solution will not be unique.

Example 121 *Consider the Cauchy problem*

$$\begin{cases} u'(t) = \sqrt{|u(t)|}, \\ u(0) = 0. \end{cases}$$

One solution is $u_1(t) \equiv 0$. To find a second assume that $u(t) = 0$ for all $t \in [0, a]$ and that $u(t) \neq 0$ for $t > a$. Then, since $u' > 0$, it follows that u is strictly increasing after a. In particular, for $t > a$,

$$\frac{u'(t)}{\sqrt{u(t)}} = 1.$$

Integrating both sides between $a + \varepsilon$ and t we get

$$\int_{a+\varepsilon}^{t} \frac{u'(s)}{\sqrt{u(s)}}\, ds = \int_{a+\varepsilon}^{t} 1\, ds = t - (a + \varepsilon).$$

Using the change of variable $y = u(s)$ gives

$$\int_{u(a+\varepsilon)}^{u(t)} \frac{1}{\sqrt{y}}\, dy = t - a - \varepsilon.$$

Hence,

$$[2\sqrt{y}]_{u(a+\varepsilon)}^{u(t)} = t - a - \varepsilon,$$

that is, $2\sqrt{u(t)} - 2\sqrt{u(a+\varepsilon)} = t - a - \varepsilon$. Since u is continuous, letting $\varepsilon \to 0^+$ gives $2\sqrt{u(t)} - 0 = t - a$. Since $u > 0$ for $t > a$, we have that

$$u(t) = \frac{1}{4}(t - a)^2.$$

Thus for every $a > 0$, the function

$$u(t) = \begin{cases} 0 & \text{for } t \leq a, \\ \frac{1}{4}(t-a)^2 & \text{for } t > a, \end{cases}$$

is a solution.

Wednesday, March 15, 2023

Part II
Functions Spaces

11 Continuous Functions

We have already studied completeness and compactness in the space of continuous functions. It remains to study density and separability.

Theorem 122 (Stone) *Let (X, d) be a compact metric space and let $\mathcal{F} \subseteq C(X)$ be a family of functions such that*

(i) $\mathcal{F}$ separates points; that is, if $x, y \in X$ with $x \neq y$, then there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$,

(ii) $\mathcal{F}$ contains the constant functions,

(iii) $\mathcal{F}$ is an algebra; that is, if $f, g \in \mathcal{F}$ and $t \in \mathbb{R}$, then $f + g$, fg, and tf belong to $\mathcal{F}$.

Then $\mathcal{F}$ is dense in $C(X)$.

We begin with some preliminary results.

Lemma 123 (Dini) *Let (X, d) be a metric space, let $K \subseteq X$ be a compact set and let $f_n : K \to \mathbb{R}$ be continuous functions such that $f_{n+1}(x) \leq f_n(x)$ for all $x \in K$ and all $n \in \mathbb{N}$. If $\{f_n\}$ converges pointwise in K to a continuous function $f : K \to \mathbb{R}$, then $\{f_n\}$ converges uniformly in K to f.*

Proof. Define $g_n := f_n - f$. Then g_n is continuous, $0 \leq g_{n+1}(x) \leq g_n(x)$ for all $x \in K$ and all $n \in \mathbb{N}$ and $\{g_n\}$ converges pointwise to zero in K. We need to prove that $\{g_n\}$ converges uniformly to zero in K. Let $\varepsilon > 0$ and consider $g_n^{-1}((-\infty, \varepsilon))$. Since g_n is continuous, there exists an open set U_n such that

$$U_n \cap K = g_n^{-1}((-\infty, \varepsilon)).$$

Moreover, since $g_{n+1} \leq g_n$, if $g_n(x) < \varepsilon$, then $g_{n+1}(x) < \varepsilon$, and so $g_n^{-1}((-\infty, \varepsilon)) \subseteq g_{n+1}^{-1}((-\infty, \varepsilon))$. Thus, we can assume that $U_n \subseteq U_{n+1}$ for every n. We claim that the family of open sets U_n covers K. Indeed, given $x \in K$, since $\lim_{n\to\infty} g_n(x) = 0$, there exists $n_{\varepsilon,x} \in \mathbb{N}$ such that $0 \leq g_n(x) < \varepsilon$ for all $n \geq n_{\varepsilon,x}$. Thus, $x \in U_n$ for all $n \geq n_{\varepsilon,x}$. This proves the claim.

Since $K \subseteq \bigcup_n U_n$, by compactness, there exists N such that $K \subseteq \bigcup_{n=1}^N U_n$. But since $U_n \subseteq U_{n+1}$, if follows that $K \subseteq U_N \subseteq U_n$ for all $n \geq N$, that is

$$0 \leq g_n(x) < \varepsilon$$

for all $x \in K$ and all $n \geq N$. This implies that

$$\sup_K |g_n| \leq \varepsilon$$

for all $n \geq N$, which is what we wanted to prove. ■

Example 124 *If the hypothesis that K is compact is dropped, then the lemma fails. Take $K = [0,1)$ and $f_n(x) = x^n$. Then f_n converges pointwise to 0, $f_n \geq f_{n+1}$, but we do not have uniform convergence, since* $\sup_{[0,1)} |f_n - 0| = \sup_{x\in[0,1)} x^n = 1$.

Example 125 *If the hypothesis that f is continuous is dropped, then the theorem fails. Take $K = [0,1]$ and $f_n(x) = x^n$. Then f_n converges pointwise to*

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x < 1, \end{cases}$$

$f_n \geq f_{n+1}$, but we do not have uniform convergence, since $\sup_{[0,1]} |f_n - f| = \sup_{x\in(0,1)} x^n = 1$.

Lemma 126 *For $n \in \mathbb{N}$ and $x \in [0,1]$ define recursively*

$$p_0(x) := 0, \quad p_n(x) := p_{n-1}(x) + \frac{1}{2}\left[x - p_{n-1}^2(x)\right].$$

Then $\{p_n\}_n$ is a sequence of polynomials converging uniformly to $f(x) := \sqrt{x}$, $x \in [0,1]$.

Proof. That each p_n is a polynomial follows by induction. We claim that

$$0 \leq p_n(x) \leq \sqrt{x}$$

for all $x \in [0,1]$ and all n. Indeed, assume this is true for $n-1$, then

$$p_n(x) = p_{n-1}(x) + \frac{1}{2}\left[x - p_{n-1}^2(x)\right] \leq \sqrt{x}$$

if and only if

$$(\sqrt{x} - p_{n-1}(x))\left[1 - \frac{1}{2}(p_{n-1}(x) + \sqrt{x})\right] \geq 0,$$

which holds since $\sqrt{x} - p_{n-1}(x) \geq 0$ and $\frac{1}{2}(p_{n-1}(x) + \sqrt{x}) \leq \sqrt{x} \leq 1$. Thus the claim holds.

Since $x - p_{n-1}^2(x) \geq 0$, it follows that $p_n(x) \geq p_{n-1}(x)$, and thus there exists

$$0 \leq \lim_{n\to\infty} p_n(x) = g(x) \leq \sqrt{x}.$$

Letting $n \to \infty$ in $p_n(x) = p_{n-1}(x) + \frac{1}{2}\left[x - p_{n-1}^2(x)\right]$ gives $g(x) = g(x) + \frac{1}{2}\left[x - g^2(x)\right]$, which shows that $g(x) = \sqrt{x}$. Since g is continuous, by Dini's theorem (your homework) applied to $\{f - p_n\}_n$ we conclude that $\{p_n\}_n$ converges uniformly to f. ■

Exercise 127 *Let (X,d) be a metric space and let $E \subseteq X$. Then $x_0 \in \overline{E}$ if and only if there exists a sequence $\{x_n\}_n$ in E such that $x_n \to x_0$*

We now turn to the proof of Theorem 122.

Proof of Theorem 122. Step 1: We claim that $\overline{\mathcal{F}}$ satisfies properties (i)-(iii). We only need to prove property (iii). Given $f, g \in \overline{\mathcal{F}}$ and $t \in \mathbb{R}$, by the previous exercise there exist $\{f_n\}_n, \{g_n\}_n$ in $\mathcal{F}$ such that $d_\infty(f_n, f) \to 0$ and $d_\infty(g_n, f) \to 0$. By property (iii), $f_n + g_n$, $f_n g_n$, and tf_n belong to $\mathcal{F}$. Since $d_\infty(f_n + g_n, f + g) \to 0$, $d_\infty(f_n g_n, fg) \to 0$, and $d_\infty(tf_n, tf) \to 0$ (exercise), it follows again by the previous exercise, that $f + g$, fg, and tf belong to $\overline{\mathcal{F}}$. It remains to show that $\overline{\mathcal{F}}. = C(X)$.

Step 2: We prove that if f belongs to $\overline{\mathcal{F}}$, then so does $|f|$. Since X is compact, by the Weierstrass theorem f is bounded by some constant $M > 0$. Define

$$g(x) := \frac{|f(x)|}{M}, \quad x \in X.$$

Then $g(x) \in [0, 1]$. In view of (iii), it suffices to show that g belongs to $\overline{\mathcal{F}}$. By the previous exercise there exists a sequence of polynomials p_n that converges uniformly in $[0, 1]$ to the function $h(t) := \sqrt{t}$, $t \in [0, 1]$. Define

$$g_n(x) := p_n\left(\left(\frac{f(x)}{M}\right)^2\right), \quad x \in X.$$

Then g_n converges uniformly in X to the function $\sqrt{\left(\frac{f}{M}\right)^2} = g$. Since $\overline{\mathcal{F}}$ is an algebra, we have that $g_n \in \mathcal{F}$. Hence, using the fact that $\overline{\mathcal{F}}$ is closed, it follows that g belongs to $\overline{\mathcal{F}}$.

Step 3: We prove that if f, g belong to $\overline{\mathcal{F}}$, then so do $\max\{f, g\}$ and $\min\{f, g\}$. It is enough to observe that

$$\max\{f, g\} = \frac{1}{2}[f + g + |f - g|],$$
$$\min\{f, g\} = \frac{1}{2}[f + g - |f - g|].$$

Step 4: We prove that if $x, y \in X$ with $x \neq y$ and $\alpha, \beta \in \mathbb{R}$, then there exists $g \in \mathcal{F}$ such that $g(x) = \alpha$ and $g(y) = \beta$. To see this, use property (i) to find $f \in \mathcal{F}$ such that $f(x) \neq f(y)$ and define

$$g(z) := \frac{\alpha(f(z) - f(y)) - \beta(f(x) - f(z))}{f(x) - f(y)}, \quad z \in X.$$

■

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Proof. Step 5: We are now ready to prove that $\overline{\mathcal{F}} = C(X)$. Let $f \in C(X)$ and $\varepsilon > 0$. By the previous step, for every $x, y \in X$ there exists a function $g_{x,y} \in \overline{\mathcal{F}}$ such that $g_{x,y}(x) = f(x)$ and $g_{x,y}(y) = f(y)$. Define

$$U_{x,y} := \{z \in X : g_{x,y}(z) < f(z) + \varepsilon\},$$
$$V_{x,y} := \{z \in X : g_{x,y}(z) > f(z) - \varepsilon\}.$$

By the continuity of $g_{x,y}$ and f we have that $U_{x,y}$ and $V_{x,y}$ are open sets containing x and y. Since $\{U_{x,y}\}_{x\in X}$ is an open cover of X, it follows by compactness that there exist $x_1^{(y)},\dots,x_{m_y}^{(y)}\in X$ such that

$$\bigcup_{i=1}^{m_y} U_{x_i^{(y)},y} = X. \tag{22}$$

Define

$$g_y := \min\{g_{x_1^{(y)},y},\dots,g_{x_{m_y}^{(y)},y}\}.$$

Then g_y belongs to $\overline{\mathcal{F}}$ by Step 3 and by (22) and the definition of $U_{x_i,y}$ and $V_{x_i,y}$,

$$g_y(z) < f(z)+\varepsilon \text{ for all } z\in X, \tag{23}$$

$$g_y(z) > f(z)-\varepsilon \text{ for all } z\in V_y := \bigcap_{i=1}^{m_y} V_{x_i,y}. \tag{24}$$

Since V_y is open and contains y, the family $\{V_y\}_{y\in X}$ is an open cover of X. Again by compactness, there exist $y_1,\dots,y_n\in X$ such that

$$\bigcup_{i=1}^{n} V_{y_i} = X.$$

Define

$$g := \max\{g_{y_1},\dots,g_{y_n}\}.$$

Then g belongs to $\overline{\mathcal{F}}$ by Step 3 and by (23) and (24),

$$f(z)-\varepsilon < g(z) < f(z)+\varepsilon \text{ for all } z\in X.$$

Hence $\max_{z\in X}|f(z)-g(z)|\le\varepsilon$. Since $g\in\overline{\mathcal{F}}$, we may find $h\in\mathcal{F}$ such that $\max_{z\in X}|h(z)-g(z)|\le\varepsilon$, and thus, by the triangle inequality, $\max_{z\in X}|f(z)-h(z)|\le 2\varepsilon$. This concludes the proof. ■

Exercise 128 (Weierstrass) *Let $K\subset\mathbb{R}^N$ be a compact set. Prove that every continuous function $f:K\to\mathbb{R}$ is the uniform limit in K of a sequence of polynomials.*

Corollary 129 *Let (X,d) be a compact metric space. Then $C(X)$ is separable.*

Proof. Since X is separable by Exercise 110, there exists a sequence $\{x_n\}_n$ in X such that $\overline{\{x_n : n\in\mathbb{N}\}}=X$. For every n define

$$f_n(x) := d(x,x_n),\quad x\in X.$$

Then f_n is continuous. We claim that $\{f_n\}_n$ separates points. Indeed, assume the contrary. Then there exist $x,y\in X$ such that $f_n(x)=f_n(y)$ for every

$n \in \mathbb{N}$. By density we may find a subsequence $\{x_{n_k}\}_k$ X of $\{x_n\}_n$ such that $x_{n_k} \to x$. Hence,

$$d(y, x_{n_k}) = f_{n_k}(y) = f_{n_k}(x) = d(x, x_{n_k}) \to 0$$

as $k \to \infty$. Thus, $x_{n_k} \to y$. By the uniqueness of limits, it follows that $x = y$. This proves the claim.

Define $f_0 := 1$ and for every $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}_0$ define

$$f_{n_1,\dots,n_k}(x) := f_{n_1}(x) \cdots f_{n_k}(x), \quad x \in X.$$

Consider the family $\mathcal{F}$ given by all finite linear combinations of functions of the form $f_{n_1,\dots,n_k}$. Then $\mathcal{F}$ satisfies the hypotheses of Stone's theorem, and so $\mathcal{F}$ is dense in $C(X)$. On the other hand, the family $\mathcal{F}'$ given by all finite rational linear combinations of functions of the form $f_{n_1,\dots,n_k}$ is countable. For every $f \in C(X)$ and $\varepsilon > 0$ we may find $g \in \mathcal{F}$ such that

$$d_\infty(f, g) \le \varepsilon.$$

Since g is a finite linear combinations of functions of the form $f_{n_1,\dots,n_k}$, using the density of the rationals in the real, we may find $h \in \mathcal{F}$ such that

$$d_\infty(h, g) \le \varepsilon.$$

This shows that $\mathcal{F}'$ is dense in $C(X)$ and, in turn, that $C(X)$ is separable. ■

Exercise 130 *Prove that $C_b(\mathbb{R})$ is not separable.*

12 L^p Spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space. For $1 \le p < \infty$, we define the space

$$M^p(X) := \left\{ f : X \to \mathbb{R} \text{ measurable and } \|f\|_{M^p(X)} < \infty \right\},$$

where

$$\|f\|_{M^p(X)} := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

For $p = \infty$, we define

$$M^\infty(X) := \{ f : X \to \mathbb{R} \text{ measurable and bounded} \},$$

where

$$\|f\|_{M^\infty(X)} := \sup_{x \in X} |f(x)|.$$

Note that property (ii) of the previous definition is satisfied. Indeed, for $1 \le p < \infty$ and for $t \in \mathbb{R}$,

$$\|tf\|_{M^p(X)} = \left(\int_X |tf|^p \, d\mu \right)^{1/p} = \left(|t|^p \int_X |f|^p \, d\mu \right)^{1/p} = |t| \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

Next we study the triangle inequality.

Let q be the *Hölder conjugate exponent* of p, i.e.,

$$q := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that, with an abuse of notation, we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the Hölder conjugate exponent of p will often be denoted by p'.

Theorem 131 (Hölder's inequality) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \le p \le \infty$, and let q be its Hölder conjugate exponent. If f, $g : X \to \mathbb{R}$ are Lebesgue measurable functions, then*

$$\int_X |fg| \, d\mu \le \left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |g|^q \, d\mu \right)^{1/q} \tag{25}$$

if $1 < p < \infty$,

$$\int_X |fg| \, d\mu \le \sup_{x \in X} |g(x)| \int_X |f| \, d\mu \tag{26}$$

if $p = 1$, and

$$\int_X |fg| \, d\mu \le \sup_{x \in X} |f(x)| \int_X |g| \, d\mu \tag{27}$$

if $p = \infty$. In particular, if $f \in M^p(X)$ and $g \in M^p(X)$ then $fg \in M^1(X)$.

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Proof. If $\|f\|_{M^p(X)} = 0$ or $\|g\|_{M^q(X)} = 0$, then $f(x)\,g(x) = 0$ for μ a.e. $x \in X$ and so there is nothing to prove. Thus assume that $\|f\|_{M^p(X)}$, $\|g\|_{M^q(X)} > 0$. If $\|f\|_{M^p(X)} = \infty$ or $\|g\|_{M^q(X)} = \infty$ then the right-hand side is ∞ and so the inequality (25) holds. Hence in what follows we consider the case $\|f\|_{M^p(X)}$, $\|g\|_{M^q(X)} \in (0, \infty)$.

Assume that $1 < p < \infty$. Since the function $t \in [0, \infty) \mapsto \ln t$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$, for any a, $b > 0$, we have

$$\ln \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) \ge \frac{1}{p} \ln a^p + \frac{1}{q} \ln b^q = \ln ab,$$

that is

$$\frac{1}{p} a^p + \frac{1}{q} b^q \ge ab,$$

which is known as *Young's inequality.*

If we take $a = |f(x)|$ and $b = |g(x)|$, we get

$$|f(x)\, g(x)| \leq \frac{1}{p} |f(x)|^p + \frac{1}{q} |g(x)|^q .$$

Upon integration, we obtain

$$\begin{aligned}\int_X |fg| \, d\mu &\leq \frac{1}{p} \int_X |f|^p \, d\mu + \frac{1}{q} \int_X |g|^q \, d\mu \\ &= \frac{1}{p} \|f\|^p_{M^p(X)} + \frac{1}{q} \|g\|^q_{M^q(X)} .\end{aligned}$$

To obtain the desired result, it suffices to replace f with tf, where $t > 0$, to obtain

$$\int_X |fg| \, d\mu \leq \frac{t^{p-1}}{p} \|f\|^p_{M^p(X)} + \frac{1}{tq} \|g\|^q_{M^q(X)} =: h(t) .$$

By minimizing the function h, we find that for

$$t = \frac{\|g\|^{q/p}_{M^q(X)}}{\|f\|_{M^p(X)}}$$

the inequality (25) holds.

If $p = 1$ and $q = \infty$, then

$$\begin{aligned}\int_X |fg| \, d\mu &\leq \int_X |f| \sup_{x \in X} |g(x)| \, d\mu \\ &= \sup_{x \in X} |g(x)| \int_X |f| \, d\mu.\end{aligned}$$

The case $p = \infty$ is similar. ■

Exercise 132 *Prove that if $f \neq 0$ and the right-hand side of* (25) *is finite, then the equality in* (25) *holds if and only if there exists $c \geq 0$ such that*

1. $|g| = c |f|^{p-1}$ *if* $1 < p < \infty$*;*
2. $|g| \leq c$ *and* $|g(x)| = c$ *whenever* $f(x) \neq 0$ *if* $p = 1$*;*
3. $|f| \leq c$ *and* $|f(x)| = c$ *whenever* $g(x) \neq 0$ *if* $p = \infty$*.*

Theorem 133 (Minkowski's inequality) *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, let $X \in \mathfrak{M}$ and let $f, g : X \to \mathbb{R}$ be measurable functions. Then,*

$$\|f + g\|_{M^p(X)} \leq \|f\|_{M^p(X)} + \|g\|_{M^p(X)} \tag{28}$$

whenever $\|f + g\|_{M^p(X)}$ is well-defined. In particular, if $f, g \in M^p(X)$, then $f + g \in M^p(X)$ and (28) *holds.*

Proof. If $\|f\|_{M^p(X)} = \infty$ or $\|g\|_{M^p(X)} = \infty$ then the right-hand side of Minkowski's inequality is ∞, and so there is nothing to prove. Thus assume that $\|f\|_{M^p(X)}, \|g\|_{M^p(X)} < \infty$.

We consider first the case $1 < p < \infty$. By the convexity of the function $t \in [0,\infty) \mapsto t^p$, for any a, $b > 0$, we have

$$(a+b)^p = 2^p \left(\frac{a+b}{2}\right)^p \le \frac{2^p}{2} a^p + \frac{2^p}{2} b^p = 2^{p-1}\left(a^p + b^p\right),$$

and so

$$\int_X |f+g|^p \, d\mu \le \int_X (|f| + |g|)^p \, d\mu \le 2^{p-1} \left(\int_X |f|^p \, d\mu + \int_X |g|^p \, d\mu \right),$$

which shows that $f + g \in M^p(X)$. To prove Minkowski's inequality, we observe that

$$\begin{aligned} \|f+g\|_{M^p}^p &= \int_X |f+g|^p \, d\mu = \int_X |f+g| \cdot |f+g|^{p-1} \, d\mu \\ &\le \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu. \end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned} \|f+g\|_{M^p(X)}^p &\le \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f+g|^{(p-1)p'} \, d\mu \right)^{\frac{1}{p'}} \\ &\quad + \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |f+g|^{(p-1)p'} \, d\mu \right)^{\frac{1}{p'}} \\ &\le \left(\|f\|_{M^p(X)} + \|g\|_{M^p(X)} \right) \|f+g\|_{M^p(X)}^{\frac{p}{p'}}, \end{aligned}$$

where we have used the fact that $(p-1)p' = p$. If $\|f+g\|_{M^p(X)} = 0$, then there is nothing to prove, thus assume that $\|f+g\|_{M^p(X)} \in (0,\infty)$. Hence, we may divide both sides of the previous inequality by $\|f+g\|_{M^p(X)}^{\frac{p}{p'}}$ to obtain

$$\|f+g\|_{M^p} \le \|f\|_{M^p} + \|g\|_{M^p},$$

where we have used the fact that $p - \frac{p}{p'} = 1$.

