Wednesday, January 19, 2022

1 Sobolev and BV spaces

Consider the differential equation

$$f''(x) = g(x), \quad x \in I$$

where I is an open interval and $g : I \to \mathbb{R}$ is a continuous function. For this ode to make sense, we need the solution f to be at least of class C^2. Consider a function $\phi \in C_c^\infty(I)$ and multiply the equation by ϕ. If we integrate by parts, we get

$$-\int_I f'(x)\phi'(x)\,dx = \int_I g(x)\phi(x)\,dx. \tag{1}$$

This integral makes sense for functions f that are less regular than C^2. For example C^1 is enough.

If we integrate by parts once more, we get

$$\int_I f(x)\phi''(x)\,dx = \int_I g(x)\phi(x)\,dx. \tag{2}$$

This integral makes sense provided $f : I \to \mathbb{R}$ is locally integrable. The integrals (1) and (2) can be considered weak formulations of the differential equation $f'' = g$.

Motivated by this discussion, we define the weak derivative of a function.

Definition 1 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \le p \le \infty$, and $f \in L^p_{\text{loc}}(\Omega)$. Given $i = 1, \ldots, N$, we say that f admits a* weak or distributional derivative in $L^p(\Omega)$ *if there exists a function $g_i \in L^p(\Omega)$ such that*

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega g_i(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}$$

for every $\phi \in C_c^\infty(\Omega)$. The function g_i is called the weak, *or* distributional, partial derivative *of f with respect to x_i and is denoted $\frac{\partial f}{\partial x_i}$.*

Remark 2 *Observe that if $f \in C^1(\Omega)$, then by the divergence theorem we can always integrate by parts to conclude that*

$$\int_\Omega f(\boldsymbol{x})\frac{\partial \phi}{\partial x_i}(\boldsymbol{x})\,d\boldsymbol{x} = -\int_\Omega \frac{\partial f}{\partial x_i}(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$. Hence, if $\frac{\partial f}{\partial x_i} \in L^p(\Omega)$, then the classical partial derivative $\frac{\partial f}{\partial x_i}$ is the weak derivative of f. We will use this fact without further notice.

Exercise 3 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \le p \le \infty$, and $L^p_{\text{loc}}(\Omega)$. Prove that if f admits a weak derivative $\frac{\partial f}{\partial x_i}$ in $L^p(\Omega)$, then the weak derivative $\frac{\partial f}{\partial x_i}$ is unique.*

Similarly, we have

Definition 4 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $f \in L^p_{\text{loc}}(\Omega)$. Given $i = 1, \ldots, N$, we say that f admits a* weak or distributional derivative in the space of measures *if there exists a signed measure $\lambda_i : \mathcal{B}(\Omega) \to \mathbb{R}$ such that*

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, d\boldsymbol{x} = -\int_\Omega \phi \, d\lambda_i$$

for all $\phi \in C_c^\infty(\Omega)$. The measure λ_i is called the weak, *or* distributional, partial derivative *of f with respect to x_i and is denoted $D_i f$.*

We can now define the Sobolev space $W^{1,p}(\Omega)$.

Definition 5 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \leq p \leq \infty$. The* Sobolev space *$W^{1,p}(\Omega)$ is the space of all functions $f \in L^p(\Omega)$ that admit all weak derivatives $\frac{\partial f}{\partial x_i}$ in $L^p(\Omega)$, endowed with the norm*

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p(\Omega)}.$$

When $p = 2$ we write $H^1(\Omega) = W^{1,2}(\Omega)$. In this case, we have an inner product, given by

$$(f, g)_{H^1(\Omega)} := (f, g)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i} \right)_{L^2(\Omega)}.$$

For $f \in W^{1,p}(\Omega)$ we set

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right).$$

Remark 6 *In $W^{1,p}(\Omega)$ we can consider the equivalent norms*

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

or

$$\|f\|_{W^{m,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega;\mathbb{R}^N)};$$

for $1 \leq p < \infty$, and

$$\|f\|_{W^{1,\infty}(\Omega)} := \max \left\{ \|f\|_{L^\infty(\Omega)}, \left\| \frac{\partial f}{\partial x_1} \right\|_{L^\infty(\Omega)}, \ldots, \left\| \frac{\partial f}{\partial x_N} \right\|_{L^\infty(\Omega)} \right\}$$

for $p = \infty$.

We define

$$W^{1,p}_{\mathrm{loc}}(\Omega) := \left\{ f \in L^p_{\mathrm{loc}}(\Omega) : \ f \in W^{1,p}(U) \text{ for all open sets } U \Subset \Omega \right\}.$$

We now show that $W^{1,p}(\Omega)$ is a Banach space.

Theorem 7 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \le p \le \infty$. Then the space $W^{1,p}(\Omega)$ is a Banach space.*

Proof. Let $\{f_n\}_n$ be a Cauchy sequence in $W^{1,p}(\Omega)$, that is,

$$\begin{aligned} 0 &= \lim_{l,n\to\infty} \|f_n - f_l\|_{W^{1,p}(\Omega)} \\ &= \lim_{l,n\to\infty} \left(\|f_n - f_l\|_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial f_n}{\partial x_i} - \frac{\partial f_l}{\partial x_i} \right\|_{L^p(\Omega)} \right). \end{aligned}$$

Then $\{f_n\}_n$ and $\left\{ \frac{\partial f_n}{\partial x_i} \right\}_n$, $i = 1, \dots, N$, are Cauchy sequences in $L^p(\Omega)$. Since $L^p(\Omega)$ is a Banach space, there exist $f, g_i \in L^p(\Omega)$, $i = 1, \dots, N$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0, \quad \lim_{n\to\infty} \left\| \frac{\partial f_n}{\partial x_i} - g_i \right\|_{L^p(\Omega)} = 0 \tag{3}$$

for all $i = 1, \dots, N$. Fix $i = 1, \dots, N$. We claim that $\frac{\partial f_n}{\partial x_i} = g_i$. To see this let $\phi \in C_c^\infty(\Omega)$ and note that

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} = -\int_\Omega f_n \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}. \tag{4}$$

Writing

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} = \int_\Omega \phi \left(\frac{\partial f_n}{\partial x_i} - g_i \right) d\boldsymbol{x} + \int_\Omega \phi g_i \, d\boldsymbol{x} =: I_n + II,$$

by Hölder's inequality we have

$$|I_n| \le \|\phi\|_{L^{p'}(\Omega)} \left\| \frac{\partial f_n}{\partial x_i} - g_i \right\|_{L^p(\Omega)} \to 0$$

as $n \to \infty$, which shows that

$$\int_\Omega \phi \frac{\partial f_n}{\partial x_i} \, d\boldsymbol{x} \to \int_\Omega \phi g_i \, d\boldsymbol{x}.$$

Similarly,

$$-\int_\Omega f_n \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x} \to -\int_\Omega f \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}.$$

Hence, letting $n \to \infty$ in (4) yields

$$\int_\Omega \phi g_i \, d\boldsymbol{x} = -\int_\Omega f \frac{\partial \phi}{\partial x_i} \, d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$, which proves the claim. Thus $f \in W^{1,p}(\Omega)$. It follows by (3) that $f_n \to f$ in $W^{1,p}(\Omega)$. Hence, $W^{1,p}(\Omega)$ is a Banach space. ■

More generally, we can define higher order Sobolev spaces.