The cases $p = 1$ and $p = \infty$ are straightforwad. ■

We recall that

Definition 134 *Given a vector space X, a norm is a function $\|\cdot\| : X \to [0,\infty)$ such that*

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$ and $x \in X$;

(iii) $\|x+y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

In view of the previous theorem we now have that for $1 \leq p < \infty$, properties (ii) and (iii) of Definition 134 are satisfied. The problem is property (i). Indeed, if

$$\|f\|_{M^p} = \left(\int_X |f|^p \, d\mu \right)^{1/p} = 0,$$

then by there exists a set $E_0 \in \mathfrak{M}$ with $\mu(E_0) = 0$ such that $f(x) = 0$ for all $x \in X \setminus E_0$. This does not imply that the function f is zero. For example, the Dirichlet function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

has exactly this property.

To circumvent this proplem, given two measurable functions $f, g : X \to \mathbb{R}$, we say that f is *equivalent* to g, and we write

$$f \sim g \text{ if } f(x) = g(x) \text{ for } \mu \text{ a.e. } x \in X. \tag{29}$$

Note that $\sim$ is an equivalence relation in the class of measurable functions. Moreover, if $f(x) = 0$ for μ a.e. $x \in X$, then $f \sim 0$, or, equivalently, f belongs to equivalence class $[0]$.

Definition 135 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space, let* $X \in \mathfrak{M}$*, and let* $1 \leq p < \infty$*. We define*

$$L^p(X) := M^p(X) \diagup \sim = \left\{ [f] : f : X \to \mathbb{R} \text{ measurable and } \|f\|_{M^p(X)} < \infty \right\}.$$

In the space $L^p(X)$ we define the norm

$$\|[f]\|_{L^p(X)} := \|f\|_{M^p(X)}.$$

Note that $\|[f]\|_{L^p}$ does not depend on the choice of the representative. We now have that $(L^p(X), \|\cdot\|_{L^p})$ is a normed space, since properties (i)-(ii) of Definition 134 are satisfied.

Indeed, if $f \in L^p([0,1])$ (with the Lebsgue measure), then after the identification f is actually an equivalence class. Hence, for example, talking about the value $f(1)$ or $f\left(\frac{1}{2}\right)$ make no sense. Indeed, given any *number* $y \in \mathbb{R}$, in the equivalence class $[f]$ there is always a function g such that $g(1) = y$. Just define

$$g(x) := \begin{cases} y & \text{if } x = 1, \\ f(x) & \text{otherwise.} \end{cases}$$

Then f and g differ only at the point 1, and so $f \sim g$.

Let's now consider the case $p = \infty$. Unlike the case $1 \leq p < \infty$, the supremum of a function changes if we change the function even at one point. Thus, we cannot take as a norm $\|[f]\|_{L^\infty(X)} := \sup_{x \in X} |f(x)|$. What we need is a notion of supremum that does not change if we modify a function on a set of measure zero.

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Given a measurable function $f : X \to \mathbb{R}$ we define the *essential supremum* $\operatorname{esssup} f$ of the function f as

$$\operatorname{esssup} f := \inf \{t \in \mathbb{R} : \ f(x) \le t \text{ for } \mu \text{ a.e. } x \in X\}.$$

Note that if $M := \operatorname{esssup} f < \infty$, then by taking $t_n := M + \frac{1}{n}$ we can find $E_n \in \mathfrak{M}$ with $\mu(E_n) = 0$ such that

$$f(x) \le M + \frac{1}{n} \text{ for all } x \in X \setminus E_n.$$

Take

$$E_\infty := \bigcup_{n=1}^{\infty} E_n.$$

Then $\mu(E_\infty) \le \sum_{n=1}^{\infty} \mu(E_n) = 0$, and if $x \in X \setminus E_\infty$, then

$$f(x) \le M + \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get that $f(x) \le M$ for all $x \in X \setminus E_\infty$. Conversely, if there are $t \in \mathbb{R}$ and $E_0 \in \mathfrak{M}$ with $\mu(E_0) = 0$ such that $f(x) \le t$ for all $x \in X \setminus E_0$, then by definition of $\operatorname{esssup} f$, we have that $\operatorname{esssup} f \le t < \infty$. This shows that $\operatorname{esssup} f < \infty$ if and only if the function f is bounded from above except on a set of measure zero.

Moreover, if $f \sim g$ then $\operatorname{esssup} f = \operatorname{esssup} g$. This leads us to the following definition.

Definition 136 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $E \in \mathfrak{M}$. We define*

$$L^\infty(X) := \{[f] : \ f : X \to \mathbb{R} \text{ measurable and } \operatorname{esssup} |f| < \infty\}.$$

In the space $L^\infty(X)$ we define the norm

$$\|[f]\|_{L^\infty} := \operatorname{esssup} |f|.$$

Indeed, properties (i) and (ii) are satisfied. To prove property (iii), note that if $[f]$ and $[g]$ belong to $L^\infty(X)$, then there exist $E_0, F \in \mathfrak{M}$ with $\mu(E_0) = \mu(F) = 0$ such that $|f(x)| \le \operatorname{esssup} |f|$ for all $x \in X \setminus E_0$ and $|g(x)| \le \operatorname{esssup} |g|$ for all $x \in X \setminus F$. Hence,

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$$

for all $x \in X \setminus (E_0 \cup F)$, which implies that $\operatorname{esssup} |f + g| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$. Thus, the triangle inequality holds.

Remark 137 *Note that in Hölder's inequality one can replace* (26) *and* (27) *with*

$$\int_X |fg| \, d\mu \leq \operatorname{esssup} |g| \int_X |f| \, d\mu$$

and

$$\int_X |fg| \, d\mu \leq \operatorname{esssup} |f| \int_X |g| \, d\mu,$$

respectively. Indeed, in the first case, since $|g(x)| \leq \operatorname{esssup} |g|$ *for all* $x \in X \setminus E_0$, *where* $E_0 \in \mathfrak{M}$ *with* $\mu(E_0) = 0$, *we have that*

$$\begin{aligned}\int_X |fg| \, d\mu &= \int_{X\setminus E_0} |f| \, |g| \, d\mu \leq \int_{X\setminus E_0} |f| \operatorname{esssup} |g| \, d\mu \\ &= \operatorname{esssup} |g| \int_{X\setminus E_0} |f| \, d\mu \leq \operatorname{esssup} |g| \int_X |f| \, d\mu.\end{aligned}$$

With an abuse of notation, from now on we identify a measurable function $f : X \to \mathbb{R}$ with its equivalence class $[f]$. Note that this is very dangerous.

Wednesday, March 22, 2023

We now turn to the relation between different L^p spaces.

Theorem 138 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $X \in \mathfrak{M}$. *Suppose that* $1 \leq p < q < \infty$. *Then*

(i) $L^p(X)$ *is not contained in* $L^q(X)$ *if and only if* X *contains measurable sets of arbitrarily small positive measure;*

(ii) $L^q(X)$ *is not contained in* $L^p(X)$ *if and only if* X *contains measurable sets of arbitrarily large finite measure.*

Proof. (i) Assume that $L^p(X)$ is not contained in $L^q(X)$. Then there exists $[f] \in L^p(X)$ such that

$$\int_X |f|^q \, d\mu = \infty. \tag{30}$$

For each $n \in \mathbb{N}$ let

$$E_n := \{x \in X : |f(x)| > n\}.$$

Then

$$\mu(E_n) \leq \frac{1}{n^p} \int_X |f|^p \, d\mu \to 0$$

as $n \to \infty$. Thus, it suffices to show that $\mu(E_n) > 0$ for all n sufficiently large. If to the contrary, $\mu(E_n) = 0$ for infinitely many n, we have that

$$\int_X |f|^q \, d\mu = \int_{\{|f| \leq n\}} |f|^q \, d\mu \leq n^{q-p} \int_{\{|f| \leq n\}} |f|^p \, d\mu < \infty,$$

which is a contradiction with (30). Hence, X contains measurable sets of arbitrarily small positive measure.

Conversely, assume that X contains measurable sets of arbitrarily small positive measure. Then it is possible to construct a sequence of pairwise disjoint sets $\{E_n\}_n$ in $\mathfrak{M}$ such that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ and

$$\mu(E_n) \searrow 0.$$

Let

$$f := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \nearrow \infty$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^q \mu(E_n) = \infty, \quad \sum_{n=1}^{\infty} c_n^p \mu(E_n) < \infty. \tag{31}$$

Then $[f] \in L^p(X) \setminus L^q(X)$.
(ii) Assume that $L^q(X)$ is not contained in $L^p(X)$. Then there exists $[f] \in L^q(X)$ such that

$$\int_X |f|^p \, d\mu = \infty. \tag{32}$$

For each $n \in \mathbb{N}$ let

$$F_n := \left\{x \in X : \frac{1}{n+1} < |f(x)| \leq \frac{1}{n}\right\}$$

and let

$$F_\infty := \{x \in X : 0 < |f(x)| \leq 1\} = \bigcup_{n=1}^{\infty} F_n.$$

If $\mu(F_\infty) < \infty$, then

$$\begin{aligned}\int_X |f|^p \, d\mu &= \int_{\{|f|\leq 1\}} |f|^p \, d\mu + \int_{\{|f|>1\}} |f|^p \, d\mu \\ &\leq \mu(F_\infty) + \int_{\{|f|>1\}} |f|^q \, d\mu < \infty,\end{aligned}$$

which contradicts (32). Hence, $\mu(F_\infty) = \infty$. On the other hand, since for every $n \in \mathbb{N}$,

$$\infty > \int_X |f|^q \, d\mu \geq \int_{\left\{\frac{1}{n+1} < |f| \leq \frac{1}{n}\right\}} |f|^q \, d\mu \geq \frac{1}{(n+1)^q} \mu(F_n),$$

it follows that X contains measurable sets of arbitrarily large finite measure. Indeed, setting

$$G_n := \bigcup_{k=1}^{n} F_k,$$

we have that $\mu(G_n) < \infty$, while by Proposition **??**(i),

$$\mu(G_n) \to \mu(F_\infty) = \infty.$$

Conversely, assume that X contains measurable sets of arbitrarily large finite measure. Then it is possible to construct a sequence of pairwise disjoint sets $\{E_n\}_n$ in $\mathfrak{M}$ of finite measure such that

$$\mu(E_n) \nearrow \infty.$$

Let

$$f := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \searrow 0$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^q \mu(E_n) < \infty, \quad \sum_{n=1}^{\infty} c_n^p \mu(E_n) = \infty. \tag{33}$$

Then $[f] \in L^q(X) \setminus L^p(X)$. ∎

Remark 139 *Note that the previous proof works also for p, $q > 0$. What about $q = \infty$?*

Exercise 140 *(i) Let $X = [0,1]$ and let μ be the Lebesgue measure. Show that for every $1 \le p < \infty$ the function*

$$f(x) = \frac{1}{x^{1/p} \log^{2/p}\left(\frac{2}{x}\right)}$$

is in $L^p([0,1])$ but not in $L^q([0,1])$ for all $q > p$.

(ii) Construct sequences $c_n \nearrow \infty$ and $c_n \searrow 0$ for which conditions (31) *and* (33) *hold, respectively.*

Corollary 141 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $X \in \mathfrak{M}$. Suppose that $1 \le p < q \le \infty$. If $\mu(X) < \infty$, then*

$$L^q(X) \subseteq L^p(X).$$

Proof. When $1 \le q < \infty$, this follows from the previous theorem. There's also a direct proof. By Hölder's inequality (with $\frac{q}{p}$ in place of p and $|f|^p$ and 1 in place of f and g)

$$\int_X |f|^p \, d\mu \le \| |f|^p \|_{L^{\frac{q}{p}}} \|1\|_{L^{\left(\frac{q}{p}\right)'}} = \left(\int_X |f|^q \, d\mu\right)^{\frac{p}{q}} \|1\|_{L^{\left(\frac{q}{p}\right)'}}$$
$$= \left(\int_X |f|^q \, d\mu\right)^{\frac{p}{q}} (\mu(X))^{\frac{q-p}{p}}.$$

■

By identifying functions with their equivalence classes $[f]$, it follows from Minkowski's inequality that $\|\cdot\|_{L^p}$ is a norm on $L^p(X)$. Next we prove that $L^p(X)$ is a complete metric space, that is, that every Cauchy sequence has a limit in $L^p(X)$.

Theorem 142 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $X \in \mathfrak{M}$. Then $L^p(X)$ is a Banach space for $1 \leq p \leq \infty$.*

Proof. Assume that $1 \leq p < \infty$, and let $\{[f_n]\}_n$ be a Cauchy sequence in $L^p(X)$. Then for every $k \in \mathbb{N}$ we can find $n_k \in \mathbb{N}$ such that

$$\|[f_n] - [f_\ell]\|_{L^p} < \frac{1}{2^k}$$

for all $n, \ell \geq n_k$. Without loss of generality, we can assume that $n_{k+1} > n_k$ for every k. For $j \in \mathbb{N}$ consider the function

$$g_j(x) := \sum_{k=1}^{j} |f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

$$g(x) := \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

By Minkowki's inequality,

$$\|[g_j]\|_{L^p} \leq \sum_{k=1}^{j} \|[f_{n_{k+1}}] - [f_{n_k}]\|_{L^p} \leq \sum_{k=1}^{\infty} \frac{1}{2^k}.$$

Letting $j \to \infty$, it follows from Fatou's lemma that

$$\|[g]\|_{L^p} \leq \liminf_{j\to\infty} \|[g_j]\|_{L^p} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

Since

$$\int_X |g|^p d\mu < \infty,$$

there exists a set $F \subseteq X$ with $\mu(F) = 0$, such that $g(x) \in \mathbb{R}$ for all $x \in X \setminus F$.
■

Friday, March 24, 2023

Proof. It follows that the partial sum

$$f_{n_1}(x) + \sum_{k=1}^{j} f_{n_{k+1}}(x) - f_{n_k}(x)$$

converges absolutely for every $x \in X \setminus F$ to a function f. Define $f(x) := 0$ for $x \in F$. We claim that $[f_n] \to [f]$ in $L^p(X)$.

To see this, fix $\varepsilon > 0$ and let n_ε be such that

$$\|[f_n] - [f_\ell]\|_{L^p} < \varepsilon$$

for all $n, \ell \geq n_\varepsilon$. In particular, if $n_k \geq n_\varepsilon$,

$$\|f_n - f_{n_k}\|_{L^p} < \varepsilon.$$

Letting $k \to \infty$, it follows from Fatou's lemma that

$$\|[f_n] - [f]\|_{L^p} \leq \liminf_{k\to\infty} \|[f_n] - [f_{n_k}]\|_{L^p} \leq \varepsilon$$

■

Exercise 143 *Prove the case $p = \infty$.*

Remark 144 *The proof of the previous theorem implies that if $\{[f_n]\}_n$ converges to $[f]$ in $L^p(X)$, then there exist a subsequence $\{f_{n_k}\}_k$ that converges pointwise to f a.e. and*

$$|f_{n_k}(x)| \leq g(x) \quad \text{for a.e. } x \in X \text{ and for all } k,$$

where $[g] \in L^p(X)$.

Next we study some density results for $L^p(X)$ spaces.

Theorem 145 *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then the family of all simple functions in $L^p(X)$ is dense in $L^p(X)$ for $1 \leq p \leq \infty$.*

Proof. Assume first that $1 \leq p < \infty$. Let $[f] \in L^p(X)$. Since f^+, f^- are masurable, there exist increasing sequences $\{s_n\}_n$ and $\{t_n\}_n$ of simple functions such that $\{s_n(x)\}_n$ converges monotonically to $f^+(x)$ for μ a.e. $x \in X$ and $\{t_n(x)\}_n$ converges monotonically to $f^-(x)$ for μ a.e. $x \in X$. Then for each $n \in \mathbb{N}$ the function $S_n := s_n - t_n$ is still simple, belongs to $L^p(X)$, and

$$\begin{aligned} |f(x) - S_n(x)|^p &= \left|f^+(x) - s_n(x) - \left(f^-(x) - t_n(x)\right)\right|^p \\ &\leq 2^{p-1}\left(f^+(x) - s_n(x)\right)^p + 2^{p-1}\left(f^-(x) - t_n(x)\right)^p \\ &\leq 2^{p-1}\left(f^+(x)\right)^p + 2^{p-1}\left(f^-(x)\right)^p \end{aligned}$$

for μ a.e. $x \in X$. Since $f(x) - S_n(x) \to 0$ as $n \to \infty$ for μ a.e. $x \in X$, we may apply the Lebesgue dominated convergence theorem to conclude that $[S_n] \to [f]$ in $L^p(X)$.

The case $p = \infty$ is left as an exercise. ■

The next result gives conditions on X and μ that ensure the density of continuous functions in $L^p(X)$.

Theorem 146 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with X a metric space and μ a Borel measure such that*

$$\mu(E) = \sup\{\mu(C) : C \text{ closed, } C \subseteq E\} = \inf\{\mu(A) : A \text{ open, } A \supseteq E\}$$

for every set $E \in \mathfrak{M}$ with finite measure. Then $L^p(X) \cap C_b(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$.

Proof. Since by Theorem 145 simple functions in $L^p(X)$ are dense in $L^p(X)$, it suffices to approximate in $L^p(X)$ functions χ_E, with $E \in \mathfrak{M}$ and $\mu(E) < \infty$, by functions in $L^p(X) \cap C_b(X)$. Thus, fix $E \in \mathfrak{M}$ with $\mu(E) < \infty$, and for any $\varepsilon > 0$ find an open set $A \supseteq E$ and a closed set $C \subseteq E$ such that

$$\mu(A \setminus C) \leq \varepsilon^p.$$

Find (exercise) a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C and $f \equiv 0$ in $X \setminus A$. Since $\operatorname{supp} f \subseteq A$ and $\mu(A) < \infty$, it follows that $[f] \in L^p(X) \cap C_b(X)$. Moreover,

$$\int_X |\chi_E - f|^p \, d\mu = \int_{A \setminus C} |\chi_E - f|^p \, d\mu \leq \mu(A \setminus C) \leq \varepsilon^p,$$

and the result follows. ■

Definition 147 *A measurable space $(X, \mathfrak{M})$ is called* separable *if there exists a sequence $\{E_n\}_n$ in $\mathfrak{M}$ such that the smallest σ-algebra that contains all the sets E_n is $\mathfrak{M}$. In this case $\mathfrak{M}$ is said to be* generated *by the sequence $\{E_n\}_n$.*

Example 148 *The σ-algebra of all Lebesgue measurable sets in $\mathbb{R}^N$ is generated by the countable family of cubes with centers in $\mathbb{Q}^N$ and rational side length.*

Exercise 149 *Prove that if X is a separable metric space and $\mathfrak{M}$ is the Borel σ-algebra, then X is a separable measurable space.*

Theorem 150 *Let $(X, \mathfrak{M})$ be a separable measurable space with $\mathfrak{M}$ generated by a sequence $\{E_n\}_n$, and assume that μ is σ-finite. Let $\mathfrak{N}$ be the smallest algebra containing $\{E_n\}$. Then simple functions of the form*

$$\sum_{i=1}^{n} c_i \chi_{F_i},$$

where $n \in \mathbb{N}$, $c_i \in \mathbb{Q}$, and $F_i \in \mathfrak{N}$, $\mu(F_i) < \infty$, $i = 1, \ldots, n$, form a countable dense subset of $L^p(X)$ for $1 \leq p < \infty$. In particular, $L^p(X)$ is separable for $1 \leq p < \infty$.

The proof will be likely an exercise in a future homework.

To study compactness in L^p spaces, we take $X = \mathbb{R}^N$ with the Lebesgue measure.

Theorem 151 *Let $1 \leq p < \infty$. A set $\mathcal{F} \subseteq L^p(\mathbb{R}^N)$ is totally bounded if and only if*

(i) $\mathcal{F}$ is bounded;

(ii) for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(\mathbf{0}, R)} |f(\boldsymbol{x})|^p d\boldsymbol{x} < \varepsilon^p$$

for all $[f] \in \mathcal{F}$

(iii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p d\boldsymbol{x} < \varepsilon^p$$

for all $\boldsymbol{h} \in \mathbb{R}^N$ with $\|\boldsymbol{h}\| < \delta$ and for all $[f] \in \mathcal{F}$.

Monday, March 27, 2023

We will use the following lemma.

Lemma 152 *Let (X, d_X) be a metric space. Assume that for every $\varepsilon > 0$ there exist $\delta > 0$, a metric space (Y, d_Y), and a function $g : X \to Y$ such that $g(X)$ is totally bounded and whenever $x, z \in X$ are such that $d_Y(g(x), g(z)) < \delta$, then $d_X(x, z) < \varepsilon$. Then X is totally bounded.*

Proof. Since $g(X)$ is totally bounded, there exist $y_1, \ldots, y_n$ such that

$$g(X) \subseteq \bigcup_{i=1}^n B_Y(y_i, \tfrac{\delta}{2}).$$

Let $U_i = g^{-1}(B_Y(y_i, \frac{\delta}{2}))$. If $x, z \in U_i$, then $d_Y(g(x), g(z)) < \delta$, and so $d_X(x, z) < \varepsilon$. Hence, if we fix $x_i \in U_i$, we have that $U_i \subseteq B_X(x_i, \varepsilon)$. Since

$$g(X) \subseteq \bigcup_{i=1}^n B_Y(y_i, \tfrac{\delta}{2}),$$

and so

$$X = \bigcup_{i=1}^n U_i \subseteq \bigcup_{i=1}^n B_X(x_i, \varepsilon).$$

■

We turn to the proof of the theorem.

Proof. Step 1: Assume that $\mathcal{F}$ satisfies items (i)–(iii). Let Q be an open cube centered at the origin and of side-length $r = \frac{\delta}{2\sqrt{N}}$. If $\boldsymbol{x} \in Q$, then $\|\boldsymbol{x}\| < \frac{\delta}{2}$. Let $Q_1, \ldots, Q_n$ be disjoint open cubes obtained by translating Q such that

$$B(\mathbf{0}, R) \subseteq \overline{\bigcup_{i=1}^n Q_i}.$$

Let

$$Y = \operatorname{span}\{\chi_{Q_1}, \ldots, \chi_{Q_n}\}$$

and let $\Pi : L^p(\mathbb{R}^N) \to Y$ be given by

$$\Pi([f])(\boldsymbol{x}) := \begin{cases} \frac{1}{r^N} \int_{Q_i} f(\boldsymbol{y})\, d\boldsymbol{y} & \text{if } \boldsymbol{x} \in Q_i,\ i = 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

From (ii), we have

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x} \leq \int_{\mathbb{R}^N \setminus B(\boldsymbol{0},R)} |f(\boldsymbol{x}) - 0|^p d\boldsymbol{x} + \sum_{i=1}^N \int_{Q_i} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x}$$
$$\leq \varepsilon^p + \sum_{i=1}^N \int_{Q_i} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x}.$$

By Hölder's inequality

$$|f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})| = \frac{1}{r^N} \left| \int_{Q_i} [f(\boldsymbol{x}) - f(\boldsymbol{y})]\, d\boldsymbol{y} \right| \leq \frac{1}{r^N} \int_{Q_i} |f(\boldsymbol{x}) - f(\boldsymbol{y})|\, d\boldsymbol{y}$$
$$\leq \frac{r^{N/p'}}{r^N} \left(\int_{Q_i} |f(\boldsymbol{x}) - f(\boldsymbol{y})|^p d\boldsymbol{y} \right)^{1/p},$$

and so

$$\int_{Q_i} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x} \leq \frac{1}{r^N} \int_{Q_i} \int_{Q_i} |f(\boldsymbol{x}) - f(\boldsymbol{y})|^p d\boldsymbol{y} d\boldsymbol{x}$$
$$\leq \frac{1}{r^N} \int_{Q_i} \int_{2Q} |f(\boldsymbol{x}) - f(\boldsymbol{x}+\boldsymbol{h})|^p d\boldsymbol{h} d\boldsymbol{x}$$
$$= \frac{1}{r^N} \int_{2Q} \int_{Q_i} |f(\boldsymbol{x}) - f(\boldsymbol{x}+\boldsymbol{h})|^p d\boldsymbol{x} d\boldsymbol{h},$$

where we used the change of variables $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{h}$ and used the fact that if $\boldsymbol{x}, \boldsymbol{y} \in Q_i$, $\boldsymbol{h} \in 2Q$. In turn, since the cubes Q_i are disjoint, by item (iii),

$$\sum_{i=1}^N \int_{Q_i} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x} \leq \frac{1}{r^N} \int_{2Q} \sum_{i=1}^N \int_{Q_i} |f(\boldsymbol{x}) - f(\boldsymbol{x}+\boldsymbol{h})|^p d\boldsymbol{x} d\boldsymbol{h}$$
$$\leq \frac{1}{r^N} \int_{2Q} \int_{\mathbb{R}^N} |f(\boldsymbol{x}) - f(\boldsymbol{x}+\boldsymbol{h})|^p d\boldsymbol{x} d\boldsymbol{h}$$
$$\leq \varepsilon^p \frac{1}{r^N} \int_{2Q} d\boldsymbol{h} \leq 2^N \varepsilon^p.$$

Hence,

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}) - \Pi([f])(\boldsymbol{x})|^p d\boldsymbol{x} \leq \varepsilon^p (1 + 2^N).$$

It follows that

$$\|[f] - \Pi([f])\|_{L^p(\mathbb{R}^N)} \leq \varepsilon (1 + 2^N)^{1/p} \tag{34}$$

and by Minkwoski's inequality,

$$\|[f]\|_{L^p(\mathbb{R}^N)} \leq \|[f] - \Pi([f])\|_{L^p(\mathbb{R}^N)} + \|\Pi([f])\|_{L^p(\mathbb{R}^N)}$$
$$\leq \varepsilon (1 + 2^N)^{1/p} + \|\Pi([f])\|_{L^p(\mathbb{R}^N)}.$$

Since Π is linear, if $[f], [g] \in \mathcal{F}$ are such that $\|\Pi([f]) - \Pi([g])\|_{L^p(\mathbb{R}^N)} \leq \varepsilon$, then

$$\|[f] - [g]\|_{L^p(\mathbb{R}^N)} \leq \varepsilon(1 + 2^N)^{1/p} + \|\Pi([f] - [g])\|_{L^p(\mathbb{R}^N)} \leq \varepsilon((1 + 2^N)^{1/p} + 1).$$

This proves that Π satisfies the ε-δ condition in the previous lemma.

Since $\mathcal{F}$ is bounded, there exists $M > 0$ such that $\|[f]\|_{L^p(\mathbb{R}^N)} \leq M$ for all $[f] \in \mathcal{F}$. It follows from (34) and Minkowski's inequality that

$$\begin{aligned}\|\Pi([f])\|_{L^p(\mathbb{R}^N)} &\leq \|[f] - \Pi([f])\|_{L^p(\mathbb{R}^N)} + \|[f]\|_{L^p(\mathbb{R}^N)} \\ &\leq \varepsilon(1 + 2^N)^{1/p} + M.\end{aligned}$$

(Actually we can show that $\|\Pi([f])\|_{L^p(\mathbb{R}^N)} \leq \|[f]\|_{L^p(\mathbb{R}^N)} \leq M$ but we don't need that here).

Consider the noremd space $(Y, \|\cdot\|_{L^p})$. Since Y is finite-dimensional, we have that all norms are equivalent. Since Y is finite-dimensional and $\Pi(\mathcal{F})$ is bounded, $\Pi(\mathcal{F})$ is totally bounded. It follows by the previous lemma that $\mathcal{F}$ is totally bounded. ■

Wednesday, March 29, 2023

Proof. Step 2: Assume that $\mathcal{F}$ is totally bounded. Given $\varepsilon > 0$, we can find $f_1, \ldots, f_n \in L^p(\mathbb{R}^N)$ such that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{n} B_{L^p}(f_i, \varepsilon).$$

Since, by the Lebesgue dominated convergence theorem,

$$\lim_{R\to\infty} \int_{\mathbb{R}^N \setminus B(\mathbf{0},R)} |f_i(\boldsymbol{x})|^p d\boldsymbol{x} = 0,$$

we can find $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(\mathbf{0},R)} |f_i(\boldsymbol{x})|^p d\boldsymbol{x} \leq \varepsilon^p$$

for all $i = 1, \ldots, n$. Hence, if $f \in \mathcal{F}$ we can find i such that $f \in B_{L^p}(f_i, \varepsilon)$. By Minkowski's inequality

$$\left(\int_{\mathbb{R}^N \setminus B(\mathbf{0},R)} |f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} \leq \left(\int_{\mathbb{R}^N \setminus B(\mathbf{0},R)} |f_i(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} + \|f - f_i\|_{L^p} \leq 2\varepsilon.$$

This proves condition (ii). To prove (iii), we use the density of $C_c(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$ to find $g_i \in C_c(\mathbb{R}^N)$ such that $\|[g_i] - [f_i]\|_{L^p} < \varepsilon$. Let $R_i > 0$ be such that $g_i = 0$ outside $B(\mathbf{0}, R_i)$. By the Lebesgue dominated convergence theorem

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \int_{\mathbb{R}^N} |g_i(\boldsymbol{x} + \boldsymbol{h}) - g_i(\boldsymbol{x})|^p d\boldsymbol{x} = \lim_{\boldsymbol{h}\to\mathbf{0}} \int_{B(\mathbf{0},2R_i)} |g_i(\boldsymbol{x} + \boldsymbol{h}) - g_i(\boldsymbol{x})|^p d\boldsymbol{x} = 0.$$

Hence, we can find $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |g_i(\boldsymbol{x}+\boldsymbol{h}) - g_i(\boldsymbol{x})|^p d\boldsymbol{x} < \varepsilon^p$$

for all $\boldsymbol{h} \in \mathbb{R}^N$ with $\|\boldsymbol{h}\| < \delta$ and all $i = 1, \ldots, n$. In turn, by Minkowski's inequality and the change of variables $\boldsymbol{x}+\boldsymbol{h} = \boldsymbol{y}$,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} &\leq \left(\int_{\mathbb{R}^N} |g_i(\boldsymbol{x}+\boldsymbol{h}) - g_i(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^N} |g_i(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}+\boldsymbol{h})|^p d\boldsymbol{x}\right)^{1/p} \\ + \left(\int_{\mathbb{R}^N} |g_i(\boldsymbol{x}) - f(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} &\leq \varepsilon + 4\varepsilon.\end{aligned}$$

■

Theorem 153 (Fréchet–Kolmogorov-Riesz) *Let $1 \leq p < \infty$. A set $\mathcal{F} \subseteq L^p(\mathbb{R}^N)$ is compact if and only if*

(i) $\mathcal{F}$ is closed, bounded;

(ii) for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(\mathbf{0},R)} |f(\boldsymbol{x})|^p d\boldsymbol{x} < \varepsilon^p$$

for all $[f] \in \mathcal{F}$

(iii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p d\boldsymbol{x} < \varepsilon^p$$

for all $\boldsymbol{h} \in \mathbb{R}^N$ with $\|\boldsymbol{h}\| < \delta$ and for all $[f] \in \mathcal{F}$.

Next we study mollifiers and L^p functions. Consider the function

$$\varphi(\boldsymbol{x}) := \begin{cases} c\exp\left(\frac{1}{\|\boldsymbol{x}\|^2-1}\right) & \text{if } \|\boldsymbol{x}\| < 1, \\ 0 & \text{if } \|\boldsymbol{x}\| \geq 1, \end{cases} \tag{35}$$

where the constant $c > 0$ is chosen so that

$$\int_{\mathbb{R}^N} \varphi(\boldsymbol{x})\, d\boldsymbol{x} = 1. \tag{36}$$

We leave as an exercise to prove that $\varphi \in C_c^\infty(\mathbb{R}^N)$. For every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^N}\varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

The functions φ_ε are called *standard mollifiers.*

Remark 154 *Fix* $\boldsymbol{x} \in \mathbb{R}^N$*. Using the change of variables* $\boldsymbol{z} = \frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}$ *we have that*

$$\int_{B(\boldsymbol{x},\varepsilon)} \varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = \frac{1}{\varepsilon^N}\int_{B(\boldsymbol{x},\varepsilon)} \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) d\boldsymbol{y}$$
$$= \frac{\varepsilon^N}{\varepsilon^N}\int_{B(\boldsymbol{0},1)} \varphi(\boldsymbol{z})\, d\boldsymbol{z} = 1.$$

Given a Lebesgue measurable set $E \subseteq \mathbb{R}^N$ and a Lebesgue integrable function $f : E \to \mathbb{R}$, we define

$$f_\varepsilon(\boldsymbol{x}) := \int_E \varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y}$$

for $\boldsymbol{x} \in \mathbb{R}^N$. Since φ_ε is bounded and continuous, and f is Lebesgue integrable, $f_\varepsilon(\boldsymbol{x})$ is well-defined. The function $f_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ is called a *mollification* of f.

Theorem 155 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set, let* φ *be a standard mollifier, let* $1 \le p \le \infty$*, and let* $[f] \in L^p(\Omega)$*.*

(i) f_ε *is well-defined;*

(ii) For every Lebesgue point $\boldsymbol{x} \in \Omega$ *of* f *(and so for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \Omega$*),* $f_\varepsilon(\boldsymbol{x}) \to f(\boldsymbol{x})$ *as* $\varepsilon \to 0^+$*. Moreover,* $f_\varepsilon(\boldsymbol{x}) \to 0$ *for every* $\boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}$*;*

(iii) For every $\varepsilon > 0$*,* $\|[f_\varepsilon]\|_{L^p(\mathbb{R}^N)} \le \|[f]\|_{L^p(\Omega)}$*;*

(iv) $\|[f_\varepsilon]\|_{L^p(\mathbb{R}^N)} \to \|f\|_{L^p(\Omega)}$ *as* $\varepsilon \to 0^+$*;*

(v) If $1 \le p < \infty$*, then*

$$\lim_{\varepsilon\to 0^+}\left(\int_\Omega |f_\varepsilon - f|^p\, d\boldsymbol{x}\right)^{\frac{1}{p}} = 0.$$

Proof. (i) Since $[\varphi_\varepsilon(\boldsymbol{x}-\cdot)] \in L^{p'}(\mathbb{R}^N)$, the fact that f_ε is well defined follows from Hölder's inequality.
(ii) Let $\boldsymbol{x} \in \Omega$ be a Lebesgue point of f, that is,

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{B(\boldsymbol{x},\varepsilon)} |f(\boldsymbol{y}) - f(\boldsymbol{x})|\, d\boldsymbol{y} = 0.$$

Since Ω is open, $B(\boldsymbol{x},\varepsilon) \subseteq \Omega$ for ε small enough. Using Remark 154,

$$f_\varepsilon(\boldsymbol{x}) - f(\boldsymbol{x}) = \int_{B(\boldsymbol{x},\varepsilon)} \varphi_\varepsilon(\boldsymbol{x}-\boldsymbol{y})\,[f(\boldsymbol{y}) - f(\boldsymbol{x})]\, d\boldsymbol{y}$$

and so

$$\begin{aligned}\left|f_{\varepsilon}(\boldsymbol{x})-f(\boldsymbol{x})\right| &\leq \frac{1}{\varepsilon^{N}} \int_{B(\boldsymbol{x},\varepsilon)} \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\left|f(\boldsymbol{y})-f(\boldsymbol{x})\right| d\boldsymbol{y} \\ &\leq \frac{\|\varphi\|_{\infty}}{\varepsilon^{N}} \int_{B(\boldsymbol{x},\varepsilon)}\left|f(\boldsymbol{y})-f(\boldsymbol{x})\right| d\boldsymbol{y} \rightarrow 0\end{aligned}$$

as $\varepsilon \rightarrow 0^{+}$.
(iii) If $1 \leq p<\infty$, by Hölder's inequality and (36) for all $\boldsymbol{x} \in \mathbb{R}^{N}$,

$$\begin{aligned}\left|f_{\varepsilon}(\boldsymbol{x})\right| &= \left|\int_{\Omega}\left(\varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})\right)^{\frac{1}{p^{\prime}}}\left(\varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})\right)^{\frac{1}{p}} f(\boldsymbol{y})\, d\boldsymbol{y}\right| \\ &\leq\left(\int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}\right)^{\frac{1}{p^{\prime}}}\left(\int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})|f(\boldsymbol{y})|^{p}\, d\boldsymbol{y}\right)^{\frac{1}{p}} \\ &\leq\left(\int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})|f(\boldsymbol{y})|^{p}\, d\boldsymbol{y}\right)^{\frac{1}{p}}\end{aligned} \tag{37}$$

and so by Fubini's theorem and (36) once more

$$\begin{aligned}\int_{\mathbb{R}^{N}}\left|f_{\varepsilon}(\boldsymbol{x})\right|^{p} d\boldsymbol{x} &\leq \int_{\mathbb{R}^{N}} \int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})|f(\boldsymbol{y})|^{p}\, d\boldsymbol{y} d\boldsymbol{x} \\ &=\int_{\Omega}|f(\boldsymbol{y})|^{p}\left(\int_{\mathbb{R}^{N}} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{x}\right) d\boldsymbol{y} \\ &=\int_{\Omega}|f(\boldsymbol{y})|^{p} d\boldsymbol{y}.\end{aligned}$$

On the other hand, if $p=\infty$, then for every $\boldsymbol{x} \in \mathbb{R}^{N}$,

$$\begin{aligned}\left|f_{\varepsilon}(\boldsymbol{x})\right| &\leq \int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})|f(\boldsymbol{y})|\, d\boldsymbol{y} \\ &\leq \|[f]\|_{L^{\infty}(\Omega)} \int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \leq \|[f]\|_{L^{\infty}(\Omega)}\end{aligned}$$

again by (36), and so item (iii) holds for all $1 \leq p \leq \infty$.
(iv) By item (iii),

$$\limsup_{\varepsilon \rightarrow 0^{+}}\left\|\left[f_{\varepsilon}\right]\right\|_{L^{p}\left(\mathbb{R}^{N}\right)} \leq\|[f]\|_{L^{p}(\Omega)} .$$

To prove the opposite inequality, assume first that $1 \leq p<\infty$. By part (ii), $f_{\varepsilon}(\boldsymbol{x}) \rightarrow f(\boldsymbol{x})$ as $\varepsilon \rightarrow 0^{+}$ for $\mathcal{L}^{N}$ a.e. $\boldsymbol{x} \in \Omega$, and so by Fatou's lemma

$$\int_{\Omega}|f(\boldsymbol{x})|^{p}\, d\boldsymbol{x}=\int_{\Omega} \lim_{\varepsilon \rightarrow 0^{+}}\left|f_{\varepsilon}(\boldsymbol{x})\right|^{p}\, d\boldsymbol{x} \leq \liminf_{\varepsilon \rightarrow 0^{+}} \int_{\mathbb{R}^{N}}\left|f_{\varepsilon}(\boldsymbol{x})\right|^{p}\, d\boldsymbol{x}.$$

If $p=\infty$, then again by part (ii) $f_{\varepsilon}(\boldsymbol{x}) \rightarrow f(\boldsymbol{x})$ as $\varepsilon \rightarrow 0^{+}$ for $\mathcal{L}^{N}$ a.e. $\boldsymbol{x} \in \Omega$. Hence

$$|f(\boldsymbol{x})|=\lim_{\varepsilon \rightarrow 0^{+}}\left|f_{\varepsilon}(\boldsymbol{x})\right| \leq \liminf_{\varepsilon \rightarrow 0^{+}}\left\|\left[f_{\varepsilon}\right]\right\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. It follows that

$$\|[f]\|_{L^\infty(\Omega)} \leq \liminf_{\varepsilon \to 0^+} \|[f_\varepsilon]\|_{L^\infty(\mathbb{R}^N)} .$$

Hence, item (iv) holds also in this case.
(v) Fix $\rho > 0$ and find a function $g \in C_c(\Omega)$ such that

$$\|[f] - [g]\|_{L^p(\Omega)} \leq \rho.$$

Since $K := \operatorname{supp} g$ is compact, it follows that for every $0 < \eta < \operatorname{dist}(K, \partial\Omega)$, the mollification g_ε of g converges to g uniformly in the compact set

$$K_\eta := \left\{\boldsymbol{x} \in \mathbb{R}^N : \operatorname{dist}(\boldsymbol{x}, K) \leq \eta\right\} .$$

Since $g_\varepsilon = g = 0$ in $\Omega \setminus K_\eta$ for $0 < \varepsilon < \eta$, we have that

$$\int_\Omega |g_\varepsilon - g|^p \, d\boldsymbol{x} = \int_{K_\eta} |g_\varepsilon - g|^p \, d\boldsymbol{x} \leq \left(\|g_\varepsilon - g\|_{C(K_\eta)}\right)^p |K_\eta| \leq \rho,$$

provided $\varepsilon > 0$ is sufficiently small. By Minkowski's inequality

$$\begin{aligned} \|[f_\varepsilon] - [f]\|_{L^p(\Omega)} &\leq \|[f_\varepsilon] - [g_\varepsilon]\|_{L^p(\Omega)} + \|[g_\varepsilon] - [g]\|_{L^p(\Omega)} + \|[g] - [f]\|_{L^p(\Omega)} \\ &\leq 2\,\|[f] - [g]\|_{L^p(\Omega)} + \|[g_\varepsilon] - [g]\|_{L^p(\Omega)} \leq 3\rho, \end{aligned}$$

where we have used item (iii) for the function $f - g$. ■

Remark 156 *Part (iv) does not hold for* $p = \infty$*, since uniform convergence of continuous functions would imply that* f *is continuous.*

Friday, March 31, 2023

In what follows given $[f] \in L^p(\Omega)$, we will write simply $f \in L^p(\Omega)$ and so, we will identify a function with an equivalence class of functions. Please be careful about this, because it can cause all kind of mistakes.

13 Sobolev Spaces

Consider the differential equation

$$f''(x) = g(x), \quad x \in I$$

where I is an open interval and $g : I \to \mathbb{R}$ is a continuous function. For this ode to make sense, we need the solution f to be at least of class C^2. Consider a function $\phi \in C_c^\infty(I)$ and multiply the equation by ϕ. If we integrate by parts, we get

$$-\int_I f'(x)\phi'(x)\,dx = \int_I g(x)\phi(x)\,dx. \tag{38}$$

This integral makes sense for functions f that are less regular than C^2. For example C^1 is enough.

If we integrate by parts once more, we get

$$\int_I f(x)\phi''(x)\,dx = \int_I g(x)\phi(x)\,dx. \tag{39}$$

This integral makes sense provided $f : I \to \mathbb{R}$ is locally integrable. The integrals (38) and (39) can be considered weak formulations of the differential equation $f'' = g$.

Motivated by this discussion, we define the weak derivative of a function. But first, let's recall integration by parts in several dimensions.

If $\Omega \subset \mathbb{R}^N$ is an open bounded set whose boundary is of class C^1 and $f, \phi \in C^1(\overline{\Omega})$, then as a corollary of the divergence theorem, we have that for every $i = 1, \dots, N$,

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x} + \int_{\partial\Omega} f\phi\nu_i\,d\mathcal{H}^{N-1},$$

where $\boldsymbol{\nu}(\boldsymbol{x}) = (\nu_1(\boldsymbol{x}), \dots, \nu_N(\boldsymbol{x}))$ is the outward unit normal to $\partial\Omega$ at $\boldsymbol{x}$. Now, if we assume that $\phi \in C_c^1(\Omega)$, then $\phi = 0$ on $\partial\Omega$, and so

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}.$$

In this case, we don't need to assume that Ω is bounded or that $\partial\Omega$ is of class C^1. To be precise, let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $f \in C^1(\Omega)$ (note that f may not be integrable) and let $\phi \in C_c^1(\Omega)$. Then there exists a compact set $K \subset \Omega$ such that $\phi = 0$ in $\Omega \setminus K$. Construct an open bounded set V with boundary of class C^1 such that $K \subset V \subset \overline{V} \subset \Omega$. Since $f \in C^1(\overline{V})$, we can apply the divergence theorem in V to obtain

$$\int_V f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_V \frac{\partial f}{\partial x_i}(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}.$$

On the other hand, since $\phi = 0$ and $\frac{\partial \phi}{\partial x_i}$ are zero in $\Omega \setminus V$, we can write

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}. \tag{40}$$

Thus, we have shown that given an open set $\Omega \subseteq \mathbb{R}^N$ and a function $f \in C^1(\Omega)$, the integration by parts formula (40) holds for all $\phi \in C_c^1(\Omega)$. We now extend the previous formula to functions f not in $C^1(\Omega)$.