Definition 8 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p \leq \infty$. The* Sobolev space $W^{m,p}(\Omega)$ *is the space of all functions $f \in L^p(\Omega)$ such that for every multi-index $\boldsymbol{\alpha}$ with $1 \leq |\boldsymbol{\alpha}| \leq m$ there exists a function $g_{\boldsymbol{\alpha}} \in L^p(\Omega)$ such that*

$$\int_\Omega f \frac{\partial^{\boldsymbol{\alpha}} \phi}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} \, d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega g_{\boldsymbol{\alpha}} \phi \, d\boldsymbol{x}$$

for all $\phi \in C_c^\infty(\Omega)$. The function $g_{\boldsymbol{\alpha}}$ is called the weak *or* distributional partial derivative *of f with respect to $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and is denoted $\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}$.*

Exercise 9 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $m \in \mathbb{N}$, and let $1 \leq p \leq \infty$. Given $f \in W^{m,p}(\Omega)$, prove that the weak derivative of f with respect to $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is unique.*

We define

$$W^{m,p}_{\text{loc}}(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) : \ f \in W^{m,p}(U) \text{ for all open sets } U \Subset \Omega \right\}.$$

Exercise 10 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$.*

(i) Prove that a subset of a separable metric space is separable.

(ii) Prove that $W^{1,p}(\Omega)$ is separable. Hint: Consider the mapping

$$\begin{aligned} W^{1,p}(\Omega) &\to L^p(\Omega) \times L^p\left(\Omega; \mathbb{R}^N\right) \\ f &\mapsto (f, \nabla f). \end{aligned}$$

Exercise 11 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Prove that $W^{1,\infty}(\Omega)$ is not separable.*

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Next we prove that smooth functions are dense in $W^{1,p}(\Omega)$

Theorem 12 (Meyers–Serrin) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \leq p < \infty$. Then the space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

We begin with an auxiliary result. We use mollifiers. Given a nonnegative function $\varphi \in C_c^\infty\left(\mathbb{R}^N\right)$ with

$$\operatorname{supp} \varphi \subseteq \overline{B(\mathbf{0}, 1)}, \quad \int_{\mathbb{R}^N} \varphi(\boldsymbol{x}) \, d\boldsymbol{x} = 1, \tag{5}$$

for every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^N} \varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

The functions φ_ε are called *mollifiers*. Given an open set $\Omega \subseteq \mathbb{R}^N$ and a function $f \in L^1_{\text{loc}}(\Omega)$, for $\boldsymbol{x} \in \Omega_\varepsilon$, we define

$$f_\varepsilon(\boldsymbol{x}) := (f * \varphi_\varepsilon)(\boldsymbol{x}) = \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} \tag{6}$$

for $\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}$. Note that if $f \in L^p(\Omega)$ for some $1 \leq p \leq \infty$, then by Hölder's inequality, $f_\varepsilon(\boldsymbol{x})$ is well-defined for all $\boldsymbol{x} \in \mathbb{R}^N$.

Theorem 13 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and let* $f \in L^p(\Omega)$, $1 \le p < \infty$*. Then for every Lebesgue point* $\boldsymbol{x} \in \Omega$ *(and so for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \Omega$*),* $f_\varepsilon(\boldsymbol{x}) \to f(\boldsymbol{x})$ *as* $\varepsilon \to 0^+$*. Moreover,*

$$\lim_{\varepsilon \to 0^+} \left(\int_\Omega |f_\varepsilon - f|^p \, d\boldsymbol{x} \right)^{\frac{1}{p}} = 0.$$

Proof. Exercise. ■

Lemma 14 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $1 \le p < \infty$*, and* $f \in W^{1,p}(\Omega)$*. For every* $\varepsilon > 0$ *define* $f_\varepsilon := \varphi_\varepsilon * f$ *in* $\mathbb{R}^N$*, where* φ_ε *is a standard mollifier. Then*

$$\lim_{\varepsilon \to 0^+} \|f_\varepsilon - f\|_{W^{1,p}(\Omega_\varepsilon)} = 0,$$

where the open set Ω_ε *is given by*

$$\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

In particular, if $U \subset \Omega$*, with* $\operatorname{dist}(U, \partial\Omega) > 0$*, then*

$$\|f_\varepsilon - f\|_{W^{m,p}(U)} \to 0 \text{ as } \varepsilon \to 0^+.$$

Proof. By differentiating under the integral sign we have that $f_\varepsilon \in C^\infty(\mathbb{R}^N)$ and for $\boldsymbol{x} \in \Omega_\varepsilon$ and for every $i = 1, \dots, N$,

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial x_i}(\boldsymbol{x}) &= \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} = -\int_\Omega \frac{\partial \varphi_\varepsilon}{\partial y_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} \\ &= \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) \frac{\partial f}{\partial y_i}(\boldsymbol{y}) \, d\boldsymbol{y} = \left(\varphi_\varepsilon * \frac{\partial f}{\partial x_i} \right)(\boldsymbol{x}), \end{aligned}$$

where we have used the definition of weak derivative and the fact that for each $\boldsymbol{x} \in \Omega_\varepsilon$ the function $\varphi_\varepsilon(\boldsymbol{x} - \cdot) \in C_c^\infty(\Omega)$, since $\operatorname{supp} \varphi_\varepsilon(\boldsymbol{x} - \cdot) \subseteq \overline{B(\boldsymbol{x}, \varepsilon)} \subset \Omega$. The result now follows from Theorem 13 applied to the functions f and $\frac{\partial f}{\partial \boldsymbol{x}_i}$, $i = 1, \dots, N$. ■

Remark 15 *Note that if* $\Omega = \mathbb{R}^N$*, then* $\Omega_\varepsilon = \mathbb{R}^N$*. Hence,* $f_\varepsilon \to f$ *in* $W^{1,p}(\mathbb{R}^N)$*.*

Exercise 16 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and* $1 \le p \le \infty$*. Prove that if* $f \in W^{1,p}(\Omega)$ *and* $\varphi \in C_c^\infty(\Omega)$*, then* $\varphi f \in W^{1,p}(\Omega)$*.*

We now turn to the proof of the Meyers–Serrin theorem.

Proof of Theorem 12. Let $\Omega_i \Subset \Omega_{i+1}$ be such that

$$\Omega = \bigcup_{i=1}^\infty \Omega_i$$

and consider a smooth partition of unity $\mathcal{F}$ subordinated to the open cover $\{\Omega_{i+1} \setminus \overline{\Omega_{i-1}}\}$, where $\Omega_{-1} = \Omega_0 := \emptyset$. For each $i \in \mathbb{N}$ let ψ_i be the sum of all the finitely many $\psi \in \mathcal{F}$ such that $\operatorname{supp}\psi \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ and that have not already been selected at previous steps $j < i$. Then $\psi_i \in C_c^\infty\left(\Omega_{i+1} \setminus \overline{\Omega_{i-1}}\right)$ and

$$\sum_{i=1}^{\infty} \psi_i = 1 \text{ in } \Omega. \tag{7}$$

Fix $\eta > 0$. For each $i \in \mathbb{N}$ we have that

$$\operatorname{supp}(\psi_i f) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}, \tag{8}$$

and so, by the previous lemma, we may find $\varepsilon_i > 0$ so small that

$$\operatorname{supp}(\psi_i f)_{\varepsilon_i} \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}} \tag{9}$$

and

$$\left\|(\psi_i f)_{\varepsilon_i} - \psi_i f\right\|_{W^{1,p}(\Omega)} \leq \frac{\eta}{2^i},$$

where we have used the previous exercise. ■

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Proof. Note that in view of (9), for every $U \Subset \Omega$ only finitely many $\Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ cover U, and so the function

$$g := \sum_{i=1}^{\infty} (\psi_i f)_{\varepsilon_i}$$

belongs to $C^\infty(\Omega)$. In particular, $g \in W^{m,p}_{\mathrm{loc}}(\Omega)$.

For $\boldsymbol{x} \in \Omega_\ell$ by (7), (8), and (9),

$$f(\boldsymbol{x}) = \sum_{i=1}^{\ell} (\psi_i f)(\boldsymbol{x}), \quad g(\boldsymbol{x}) = \sum_{i=1}^{\ell} (\psi_i f)_{\varepsilon_i}(\boldsymbol{x}). \tag{10}$$

Hence

$$\|f - g\|_{W^{m,p}(\Omega_\ell)} \leq \sum_{i=1}^{\ell} \left\|(\psi_i f)_{\varepsilon_i} - \psi_i f\right\|_{W^{m,p}(\Omega)} \leq \sum_{i=1}^{\ell} \frac{\eta}{2^i} \leq \eta. \tag{11}$$

Letting $\ell \to \infty$ it follows from the Lebesgue dominated convergence theorem that $\|f - g\|_{W^{m,p}(\Omega)} \leq \eta$. This also implies that $f - g$ (and, in turn, g) belongs to the space $W^{m,p}(\Omega)$. ■

Remark 17 *Note that we can adapt the proof of the Meyers-Serrin theorem to show that if $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $\nabla f \in L^p\left(\Omega; \mathbb{R}^N\right)$ then for every $\varepsilon > 0$ there exists a function $g \in C^\infty(\Omega) \cap W^{1,p}_{\mathrm{loc}}(\Omega)$ such that*

$$\|f - g\|_{W^{1,p}(\Omega)} \leq \varepsilon,$$

despite the fact that neither f nor g need belong to $W^{1,p}(\Omega)$.

Exercise 18 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $f : \Omega \to \mathbb{R}$ be a locally Lipschitz continuous function (that is, f is Lipschitz continuous in each compact set $K \subset \Omega$). Prove that $f \in W^{1,p}_{\text{loc}}(\Omega)$ and that the classical derivatives of f are the weak derivatives.*

Exercise 19 *Prove that the function $f(x) := |x|$ belongs to $W^{1,\infty}(-1,1)$ but not to the closure of $C^\infty(-1,1) \cap W^{1,\infty}(-1,1)$.*

The previous exercise shows that the Meyers–Serrin theorem is false for $p = \infty$. This is intuitively clear, since if $\Omega \subseteq \mathbb{R}^N$ is an open set and $\{f_n\} \subset C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$ is such that $\|f_n - f\|_{W^{1,\infty}(\Omega)} \to 0$, then $f \in C^1(\Omega)$ (why?).

Next we define the space of functions of bounded variation.

Definition 20 *Given an open set $\Omega \subseteq \mathbb{R}^N$, the space $C_0(\Omega)$ is the space of all continuous functions $f : \Omega \to \mathbb{R}$ with the property that for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that $|f(\boldsymbol{x})| \le \varepsilon$ for all $\boldsymbol{x} \in \Omega \setminus K_\varepsilon$. We endow $C_0(\Omega)$ with the supremum norm $\|\cdot\|_\infty$.*

Exercise 21 *Prove that $C_0(\Omega)$ is the closure of $C_c(\Omega)$ in the space $C_b(\Omega)$ of all continuous and bounded functions with the supremum norm $\|\cdot\|_\infty$.*

It turns out that the dual of $C_0(\Omega)$ can be identified with the space of signed measures.

Theorem 22 (Riesz representation theorem) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. For every linear and continuous function $T : C_0(\Omega) \to \mathbb{R}$, there exists a unique signed measure $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}$ such that*

$$T(\varphi) = \int_\Omega \varphi \, d\lambda \quad \text{for all } \varphi \in C_0(\Omega),$$

with

$$\|T\|_{(C_0(\Omega))'} = |\lambda|(\Omega),$$

where $|\lambda| = \lambda^+ + \lambda^-$. Conversely, for every signed measure $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}$, the function

$$T_\lambda(\varphi) := \int_\Omega \varphi \, d\lambda \quad \text{for all } \varphi \in C_0(\Omega),$$

is linear and continuous.

For the decomposition $\lambda = \lambda^+ - \lambda^-$ see the Hahn theorem and the Jordan decomposition theorem. Hence, if we identify T_λ with λ, then we can say that $(C_0(\Omega))'$ is the space of signed measures.

Recalling Definition 4, we can define the space of functions of bounded variation.

Definition 23 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We define the* space of functions of bounded variation $BV(\Omega)$ *as the space of all functions $f \in L^1(\Omega)$ such that, for all $i = 1, \ldots, N$ the i-th weak derivative $D_i f$ is a signed measure. We endow $BV(\Omega)$ with the norm*

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \sum_{i=1}^{n} \|D_i f\|_{(C_0(\Omega))'}.$$

Exercise 24 *Prove that $BV(\Omega)$ is a Banach space.*

Since every function $f \in C^\infty(\Omega) \cap BV(\Omega)$ belongs to $W^{1,1}(\Omega)$ (why?) and

$$\|D_i f\|_{(C_0(\Omega))'} = \int_\Omega \left| \frac{\partial f}{\partial x_i} \right| d\boldsymbol{x}$$

for every $i = 1, \ldots, N$, the closure of $C^\infty(\Omega) \cap BV(\Omega)$ in $BV(\Omega)$ is $W^{1,1}(\Omega)$. Thus, we cannot expect the Meyers–Serrin theorem (see Theorem 12) to hold in $BV(\Omega)$. However, the following weaker version holds.