Remark 157 (Important) *From now on, instead of writing* $[f] \in L^p(\Omega)$, *I will write* $f \in L^p(\Omega)$.

Definition 158 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $1 \le p \le \infty$, *and* $f \in L^p_{\mathrm{loc}}(\Omega)$. *Given* $i = 1, \dots, N$, *we say that* f *admits a* weak or distributional derivative in $L^p(\Omega)$ *if there exists a function* $g_i \in L^p(\Omega)$ *such that*

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega g_i(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}$$

for every $\phi \in C_c^\infty(\Omega)$*. The function* g_i *is called the* weak*, or* distributional, partial derivative *of* f *with respect to* x_i *and is denoted* $\frac{\partial f}{\partial x_i}$*.*

Remark 159 *Observe that if* $f \in C^1(\Omega)$*, then by the divergence theorem we can always integrate by parts to conclude that*

$$\int_\Omega f(\boldsymbol{x}) \frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\, d\boldsymbol{x} = -\int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \phi(\boldsymbol{x})\, d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$*. Hence, if* $\frac{\partial f}{\partial x_i} \in L^p(\Omega)$*, then the classical partial derivative* $\frac{\partial f}{\partial x_i}$ *is the weak derivative of* f*. We will use this fact without further notice.*

Exercise 160 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $1 \le p \le \infty$*, and* $L^p_{\text{loc}}(\Omega)$*. Prove that if* f *admits a weak derivative* $\frac{\partial f}{\partial x_i}$ *in* $L^p(\Omega)$*, then the weak derivative* $\frac{\partial f}{\partial x_i}$ *is unique.*

We can now define the Sobolev space $W^{1,p}(\Omega)$.

Definition 161 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p \le \infty$*. The* Sobolev space $W^{1,p}(\Omega)$ *is the space of all functions* $f \in L^p(\Omega)$ *that admit all weak derivatives* $\frac{\partial f}{\partial x_i}$ *in* $L^p(\Omega)$*, endowed with the norm*

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(\Omega)}.$$

When $p = 2$ we write $H^1(\Omega) = W^{1,2}(\Omega)$. In this case, we have an inner product, given by

$$(f, g)_{H^1(\Omega)} := (f, g)_{L^2(\Omega)} + \sum_{i=1}^N \left(\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right)_{L^2(\Omega)}.$$

For $f \in W^{1,p}(\Omega)$ we set

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N} \right).$$

Remark 162 *In* $W^{1,p}(\Omega)$ *we can consider the equivalent norms*

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

or

$$\|f\|_{W^{m,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega;\mathbb{R}^N)};$$

for $1 \le p < \infty$*, and*

$$\|f\|_{W^{1,\infty}(\Omega)} := \max \left\{ \|f\|_{L^\infty(\Omega)}, \left\| \frac{\partial f}{\partial x_1} \right\|_{L^\infty(\Omega)}, \dots, \left\| \frac{\partial f}{\partial x_N} \right\|_{L^\infty(\Omega)} \right\}$$

for $p = \infty$*.*

We define

$$W^{1,p}_{\text{loc}}(\Omega) := \left\{ f \in L^p_{\text{loc}}(\Omega) : \ f \in W^{1,p}(U) \text{ for all open sets } U \Subset \Omega \right\}.$$

Monday, April 3, 2023

We now show that $W^{1,p}(\Omega)$ is a Banach space.

Theorem 163 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \le p \le \infty$. Then the space $W^{1,p}(\Omega)$ is a Banach space.*

Proof. Let $\{f_n\}_n$ be a Cauchy sequence in $W^{1,p}(\Omega)$, that is,

$$\begin{aligned} 0 &= \lim_{l,n\to\infty} \|f_n - f_l\|_{W^{1,p}(\Omega)} \\ &= \lim_{l,n\to\infty} \left(\|f_n - f_l\|_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial f_n}{\partial x_i} - \frac{\partial f_l}{\partial x_i} \right\|_{L^p(\Omega)} \right). \end{aligned}$$

Then $\{f_n\}_n$ and $\left\{\frac{\partial f_n}{\partial x_i}\right\}_n$, $i = 1, \ldots, N$, are Cauchy sequences in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space, there exist $f, g_{\boldsymbol{i}} \in L^p(\Omega)$, $i = 1, \ldots, N$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0, \quad \lim_{n\to\infty} \left\| \frac{\partial f_n}{\partial x_i} - g_i \right\|_{L^p(\Omega)} = 0 \tag{41}$$

for all $i = 1, \ldots, N$. Fix $i = 1, \ldots, N$. We claim that $\frac{\partial f_n}{\partial x_i} = g_i$. To see this let $\phi \in C_c^\infty(\Omega)$ and note that

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} = - \int_\Omega f_n \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}. \tag{42}$$

Writing

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} = \int_\Omega \phi \left(\frac{\partial f_n}{\partial x_i} - g_i \right) d\boldsymbol{x} + \int_\Omega \phi g_i \, d\boldsymbol{x} =: I_n + II,$$

by Hölder's inequality we have

$$|I_n| \le \|\phi\|_{L^{p'}(\Omega)} \left\| \frac{\partial f_n}{\partial x_i} - g_i \right\|_{L^p(\Omega)} \to 0$$

as $n \to \infty$, which shows that

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} \to \int_\Omega \phi g_i \, d\boldsymbol{x}.$$

Similarly,

$$- \int_\Omega f_n \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x} \to - \int_\Omega f \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}.$$

Hence, letting $n \to \infty$ in (42) yields

$$\int_\Omega \phi g_i \, d\boldsymbol{x} = -\int_\Omega f \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$, which proves the claim. Thus $f \in W^{1,p}(\Omega)$. It follows by (41) that $f_n \to f$ in $W^{1,p}(\Omega)$. Hence, $W^{1,p}(\Omega)$ is a Banach space. ■

More generally, we can define higher order Sobolev spaces.

Definition 164 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p \leq \infty$. The* Sobolev space *$W^{m,p}(\Omega)$ is the space of all functions $f \in L^p(\Omega)$ such that for every multi-index $\boldsymbol{\alpha}$ with $1 \leq |\boldsymbol{\alpha}| \leq m$ there exists a function $g_{\boldsymbol{\alpha}} \in L^p(\Omega)$ such that*

$$\int_\Omega f \frac{\partial^{\boldsymbol{\alpha}} \phi}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} \, d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega g_{\boldsymbol{\alpha}} \phi \, d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$. The function $g_{\boldsymbol{\alpha}}$ is called the weak *or* distributional partial derivative *of f with respect to $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and is denoted $\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}$.*

Exercise 165 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p \leq \infty$. Given $f \in W^{m,p}(\Omega)$, prove that the weak derivative of f with respect to $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is unique.*

We define

$$W^{m,p}_{\mathrm{loc}}(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \ f \in W^{m,p}(U) \text{ for all open sets } U \Subset \Omega \right\}.$$

Exercise 166 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$.*

(i) Prove that a subset of a separable metric space is separable.

(ii) Prove that $W^{1,p}(\Omega)$ is separable. Hint: Consider the mapping

$$\begin{aligned} W^{1,p}(\Omega) &\to L^p(\Omega) \times L^p\left(\Omega; \mathbb{R}^N\right) \\ f &\mapsto (f, \nabla f). \end{aligned}$$

Exercise 167 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Prove that $W^{1,\infty}(\Omega)$ is not separable.*

Next we prove that smooth functions are dense in $W^{1,p}(\Omega)$

Theorem 168 (Meyers–Serrin) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \leq p < \infty$. Then the space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Lemma 169 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \leq p < \infty$, and $f \in W^{1,p}(\Omega)$. For every $\varepsilon > 0$ define $f_\varepsilon := \varphi_\varepsilon * f$ in $\mathbb{R}^N$, where φ_ε is a standard mollifier. Then*

$$\lim_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_{W^{1,p}(\Omega_\varepsilon)} = 0,$$

where the open set Ω_ε *is given by*

$$\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

In particular, if $U \subset \Omega$*, with* $\operatorname{dist}(U, \partial\Omega) > 0$*, then*

$$\|f_\varepsilon - f\|_{W^{1,p}(U)} \to 0 \text{ as } \varepsilon \to 0^+.$$

Proof. By differentiating under the integral sign we have that $f_\varepsilon \in C^\infty(\mathbb{R}^N)$ and for $\boldsymbol{x} \in \Omega_\varepsilon$ and for every $i = 1, \dots, N$,

$$\begin{aligned}\frac{\partial f_\varepsilon}{\partial x_i}(\boldsymbol{x}) &= \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} = -\int_\Omega \frac{\partial \varphi_\varepsilon}{\partial y_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) \frac{\partial f}{\partial y_i}(\boldsymbol{y})\, d\boldsymbol{y} = \left(\varphi_\varepsilon * \frac{\partial f}{\partial x_i}\right)(\boldsymbol{x}),\end{aligned}$$

where we have used the definition of weak derivative and the fact that for each $\boldsymbol{x} \in \Omega_\varepsilon$ the function $\varphi_\varepsilon(\boldsymbol{x} - \cdot) \in C_c^\infty(\Omega)$, since $\operatorname{supp} \varphi_\varepsilon(\boldsymbol{x} - \cdot) \subseteq \overline{B(\boldsymbol{x}, \varepsilon)} \subset \Omega$. The result now follows from Theorem 155 applied to the functions f and $\frac{\partial f}{\partial \boldsymbol{x}_i}$, $i = 1, \dots, N$. ■

Remark 170 *Note that if* $\Omega = \mathbb{R}^N$*, then* $\Omega_\varepsilon = \mathbb{R}^N$*. Hence,* $f_\varepsilon \to f$ *in* $W^{1,p}(\mathbb{R}^N)$*.*

Exercise 171 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p \le \infty$*. Prove that if* $f \in W^{1,p}(\Omega)$ *and* $\varphi \in C_c^\infty(\Omega)$*, then* $\varphi f \in W^{1,p}(\Omega)$*.*

We now turn to the proof of the Meyers–Serrin theorem.

Proof of Theorem 168. Let $\Omega_i \Subset \Omega_{i+1}$ be such that

$$\Omega = \bigcup_{i=1}^\infty \Omega_i$$

and consider a smooth partition of unity $\mathcal{F}$ subordinated to the open cover $\{\Omega_{i+1} \setminus \overline{\Omega_{i-1}}\}$, where $\Omega_{-1} = \Omega_0 := \emptyset$. For each $i \in \mathbb{N}$ let ψ_i be the sum of all the finitely many $\psi \in \mathcal{F}$ such that $\operatorname{supp} \psi \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ and that have not already been selected at previous steps $j < i$. Then $\psi_i \in C_c^\infty(\Omega_{i+1} \setminus \overline{\Omega_{i-1}})$ and

$$\sum_{i=1}^\infty \psi_i = 1 \text{ in } \Omega. \tag{43}$$

Fix $\eta > 0$. For each $i \in \mathbb{N}$ we have that

$$\operatorname{supp}(\psi_i f) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}, \tag{44}$$

and so, by the previous lemma, we may find $\varepsilon_i > 0$ so small that

$$\operatorname{supp}(\psi_i f)_{\varepsilon_i} \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}} \tag{45}$$

and

$$\left\|(\psi_i f)_{\varepsilon_i} - \psi_i f\right\|_{W^{1,p}(\Omega)} \leq \frac{\eta}{2^i},$$

where we have used the previous exercise. ■

Proof. Note that in view of (45), for every $U \Subset \Omega$ only finitely many $\Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ cover U, and so the function

$$g := \sum_{i=1}^{\infty} (\psi_i f)_{\varepsilon_i}$$

belongs to $C^\infty(\Omega)$. In particular, $g \in W^{m,p}_{\text{loc}}(\Omega)$.

For $\boldsymbol{x} \in \Omega_\ell$ by (43), (44), and (45),

$$f(\boldsymbol{x}) = \sum_{i=1}^{\ell} (\psi_i f)(\boldsymbol{x}), \quad g(\boldsymbol{x}) = \sum_{i=1}^{\ell} (\psi_i f)_{\varepsilon_i}(\boldsymbol{x}). \tag{46}$$

Hence

$$\|f - g\|_{W^{1,p}(\Omega_\ell)} \leq \sum_{i=1}^{\ell} \left\|(\psi_i f)_{\varepsilon_i} - \psi_i f\right\|_{W^{1,p}(\Omega)} \leq \sum_{i=1}^{\ell} \frac{\eta}{2^i} \leq \eta. \tag{47}$$

Letting $\ell \to \infty$ it follows from the Lebesgue dominated convergence theorem that $\|f - g\|_{W^{1,p}(\Omega)} \leq \eta$. This also implies that $f - g$ (and, in turn, g) belongs to the space $W^{m,p}(\Omega)$. ■

Remark 172 *Note that we can adapt the proof of the Meyers-Serrin theorem to show that if $f \in W^{1,p}_{\text{loc}}(\Omega)$ with $\nabla f \in L^p(\Omega; \mathbb{R}^N)$ then for every $\varepsilon > 0$ there exists a function $g \in C^\infty(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega)$ such that*

$$\|f - g\|_{W^{1,p}(\Omega)} \leq \varepsilon,$$

despite the fact that neither f nor g need belong to $W^{1,p}(\Omega)$.

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Exercise 173 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $f : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function (that is, f is Lipschitz continuous in each compact set $K \subset \Omega$). Prove that $f \in W^{1,p}_{\text{loc}}(\Omega)$ and that the classical derivatives of f are the weak derivatives.*

Exercise 174 *Prove that the function $f(x) := |x|$ belongs to $W^{1,\infty}(-1,1)$ but not to the closure of $C^\infty(-1,1) \cap W^{1,\infty}(-1,1)$.*

The previous exercise shows that the Meyers–Serrin theorem is false for $p = \infty$. This is intuitively clear, since if $\Omega \subseteq \mathbb{R}^N$ is an open set and $\{f_n\} \subset C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$ is such that $\|f_n - f\|_{W^{1,\infty}(\Omega)} \to 0$, then $f \in C^1(\Omega)$ (why?).

Exercise 175 *Let* $\Omega = B(\mathbf{0},1) \setminus \{\boldsymbol{x} \in \mathbb{R}^N : x_N = 0\}$*. Show that the function* $f : \Omega \to \mathbb{R}$*, defined by*

$$f(\boldsymbol{x}) = f(x_1, \ldots, x_N) := \begin{cases} 1 & \text{if } x_N > 0, \\ 0 & \text{if } x_N < 0, \end{cases}$$

belongs to $W^{1,p}(\Omega)$ *for all* $1 \le p \le \infty$*, but cannot be approximated by functions in* $C^\infty(\overline{\Omega})$*.*

Definition 176 *Given an open set* $\Omega \subseteq \mathbb{R}^N$*, we denote by* $C^\infty(\overline{\Omega})$ *the space of all functions* $f \in C^\infty(\Omega)$ *that can be extended to a function in* $C^\infty(\mathbb{R}^N)$*.*

The previous exercise shows that in the Meyers–Serrin theorem for general open sets Ω we may not replace $C^\infty(\Omega)$ with $C^\infty(\overline{\Omega})$.

Theorem 177 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set with boundary of class* C^0 *and let* $1 \le p < \infty$*. Then* $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ *is dense in* $W^{1,p}(\Omega)$*.*

Exercise 178 *Let* $\Omega, U \subseteq \mathbb{R}^N$ *be open sets, let* $\Psi : U \to \Omega$ *be invertible, with* Ψ *and* Ψ^{-1} *Lipschitz functions of class* C^1*, and let* $f \in W^{1,p}(\Omega)$*,* $1 \le p < \infty$*. Then* $f \circ \Psi \in W^{1,p}(U)$ *and for all* $i = 1, \ldots, N$ *and for* $\mathcal{L}^N$*-a.e.* $\boldsymbol{y} \in U$*,*

$$\frac{\partial (f \circ \Psi)}{\partial y_i}(\boldsymbol{y}) = \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(\Psi(\boldsymbol{y})) \frac{\partial \Psi_j}{\partial y_i}(\boldsymbol{y}).$$

14 Absolute Continuity on Lines

The next theorem relates weak partial derivatives with the (classical) partial derivatives. Given $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $i \in \{1, \ldots, N\}$ we denote by $\boldsymbol{x}_i'$ the vector of $\mathbb{R}^{N-1}$ obtained from $\boldsymbol{x}$ by removing the i-th component x_i. With a slight abuse of notation we write

$$\boldsymbol{x} = (\boldsymbol{x}_i', x_i) \in \mathbb{R}^{N-1} \times \mathbb{R}. \tag{48}$$

Given a set $E \subseteq \mathbb{R}^N$ and $\boldsymbol{x}_i' \in \mathbb{R}^{N-1}$, we denote by $E_{\boldsymbol{x}_i'}$ the section

$$E_{\boldsymbol{x}_i'} := \{x_i \in \mathbb{R} : (\boldsymbol{x}_i', x_i) \in E\}.$$

To state the following theorem, we will work with equivalences classes of functions, and so we will use $[f] \in L^p(\Omega)$

Theorem 179 (Absolute Continuity on Lines) *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and let* $1 \le p < \infty$*. Then* $[f] \in L^p(\Omega)$ *belongs to the space* $W^{1,p}(\Omega)$ *if and only if* $f \sim g$*, where* $g : \Omega \to \mathbb{R}$ *has the property that for every* $i = 1, \ldots, N$*, there exists a Lebesgue measurable set* $M_i \subset \mathbb{R}^{N-1}$ *with* $\mathcal{L}^{N-1}(M_i) = 0$ *such that* *for every* $\boldsymbol{x}_i' \in \mathbb{R}^{N-1} \setminus M_i$ *for which the section* $\Omega_{\boldsymbol{x}_i'}$ *is nonempty, the function* $g(\boldsymbol{x}_i', \cdot)$ *is absolutely continuous on each maximal interval* $I \subseteq \Omega_{\boldsymbol{x}_i'}$ *and*

$$\int_\Omega \left| \frac{\partial g}{\partial x_i}(\boldsymbol{x}) \right| d\boldsymbol{x} < \infty.$$

Moreover, $\left[\frac{\partial g}{\partial x_i}\right]$ *is the weak ith derivative of* $[f]$.

Proof. Step 1: Assume that $[f] \in W^{1,p}(\Omega)$. Consider a sequence of standard mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$ and for every $\varepsilon > 0$ define $f_\varepsilon := f * \varphi_\varepsilon$ in $\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}$. By Lemma 169,

$$\lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} \|\nabla f_\varepsilon(\boldsymbol{x}) - \nabla f(\boldsymbol{x})\|^p d\boldsymbol{x} = 0.$$

It follows by Fubini's theorem that for all $i = 1, \ldots, N$,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{N-1}} \left(\int_{(\Omega_\varepsilon)_{x_i}} \|\nabla f_\varepsilon(\boldsymbol{x}_i, x_i) - \nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i \right) d\boldsymbol{x}_i = 0,$$

where $(\Omega_\varepsilon)_{\boldsymbol{x}_i} := \{x_i \in \mathbb{R} : (\boldsymbol{x}_i, x_i) \in \Omega_\varepsilon\}$, and so, by Remark 144, we may find a subsequence $\{\varepsilon_n\}_n$ such that for all $i = 1, \ldots, N$ and for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$,

$$\lim_{n \to \infty} \int_{(\Omega_{\varepsilon_n})_{\boldsymbol{x}_i}} \|\nabla f_{\varepsilon_n}(\boldsymbol{x}_i, x_i) - \nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i = 0. \tag{49}$$

Set $f_n := f_{\varepsilon_n}$ and

$$E := \left\{ x \in \Omega : \lim_{n \to \infty} f_n(x) \text{ exists in } \mathbb{R} \right\}.$$

Since E contains every Lebesgue points of f, we have that $\mathcal{L}^N(\Omega \setminus E) = 0$. Define

$$g(\boldsymbol{x}) := \begin{cases} \lim_{n \to \infty} f_n(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The function g is a representative of $[f]$, since by Theorem 155, $\{f_n\}_n$ converges pointwise at every Lebesgue point of f. It remains to prove that g has the desired properties.

By Fubini's theorem for every $i = 1, \ldots, N$ we have that

$$\int_{\mathbb{R}^{N-1}} \left(\int_{\Omega_{\boldsymbol{x}_i}} \|\nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i \right) d\boldsymbol{x}_i < \infty$$

and

$$\int_{\mathbb{R}^{N-1}} \mathcal{L}^1(\{x_i \in \Omega_{\boldsymbol{x}_i} : (\boldsymbol{x}_i, x_i) \notin E\})\, d\boldsymbol{x}_i = 0,$$

where $\Omega_{\boldsymbol{x}_i} := \{x_i \in \mathbb{R} : (\boldsymbol{x}_i, x_i) \in \Omega\}$, and so we may find a set $N_i \subset \mathbb{R}^{N-1}$, with $\mathcal{L}^{N-1}(N_i) = 0$, such that for all $\boldsymbol{x}_i \in \mathbb{R}^{N-1} \setminus N_i$ for which $\Omega_{\boldsymbol{x}_i}$ is nonempty we have that

$$\int_{\Omega_{\boldsymbol{x}_i}} \|\nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i < \infty, \tag{50}$$

(49) holds for all $i = 1, \ldots, N$ and $(\boldsymbol{x}_i, x_i) \in E$ for $\mathcal{L}^1$ a.e. $x_i \in \Omega_{\boldsymbol{x}_i}$. Fix any such $\boldsymbol{x}_i$ and let $I \subseteq \Omega_{\boldsymbol{x}_i}$ be a maximal interval. Fix $t_0 \in I$ such that $(\boldsymbol{x}_i, t_0) \in E$ and let $t \in I$. For all n large, the interval of endpoints t and t_0 is contained in $(\Omega_{\varepsilon_n})_{\boldsymbol{x}_i}$ and so, since $f_n \in C^\infty(\Omega_{\varepsilon_n})$, by the fundamental theorem of calculus,

$$f_n(\boldsymbol{x}_i, t) = f_n(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s)\, ds.$$

Since $(\boldsymbol{x}_i, t_0) \in E$. Then $f_n(\boldsymbol{x}_i, t_0) \to g(\boldsymbol{x}_i, t_0) \in \mathbb{R}$. On the other hand, by (49)

$$\lim_{n\to\infty} \int_{t_0}^{t} \left| \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s) - \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s) \right| ds = 0. \tag{51}$$

Hencewe have that there exists the limit

$$\begin{aligned} \lim_{n\to\infty} f_n(\boldsymbol{x}_i, t) &= \lim_{n\to\infty} \left(f_n(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s)\, ds \right) \\ &= g(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, s)\, ds. \end{aligned}$$

Note that by the definition of E and g, this implies, in particular, that

$$(\boldsymbol{x}_i, t) \in E \tag{52}$$

and that

$$g(\boldsymbol{x}_i, t) = g(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, s)\, ds \tag{53}$$

for *all* $t \in I$. Since $g(\boldsymbol{x}_i, \cdot)$ satisfies the fundamental theorem of calculus, it is locally absolutely continuous in I and $\frac{\partial g}{\partial x_N}(\boldsymbol{x}_i, t) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, t)$ for $\mathcal{L}^1$ a.e. $t \in I$. We can now apply exercise 180 to conclude that $g(\boldsymbol{x}_i, \cdot)$ is absolutely continuous in I. ■

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Proof. Step 2: Assume that $[f]$ admits a representative g that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Fix $i = 1, \ldots, N$ and let $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$ be such that $\overline{f}(\boldsymbol{x}_i, \cdot)$ is absolutely continuous on the open set $\Omega_{\boldsymbol{x}_i}$. Then for every function $\varphi \in C_c^\infty(\Omega)$, by the integration by parts formula for absolutely continuous functions, we have

$$\int_{\Omega_{\boldsymbol{x}_i}} g(\boldsymbol{x}_i, t) \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}_i, t)\, dt = -\int_{\Omega_{\boldsymbol{x}_i}} \frac{\partial g}{\partial x_i}(\boldsymbol{x}_i, t) \varphi(\boldsymbol{x}_i, t)\, dt.$$

Since this holds for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$, integrating over $\mathbb{R}^{N-1}$ and using Fubini's theorem yields

$$\int_{\Omega} g(\boldsymbol{x}) \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x})\, d\boldsymbol{x} = -\int_{\Omega} \frac{\partial g}{\partial x_i}(\boldsymbol{x}) \varphi(\boldsymbol{x})\, d\boldsymbol{x},$$

which implies that $\left[\frac{\partial g}{\partial x_i}\right] \in L^p(\Omega)$ is the weak partial derivative of $[f]$ with respect to x_i . This shows that $[f] \in W^{1,p}(\Omega)$. ■

Exercise 180 *Let $I \subseteq \mathbb{R}$ and let $f : I \to \mathbb{R}$ be locally absolutely continuous with $f' \in L^p(I)$, $1 \leq p \leq \infty$. Prove that f is absolutely continuous.*

Corollary 181 *Let $I \subseteq \mathbb{R}$ be an open interval, let $1 \leq p < \infty$, and let $[f] \in L^p(I)$. Then $[f] \in W^{1,p}(I)$ if and only if there exists $g : I \to \mathbb{R}$ with $g \sim f$ such that g is absolutely continuous and*

$$\int_I |g'(x)|^p dx < \infty.$$

Remark 182 *In view of the previous corollary, for $N = 1$, we could have defined $W^{1,p}(I)$ as the set of all absolutely continuous functions $g : I \to \mathbb{R}$ such that*

$$\int_I |g(x)|^p dx + \int_I |g'(x)|^p dx < \infty.$$

Using the previous corollary, we can prove the following embedding theorems.

Corollary 183 *Let $I \subseteq \mathbb{R}$ be an open interval, let $1 \leq p < \infty$, and let $[f] \in W^{1,p}(I)$.*

(i) If $1 < p < \infty$, then the absolutely continuous representative g of $[f]$ is Hölder continuous with exponent $\frac{1}{p'}$.

(ii) If $p = 1$ and I is unbounded then $\|[f]\|_{L^\infty(I)} \leq \|[f']\|_{L^\infty(I)}$

Proof. (i) Assume $1 < p < \infty$ and let g be the absolutely continuous representative g of $[f]$. By the fundamental theorem of calculus, for $x, y \in I$ with $x < y$, we have

$$g(y) - g(x) = \int_x^y g'(t)\, dt.$$

Using Hölder's inequality, we get

$$\begin{aligned} |g(x) - g(y)| &\leq \int_x^y 1|g'(t)|\, dt \leq \left(\int_x^y 1^{p'} dt\right)^{1/p'} \left(\int_x^y |g'(t)|^p dt\right)^{1/p} \\ &\leq |y - x|^{1/p'} \|[f']\|_{L^p(I)}, \end{aligned}$$

which proves that g is Hölder continuous.

(ii) Assume that $\sup I = \infty$ and let g be the absolutely continuous representative g of $[f]$. Since $\int_I |g(x)|\, dx < \infty$, necessarily

$$\liminf_{x\to\infty} |g(x)| = 0,$$

since otherwise, we would be able to find $C > 0$ such that $|g(x)| \geq C$ for all x large, which would contradict the fact that g is integrable. Hence, we can find $x_n \in I$, $x_n \to \infty$ such that $g(x_n) \to \infty$. By the fundamental theorem of calculus, for $x \in I$, we have

$$g(x) = g(x_n) + \int_{x_n}^{x} g'(t)\, dt.$$

Let n be so large that $x_n > x$. Then

$$|g(x)| \leq |g(x_n)| + \int_{x}^{x_n} |g'(t)|\, dt \leq |g(x_n)| + \|[f']\|_{L^1(I)}.$$

Letting $n \to \infty$, we get

$$|g(x)| \leq \|[f']\|_{L^1(I)}$$

for all $x \in I$. ■

As a consequence of Theorem 179 and of the properties of absolutely continuous functions we have the following results.

Exercise 184 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$. Using Theorem 179 prove the following results.*

(i) ***(Chain rule)*** *Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz and let $f \in W^{1,p}(\Omega)$. Assume that $h(0) = 0$ if Ω has infinite measure. Then $h \circ f \in W^{1,p}(\Omega)$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$,*

$$\frac{\partial (h \circ f)}{\partial x_i}(\boldsymbol{x}) = h'\left(\overline{f}(\boldsymbol{x})\right) \frac{\partial f}{\partial x_i}(\boldsymbol{x}),$$

where $h'\left(\overline{f}(\boldsymbol{x})\right) \frac{\partial f}{\partial x_i}(\boldsymbol{x})$ is interpreted to be zero whenever $\frac{\partial f}{\partial x_i}(\boldsymbol{x}) = 0$. What can you say about the case $p = \infty$?

(ii) ***(Product rule)*** *Let $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $fg \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$,*

$$\frac{\partial (fg)}{\partial x_i}(\boldsymbol{x}) = g(\boldsymbol{x}) \frac{\partial f}{\partial x_i}(\boldsymbol{x}) + f(\boldsymbol{x}) \frac{\partial g}{\partial x_i}(\boldsymbol{x}).$$

What can you say about the case $p = \infty$?

(iii) ***(Reflection)*** *Let $\Omega = \mathbb{R}^N_+ := \left\{(\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\right\}$ and let $f \in W^{1,p}\left(\mathbb{R}^N_+\right)$. Then the function*

$$g(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } x_N > 0, \\ f(\boldsymbol{x}', -x_N) & \text{if } x_N < 0 \end{cases}$$

belongs to $W^{1,p}\left(\mathbb{R}^N\right)$ *and for all* $i = 1, \ldots, N$ *and for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \mathbb{R}^N$

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \dfrac{\partial f}{\partial x_i}(\boldsymbol{x}) & \text{if } x_N > 0, \\ (-1)^{\delta_{iN}} \dfrac{\partial f}{\partial x_i}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0. \end{cases}$$

(iv) Let $E \subset \mathbb{R}$ *be such that* $\mathcal{L}^1(E) = 0$, *let* $f \in W^{1,1}_{\text{loc}}(\Omega)$, *and let* $\overline{f}$ *be its precise representative given in Theorem 179. Prove that* $\nabla f(\boldsymbol{x}) = 0$ *for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \left(\overline{f}\right)^{-1}(E)$.

15 Embeddings: $1 \leq p < N$

Consider a function $f \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ such that its weak gradient ∇f belongs to $L^p\left(\mathbb{R}^N; \mathbb{R}^N\right)$ for some $1 \leq p < \infty$. We are interested in finding an exponent q such that $f \in L^q\left(\mathbb{R}^N\right)$, and so we are after an inequality of the type

$$\|f\|_{L^q(\mathbb{R}^N)} \leq c \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}, \tag{54}$$

which should hold for all such f.

Assume for simplicity that $f \in C^1_c\left(\mathbb{R}^N\right)$ and for $r > 0$ define the rescaled function

$$f_r(\boldsymbol{x}) := f(r\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

Applying the previous inequality to f_r we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f(r\boldsymbol{x})|^q \, d\boldsymbol{x}\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^N} |f_r(\boldsymbol{x})|^q \, d\boldsymbol{x}\right)^{\frac{1}{q}} \\ &\leq c\left(\int_{\mathbb{R}^N} \|\nabla f_r(\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} = c\left(r^p \int_{\mathbb{R}^N} \|\nabla f(r\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},\end{aligned}$$

or, equivalently, after the change of variables $\boldsymbol{y} := r\boldsymbol{x}$,

$$\left(\frac{1}{r^N}\int_{\mathbb{R}^N} |f(\boldsymbol{y})|^q \, d\boldsymbol{y}\right)^{\frac{1}{q}} \leq c\left(\frac{r^p}{r^N}\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p d\boldsymbol{y}\right)^{\frac{1}{p}},$$

that is,

$$\left(\int_{\mathbb{R}^N} |f(\boldsymbol{y})|^q \, d\boldsymbol{y}\right)^{\frac{1}{q}} \leq c r^{1-\frac{N}{p}+\frac{N}{q}} \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p d\boldsymbol{y}\right)^{\frac{1}{p}}.$$

If $1 - \frac{N}{p} + \frac{N}{q} > 0$, let $r \to 0^+$ to conclude that $f \equiv 0$, while if $1 - \frac{N}{p} + \frac{N}{q} < 0$, let $r \to \infty$ to conclude again that $f \equiv 0$. Hence, the only possible case is when

$$\frac{N}{q} = \frac{N}{p} - 1.$$

So in order for q to be positive, we need $p < N$ in which case

$$q = p^* := \frac{Np}{N-p}.$$

The number p^* is called *Sobolev critical exponent.*

Theorem 185 (Sobolev–Gagliardo–Nirenberg Embedding) *Let $1 \leq p < N$. Then for every $f \in W^{1,p}\left(\mathbb{R}^N\right)$,*

$$\left(\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^{p^*}\, d\boldsymbol{x}\right)^{\frac{1}{p^*}} \leq C\left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},$$

where $C = C(N,p) > 0$. In particular, $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^{p^*}\left(\mathbb{R}^N\right)$.

The proof makes use of the following result, which follows from Hölder's inequality.

Exercise 186 *Let $1 \leq p_1, \ldots, p_n, p \leq \infty$, with $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{p}$, and $f_i \in L^{p_i}\left(\mathbb{R}^N\right)$, $i = 1, \ldots, n$. Prove that*

$$\left\|\prod_{i=1}^n f_i\right\|_{L^p} \leq \prod_{i=1}^n \|f_i\|_{L^{p_i}}.$$

Exercise 187 *Prove that if $g : \mathbb{R} \to \mathbb{R}$ is measurable with $\int_{\mathbb{R}} |g(t)|^p\, dt < \infty$ for some $p > 0$, then*

$$\liminf_{x \to -\infty} |g(x)| = 0, \quad \liminf_{x \to \infty} |g(x)| = 0$$

and that in general one cannot replace the limit inferiors with actual limits.

In what follows, we use the notation (48).

Lemma 188 *Let $N \geq 2$ and let $f_i \in L^{N-1}\left(\mathbb{R}^{N-1}\right)$, $i = 1, \ldots, N$. Then the function*

$$f(\boldsymbol{x}) := f_1(\boldsymbol{x}_1')\, f_2(\boldsymbol{x}_2') \cdots f_N(\boldsymbol{x}_N'), \quad \boldsymbol{x} \in \mathbb{R}^N,$$

belongs to $L^1\left(\mathbb{R}^N\right)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^N \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

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Proof. The proof is by induction on N. If $N = 2$, then

$$f(\boldsymbol{x}) := f_1(x_2)\, f_2(x_1), \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Integrating both sides with respect to $\boldsymbol{x}$ and using Fubini's theorem, we get

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})| \, d\boldsymbol{x} = \int_{\mathbb{R}} |f_1(x_2)| \, dx_2 \int_{\mathbb{R}} |f_2(x_1)| \, dx_1.$$

Assume next that the result is true for N and let's prove it for $N+1$. Let

$$f(\boldsymbol{x}) := f_1(\boldsymbol{x}_1') f_2(\boldsymbol{x}_2') \cdots f_{N+1}(\boldsymbol{x}_{N+1}'), \quad \boldsymbol{x} \in \mathbb{R}^{N+1},$$

where $f_i \in L^N(\mathbb{R}^N)$, $i = 1, \ldots, N+1$. Fix $x_{N+1} \in \mathbb{R}$. Integrating both sides with respect to $x_1, \ldots, x_N$ and using Hölder's inequality we get

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, dx_1 \cdots dx_N \\ &\leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\boldsymbol{x}_i')|^{\frac{N}{N-1}} \, dx_1 \cdots dx_N \right)^{\frac{N-1}{N}}.\end{aligned}$$

For every $i = 1, \ldots, N$ we denote by $\boldsymbol{x}_i''$ the $N-1$ dimensional vector obtained by removing the last component from $\boldsymbol{x}_i'$ and with an abuse of notation we write $\boldsymbol{x}_i' = (\boldsymbol{x}_i'', x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Since x_{N+1} is fixed, by the induction hypothesis applied to the functions $g_i(\boldsymbol{x}_i'') := |f_i(\boldsymbol{x}_i'', x_{N+1})|^{\frac{N}{N-1}}$, $\boldsymbol{x}_i'' \in \mathbb{R}^{N-1}$, $i = 1, \ldots, N$, we obtain that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\boldsymbol{x}_i')|^{\frac{N}{N-1}} \, dx_1 \cdots dx_N \leq \prod_{i=1}^{N} \|g_i\|_{L^{N-1}(\mathbb{R}^{N-1})},$$

and so

$$\begin{aligned}&\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, dx_1 \cdots dx_N \\ &\leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \prod_{i=1}^{N} \left(\int_{\mathbb{R}^{N-1}} |f_i(\boldsymbol{x}_i'', x_{N+1})|^N \, d\boldsymbol{x}_i'' \right)^{\frac{1}{N}}.\end{aligned}$$

Integrating both sides with respect to x_{N+1} and using Fubini's theorem and the extended Hölder's inequality (see the previous exercise), with

$$1 = \underbrace{\frac{1}{N} + \cdots + \frac{1}{N}}_{N},$$

we get

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, d\boldsymbol{x} \leq \prod_{i=1}^{N+1} \|f_i\|_{L^N(\mathbb{R}^N)},$$

which concludes the proof. ■

We now turn to the proof of the Sobolev–Gagliardo–Nirenberg embedding theorem.

Proof. Step 1: Assume first that $p = 1$. By mollification we can assume that $f \in C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$. Fix $i = 1, \dots, N$. By Fubini's theorem for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i' \in \mathbb{R}^{N-1}$ we have that the function $g(t) := f(\boldsymbol{x}_i', t)$, $t \in \mathbb{R}$, belongs to $L^p(\mathbb{R}) \cap C^1(\mathbb{R})$ with $g' \in L^1(\mathbb{R})$. By the previous exercise

$$\liminf_{t \to -\infty} |g(t)| = 0,$$

and so we may find a sequence $t_n \to -\infty$ such that $g(t_n) \to 0$. Hence, for every $t \in \mathbb{R}$ we have that

$$g(t) = g(t_n) + \int_{t_n}^{t} g'(s)\, ds.$$

Letting $n \to \infty$ and using the fact that $g' \in L^1(\mathbb{R})$, by Lebesgue dominated convergence theorem we conclude that for each $i = 1, \dots, N$ and $\boldsymbol{x} \in \mathbb{R}^N$ we have

$$f(\boldsymbol{x}) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i)\, dy_i,$$

and so

$$|f(\boldsymbol{x})| \le \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i) \right| dy_i$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Multiplying these N inequalities and raising to power $\frac{1}{N-1}$, we get

$$|f(\boldsymbol{x})|^{\frac{N}{N-1}} \le \prod_{i=1}^{N} \left(\int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i) \right| dy_i \right)^{\frac{1}{N-1}} =: \prod_{i=1}^{N} w_i(\boldsymbol{x}_i')$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. We now apply the previous lemma to the function

$$w(\boldsymbol{x}) := \prod_{i=1}^{N} w_i(\boldsymbol{x}_i'), \quad \boldsymbol{x} \in \mathbb{R}^N,$$

to obtain that

$$\begin{aligned} \int_{\mathbb{R}^N} |f(\boldsymbol{x})|^{\frac{N}{N-1}}\, d\boldsymbol{x} &\le \int_{\mathbb{R}^N} |w(\boldsymbol{x})|\, d\boldsymbol{x} \le \prod_{i=1}^{N} \|w_i\|_{L^{N-1}(\mathbb{R}^{N-1})} \\ &= \prod_{i=1}^{N} \left(\int_{\mathbb{R}^N} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right| d\boldsymbol{x} \right)^{\frac{1}{N-1}} \le \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|\, d\boldsymbol{x} \right)^{\frac{N}{N-1}}, \end{aligned}$$

where we have used Fubini's theorem. This gives the desired inequality for $p = 1$.

Note that Step 1 continues to hold if we assume that $f \in L^q(\mathbb{R}^N)$ for some $q \ge 1$ and $\nabla f \in L^1(\mathbb{R}^N; \mathbb{R}^N)$. ■

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Proof. Step 2: Assume next that $1 < p < N$ and that $f \in L^{p^*}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$. Again by mollification we can assume that $f \in C^1(\mathbb{R}^N)$. Define

$$g := |f|^q, \quad q := \frac{p(N-1)}{N-p}.$$

Note that since $q > 1$, we have that $g \in C^1(\mathbb{R}^N)$. Moreover, $\nabla g \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ (see below), while $g \in L^{1^*}(\mathbb{R}^N)$. Applying Step 1 to the function g we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} &= \left(\int_{\mathbb{R}^N} |g|^{\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} \\ &\leq \int_{\mathbb{R}^N} \|\nabla g\|\, d\boldsymbol{x} \leq q \int_{\mathbb{R}^N} |f|^{q-1} \|\nabla f\|\, d\boldsymbol{x} \\ &\leq q \left(\int_{\mathbb{R}^N} |f|^{(q-1)p'}\, d\boldsymbol{x}\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Since

$$(q-1)\, p' = p^*,$$

if $f \neq 0$ we obtain

$$\left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N} - \frac{p-1}{p}} = \left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-p}{Np}} \leq q \left(\int_{\mathbb{R}^N} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},$$

which proves the result. Note that here it was important to know that $f \in L^{p^*}(\mathbb{R}^N)$, since we divided by $\left(\int_{\mathbb{R}^N} |f|^{(q-1)p'}\, d\boldsymbol{x}\right)^{\frac{1}{p'}}$.

Step 3: Assume that $f \in W^{1,p}(\mathbb{R}^N)$. For $n \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^N$ define

$$g_n(\boldsymbol{x}) := \begin{cases} |f(\boldsymbol{x})| - \frac{1}{n} & \text{if } \frac{1}{n} \leq |f(\boldsymbol{x})| \leq n, \\ 0 & \text{if } |f(\boldsymbol{x})| < \frac{1}{n}, \\ n - \frac{1}{n} & \text{if } |f(\boldsymbol{x})| > \frac{1}{n}. \end{cases}$$

By the chain rule (see Exercise 184 (i) and (vi)) for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$

$$\|\nabla g_n(\boldsymbol{x})\| = \begin{cases} \|\nabla f(\boldsymbol{x})\| & \text{if } \frac{1}{n} < |f(\boldsymbol{x})| < n, \\ 0 & \text{otherwise,} \end{cases}$$

and so $\nabla g_n \in L^p(\mathbb{R}^N; \mathbb{R}^N)$, while for every $s \geq 1$,

$$\begin{aligned}\int_{\mathbb{R}^N} |g_n|^s\, d\boldsymbol{x} &= \int_{\left\{|f| > \frac{1}{n}\right\}} |g_n|^s\, d\boldsymbol{x} \\ &\leq \left(n - \frac{1}{n}\right)^s \mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : |f(\boldsymbol{x})| > \frac{1}{n}\right\}\right) < \infty,\end{aligned}$$

since $f \in L^p(\mathbb{R})$. Hence, $g_n \in L^{p^*}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and so by the previous step

$$\left(\int_{\left\{\frac{1}{n} \leq |f| \leq n\right\}} \left(|f(\boldsymbol{x})| - \frac{1}{n}\right)^{\frac{pN}{N-p}} d\boldsymbol{x}\right)^{\frac{N-p}{Np}} \leq \left(\int_{\mathbb{R}^N} |g_n|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-p}{Np}}$$

$$\leq q \left(\int_{\mathbb{R}^N} \|\nabla g_n\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} = q \left(\int_{\left\{\frac{1}{n} \leq |f| \leq n\right\}} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \leq q \left(\int_{\mathbb{R}^N} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}}.$$

Letting first $n \to \infty$ and using Fatou's lemma we obtain the desired result. ■

Exercise 189 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp < N$. Prove that*

(i) $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $W^{j,q}\left(\mathbb{R}^N\right)$ for all $j \in \mathbb{N}$ and for all $p \leq q \leq \frac{Np}{N-kp}$,

(ii) $W^{k,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q \leq \frac{Np}{N-kp}$.

Remark 190 *Note that in the last step of the proof of the previous theorem we only used the fact that f vanishes at infinity and its distributional gradient $\nabla f \in L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)$. In particular, it holds if we assume that $f \in L^q(\mathbb{R}^N)$ for some $1 \leq q < \infty$ and the distributional gradiend $\nabla f \in L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)$.*

Remark 191 *In view of Theorem 177 in Step 1 and 2 we could have assumed that $f \in C_c^1(\mathbb{R}^N)$ and so avoid Step 3. However, see the previous remark.*

Next we discuss the validity of the Sobolev–Gagliardo–Nirenberg embedding theorem for arbitrary domains.