Theorem 25 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $f \in BV(\Omega)$. Then there exists a sequence $\{f_n\} \subset C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and*

$$\lim_{n \to \infty} \int_\Omega \left| \frac{\partial f_n}{\partial x_i} \right| d\boldsymbol{x} = \|D_i f\|_{(C_0(\Omega))'}$$

for every $i = 1, \ldots, N$.

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Exercise 26 *Let $\Omega = B(\mathbf{0}, 1) \setminus \left\{ \boldsymbol{x} \in \mathbb{R}^N : x_N = 0 \right\}$. Show that the function $f : \Omega \to \mathbb{R}$, defined by*

$$f(\boldsymbol{x}) = f(x_1, \ldots, x_N) := \begin{cases} 1 & \text{if } x_N > 0, \\ 0 & \text{if } x_N < 0, \end{cases}$$

belongs to $W^{1,p}(\Omega)$ for all $1 \le p \le \infty$, but cannot be approximated by functions in $C^\infty\left(\overline{\Omega}\right)$.

Definition 27 *Given an open set $\Omega \subseteq \mathbb{R}^N$, we denote by $C^\infty(\overline{\Omega})$ the space of all functions $f \in C^\infty(\Omega)$ that can be extended to a function in $C^\infty(\mathbb{R}^N)$.*

The previous exercise shows that in the Meyers–Serrin theorem for general open sets Ω we may not replace $C^\infty(\Omega)$ with $C^\infty\left(\overline{\Omega}\right)$.

Next we show that if Ω has continuous boundary, then $C^\infty\left(\overline{\Omega}\right) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$. We recall that a rigid motion $\boldsymbol{T} : \mathbb{R}^N \to \mathbb{R}^N$ is an affine function given by $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{c} + R(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^N$, where R is a rotation and $\boldsymbol{c} \in \mathbb{R}^N$.

Definition 28 *Given an open set* $\Omega \subseteq \mathbb{R}^N$ *we say that its boundary* $\partial\Omega$ *is* Lipschitz continuous *if for all* $\boldsymbol{x}_0 \in \partial\Omega$ *there exist a rigid motion* $\boldsymbol{T} : \mathbb{R}^N \to \mathbb{R}^N$, *with* $\boldsymbol{T}(\boldsymbol{x}_0) = 0$, *a Lipschitz continuous function* $h : \mathbb{R}^{N-1} \to \mathbb{R}$, *with* $h(\boldsymbol{0}) = 0$, *and* $r > 0$ *such that, setting* $\boldsymbol{y} := \boldsymbol{T}(\boldsymbol{x})$, *we have*

$$\boldsymbol{T}(\Omega \cap B(\boldsymbol{x}_0, r)) = \{\boldsymbol{y} \in B(\boldsymbol{0}, r) : y_N > h(\boldsymbol{y}')\}. \tag{12}$$

We say that $\partial\Omega$ *is of class* C^m, $m \in \mathbb{N}_0$, *if the functions* h *are of class* C^m.

Observe that R, h, and r depend on $\boldsymbol{x}_0$. The coordinates $\boldsymbol{x}$ are called *background coordinates* while the coordinates $\boldsymbol{y}$ are called *local coordinates.*

Remark 29 *Without loss of generality, in the previous definition one can replace the ball* $B(\boldsymbol{x}_0, r)$ *with any small (open) neighborhood of* $\boldsymbol{x}_0$. *We will use this fact without further notice.*

Theorem 30 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set with boundary of class* C^0 *and let* $1 \le p < \infty$. *Then* $C^\infty\left(\overline{\Omega}\right) \cap W^{1,p}(\Omega)$ *is dense in* $W^{1,p}(\Omega)$.

To prove the theorem we need an auxiliary result.

Lemma 31 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set, let* $1 \le p < \infty$, *and let* $f \in L^p(\Omega)$. *Extend* f *by zero outside* Ω. *Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\int_\Omega |f(\boldsymbol{x} + \boldsymbol{\xi}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} \le \varepsilon$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$, *with* $\|\boldsymbol{\xi}\| \le \delta$.

Proof. Exercise. ■

Proposition 32 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and let* $1 \le p < \infty$. *Then for every* $f \in W^{1,p}(\Omega)$ *there exists a sequence of functions* f_n *in* $W^{1,p}(\Omega)$ *such that* $f_n \to f$ *in* $W^{1,p}(\Omega)$ *as* $n \to \infty$ *and* $f_n = 0$ *in* $\Omega \setminus B(\boldsymbol{0}, r_n)$ *for some (large)* $r_n > 0$.

Proof. Consider a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi \subseteq \overline{B(\boldsymbol{0}, 2)}$, $\varphi = 1$ in $B(\boldsymbol{0}, 1)$ and $0 \le \varphi \le 1$. For $n \in \mathbb{N}$, define

$$f_n(\boldsymbol{x}) := \varphi_n(\boldsymbol{x}) f(\boldsymbol{x}), \quad \varphi_n(\boldsymbol{x}) := \varphi(\boldsymbol{x}/n), \quad \boldsymbol{x} \in \Omega.$$

By the Lebesgue dominated convergence theorem, we have that $f_n \to f$ in $L^p(\Omega)$ as $n \to \infty$, while by Exercise **??**,

$$\frac{\partial f_n}{\partial x_i}(\boldsymbol{x}) = \varphi_n(\boldsymbol{x}) \frac{\partial f}{\partial x_i}(\boldsymbol{x}) + f(\boldsymbol{x}) \frac{\partial \varphi_n}{\partial x_i}(\boldsymbol{x}).$$

Again by the Lebesgue dominated convergence theorem, $\varphi_n \frac{\partial f}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ in $L^p(\Omega)$ as $n \to \infty$, while

$$\int_\Omega \left| f(\boldsymbol{x}) \frac{\partial \varphi_n}{\partial x_i}(\boldsymbol{x}) \right|^p d\boldsymbol{x} = \frac{1}{n^p} \int_\Omega \left| f(\boldsymbol{x}) \frac{\partial \varphi}{\partial x_i}\left(\frac{x}{n}\right) \right|^p d\boldsymbol{x} \le \frac{C}{n^p} \int_\Omega |f(\boldsymbol{x})|^p d\boldsymbol{x} \to 0$$

as $n \to \infty$. This concludes the proof. ■

When studing regular domains, the standard strategy is to consider the following.

- The flat case, that is $\Omega = \mathbb{R}^N_+$
- The case of a supergraph,

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^N : x_N > h(\boldsymbol{x}')\},$$

- The general case, using partitions of unity.

Lemma 33 *Let $\Omega = \mathbb{R}^N_+$ and $1 \le p < \infty$. Then the space $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

Proof. Let $f \in W^{1,p}(\mathbb{R}^N_+)$. Given $\delta > 0$, consider the function $f_\delta : \mathbb{R}^{N-1} \times (-\delta, \infty) \to \mathbb{R}$, given by

$$f_\delta(\boldsymbol{x}) := f(\boldsymbol{x}', x_N + \delta).$$

By Lemma 31,

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^N_+} |f(\boldsymbol{x}', x_N + \delta) - f(\boldsymbol{x})|^p = 0, \quad \lim_{\delta \to 0^+} \int_{\mathbb{R}^N_+} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}', x_N + \delta) - \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p = 0.$$

We leave as an exercise to check that $f_\delta \in W^{1,p}(\mathbb{R}^{N-1} \times (-\delta, \infty))$ and that $\frac{\partial f_\delta}{\partial x_i}(\boldsymbol{x}) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}', x_N + \delta)$. Since $\operatorname{dist}\left(\mathbb{R}^N_+, \partial(\mathbb{R}^{N-1} \times (-\delta, \infty))\right) = \delta > 0$, by Lemma 14,

$$\|(f_\delta) * \varphi_\varepsilon - f_\delta\|_{W^{1,p}(\mathbb{R}^N_+)} \to 0 \text{ as } \varepsilon \to 0^+.$$

Note that $(f_\delta) * \varphi_\varepsilon \in C^\infty(\mathbb{R}^N)$. ■

Friday, January, 28, 2022

Lemma 34 *Let $h : \mathbb{R}^{N-1} \to \mathbb{R}$ be a continuous function,*

$$\Omega := \{\boldsymbol{x} \in \mathbb{R}^N : x_N > h(\boldsymbol{x}')\},$$

and let $1 \le p < \infty$. Then the space $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

Proof. Let $f \in W^{1,p}(\Omega)$. By Proposition 32 we can assume that there exists $r > 0$ such that $f = 0$ in $\Omega \setminus B(\mathbf{0}, r)$. Given $0 < \delta << r$, consider the set

$$\Omega_\delta := \{\boldsymbol{x} \in \mathbb{R}^N : x_N > h(\boldsymbol{x}') - \delta\}$$

and the function $f_\delta : \Omega_\delta \to \mathbb{R}$, given by

$$f_\delta(\boldsymbol{x}) := f(\boldsymbol{x}', x_N + \delta).$$

By Lemma 31,

$$\lim_{\delta \to 0^+} \int_\Omega |f(\boldsymbol{x}', x_N + \delta) - f(\boldsymbol{x})|^p = 0, \quad \lim_{\delta \to 0^+} \int_\Omega \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}', x_N + \delta) - \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p = 0.$$

We leave as an exercise to check that $f_\delta \in W^{1,p}(\Omega_\delta)$ and that $\frac{\partial f_\delta}{\partial x_i}(\boldsymbol{x}) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}', x_N + \delta)$. Since $\operatorname{dist}(\Omega \cap B(\mathbf{0}, 2r), \partial(\Omega_\delta \cap B(\mathbf{0}, 4r))) > 0$, by Lemma 14,

$$\|(f_\delta) * \varphi_\varepsilon - f_\delta\|_{W^{1,p}(\Omega \cap B(\mathbf{0},2r))} \to 0 \text{ as } \varepsilon \to 0^+.$$

Since $f_\delta = 0$ and $(f_\delta) * \varphi_\varepsilon = 0$ outside $\Omega \setminus B(\mathbf{0}, 2r)$ for $0 < \varepsilon < \delta << r$, we have that

$$\|(f_\delta) * \varphi_\varepsilon - f_\delta\|_{W^{1,p}(\Omega)} = \|(f_\delta) * \varphi_\varepsilon - f_\delta\|_{W^{1,p}(\Omega \cap B(\mathbf{0},2r))} \to 0$$

as $\varepsilon \to 0^+$. Note that $(f_\delta) * \varphi_\varepsilon \in C^\infty(\mathbb{R}^N)$. ■

Remark 35 *Observe that if $f = 0$ in $\Omega \setminus B(\boldsymbol{x}_0, R)$, then by taking ε and δ sufficiently small, we can assume that $(f_\delta) * \varphi_\varepsilon = 0$ in $\Omega \setminus B(\boldsymbol{x}_0, 2R)$.*

We now turn to the proof of Theorem 30.

Proof. Fix $f \in W^{1,p}(\Omega)$. By the Meyers–Serrin theorem without loss of generality, we may assume that $f \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. Moreover, by Proposition 32 we can assume that there exists $r > 0$ such that $f = 0$ in $\Omega \setminus B(\mathbf{0}, r)$.

For every $\boldsymbol{x}_0 \in \partial\Omega$ there exist a rigid motion $\boldsymbol{T}_{\boldsymbol{x}_0} : \mathbb{R}^N \to \mathbb{R}^N$, with $\boldsymbol{T}_{\boldsymbol{x}_0}(\boldsymbol{x}_0) = 0$, a continuous function $h_{\boldsymbol{x}_0} : \mathbb{R}^{N-1} \to \mathbb{R}$, with $h_{\boldsymbol{x}_0}(\mathbf{0}) = 0$, and $r_{\boldsymbol{x}_0} > 0$ such that in local coordinates is given by

$$\boldsymbol{T}_{\boldsymbol{x}_0}(\Omega \cap B(\boldsymbol{x}_0, 2r_{\boldsymbol{x}_0})) = \{\boldsymbol{y} \in B(\mathbf{0}, 2r_{\boldsymbol{x}_0}) : \, y_N > h(\boldsymbol{y}')\}. \tag{13}$$

If the set $\Omega \setminus \bigcup_{\boldsymbol{x} \in \partial\Omega} B(\boldsymbol{x}, r_{\boldsymbol{x}})$ is nonempty, for every $\boldsymbol{x}_0 \in \Omega \setminus \bigcup_{\boldsymbol{x} \in \partial\Omega} B(\boldsymbol{x}, r_{\boldsymbol{x}})$ let $B(\boldsymbol{x}_0, 2r_{\boldsymbol{x}_0})$ be any open ball contained in Ω. The family $\{B(\boldsymbol{x}, r_{\boldsymbol{x}})\}_{\boldsymbol{x} \in \overline{\Omega}}$ is an open cover of $\overline{\Omega}$. Since $f = 0$ outside $B(\mathbf{0}, r)$, we have that $\overline{\Omega} \cap \overline{B(\mathbf{0}, r)}$ is compact. Hence, there is a finite number of balls $B_1, \ldots, B_\ell$, where $B_n := B(\boldsymbol{x}_n, r_n)$, that covers $\overline{\Omega} \cap K$. Let $\{\psi_n\}_{n=1}^\ell$ be a smooth partition of unity subordinated to $B_1, \ldots, B_\ell$ (Exercise).