Exercise 192 (Room and Passages) *Let $\{h_n\}$ and $\{\delta_{2n}\}$ be two sequences of positive numbers such that*

$$\sum_{n=1}^{\infty} h_n = \ell < \infty, \quad 0 < const. \leq \frac{h_{n+1}}{h_n} \leq 1, \quad 0 < \delta_{2n} \leq h_{2n+1},$$

and for $n \in \mathbb{N}$ let

$$c_n := \sum_{i=1}^{n} h_i.$$

Define $\Omega \subset \mathbb{R}^2$ to be the union of all sets of the form

$$R_j := (c_j - h_j, c_j) \times \left(-\frac{1}{2}h_j, \frac{1}{2}h_j\right),$$

$$P_{j+1} := [c_j, c_j + h_{j+1}] \times \left(-\frac{1}{2}\delta_{j+1}, \frac{1}{2}\delta_{j+1}\right),$$

for $j = 1, 3, 5, \ldots$,

(i) Prove that $\partial\Omega$ is a rectifiable curve but Ω is not of class C.

(ii) Let

$$h_n := \frac{1}{n^{\frac{3}{2}}}, \quad \delta_{2n} := \frac{1}{n^{\frac{5}{2}}},$$

and for $j = 1, 3, 5, \ldots$,

$$f(x,y) := \begin{cases} \dfrac{j}{\log 2j} =: K_j & \text{in } R_j, \\ K_j + (K_{j+2} - K_j)\dfrac{x - c_j}{h_{j+1}} & \text{in } P_{j+1}. \end{cases}$$

Prove that $f \in W^{1,2}(\Omega)$ but $f \notin L^q(\Omega)$ for any $q > 2$.

(iii) *Let $p > 1$, $q \geq \frac{1}{2}(2p-1)$,*

$$h_{2n-1} = h_{2n} := \frac{1}{n^p}, \quad \delta_{2n} := \frac{1}{3^p n^{2q+p}},$$

and for $n \in \mathbb{N}$,

$$f(x,y) := \frac{1}{n^p} \text{ in } R_{2n-1},$$

and

$$\nabla f(x,y) := \left(\frac{(n+1)^q - n^q}{\frac{1}{n^p}}, 0 \right) \text{ in } P_{2n}.$$

Prove that $\nabla f \in L^2\left(\Omega; \mathbb{R}^{2\times 2}\right)$ but $f \notin L^2(\Omega)$.

16 Embeddings: $p = N$

The argument at the beginning of the previous section shows that when $p \geq N$ we cannot expect an inequality of the form

$$\|f\|_{L^q(\mathbb{R}^N)} \leq c \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

However, we could still have embeddings of the type

$$W^{1,p}\left(\mathbb{R}^N\right) \to L^q\left(\mathbb{R}^N\right)$$
$$f \mapsto f$$

that is, inequalities of the type

$$\|f\|_{L^q(\mathbb{R}^N)} \leq c \|f\|_{W^{1,p}(\mathbb{R}^N)}.$$

We now show that this is the case when $p = N$. We begin by observing that when $p \nearrow N$, then $p^* \nearrow \infty$, and so one would be tempted to say that if $f \in W^{1,N}\left(\mathbb{R}^N\right)$, then $f \in L^\infty\left(\mathbb{R}^N\right)$. For $N = 1$ this is true since if $f \in W^{1,1}(\mathbb{R})$, then a representative $\overline{f}$ is absolutely continuous in $\mathbb{R}$ so that

$$\overline{f}(x) = \overline{f}(0) + \int_0^x \overline{f}'(s)\, ds$$

and since $\overline{f}' = f' \in L^1(\mathbb{R})$, we have that $\overline{f}$ is bounded and continuous. For $N > 1$ this is not the case, as the next exercise shows.

Exercise 193 *Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $N > 1$, and show that the function*

$$f(\boldsymbol{x}) := \log\left(\log\left(1 + \frac{1}{\|\boldsymbol{x}\|}\right)\right), \quad \boldsymbol{x} \in B(0,1) \setminus \{0\},$$

belongs to $W^{1,N}(B(0,1))$ but not to $L^\infty(B(0,1))$.

However, we have the following result.

Theorem 194 *The space* $W^{1,N}\left(\mathbb{R}^N\right)$ *is continuously embedded in the space* $L^q\left(\mathbb{R}^N\right)$ *for all* $N \le q < \infty$.

Proof. Let $f \in W^{1,N}\left(\mathbb{R}^N\right)$. Define $g := |f|^t$, where $t > 1$ will be determined so that $g \in L^r\left(\mathbb{R}^N\right)$ and $\nabla g \in L^1(\mathbb{R}^N;\mathbb{R}^N)$. By the Sobolev–Gagliardo–Nirenberg embedding theorem with $p = 1$ and Remark 190,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{tN}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} &= \left(\int_{\mathbb{R}^N} |g|^{\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} \\ &\le \int_{\mathbb{R}^N} \|\nabla g\|\, d\boldsymbol{x} \le t\int_{\mathbb{R}^N} |f|^{t-1}\|\nabla f\|\, d\boldsymbol{x} \\ &\le t\left(\int_{\mathbb{R}^N} |f|^{(t-1)N'}\, d\boldsymbol{x}\right)^{\frac{1}{N'}}\left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}},\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Hence,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{tN}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{Nt}} &\le C\left(\int_{\mathbb{R}^N} |f|^{(t-1)\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{tN}}\left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{Nt}} \\ &\le C\left[\left(\int_{\mathbb{R}^N} |f|^{(t-1)\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}\frac{1}{t-1}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right],\end{aligned} \tag{55}$$

where we have used Young's inequality $ab \le a^t + b^{t'}$ for $a, b \ge 0$. Taking $t = N$ yields

$$\left(\int_{\mathbb{R}^N} |f|^{\frac{N^2}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N^2}} \le C\left[\left(\int_{\mathbb{R}^N} |f|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right],$$

so that $f \in L^{\frac{N^2}{N-1}}\left(\mathbb{R}^N\right)$ with continuous embedding. In turn by Theorem **??**, we conclude that

$$\|f\|_{L^q(\mathbb{R}^N)} \le C\,\|f\|_{W^{1,N}(\mathbb{R}^N)}$$

for all $N \le q \le \frac{N^2}{N-1}$.

Taking $t = N + 1 \le \frac{N^2}{N-1}$ in (55) and using what we just proved gives

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |f|^{\frac{N(N+1)}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N(N+1)}} \\ &\qquad \le C\left[\left(\int_{\mathbb{R}^N} |f|^{\frac{N^2}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N^2}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right] \\ &\qquad \le C\,\|f\|_{W^{1,N}(\mathbb{R}^N)},\end{aligned}$$

and so the embedding

$$\begin{aligned}W^{1,p}\left(\mathbb{R}^N\right) &\to L^q\left(\mathbb{R}^N\right) \\ f &\mapsto f\end{aligned}$$

is continuous for all $N \leq q \leq \frac{N(N+1)}{N-1}$. We proceed in this fashion taking $t = N+2, N+3$, etc. ■

Exercise 195 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp = N$. Prove that*

(i) $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $W^{j,q}\left(\mathbb{R}^N\right)$ for all $j \in \mathbb{N}$ and for all $p \leq q < \infty$,

(ii) $W^{k,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q < \infty$.

Exercise 196 *Prove that for every function $f \in W^{N,1}\left(\mathbb{R}^N\right)$,*

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq \left\| \frac{\partial^N f}{\partial x_1 \cdots \partial x_N} \right\|_{L^N(\mathbb{R}^N)}.$$

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17 Embeddings: $p > N$

We recall that, given an open set $\Omega \subseteq \mathbb{R}^N$, a function $f : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq C\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. We define the space $C^{0,\alpha}\left(\overline{\Omega}\right)$ as the space of all bounded functions that are Hölder continuous with exponent α.

Exercise 197 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\alpha > 0$.*

(i) Prove that if $\alpha > 1$ and Ω is connected, then any function that is Hölder continuous with exponent α is constant.

(ii) Prove that the space $C^{0,\alpha}\left(\overline{\Omega}\right)$, $0 < \alpha \leq 1$, is a Banach space with the norm

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\boldsymbol{x} \in \Omega} |f(\boldsymbol{x})| + \sup_{\boldsymbol{x},\, \boldsymbol{y} \in \Omega,\, \boldsymbol{x} \neq \boldsymbol{y}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha}.$$

Note that if Ω is bounded, then every function $f : \Omega \to \mathbb{R}$ that is Hölder continuous with exponent $\alpha > 0$ is uniformly continuous and thus it can be uniquely extended to a bounded continuous function on $\mathbb{R}^N$. Thus, in the definition of $C^{0,\alpha}\left(\overline{\Omega}\right)$ one can drop the requirement that the functions are bounded.

The next theorem shows that if $p > N$ a function $f \in W^{1,p}\left(\mathbb{R}^N\right)$ has a representative in the space $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$.

Theorem 198 (Morrey) *Let $N < p < \infty$. Then the space $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$. Moreover, if $f \in W^{1,p}\left(\mathbb{R}^N\right)$ and $\bar{f}$ is its representative in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, then*

$$\lim_{\|\boldsymbol{x}\|\to\infty} \bar{f}(\boldsymbol{x}) = 0.$$

Proof. Let $f \in W^{1,p}\left(\mathbb{R}^N\right) \cap C^\infty\left(\mathbb{R}^N\right)$ and let Q_r be any cube with sides of length r parallel to the axes. Fix $\boldsymbol{x}, \boldsymbol{y} \in Q_r$ and let

$$g(t) := f(t\boldsymbol{x} + (1-t)\boldsymbol{y}), \quad 0 \le t \le 1.$$

By the fundamental theorem of calculus

$$\begin{aligned} f(\boldsymbol{x}) - f(\boldsymbol{y}) = g(1) - g(0) &= \int_0^1 g'(t)\, dt \\ &= \int_0^1 \nabla f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\, dt. \end{aligned}$$

Averaging in the $\boldsymbol{x}$ variable over Q_r yields

$$f_{Q_r} - f(\boldsymbol{y}) = \frac{1}{r^N} \int_{Q_r} \int_0^1 \nabla f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\, dt\, d\boldsymbol{x},$$

where f_{Q_r} is the integral average of f over Q_r, that is,

$$f_{Q_r} := \frac{1}{r^N} \int_{Q_r} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Hence,

$$\begin{aligned} |f_{Q_r} - f(\boldsymbol{y})| &\le \sum_{i=1}^N \frac{1}{r^N} \int_{Q_r} \int_0^1 \left| \frac{\partial f}{\partial x_i}(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \right| |x_i - \boldsymbol{y}_i|\, dt\, d\boldsymbol{x} \\ &\le \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \int_{Q_r} \left| \frac{\partial f}{\partial x_i}(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \right| d\boldsymbol{x}\, dt \\ &= \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \frac{1}{t^N} \int_{(1-t)\boldsymbol{y} + Q_{rt}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{z}) \right| d\boldsymbol{z}\, dt, \end{aligned}$$

where we have used the fact that $|x_i - y_i| \le r$ in Q_r, Tonelli's theorem, and the change of variables $\boldsymbol{z} = t\boldsymbol{x} + (1-t)\boldsymbol{y}$ (so that $d\boldsymbol{z} = t^N d\boldsymbol{x}$). By Hölder's inequality and the fact that $(1-t)\boldsymbol{y} + Q_{rt} \subset Q_r$, we now have

$$\begin{aligned} |f_{Q_r} - f(\boldsymbol{y})| &\le \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \frac{(rt)^{\frac{N}{p'}}}{t^N} \left(\int_{(1-t)\boldsymbol{y} + Q_{rt}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{z}) \right|^p d\boldsymbol{z} \right)^{\frac{1}{p}} dt \\ &\le N \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)} \frac{r^{N-\frac{N}{p}}}{r^{N-1}} \int_0^1 \frac{t^{N-\frac{N}{p}}}{t^N}\, dt \qquad (56) \\ &= \frac{Np}{p-N} r^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)}. \end{aligned}$$

Since this is true for all $\boldsymbol{y} \in Q_r$, if $\boldsymbol{x}, \boldsymbol{y} \in Q_r$, then

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &\le |f(\boldsymbol{x}) - f_{Q_r}| + |f(\boldsymbol{y}) - f_{Q_r}| \\ &\le \frac{2Np}{p-N} r^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)} . \end{aligned}$$

Now if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, consider a cube Q_r containing $\boldsymbol{x}$ and $\boldsymbol{y}$ and of side length $r := 2\|\boldsymbol{x} - \boldsymbol{y}\|$. Then the previous inequality yields

$$\begin{aligned} |f(\boldsymbol{x}) - f(\boldsymbol{y})| &\le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)} \\ &\le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} . \end{aligned} \tag{57}$$

Hence, f is Hölder continuous of exponent $1-\frac{N}{p}$. To prove that $f \in C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, it remains to show that f is bounded. Let $\boldsymbol{x} \in \mathbb{R}^N$ and consider a cube Q_1 containing $\boldsymbol{x}$ and of side length one. By (56) we get

$$\begin{aligned} |f(\boldsymbol{x})| &\le |f_{Q_1}| + |f(\boldsymbol{x}) - f_{Q_1}| \le \left| \int_{Q_1} f(\boldsymbol{x})\, d\boldsymbol{x} \right| + C\|\nabla f\|_{L^p(Q_1;\mathbb{R}^N)} \\ &\le \|f\|_{L^p(Q_1)} + C\|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\mathbb{R}^N)} , \end{aligned} \tag{58}$$

where we have used Hölder's inequality.

Next we remove the extra hypothesis that $f \in C^\infty\left(\mathbb{R}^N\right)$. Given any $f \in W^{1,p}\left(\mathbb{R}^N\right)$, let $\bar{f}$ be a representative of f and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ be two Lebesgue points of $\bar{f}$ and let $f_\varepsilon := f * \varphi_\varepsilon$, where φ_ε is a standard mollifier. By (57) we have that

$$|f_\varepsilon(\boldsymbol{x}) - f_\varepsilon(\boldsymbol{y})| \le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f_\varepsilon\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} .$$

Since $\{f_\varepsilon\}$ converge at every Lebesgue point by Theorem 155 and $\nabla f_\varepsilon = (\nabla f)_\varepsilon \to \nabla f$ in $L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)$ by Theorems 155, letting $\varepsilon \to 0^+$, we get

$$\left|\bar{f}(\boldsymbol{x}) - \bar{f}(\boldsymbol{y})\right| \le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \tag{59}$$

for all Lebesgue points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ of $\bar{f}$. This implies that

$$\bar{f} : \{\text{Lebesgue points of } f\} \to \mathbb{R}$$

can be uniquely extended to $\mathbb{R}^N$ as a Hölder continuous function $\bar{f}$ of exponent $1 - \frac{N}{p}$ in such a way that (59) holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$.

With a similar argument from (58) we conclude that

$$\left|\bar{f}(\boldsymbol{x})\right| \le C\|f\|_{W^{1,p}(\mathbb{R}^N)} \tag{60}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Hence,

$$\begin{aligned} \left\|\bar{f}\right\|_{C^{0,1-\frac{N}{p}}(\mathbb{R}^N)} &= \sup_{\boldsymbol{x} \in \mathbb{R}^N} \left|\bar{f}(\boldsymbol{x})\right| + \sup_{\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N, \boldsymbol{x} \neq \boldsymbol{y}} \frac{\left|\bar{f}(\boldsymbol{x}) - \bar{f}(\boldsymbol{y})\right|}{\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}}} \\ &\le C\|f\|_{W^{1,p}(\mathbb{R}^N)} . \end{aligned}$$

Finally, we prove that $\bar{f}(\boldsymbol{x}) \to 0$ as $\|\boldsymbol{x}\| \to \infty$. Let $\{f_n\} \subset C_c^\infty(\mathbb{R}^N)$ be any sequence that converges to f in $W^{1,p}(\mathbb{R}^N)$. The inequality (60) implies, in particular, that $f \in L^\infty(\mathbb{R}^N)$, with

$$\|f\|_{L^\infty(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\mathbb{R}^N)}.$$

Replacing f with $f - f_n$ gives

$$\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \le C\|f - f_n\|_{W^{1,p}(\mathbb{R}^N)},$$

and so $\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$ and find $\bar{n} \in \mathbb{N}$ such that

$$\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \le \varepsilon$$

for all $n \ge \bar{n}$. Since $f_{\bar{n}} \in C_c^\infty(\mathbb{R}^N)$, there exists $R_{\bar{n}} > 0$ such that $f_{\bar{n}}(\boldsymbol{x}) = 0$ for all $\|\boldsymbol{x}\| \ge R_{\bar{n}}$. Hence, for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$ with $\|\boldsymbol{x}\| \ge R_{\bar{n}}$ we get

$$\left|\bar{f}(\boldsymbol{x})\right| = \left|\bar{f}(\boldsymbol{x}) - f_{\bar{n}}(\boldsymbol{x})\right| \le \|f - f_n\|_{L^\infty(\mathbb{R}^N)} \le \varepsilon,$$

and, since $\bar{f}$ is continuous, we get that the previous inequality actually holds for all $\boldsymbol{x} \in \mathbb{R}^N$ with $\|\boldsymbol{x}\| \ge R_{\bar{n}}$. ■

Wednesday, April 19, 2023

18 Extension Domains

You have seen in recitations that there are open sets $\Omega \subset \mathbb{R}^N$ and functions $f \in W^{1,p}(\Omega)$, $1 < p < N$, such that $f \notin L^q(\Omega)$ for all $q > p$. This means that the Sobolev–Gagliardo–Nirenberg theorem fails in "bad" open sets. In this section, we are going to prove that if $\partial\Omega$ is sufficiently regular, then we can extend a function $f \in W^{1,p}(\Omega)$ to a function $g \in W^{1,p}(\mathbb{R}^N)$.

We begin with the case in which Ω is the half space $\mathbb{R}^N_+$.

Theorem 199 *Let $1 \le p \le \infty$ and let $f \in W^{1,p}(\mathbb{R}^N_+)$. Then there exists $g \in W^{1,p}(\mathbb{R}^N)$ such that $g(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N_+$ and*

$$\|g\|_{L^p(\mathbb{R}^N)} \le 2\|f\|_{L^p(\mathbb{R}^N_+)}, \quad \left\|\frac{\partial g}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} \le 2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(\mathbb{R}^N_+)},$$

for all $i = 1, \ldots, N$.

Proof. We only do the case $p < \infty$. Given $[f] \in W^{1,p}(\mathbb{R}^N_+)$, by Theorem 179, there exists a representative f such that $f(\boldsymbol{x}_i', \cdot)$ is absolutely continuous in $\mathbb{R}$ for $\mathcal{L}^{N-1}$-a.e. $\boldsymbol{x}_i' \in \mathbb{R}^{N-1}_+$ when $i = 1, \ldots, N-1$, and $f(\boldsymbol{x}', \cdot)$ is absolutely continuous in $\mathbb{R}_+$ for $\mathcal{L}^{N-1}$-a.e. $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. Define

$$g(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ f(\boldsymbol{x}) & \text{if } x_N \ge 0, \end{cases}$$

The $g \in C(\mathbb{R}^N)$ and absolutely continuous on $\mathcal{L}^{N-1}$ every line parallel to the axes with

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial f}{\partial x_i}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ \frac{\partial f}{\partial x_i}(\boldsymbol{x}) & \text{if } x_N > 0, \end{cases}$$

if $i = 1, \ldots, N-1$, while

$$\frac{\partial g}{\partial x_N}(\boldsymbol{x}) = \begin{cases} -\frac{\partial f}{\partial x_N}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ \frac{\partial f}{\partial x_N}(\boldsymbol{x}) & \text{if } x_N > 0. \end{cases}$$

It follows by Theorem 179 that $g \in W^{1,p}(\mathbb{R}^N)$. By a change of variables we have that

$$\|g\|_{L^p(\mathbb{R}^N)} = 2\|f\|_{L^p(\mathbb{R}^N_+)}, \left\|\frac{\partial g}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} = 2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(\mathbb{R}^N_+)}.$$

■

Note that $\frac{\partial g}{\partial x_N}$ is discontinuous at $x_N = 0$ and so we cannot use this extension for function $f \in W^{m,p}(\mathbb{R}^N_+)$ for $m \geq 2$.

Exercise 200 *Given $m \in \mathbb{N}$, and $1 \leq p \leq \infty$, let $f \in W^{m,p}(\mathbb{R}^N_+)$. Prove that there exist $c_1, \ldots, c_{m+1} \in \mathbb{R}$ such that the function*

$$g(\boldsymbol{x}) := \begin{cases} \sum_{n=1}^{m+1} c_n f(\boldsymbol{x}', -n x_N) & \text{if } x_N < 0, \\ f(\boldsymbol{x}) & \text{if } x_N > 0, \end{cases}$$

is well-defined and belongs to $W^{m,p}(\mathbb{R}^N)$. Prove also that for every $0 \leq k \leq m$, $\|\nabla^k g\|_{L^p(\mathbb{R}^N)} \leq c\|\nabla^k f\|_{L^p(\mathbb{R}^N_+)}$ for some constant $c = c(m, N, p) > 0$.

Next we consider the important special case in which Ω lies above the graph of a Lipschitz continuous function.

Theorem 201 *Let $h : \mathbb{R}^{N-1} \to \mathbb{R}$ be a Lipschitz continuous function of class C^1 and let*

$$\Omega := \{(\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > h(\boldsymbol{x}')\}. \tag{61}$$

Let $1 \leq p \leq \infty$ and let $f \in W^{1,p}(\Omega)$. Then there exists $g \in W^{1,p}(\mathbb{R}^N)$ such that $g(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$ and

$$\|g\|_{L^p(\mathbb{R}^N)} \leq 2\|f\|_{L^p(\Omega)}, \quad \left\|\frac{\partial g}{\partial x_N}\right\|_{L^p(\mathbb{R}^N)} \leq 2\|\partial_N f\|_{L^p(\Omega)}, \tag{62}$$

$$\left\|\frac{\partial g}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} \leq 2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(\Omega)} + \operatorname{Lip} h \left\|\frac{\partial f}{\partial x_N}\right\|_{L^p(\Omega)} \tag{63}$$

for all $i = 1, \ldots, N$.

Proof. The idea of the proof is to first flatten the boundary to reduce to the case in which $\Omega = \mathbb{R}^N_+$ and then use the previous theorem. We only prove the case $1 \le p < \infty$ and leave the easier case $p = \infty$ as an exercise. Consider the transformation $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ given by $\Psi(\boldsymbol{y}) := (\boldsymbol{y}', y_N + h(\boldsymbol{y}'))$. Note that Ψ is invertible, of class C^1, with inverse of class C^1 given by $\Psi^{-1}(\boldsymbol{x}) = (\boldsymbol{x}', x_N - h(\boldsymbol{x}'))$. Moreover, for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^N$,

$$\begin{aligned}\|\Psi(\boldsymbol{y}) - \Psi(\boldsymbol{z})\| &= \|(\boldsymbol{y}' - \boldsymbol{z}', h(\boldsymbol{y}') - h(\boldsymbol{z}') + y_N - z_N)\| \\ &\le \sqrt{\|\boldsymbol{y}' - \boldsymbol{z}'\|^2 + (\operatorname{Lip} h \, \|\boldsymbol{y}' - \boldsymbol{z}'\| + |y_N - z_N|)^2} \\ &\le \operatorname{Lip} h \|\boldsymbol{y} - \boldsymbol{z}\|,\end{aligned}$$

which shows that Ψ (and similarly Ψ^{-1}) is Lipschitz continuous. Since h is of class C^1, we have

$$J_\Psi(\boldsymbol{y}) = \begin{pmatrix} I_{N-1} & 0 \\ \nabla_{\boldsymbol{y}'} h(\boldsymbol{y}') & 1 \end{pmatrix},$$

which implies that $\det J_\Psi(\boldsymbol{y}) = 1$. Note that $\Psi(\mathbb{R}^N_+) = \Omega$.

Given a function $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$, define the function

$$w(\boldsymbol{y}) := f(\Psi(\boldsymbol{y})) = f(\boldsymbol{y}', y_N + h(\boldsymbol{y}')), \quad \boldsymbol{y} \in \mathbb{R}^N_+.$$

By Exercise 178 the function w belongs to $W^{1,p}(\mathbb{R}^N_+)$ and the usual chain rule formula for the partial derivatives holds. By the previous theorem the function $\hat{w} : \mathbb{R}^N \to \mathbb{R}$, defined by

$$\hat{w}(\boldsymbol{y}) := \begin{cases} w(\boldsymbol{y}) & \text{if } y_N > 0, \\ w(\boldsymbol{y}', -y_N) & \text{if } y_N < 0, \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$ and the usual chain rule formula for the partial derivatives holds.

Define the function $v : \mathbb{R}^N \to \mathbb{R}$ by

$$g(\boldsymbol{x}) := (\hat{w} \circ \Psi^{-1})(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \text{if } x_N > h(\boldsymbol{x}'), \\ f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N) & \text{if } x_N < h(\boldsymbol{x}'). \end{cases} \tag{64}$$

Again by Exercise 178, we have that $g \in W^{1,p}(\mathbb{R}^N)$ and the usual chain rule formula for the partial derivatives holds.

By a change variables and the fact that $\det \nabla\Psi = \det \nabla\Psi^{-1} = 1$, we have that

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} |g(\boldsymbol{x})|^p d\boldsymbol{x} = \int_{\mathbb{R}^N \setminus \overline{\Omega}} |f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)|^p d\boldsymbol{x} = \int_\Omega |f(\boldsymbol{y})|^p d\boldsymbol{y}.$$

Since for all $i = 1, \dots, N-1$ and for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}$,

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N) + \frac{\partial f}{\partial x_N}(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)\frac{\partial h}{\partial x_i}(\boldsymbol{x}'), \tag{65}$$

again by a change variables we have that

$$\left(\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left|\frac{\partial g}{\partial x_i}(\boldsymbol{x})\right|^p d\boldsymbol{x}\right)^{1/p} \leq \left(\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left|\frac{\partial f(}{\partial x_i}\boldsymbol{x}', 2h(\boldsymbol{x}')-x_N)\right|^p d\boldsymbol{x}\right)^{1/p}$$

$$+\operatorname{Lip} h\left(\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left|\frac{\partial f}{\partial x_N}(\boldsymbol{x}', 2h(\boldsymbol{x}')-x_N)\right|^p d\boldsymbol{x}\right)^{1/p}$$

$$\leq \left(\int_{\Omega}\left|\frac{\partial f}{\partial x_i}(\boldsymbol{y})\right|^p d\boldsymbol{y}\right)^{1/p} + \operatorname{Lip} h\left(\int_{\Omega}\left|\frac{\partial f}{\partial x_N}(\boldsymbol{y})\right|^p d\boldsymbol{y}\right)^{1/p}.$$

Similarly, using the fact that $\frac{\partial g}{\partial x_N}(\boldsymbol{x}) = -\frac{\partial f}{\partial x_N}(\boldsymbol{x}', 2h(\boldsymbol{x}')-x_N)$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x}\in\mathbb{R}^N\setminus\overline{\Omega}$, we obtain

$$\int_{\mathbb{R}^N\setminus\overline{\Omega}}\left|\frac{\partial g}{\partial x_N}(\boldsymbol{x})\right|^p d\boldsymbol{x} = \int_{\mathbb{R}^N\setminus\overline{\Omega}}\left|\frac{\partial f}{\partial x_N}(\boldsymbol{x}', 2h(\boldsymbol{x}')-x_N)\right|^p d\boldsymbol{x}$$

$$= \int_{\Omega}\left|\frac{\partial f}{\partial x_N}(\boldsymbol{y})\right|^p d\boldsymbol{y}.$$

■

Friday, April 21, 2023

Next we study the case of open bounded sets with regular boundary.

Definition 202 *Given an open set $\Omega\subseteq\mathbb{R}^N$ we say that its boundary $\partial\Omega$ is of* class C^m, $m\in\mathbb{N}$ *if for every $\boldsymbol{x}_0\in\partial\Omega$ there exist $i\in\{1,\dots,N\}$, $r>0$, and a function $h:\mathbb{R}^{N-1}\to\mathbb{R}$ of class C^m such that, writing $\boldsymbol{x}=(\boldsymbol{x}_i, x_i)$, we have either*

$$\Omega\cap B(\boldsymbol{x}_0,r) := \{\boldsymbol{x}\in B(\boldsymbol{x}_0,r):\ h(\boldsymbol{x}_i) < x_i\}$$

or

$$\Omega\cap B(\boldsymbol{x}_0,r) := \{\boldsymbol{x}\in B(\boldsymbol{x}_0,r):\ h(\boldsymbol{x}_i) > x_i\}.$$

Theorem 203 *Let $\Omega\subset\mathbb{R}^N$ be an open bounded open set with $\partial\Omega$ of class C^1. Let $1\leq p\leq\infty$ and let $f\in W^{1,p}(\Omega)$. Then there exists $g\in W^{1,p}(\mathbb{R}^N)$ such that $g(\boldsymbol{x})=f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x}\in\Omega$ and*

$$\|g\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^p(\Omega)},$$
$$\|\nabla g\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{W^{1,p}(\Omega)}$$

for some constant $C=C(N,p,\Omega)>0$.

We begin with two auxiliary lemmas.

Lemma 204 *Let $\Omega\subseteq\mathbb{R}^N$ be an open set, let $1\leq p\leq\infty$, and let $f\in W^{1,p}(\Omega)$. Given $\boldsymbol{x}_0\in\Omega$ let $r>0$ be such that $B(\boldsymbol{x}_0,2r)\subseteq\Omega$. Given $\psi\in C_c^\infty(\mathbb{R}^N)$ with* $\operatorname{supp}\psi\subseteq B(\boldsymbol{x}_0,r)$, *the function $g:\mathbb{R}^N\to\mathbb{R}$, defined by*

$$g(\boldsymbol{x}) := \begin{cases} (f\psi)(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega,\\ 0 & \text{if } \boldsymbol{x}\in\mathbb{R}^N\setminus\overline{\Omega},\end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$*, with weak derivatives*

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial (f\psi)}{\partial x_i}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega, \\ 0 & \text{if } \boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega} \end{cases}$$

for $i = 1, \ldots, N$.

Proof. Construct a function $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi = 1$ in $B(\boldsymbol{x}_0, r)$ and $\phi = 0$ outside $B(\boldsymbol{x}_0, 2r)$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and $i = 1, \ldots, N$. Since $\phi = 1$ in $B(\boldsymbol{x}_0, r)$, we have that $\frac{\partial \varphi}{\partial x_i} = \frac{\partial(\phi\varphi)}{\partial x_i}$ in $B(\boldsymbol{x}_0, r)$. Using the fact that $\operatorname{supp} \psi \subseteq B(\boldsymbol{x}_0, r) \subset \Omega$, we can write

$$\begin{aligned} \int_{\mathbb{R}^N} g \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} &= \int_{B(\boldsymbol{x}_0,r)} f\psi \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} = \int_{B(\boldsymbol{x}_0,r)} f\psi \frac{\partial(\phi\varphi)}{\partial x_i} \, d\boldsymbol{x} \\ &= \int_\Omega f\psi \frac{\partial(\phi\varphi)}{\partial x_i} \, d\boldsymbol{x}. \end{aligned}$$

The function $\phi\varphi$ has support contained in $B(\boldsymbol{x}_0, 2r) \subseteq \Omega$. Hence, $\phi\varphi \in C_c^\infty(\Omega)$ and so we can integrate by parts to obtain that the right-hand side equals to

$$-\int_\Omega \frac{\partial(f\psi)}{\partial x_i} \phi\varphi \, d\boldsymbol{x} = -\int_{B(\boldsymbol{x}_0,r)} \frac{\partial(f\psi)}{\partial x_i} \phi\varphi \, d\boldsymbol{x} = -\int_{B(\boldsymbol{x}_0,r)} \frac{\partial(f\psi)}{\partial x_i} 1\varphi \, d\boldsymbol{x} = -\int_\Omega \frac{\partial(f\psi)}{\partial x_i} \varphi \, d\boldsymbol{x}.$$

This shows that the weak ith derivative of g is

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial (f\psi)}{\partial x_i}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega, \\ 0 & \text{if } \boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

■

Lemma 205 *Let* $\Omega, U \subseteq \mathbb{R}^N$ *be open sets, with*

$$U = \{(\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \, x_N > h(\boldsymbol{x}')\},$$

where $h : \mathbb{R}^{N-1} \to \mathbb{R}$ *is Lipschitz continuous and of class* C^1*. Assume that there exist* $\boldsymbol{x}_0 \in \partial\Omega$ *and* $r > 0$ *such that*

$$\Omega \cap B(\boldsymbol{x}_0, 2r) = U \cap B(\boldsymbol{x}_0, 2r).$$

Let $1 \le p \le \infty$ *and* $f \in W^{1,p}(\Omega)$*. Given* $\psi \in C_c^\infty(\mathbb{R}^N)$ *with* $\operatorname{supp} \psi \subseteq B(\boldsymbol{x}_0, r)$*, the function* $g : \mathbb{R}^N \to \mathbb{R}$*, defined by*

$$g(\boldsymbol{x}) := \begin{cases} (f\psi)(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U \cap B(\boldsymbol{x}_0, r), \\ 0 & \text{if } \boldsymbol{x} \in U \setminus \overline{B(\boldsymbol{x}_0, r)}, \end{cases}$$

belongs to $W^{1,p}(U)$*, with weak derivatives*

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial (f\psi)}{\partial x_i}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U \cap B(\boldsymbol{x}_0, r), \\ 0 & \text{if } \boldsymbol{x} \in U \setminus \overline{B(\boldsymbol{x}_0, r)} \end{cases}$$

for $i = 1, \ldots, N$.

Proof. Construct a function $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi = 1$ in $B(\boldsymbol{x}_0, r)$ and $\phi = 0$ outside $B(\boldsymbol{x}_0, 2r)$. Let $\varphi \in C_c^\infty(U)$ and $i = 1, \dots, N$. Since $\phi = 1$ in $B(\boldsymbol{x}_0, r)$, we have that $\frac{\partial \varphi}{\partial x_i} = \frac{\partial(\phi\varphi)}{\partial x_i}$ in $B(\boldsymbol{x}_0, r) \cap U$. Using the fact that $\operatorname{supp} \psi \subseteq B(\boldsymbol{x}_0, r)$, we can write

$$\begin{aligned}\int_U g \frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} &= \int_{B(\boldsymbol{x}_0,r)\cap U} f\psi \frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} = \int_{B(\boldsymbol{x}_0,r)\cap \Omega} f\psi \frac{\partial(\phi\varphi)}{\partial x_i}\, d\boldsymbol{x} \\ &= \int_\Omega f\psi \frac{\partial(\phi\varphi)}{\partial x_i}\, d\boldsymbol{x}.\end{aligned}$$

The function $\phi\varphi$ has support contained in $B(\boldsymbol{x}_0, 2r) \cap U = B(\boldsymbol{x}_0, 2r) \cap \Omega$. Hence, $\phi\varphi \in C_c^\infty(U)$ and so we can integrate by parts to obtain that the right-hand side equals to

$$-\int_\Omega \frac{\partial(f\psi)}{\partial x_i}\phi\varphi\, d\boldsymbol{x} = -\int_{B(\boldsymbol{x}_0,r)\cap\Omega} \frac{\partial(f\psi)}{\partial x_i}\phi\varphi\, d\boldsymbol{x} = -\int_{B(\boldsymbol{x}_0,r)\cap U} \frac{\partial(f\psi)}{\partial x_i} 1\varphi\, d\boldsymbol{x}.$$

This shows that the weak ith derivative of g in U is

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial(f\psi)}{\partial x_i}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in U \cap B(\boldsymbol{x}_0, r), \\ 0 & \text{if } \boldsymbol{x} \in U \setminus \overline{B(\boldsymbol{x}_0, r)}. \end{cases}$$

■

We turn to the proof of Theorem 203.

Proof of Theorem 203. Let $f \in W^{1,p}(\Omega)$. for every $\boldsymbol{x}_0 \in \partial\Omega$ there exist $i \in \{1, \dots, N\}$, $r > 0$, and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^m such that, writing $\boldsymbol{x} = (\boldsymbol{x}_i, x_i)$, we have either

$$\Omega \cap B(\boldsymbol{x}_0, 2r) := \{\boldsymbol{x} \in B(\boldsymbol{x}_0, 2r) : \, h(\boldsymbol{x}_i) < x_i\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, 2r) := \{\boldsymbol{x} \in B(\boldsymbol{x}_0, 2r) : \, h(\boldsymbol{x}_i) > x_i\}.$$

If the set $\Omega \setminus \bigcup_{\boldsymbol{x}\in\partial\Omega} B(\boldsymbol{x}, r_{\boldsymbol{x}})$ is nonempty, for every $\boldsymbol{x}_0 \in \Omega \setminus \bigcup_{\boldsymbol{x}\in\partial\Omega} B(\boldsymbol{x}, r_{\boldsymbol{x}})$ let $B(\boldsymbol{x}_0, r_{\boldsymbol{x}_0}) \subseteq \Omega$. The family $\{B(\boldsymbol{x}, r_{\boldsymbol{x}})\}_{\boldsymbol{x}\in\overline{\Omega}}$ is an open cover of $\overline{\Omega}$. Since $\overline{\Omega}$ is compact, there is a finite number of balls $B_1, \dots, B_\ell$, where $B_n := B(\boldsymbol{x}_n, r_{\boldsymbol{x}_n})$, that covers $\overline{\Omega}$. Let $\{\psi_n\}_{n=1}^\ell$ be a smooth partition of unity subordinated to B_1, $\dots$, B_ℓ, with $\operatorname{supp}\psi_n \subseteq B_n$. Then $\sum_{n=1}^\ell \psi_n = 1$ in $\overline{\Omega}$.

Fix $n \in \{1, \dots, \ell\}$. by Exercise 184, the function $f\psi_n$ belongs to $W^{1,p}(\Omega)$. There are two cases. If $\operatorname{supp}\psi_n \subseteq B_n \subseteq \Omega$, then if we extend $f\psi_n$ by zero outside Ω, the resulting function, denoted by g_n, belongs to $W^{1,p}(\mathbb{R}^N)$ by Lemma 204, with

$$\|g_n\|_{W^{1,p}(\mathbb{R}^N)} = \|f\psi_n\|_{W^{1,p}(\Omega)} \le C_n \|f\|_{W^{1,p}(\Omega)}. \tag{66}$$

If $\operatorname{supp}\psi_n$ is not contained in Ω, let $\boldsymbol{x}_n \in \partial\Omega$ be such $B_n = B(\boldsymbol{x}_n, r_n)$. Then writing $\boldsymbol{x} = (\boldsymbol{x}_i, x_i)$, we have either

$$\Omega \cap B(\boldsymbol{x}_n, 2r_n) := \{\boldsymbol{x} \in B(\boldsymbol{x}_n, 2r_n) : \, h_n(\boldsymbol{x}_i) < x_i\} \tag{67}$$

or

$$\Omega \cap B(\boldsymbol{x}_n, 2r_n) := \{\boldsymbol{x} \in B(\boldsymbol{x}_n, 2r_n) : h_n(\boldsymbol{x}_i) > x_i\},$$

where $h_n : \mathbb{R}^{N-1} \to \mathbb{R}$ is Lipschitz continuous and of class C^1. Assume that (67) holds. Let

$$\Omega_n := \left\{\boldsymbol{x} \in \mathbb{R}^N : h_n(\boldsymbol{x}_i) < x_i\right\}.$$

Let f_n be the function obtained by extending $f\psi_n$ to be zero in $\Omega_n \setminus (\Omega \cap B(\boldsymbol{x}_n, r_n))$. By Lemma 205, we have that $f_n \in W^{s,p}(\Omega_n)$, with

$$\|f_n\|_{W^{1,p}(\Omega_n)} = \|f\psi_n\|_{W^{1,p}(\Omega)} \leq C_n \|f\|_{W^{1,p}(\Omega)}.$$

By the previous theorem, we can extend f_n to a function $g_n \in W^{1,p}(\mathbb{R}^N)$ with

$$\|g_n\|_{W^{s,p}(\mathbb{R}^N)} \leq C_n \|f_n\|_{W^{1,p}(\Omega_n)} \leq C_n \|f\|_{W^{1,p}(\Omega)}. \tag{68}$$

Define $g := \sum_{n=1}^{n} g_n$. If $\boldsymbol{x} \in \Omega$, then

$$g(\boldsymbol{x}) = \sum_{n=1}^{n} g_n(\boldsymbol{x}) = \sum_{n=1}^{n} f_n(\boldsymbol{x}) = f(x) \sum_{n=1}^{n} \psi_n(\boldsymbol{x}) = f(\boldsymbol{x}).$$

Moreover, since the mapping $f_n \mapsto g_n$ given by Theorem 201 is linear, so is the mapping $f \mapsto g$. Finally, by (66) and (68),

$$\|g\|_{W^{1,p}(\mathbb{R}^N)} \leq \sum_{n=1}^{n} \|g_n\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|f\|_{W^{1,p}(\Omega)}.$$

■

Monday, April 24, 2023

Corollary 206 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded set with $\partial\Omega$ of class C^1 boundary, let $1 \leq p < \infty$, and let $f \in W^{1,p}(\Omega)$.*

(i) If $1 \leq p < N$, then $f \in L^q(\Omega)$ for all $1 \leq q \leq p^*$, with

$$\|f\|_{L^q(\Omega)} \leq C \|f\|_{W^{1,p}(\Omega)},$$

where $C = C(N, p, q, \Omega) > 0$;

(ii) If $p = N$, then $f \in L^q(\Omega)$ for all $1 \leq q < \infty$, with

$$\|f\|_{L^q(\Omega)} \leq C \|f\|_{W^{1,p}(\Omega)},$$

where $C = C(N, p, q, \Omega) > 0$;

(iii) If $p > N$, then f has a representative g wich is bounded and Hölder continuous with exponent $1 - N/p$, with

$$\|g\|_\infty \leq C \|f\|_{W^{1,p}(\Omega)}, \quad |g|_{C^{0,1-N/p}} \leq C \|f\|_{W^{1,p}(\Omega)},$$

where $C = C(N, p, \Omega) > 0$.

Proof. Since Ω satisfies the hypotheses of Theorem 203, there exists a function $h \in W^{1,p}(\mathbb{R}^N)$ such that $h = f$ in Ω and

$$\|h\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

If $p < N$, we can apply the Sobolev–Gagliardo–Nirenberg embedding theorem to h to get

$$\|h\|_{L^{p*}(\mathbb{R}^N)} \le C\|\nabla h\|_{L^p(\mathbb{R}^N)}.$$

Since $f = h$ in Ω, we obtain

$$\|f\|_{L^{p*}(\Omega)} = \|h\|_{L^{p*}(\Omega)} \le \|h\|_{L^{p*}(\mathbb{R}^N)} \le C\|h\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

In turn, if $1 \le q < p^*$, we can apply Hölder inequality with exponent $\frac{p^*}{q}$ to get

$$\int_\Omega |f|^q d\boldsymbol{x} \le \left(\int_\Omega |f|^{p^*} d\boldsymbol{x}\right)^{q/p^*} \left(\mathcal{L}^N(\Omega)\right)^{1/(p^*/q)'}.$$

Similarly, if $p = N$, we can apply Theorem **??** to obtain that for every $N \le q < \infty$,

$$\|h\|_{L^q(\mathbb{R}^N)} \le C\|h\|_{W^{1,p}(\mathbb{R}^N)}.$$

Since $f = h$ in Ω, we obtain

$$\|f\|_{L^q(\Omega)} = \|h\|_{L^q(\Omega)} \le \|h\|_{L^q(\mathbb{R}^N)} \le C\|h\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

If $1 \le q < p$, we can apply Hölder inequality with exponent p/q. We omit the details.