Fix $n \in \{1, \ldots, \ell\}$ and define $f_n := f\psi_n \in W^{1,p}(\Omega)$ (see Exercise 16), where we extend f_n to be zero outside $\operatorname{supp} \psi_n$. There are two cases.

If $\operatorname{supp} \psi_n$ is contained in Ω, then we set $g_n := \psi_n f \in C_c^\infty(\mathbb{R}^N)$. If $\operatorname{supp} \psi_n$ is not contained in Ω, then $\boldsymbol{x}_n \in \partial\Omega$. Since $\operatorname{supp} \psi_n \subset B(\boldsymbol{x}_n, r_n)$, if we consider the function $f_n \circ \boldsymbol{T}_n^{-1}$ defined in

$$\Omega_n := \{\boldsymbol{y} \in \mathbb{R}^N : \, y_N > h_n(\boldsymbol{y}')\},$$

we have that $f_n \circ \boldsymbol{T}_n^{-1} \in W^{1,p}(\Omega_n)$ (exercise). By the previous lemma we can find a function $G_n \in C^\infty(\mathbb{R}^N)$ such that G_n restricted to Ω_n belongs to $W^{1,p}(\Omega_n)$ and

$$\|G_n - f_n \circ \boldsymbol{T}_n^{-1}\|_{W^{1,p}(\Omega_n)} \le \eta/[(1 + L_n)2^n], \tag{14}$$

where $L_n := \|D\boldsymbol{T}_n\|_\infty$. Moreover, in view of Remark 35, we can assume that $G_n = 0$ outside $B(\mathbf{0}, 2r_n)$. Then $g_n := G_n \circ \boldsymbol{T}_n$ belongs to $C^\infty(\mathbb{R}^N)$ and to $W^{1,p}(\boldsymbol{T}_n^{-1}(\Omega_n))$ (exercise), with

$$\begin{aligned}\|g_n - f_n\|_{W^{1,p}(\Omega)} &= \|g_n - f_n\|_{W^{1,p}(\Omega \cap B(\boldsymbol{x}_n, 2r_n))} = \|g_n - f_n\|_{W^{1,p}(\boldsymbol{T}_n^{-1}(\Omega_n \cap B(\mathbf{0}, 2r_n)))} \\ &= \|g_n - f_n\|_{W^{1,p}(\boldsymbol{T}_n^{-1}(\Omega_n)))} \le L_n \|G_n - f_n \circ \boldsymbol{T}_n^{-1}\|_{W^{1,p}(\Omega_n)} \le \eta/2^n,\end{aligned}$$

where we used (13), the facts that $f_n = 0$ outside $B(\boldsymbol{x}_n, r_n)$ and $G_n = 0$ outside $B(\mathbf{0}, 2r_n)$. Define the function $g := \sum_{n=1}^{\ell} g_n$. Then $g \in C^\infty(\mathbb{R}^N)$ and $g \in W^{1,p}(\Omega)$. Moreover,

$$\|f - g\|_{W^{1,p}(\Omega)} \le \sum_{i=1}^{\ell} \|\psi_n f - g_n\|_{W^{1,p}(\Omega)} \le \eta \sum_{i=1}^{\ell} 2^{-i} \le \eta.$$

This concludes the proof. ■

Monday, January 31, 2022

Exercise 36 *Let $\Omega, U \subseteq \mathbb{R}^N$ be open sets, let $\Psi : U \to \Omega$ be invertible, with Ψ and Ψ^{-1} Lipschitz functions, and let $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$. Then $f \circ \Psi \in W^{1,p}(U)$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $\boldsymbol{y} \in U$,*

$$\frac{\partial (f \circ \Psi)}{\partial y_i}(\boldsymbol{y}) = \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(\Psi(\boldsymbol{y})) \frac{\partial \Psi_j}{\partial y_i}(\boldsymbol{y}).$$

2 Absolute Continuity on Lines

We recall some facts about absolute continuous functions.

Definition 37 *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be* absolutely continuous *on I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{\ell} |f(b_k) - f(a_k)| \le \varepsilon \tag{15}$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subseteq I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

The space of all absolutely continuous functions $f : I \to \mathbb{R}^N$ is denoted by $AC(I)$.

Theorem 38 *Let $g : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function and let*

$$f(x) := \int_a^x g(t)\, dt.$$

Then f is absolutely continuous and $f'(x) = g(x)$ for $\mathcal{L}^1$ a.e. $x \in [a, b]$.

Theorem 39 (Fundamental Theorem of Calculus) *Let $f : [a, b] \to \mathbb{R}$. Then f is absolutely continuous in $[a, b]$ if and only if f is differentiable $\mathcal{L}^1$-a.e. in $[a, b]$, f' is Lebesgue integrable, and the fundamental theorem of calculus is valid, that is, for all $x, x_0 \in [a, b]$,*

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\, dt. \tag{16}$$

Definition 40 *Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$. We say that f satisfies the* Lusin (N) property *if*

$$\mathcal{L}^1(f(D)) = 0$$

for every set $D \subseteq E$ with $\mathcal{L}^1(D) = 0$.

Theorem 41 (Chain rule) *Let $I, J \subseteq \mathbb{R}$ be two intervals and let $f : J \to \mathbb{R}$ and $g : I \to J$ be such that f, g, and $f \circ g$ are differentiable $\mathcal{L}^1$-a.e. in their respective domains. If f satisfies the Lusin (N) property, then for $\mathcal{L}^1$-a.e. $x \in I$,*

$$(f \circ g)'(x) = f'(g(x))g'(x), \tag{17}$$

where $f'(g(x))g'(x)$ is interpreted to be zero whenever $g'(x) = 0$ (even if f is not differentiable at $g(x)$).

The next theorem relates weak partial derivatives with the (classical) partial derivatives. Given $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $i \in \{1, \dots, N\}$ we denote by $\boldsymbol{x}_i'$ the vector of $\mathbb{R}^{N-1}$ obtained from $\boldsymbol{x}$ by removing the i-th component x_i. With a slight abuse of notation we write

$$\boldsymbol{x} = (\boldsymbol{x}_i', x_i) \in \mathbb{R}^{N-1} \times \mathbb{R}. \tag{18}$$

Theorem 42 (Absolute Continuity on Lines) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. A function $f \in L^p(\Omega)$ belongs to the space $W^{1,p}(\Omega)$ if and only if it has a representative $\overline{f}$ that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{f}$ agree $\mathcal{L}^N$ a.e. with the weak derivatives of f.*

Proof. Step 1: Assume that $f \in W^{1,p}(\Omega)$. Consider a sequence of standard mollifiers $\{\varphi_\varepsilon\}_{\varepsilon>0}$ and for every $\varepsilon > 0$ define $f_\varepsilon := f * \varphi_\varepsilon$ in $\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}$. By Lemma 14,

$$\lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon} \|\nabla f_\varepsilon(\boldsymbol{x}) - \nabla f(\boldsymbol{x})\|^p d\boldsymbol{x} = 0.$$

It follows by Fubini's theorem that for all $i = 1, \dots, N$,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{N-1}} \left(\int_{(\Omega_\varepsilon)_{x_i}} \|\nabla f_\varepsilon(\boldsymbol{x}_i, x_i) - \nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i \right) d\boldsymbol{x}_i = 0,$$

where $(\Omega_\varepsilon)_{\boldsymbol{x}_i} := \{x_i \in \mathbb{R} : (\boldsymbol{x}_i, x_i) \in \Omega_\varepsilon\}$, and so we may find a subsequence $\{\varepsilon_n\}$ such that for all $i = 1, \dots, N$ and for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$,

$$\lim_{n \to \infty} \int_{(\Omega_{\varepsilon_n})_{\boldsymbol{x}_i}} \|\nabla f_{\varepsilon_n}(\boldsymbol{x}_i, x_i) - \nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i = 0. \tag{19}$$

Set $f_n := f_{\varepsilon_n}$ and

$$E := \left\{ x \in \Omega : \lim_{n \to \infty} f_n(x) \text{ exists in } \mathbb{R} \right\}.$$

Since E contains every Lebesgue points of f, we have that $\mathcal{L}^N(\Omega \setminus E) = 0$. Define

$$\overline{f}(\boldsymbol{x}) := \begin{cases} \lim\limits_{n\to\infty} f_n(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\overline{f}$ is a representative of f, since by Theorem 13, $\{f_n\}$ converges pointwise at every Lebesgue point of f. It remains to prove that $\overline{f}$ has the desired properties.

By Fubini's theorem for every $i = 1, \ldots, N$ we have that

$$\int_{\mathbb{R}^{N-1}} \left(\int_{\Omega_{\boldsymbol{x}_i}} \|\nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i \right) d\boldsymbol{x}_i < \infty$$

and

$$\int_{\mathbb{R}^{N-1}} \mathcal{L}^1(\{x_i \in \Omega_{\boldsymbol{x}_i} : (\boldsymbol{x}_i, x_i) \notin E\})\, d\boldsymbol{x}_i = 0,$$

where $\Omega_{\boldsymbol{x}_i} := \{x_i \in \mathbb{R} : (\boldsymbol{x}_i, x_i) \in \Omega\}$, and so we may find a set $N_i \subset \mathbb{R}^{N-1}$, with $\mathcal{L}^{N-1}(N_i) = 0$, such that for all $\boldsymbol{x}_i \in \mathbb{R}^{N-1} \setminus N_i$ for which $\Omega_{\boldsymbol{x}_i}$ is nonempty we have that

$$\int_{\Omega_{\boldsymbol{x}_i}} \|\nabla f(\boldsymbol{x}_i, x_i)\|^p dx_i < \infty, \tag{20}$$

(19) holds for all $i = 1, \ldots, N$ and $(\boldsymbol{x}_i, x_i) \in E$ for $\mathcal{L}^1$ a.e. $x_i \in \Omega_{\boldsymbol{x}_i}$. ■

Wednesday, February 2, 2022

Proof. Fix any such $\boldsymbol{x}_i$ and let $I \subseteq \Omega_{\boldsymbol{x}_i}$ be a maximal interval. Fix $t_0 \in I$ such that $(\boldsymbol{x}_i, t_0) \in E$ and let $t \in I$. For all n large, the interval of endpoints t and t_0 is contained in $(\Omega_{\varepsilon_n})_{\boldsymbol{x}_i}$ and so, since $f_n \in C^\infty(\Omega_{\varepsilon_n})$, by the fundamental theorem of calculus,

$$f_n(\boldsymbol{x}_i, t) = f_n(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s)\, ds.$$

Since $(\boldsymbol{x}_i, t_0) \in E$. Then $f_n(\boldsymbol{x}_i, t_0) \to \overline{f}(\boldsymbol{x}_i, t_0) \in \mathbb{R}$. On the other hand, by (19)

$$\lim_{n\to\infty} \int_{t_0}^{t} \left| \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s) - \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s) \right| ds = 0. \tag{21}$$

Hencewe have that there exists the limit

$$\begin{aligned} \lim_{n\to\infty} f_n(\boldsymbol{x}_i, t) &= \lim_{n\to\infty} \left(f_n(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}_i, s)\, ds \right) \\ &= \overline{f}(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, s)\, ds. \end{aligned}$$

Note that by the definition of E and $\overline{f}$, this implies, in particular, that

$$(\boldsymbol{x}_i, t) \in E \tag{22}$$

and that

$$\overline{f}(\boldsymbol{x}_i, t) = \overline{f}(\boldsymbol{x}_i, t_0) + \int_{t_0}^{t} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, s)\, ds \tag{23}$$

for *all* $t \in I$. Since $\overline{f}(\boldsymbol{x}_i, \cdot)$ satisfies the fundamental theorem of calculus, it is locally absolutely continuous in I and $\frac{\partial \overline{f}}{\partial x_N}(\boldsymbol{x}_i, t) = \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i, t)$ for $\mathcal{L}^1$ a.e. $t \in I$. We can now apply exercise 43 to conclude that $\overline{f}(\boldsymbol{x}_i, \cdot)$ is absolutely continuous in I.

Step 2: Assume that f admits a representative $\overline{f}$ that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Fix $i = 1, \ldots, N$ and let $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$ be such that $\overline{f}(\boldsymbol{x}_i, \cdot)$ is absolutely continuous on the open set $\Omega_{\boldsymbol{x}_i}$. Then for every function $\varphi \in C_c^\infty(\Omega)$, by the integration by parts formula for absolutely continuous functions, we have

$$\int_{\Omega_{\boldsymbol{x}_i}} \overline{f}(\boldsymbol{x}_i, t) \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}_i, t)\, dt = -\int_{\Omega_{\boldsymbol{x}_i}} \frac{\partial \overline{f}}{\partial x_i}(\boldsymbol{x}_i, t)\, \varphi(\boldsymbol{x}_i, t)\, dt.$$

Since this holds for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i \in \mathbb{R}^{N-1}$, integrating over $\mathbb{R}^{N-1}$ and using Fubini's theorem yields

$$\int_\Omega \overline{f}(\boldsymbol{x}) \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x})\, d\boldsymbol{x} = -\int_\Omega \frac{\partial \overline{f}}{\partial x_i}(\boldsymbol{x})\, \varphi(\boldsymbol{x})\, d\boldsymbol{x},$$

which implies that $\frac{\partial \overline{f}}{\partial x_i} \in L^p(\Omega)$ is the weak partial derivative of $\overline{f}$ with respect to x_i. This shows that $f \in W^{1,p}(\Omega)$. ■

Exercise 43 *Let $I \subseteq \mathbb{R}$ and let $f : I \to \mathbb{R}$ be locally absolutely continuous with $f' \in L^p(I)$, $1 \le p \le \infty$. Prove that f is absolutely continuous.*

As a consequence of Theorem 42 and of the properties of absolutely continuous functions we have the following results.