Finally, if $p > N$, we can apply Morrey's embedding theorem to find a representative g such that

$$\|g\|_{C^0(\mathbb{R}^N)} \le C\|h\|_{W^{1,p}(\mathbb{R}^N)}, \quad |g|_{C^{0,1-N/p}(\mathbb{R}^N)} \le C\|\nabla h\|_{L^p(\mathbb{R}^N)}.$$

Then g restricted to Ω is a representative of f, and

$$\|g\|_{C^0(\Omega)} \le \|g\|_{C^0(\mathbb{R}^N)} \le C\|h\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)},$$
$$|g|_{C^{0,1-N/p}(\overline{\Omega})} \le |g|_{C^{0,1-N/p}(\mathbb{R}^N)} \le C\|\nabla h\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

■

19 Compactness

Given a normed space $(X, \|\cdot\|)$, the *dual of* X, is the space X' of all continuous linear functions $L : X \to \mathbb{R}$. It is a normed space, endowed with the norm

$$\|L\|_{X'} := \sup\left\{\frac{|L(x)|}{\|x\|} : x \in X \setminus \{0\}\right\}.$$

The dual of $(X', \|\cdot\|_{X'})$ is called the *bidual of* X and is denoted X''. It can be shown that the linear function

$$J : (X, \|\cdot\|) \to (X'', \|\cdot\|_{X''})$$

defined by

$$J(x)(L) := L(x), \quad L \in X'$$

has the property that

$$\|J(x)\|_{X''} = \|x\| \quad \text{for all } x \in X. \tag{69}$$

Thus, we can identify X with $J(X)$. We say that a space X is *reflexive* if $J(X) = X$.

We say that a sequence $\{x_n\}_n$ in X *converges weakly* to $x \in X$, and we write $x_n \rightharpoonup x$ if $L(x_n) \to L(x)$ for every $L \in X'$. One of the most important theorems in functional analysis is the following.

Theorem 207 *A Banach space* $(X, \|\cdot\|)$ *is reflexive if and only if for every bounded sequence* $\{x_n\}_n$ *there exist a subsequence* $\{x_{n_k}\}_k$ *and* $x \in X$ *such that* $x_{n_k} \rightharpoonup x$.

Given a Lebesgue measurable set $E \subseteq \mathbb{R}^N$ and $1 < p < \infty$, let $[g] \in L^{p'}(E)$, where $\frac{1}{p'} + \frac{1}{p} = 1$, so that $p' := \frac{p}{p-1} \in (1, \infty)$, and consider the linear function $L_{[g]} : L^p(E) \to \mathbb{R}$ defined by

$$L_{[g]}([f]) := \int_E f(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}, \quad [f] \in L^p(E).$$

Note that by Hölder's inequality,

$$\left|L_{[g]}([f])\right| = \left|\int_E f(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}\right| \le \|[f]\|_{L^p(E)} \|[g]\|_{L^{p'}(E)}.$$

Hence, if $[f] \neq [0]$, we can divide by $\|[f]\|_{L^p(E)}$ to get

$$\|L_{[g]}\|_{(L^p(E))'} = \sup\left\{\frac{\left|L_{[g]}([f])\right|}{\|[f]\|_{L^p(E)}} : [f] \in L^{p'}(E) \setminus \{[0]\}\right\} \le \|[g]\|_{L^{p'}(E)}.$$

One can actually prove that there is equality, that is,

$$\|L_{[g]}\|_{(L^p(E))'} = \|[g]\|_{L^{p'}(E)}.$$

Theorem 208 (Riesz representation theorem) *Given a Lebesgue measurable set* $E \subseteq \mathbb{R}^N$ *and* $1 \le p < \infty$, *for every* $L \in (L^p(E))'$ *there exists a unique function* $[g] \in L^{p'}(E)$ *such that*

$$L([f]) = \int_E f(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}, \quad [f] \in L^p(E)$$

and

$$\|L\|_{(L^p(E))'} = \|[g]\|_{L^{p'}(E)}.$$

It follows that the function

$$T : L^{p'}(E) \to (L^p(E))'$$
$$[g] \mapsto L_{[g]}$$

is one-to-one, onto, and preserves the norm. We say that T is an isomorphism between Banach spaces. Thus, one can identify the dual of $L^p(E)$ with $L^{p'}(E)$.

It follows that a sequence $\{[f_n]\}_n$ in $L^p(E)$ converges weakly to $[f]$ if for every $[g] \in L^{p'}(E)$,

$$\lim_{n\to\infty} \int_E f_n(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x} = \int_E f(\boldsymbol{x})\, g(\boldsymbol{x})\, d\boldsymbol{x}.$$

Observe that

$$(L^p(E))'' \cong (L^{p'}(E))' \cong L^p(E),$$

so the bidual of $L^p(E)$ can be identified with $L^p(E)$ itself. Hence, $L^p(E)$ is reflexive.

It follows from Theorem 207 that if $\{[f_n]\}_n$ is bounded in $L^p(E)$, $1 < p < \infty$, then there exist a subsequence $\{[f_{n_k}]\}_k$ and $[f] \in L^p(E)$ such that $[f_{n_k}] \rightharpoonup [f]$ as $k \to \infty$.

For $p = 1$, fix $[g] \in L^\infty(E)$, and consider the linear function $L_g : L^1(E) \to \mathbb{R}$ defined by

$$L_{[g]}([f]) := \int_E f(x)\, g(x)\, dx, \quad [f] \in L^1(E).$$

Note that by Hölder's inequality,

$$\left|L_{[g]}([f])\right| \le \int_E |f(x)|\,|g(x)|\, dx \le \|[f]\|_{L^1(E)} \operatorname*{esssup}_E |g|.$$

Hence, if $[f] \ne [0]$, we can divide by $\|[f]\|_{L^1(E)}$ to get

$$\|L_{[g]}\|_{(L^1(E))'} = \sup\left\{\frac{\left|L_{[g]}([f])\right|}{\|[f]\|_{L^1(E)}} : [f] \in L^1(E) \setminus \{[0]\}\right\} \le \|[g]\|_{L^\infty(E)}.$$

One can actually prove that there is equality, that is,

$$\|L_{[g]}\|_{(L^1(E))'} = \|[g]\|_{L^\infty(E)}.$$

Conversely, given $L : L^1(E) \to \mathbb{R}$ linear and continuous, the Riesz representation theorem (which we will not prove) gives a unique function $[g] \in L^\infty(E)$ such that

$$L([f]) = \int_E f(x)\, g(x)\, dx, \quad [f] \in L^1(E)$$

and

$$\|L\|_{(L^1(E))'} = \|[g]\|_{L^\infty(E)}.$$

Thus, the function

$$T : L^\infty(E) \to (L^1(E))'$$
$$[g] \mapsto L_{[g]}$$

is one-to-one, onto and preserves the norm. Thus one can identify the dual of $L^1(E)$ may be identified with $L^\infty(E)$.

It turns out that the dual of $L^\infty(E)$ is not $L^1(E)$, so $L^1(E)$ is not reflexive.

Wednesday, April 26, 2023

Theorem 209 (Rellich-Kondrachov) *Let $1 \le p < \infty$ and let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and a function $f \in L^p(\mathbb{R}^N)$ such that $f_{n_k} \to f$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Moreover, $f \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$.*

Proof. Step 1: We claim that for all $f \in W^{1,p}(\mathbb{R}^N)$ and for all $\boldsymbol{h} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$,

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} \le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p \, d\boldsymbol{x}.$$

Assume that $f \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. For $\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{h} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$ by the fundamental theorem of calculus we have that

$$\begin{aligned} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})| &= \left| \int_0^1 \frac{d}{dt} (f(\boldsymbol{x}+t\boldsymbol{h})) \, dt \right| \\ &\le \|\boldsymbol{h}\| \int_0^1 \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\| \, dt. \end{aligned}$$

Raising to power p and integrating over $\mathbb{R}^N$, by Hölder's inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} &\le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \left(\int_0^1 \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\| \, dt \right)^p d\boldsymbol{x} \\ &\le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \left(\int_0^1 \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\|^p \, dt \right) d\boldsymbol{x} \\ &= \|\boldsymbol{h}\|^p \int_0^1 \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\|^p \, d\boldsymbol{x} \right) dt \\ &= \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p \, d\boldsymbol{y}, \end{aligned}$$

where we have used Fubini's Theorem and the cange of variables $\boldsymbol{y} = \boldsymbol{x} + t\boldsymbol{h}$.

To remove the additional hypothesis that $f \in C^\infty(\mathbb{R}^N)$, it suffices to apply the previous inequality to $f_\varepsilon := \varphi_\varepsilon * f$, where φ_ε is a standard mollifier and let $\varepsilon \to 0^+$ (see Theorem 155 and Lemma 169).

Step 2: Let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. In view of Step 1, for all n and $\boldsymbol{h} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}$,

$$\int_{\mathbb{R}^N} |f_n(\boldsymbol{x}+\boldsymbol{h}) - f_n(\boldsymbol{x})|^p \, d\boldsymbol{x} \le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f_n(\boldsymbol{x})\|^p \, d\boldsymbol{x} \le M \|\boldsymbol{h}\|^p.$$

In view of the Kolmogorov–Riesz–Fréchet compactness theorem for every Lebesgue measurable set $E \subset \mathbb{R}^N$ of finite measure, the sequence $\{f_n\}_n$ restricted to E is relatively compact in $L^p(E)$. We now use a *diagonal argument*. Take $E = B(\mathbf{0}, 1)$. Since $\{f_n\}_n$ restricted to $B(\mathbf{0}, 1)$ is relatively compact in $L^p(B(\mathbf{0}, 1))$, we can find a subsequence $\{f_{n,1}\}_n$ of $\{f_n\}_n$ and a function $g_1 \in L^p(B(\mathbf{0}, 1))$ such that $f_{n,1} \to g_1$ in $L^p(B(\mathbf{0}, 1))$ and pointwise $\mathcal{L}^N$-a.e. in $B(\mathbf{0}, 1)$ as $n \to \infty$. Next, take $E = B(\mathbf{0}, 2)$. Since $\{f_{n,1}\}_n$ restricted to $B(\mathbf{0}, 2)$ is relatively compact in $L^p(B(\mathbf{0}, 2))$, we can find a subsequence $\{f_{n,2}\}_n$ of $\{f_{n,1}\}_n$ and a function $g_2 \in L^p(B(\mathbf{0}, 2))$ such that $f_{n,2} \to g_2$ in $L^p(B(\mathbf{0}, 2))$ and pointwise $\mathcal{L}^N$-a.e. in $B(\mathbf{0}, 2)$ as $n \to \infty$. By the uniqueness of limits, we have that $g_2 = g_1$ in $B(\mathbf{0}, 1)$. Inductively, assume we have found a subsequence $\{f_{n,k}\}_n$ of $\{f_{n,k-1}\}_n$ and a function $g_k \in L^p(B(\mathbf{0}, k))$ such that $f_{n,k} \to g_k$ in $L^p(B(\mathbf{0}, k))$ and pointwise $\mathcal{L}^N$-a.e. in $B(\mathbf{0}, k)$ as $n \to \infty$. Consider $E = B(\mathbf{0}, k+1)$. Since $\{f_{n,k}\}_n$ restricted to $B(\mathbf{0}, k+1)$ is relatively compact in $L^p(B(\mathbf{0}, k+1))$, we can find a subsequence $\{f_{n,k+1}\}_n$ of $\{f_{n,k}\}_n$ and a function $g_{k+1} \in L^p(B(\mathbf{0}, k+1))$ such that $f_{n,k+1} \to g_{k+1}$ in $L^p(B(\mathbf{0}, k+1))$ and pointwise $\mathcal{L}^N$-a.e. in $B(\mathbf{0}, k+1)$ as $n \to \infty$. By the uniqueness of limits, we have that $g_{k+1} = g_k$ in $B(\mathbf{0}, k)$. Let $f_{n_k} := f_{k,k}$ and define the function $f : \mathbb{R}^N \to \mathbb{R}$ as follows. Given $\boldsymbol{x} \in \mathbb{R}^N$ find k so large that $\boldsymbol{x} \in B(\mathbf{0}, k)$ and set $f(\boldsymbol{x}) := g_k(\boldsymbol{x})$. Then $f \in L^p_{\text{loc}}(\mathbb{R}^N)$. Moreover, for every $j \in \mathbb{N}$, we have that $\{f_{n_k}\}_{k \geq j}$ is a subsequence of $\{f_{n,j}\}_n$, and so, $f_{n_k} \to g_j = f$ in $L^p(B(\mathbf{0}, j))$. By the arbitrariness of j, this shows that $f_{n_k} \to f$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ and pointwise $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$ as $k \to \infty$.

Since $\{f_{n_k}\}_k$ is bounded in $L^p(\mathbb{R}^N)$, there exists $C > 0$ such that

$$\|f_{n_k}\|_{L^p(\mathbb{R}^N)} \leq C$$

for every k. Using Fatou's lemma, it follows that for every j,

$$\|f\|_{L^p(\mathbb{R}^N)} \leq \liminf_{k \to \infty} \|f_{n_k}\|_{L^p(\mathbb{R}^N)} \leq C.$$

Step 3: Assume now that $p > 1$. We claim that $f \in W^{1,p}(\mathbb{R}^N)$. Since $\{f_{n_k}\}_k$ is bounded in $L^p(\mathbb{R}^N)$, which is reflexive, by Theorem 207, there exist a subsequence $\{f_{n_{k_j}}\}_j$ and $g \in L^p(\mathbb{R}^N)$ such that $f_{n_{k_j}} \rightharpoonup g$ in $L^p(\mathbb{R}^N)$. By selecting N further subsequences (not relabelled) and using the fact that ince $\{\frac{\partial f_{n_k}}{\partial x_i}\}_k$ is bounded in $L^p(\mathbb{R}^N)$, we can assume that $\frac{\partial f_{n_k}}{\partial x_i} \rightharpoonup g_i$ in $L^p(\mathbb{R}^N)$. Let's prove that $g \in W^{1,p}(\mathbb{R}^N)$. For all $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} f_{n_{k_j}} \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} = -\int_{\mathbb{R}^N} \varphi \frac{\partial f_{n_{k_j}}}{\partial x_i} \, d\boldsymbol{x}.$$

Since φ has compact support, we have that $\varphi, \frac{\partial \varphi}{\partial x_i} \in L^p(\mathbb{R}^N)$, and so we can let $j \to \infty$ and use wek convergence to get

$$\int_{\mathbb{R}^N} g \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} = -\int_{\mathbb{R}^N} \varphi g_i \, d\boldsymbol{x},$$

which shows that g_i is the ih weak derivative if g. Thus, $g \in W^{1,p}(\mathbb{R}^N)$.

To see that $f = g$, let $\psi \in L^{p'}(\mathbb{R}^N)$ be such that $\psi = 0$ outside $B(\mathbf{0}, R)$. Then by weak convergence

$$\int_{B(\mathbf{0},R)} f_{n_{k_j}} \psi \, d\boldsymbol{x} = \int_{\mathbb{R}^N} f_{n_{k_j}} \psi \, d\boldsymbol{x} \to \int_{\mathbb{R}^N} g\psi \, d\boldsymbol{x} = \int_{B(\mathbf{0},R)} g\psi \, d\boldsymbol{x}.$$

On the other hand, since $f_{n_{k_j}} \to f$ in $L^p(B(\mathbf{0}, R))$, we have that

$$\int_{B(\mathbf{0},R)} f_{n_{k_j}} \psi \, d\boldsymbol{x} \to \int_{B(\mathbf{0},R)} f\psi \, d\boldsymbol{x}.$$

Indeed, by Hölder's inequality

$$\left| \int_{B(\mathbf{0},R)} (f_{n_{k_j}} - f)\psi \, d\boldsymbol{x} \right| \leq \|f_{n_{k_j}} - f\|_{L^p(B(\mathbf{0},R))} \|\psi\|_{L^{p'}(B(\mathbf{0},R))} \to 0.$$

Hence,

$$\int_{B(\mathbf{0},R)} g\psi \, d\boldsymbol{x} = \int_{B(\mathbf{0},R)} f\psi \, d\boldsymbol{x}$$

$\psi \in L^{p'}(\mathbb{R}^N)$ be such that $\psi = 0$ outside $B(\mathbf{0}, R)$. This implies that $f = g$ $\mathcal{L}^N$-a.e. in $B(\mathbf{0}, R)$. Letting $R \to \infty$, we obtain that $f = g$ $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. ■

Corollary 210 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $\partial\Omega$ of class C^1, let $1 \leq p < \infty$, and let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}(\Omega)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and a function $f \in L^p(\Omega)$ such that $f_{n_k} \to f$ in $L^p(\Omega)$. Moreover, $f \in W^{1,p}(\Omega)$ if $p > 1$.*

Friday, April 28, 2023

20 Poincaré Inequalities

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. Poincaré's inequality is the following

$$\int_\Omega |f(\boldsymbol{x}) - f_E|^p \, d\boldsymbol{x} \leq C \int_\Omega \|\nabla f\|^p d\boldsymbol{x},$$

where $E \subseteq \Omega$ is a measurable set of finite positive measure and

$$f_E := \frac{1}{|E|} \int_E f(\boldsymbol{x}) \, d\boldsymbol{x}. \tag{70}$$

Theorem 211 (Poincaré Inequality) *Let $1 \leq p < \infty$, let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with boundary of class C^1, and let $E \subseteq \Omega$ be a measurable set with positive measure. Then there exists a constant $C = C(p, \Omega, E) > 0$ such that for all $f \in W^{1,p}(\Omega)$,*

$$\int_\Omega |f(\boldsymbol{x}) - f_E|^p \, d\boldsymbol{x} \leq C \int_\Omega \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Proof. Assume by contradiction that the result is false. Then we may find a sequence $\{f_n\}_n$ in $W^{1,p}(\Omega)$ such that

$$\int_\Omega |f_n(\boldsymbol{x}) - (f_n)_E|^p \, d\boldsymbol{x} \geq n \int_\Omega \|\nabla f_n(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Define

$$g_n := \frac{f_n - (f_n)_E}{\|f_n - (f_n)_E\|_{L^p(\Omega)}}.$$

Then $g_n \in W^{1,p}(\Omega)$ with

$$\|g_n\|_{L^p(\Omega)} = 1, \quad (g_n)_E = 0, \quad \int_\Omega \|\nabla g_n\|^p d\boldsymbol{x} \leq \frac{1}{n}.$$

Extend g_n to a function $G_n \in W^{1,p}(\mathbb{R}^N)$ with

$$\|G_n\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|g_n\|_{W^{1,p}(\Omega)}.$$

Then $\{G_n\}_n$ is bounded in $W^{1,p}(\mathbb{R}^N)$. By the Rellich-Kondrachov theorem there exist a subsequence $\{G_{n_k}\}_k$ and a function $G \in L^p(\mathbb{R}^N)$ such that $G_{n_k} \to G$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Let g be the restriction of G to Ω. Since Ω is bounded, we have that $g_{n_k} \to g$ in $L^p(\Omega)$. It follows that

$$\|g\|_{L^p(\Omega)} = 1, \quad g_E = 0.$$

Moreover, for every $\psi \in C^1_c(\Omega)$ and $i = 1, \ldots, N$, by Hölder's inequality

$$\begin{aligned}\left|\int_\Omega g \frac{\partial \psi}{\partial x_i} \, d\boldsymbol{x}\right| &= \lim_{k\to\infty} \left|\int_\Omega g_{n_k} \frac{\partial \psi}{\partial x_i} \, d\boldsymbol{x}\right| = \lim_{k\to\infty} \left|\int_\Omega \psi \frac{\partial g_{n_k}}{\partial x_i} \, d\boldsymbol{x}\right| \\ &\leq \lim_{k\to\infty} \left(\int_\Omega \|\nabla g_{n_k}\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \left(\int_\Omega |\psi|^{p'} \, d\boldsymbol{x}\right)^{\frac{1}{p'}} = 0\end{aligned}$$

and so $g \in W^{1,p}(\Omega)$ with $\nabla g = 0$. Since Ω is connected, this implies that g is constant (exercise), but since $g_E = 0$, then, necessarily, $g = 0$. This contradicts the fact that $\|g\|_{L^p(\Omega)} = 1$ and completes the proof. ■

The space $W^{1,p}_0(\Omega)$ is defined as the closure of $C^\infty_c(\Omega)$ with respect to the norm in $W^{1,p}(\Omega)$.

Theorem 212 (Poincaré inequality in $W^{1,p}_0$) *Let $\Omega \subset \mathbb{R}^N$ be an open set with* finite width*, that is, Ω lies between two parallel hyperplanes, and let $1 \leq p < \infty$. Then there exists a constant $c = c(N,p) > 0$ such that for all $f \in W^{1,p}_0(\Omega)$,*

$$\int_\Omega |f(\boldsymbol{x})|^p d\boldsymbol{x} \leq c\frac{d^p}{p} \int_\Omega \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Proof. Without loss of generality, up to a rotation and translation, we may assume that Ω lies between the two parallel hyperplanes $x_N = 0$ and $x_N = d > 0$. For $f \in C_c^\infty(\Omega)$, by the fundamental theorem of calculus and Hölder's inequality, we have

$$
\begin{aligned}
|f(\boldsymbol{x}', x_N)| = |f(\boldsymbol{x}', x_N) - f(\boldsymbol{x}', 0)| &= \left| \int_0^{x_N} \frac{\partial f}{\partial x_N}(\boldsymbol{x}', t)\, dt \right| \\
&\leq x_N^{1/p'} \left(\int_0^d \left| \frac{\partial f}{\partial x_N}(\boldsymbol{x}', t) \right|^p dt \right)^{1/p}.
\end{aligned}
$$

Extend f to be zero outside $\mathbb{R}^N \setminus \Omega$. Raising to the power p and integrating over $\mathbb{R}^{N-1} \times [0, d]$, by Tonelli's theorem we get

$$
\begin{aligned}
\int_\Omega |f(\boldsymbol{x})|^p d\boldsymbol{x} &= \int_{\mathbb{R}^{N-1} \times [0,d]} |f(\boldsymbol{x}', x_N)|^p d\boldsymbol{x} \\
&\leq \int_{\mathbb{R}^{N-1}} \int_0^d x_N^{p/p'} \int_0^d \left| \frac{\partial f}{\partial x_N}(\boldsymbol{x}', t) \right|^p dt dx_N d\boldsymbol{x}' \\
&= \int_\Omega \left| \frac{\partial f}{\partial x_N}(\boldsymbol{y}) \right|^p d\boldsymbol{y} \int_0^d x_N^{p-1} dx_N = \frac{d^p}{p} \int_\Omega \left| \frac{\partial f}{\partial x_N}(\boldsymbol{y}) \right|^p d\boldsymbol{y}.
\end{aligned}
$$

■