Exercise 44 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. Using Theorem 42 prove the following results.*

(i) ***(Chain rule)*** *Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz and let $f \in W^{1,p}(\Omega)$. Assume that $h(0) = 0$ if Ω has infinite measure. Then $h \circ f \in W^{1,p}(\Omega)$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$,*

$$\frac{\partial (h \circ f)}{\partial x_i}(\boldsymbol{x}) = h'\left(\overline{f}(\boldsymbol{x})\right) \frac{\partial f}{\partial x_i}(\boldsymbol{x}),$$

where $h'\left(\overline{f}(\boldsymbol{x})\right) \dfrac{\partial f}{\partial x_i}(\boldsymbol{x})$ is interpreted to be zero whenever $\dfrac{\partial f}{\partial x_i}(\boldsymbol{x}) = 0$. What can you say about the case $p = \infty$?

(ii) ***(Product rule)*** *Let* $f, g \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$*. Then* $fg \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *for all* $i = 1, \ldots, N$ *and for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \Omega$,

$$\frac{\partial (fg)}{\partial x_i}(\boldsymbol{x}) = g(\boldsymbol{x}) \frac{\partial f}{\partial x_i}(\boldsymbol{x}) + f(\boldsymbol{x}) \frac{\partial g}{\partial x_i}(\boldsymbol{x}).$$

What can you say about the case $p = \infty$*?*

(iii) ***(Reflection)*** *Let* $\Omega = \mathbb{R}^N_+ := \left\{(\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\right\}$ *and let* $f \in W^{1,p}\left(\mathbb{R}^N_+\right)$*. Then the function*

$$g(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}) & \text{if } x_N > 0, \\ f(\boldsymbol{x}', -x_N) & \text{if } x_N < 0 \end{cases}$$

belongs to $W^{1,p}\left(\mathbb{R}^N\right)$ *and for all* $i = 1, \ldots, N$ *and for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \mathbb{R}^N$

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \dfrac{\partial f}{\partial x_i}(\boldsymbol{x}) & \text{if } x_N > 0, \\ (-1)^{\delta_{iN}} \dfrac{\partial f}{\partial x_i}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0. \end{cases}$$

(iv) *Let* $E \subset \mathbb{R}$ *be such that* $\mathcal{L}^1(E) = 0$*, let* $f \in W^{1,1}_{\text{loc}}(\Omega)$*, and let* $\overline{f}$ *be its precise representative given in Theorem 42. Prove that* $\nabla f(\boldsymbol{x}) = 0$ *for* $\mathcal{L}^N$ *a.e.* $\boldsymbol{x} \in \left(\overline{f}\right)^{-1}(E)$*.*

Friday, February 4, 2022

3 Difference Quotients

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and for every $i = 1, \ldots, N$ and $h > 0$, let

$$\Omega_{h,i} := \{x \in \Omega : x + te_i \in \Omega \quad \text{for all } 0 < t \leq h\}.$$

Theorem 45 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set,* $1 < p < \infty$*, and* $f \in L^p(\Omega)$ *be such that*

$$\liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x} < \infty \tag{24}$$

for every $i = 1, \ldots, N$*, then* $f \in W^{1,p}(\Omega)$ *and*

$$\int_\Omega \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p d\boldsymbol{x} \leq \liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x}.$$

Definition 46 *Let* $E \subseteq \mathbb{R}^N$ *be a Lebesgue measurable set,* $1 \leq p < \infty$*, and* $f_n, f \in L^p(E)$*. We say that the sequence* $\{f_n\}_n$ converges weakly in $L^p(E)$ *to* f*, and we write* $f_n \rightharpoonup f$ *in* $L^p(E)$*, if for every* $g \in L^{p'}(E)$*,*

$$\lim_{n \to \infty} \int_E f_n(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x} = \int_E f(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}.$$

The following compactness theorem is crucial.

Theorem 47 *Let $E \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, $1 < p < \infty$, and let $\{f_n\}_n$ be a bounded sequence in $L^p(E)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and $f \in L^p(E)$ such that $f_{n_k} \rightharpoonup f$ in $L^p(E)$.*

The previous theorem is a consequence of the fact that when $1 < p < \infty$, the space $L^p(E)$ is reflexive, which means that the bidual of $L^p(E)$ can be identified with $L^p(E)$.

Theorem 48 (Riesz representation theorem) *Let $E \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, $1 < p < \infty$. For every linear and continuous function $T : L^p(E) \to \mathbb{R}$, there exists a unique function $g \in L^{p'}(E)$ such that*

$$T(f) = \int_E f(\boldsymbol{x})g(\boldsymbol{x})\,d\boldsymbol{x} \quad \text{for every } f \in L^p(E),$$

with

$$\|T\|_{(L^p(E))'} = \|g\|_{L^{p'}(E)}.$$

Conversely, for every $g \in L^{p'}(\Omega)$, the functional

$$T_g(f) = \int_E f(\boldsymbol{x})g(\boldsymbol{x})\,d\boldsymbol{x} \quad \text{for every } f \in L^p(E)$$

is linear and continuous.

Hence, by identifying T_g with g, we can identify the dual of $L^p(E)$ with $L^{p'}(E)$. Since $1 < p' < \infty$, we also have that $(L^{p'}(E))'$ can be identified with $L^p(E)$. Thus,

$$(L^p(E))'' = ((L^p(E))')' \cong (L^{p'}(E))' \cong L^p(E),$$

which shows that $L^p(E)$ is reflexive when $1 < p < \infty$. This is no longer true for $p = 1$ and $p = \infty$.

We begin with a useful compactness result.

Lemma 49 (Compactness) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 < p < \infty$. Assume that $\{f_n\}_n$ is bounded $W^{1,p}(\Omega)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{u_n\}_n$ and $f \in W^{1,p}(\Omega)$ such that $f_{n_k} \rightharpoonup f$ in $L^p(\Omega)$ and $\frac{\partial f_{n_k}}{\partial x_i} \rightharpoonup \frac{\partial f}{\partial x_i}$ in $L^p(\Omega)$*

Proof. Since $\{f_n\}_n$ and $\{\nabla f_n\}_n$ are bounded in the reflexive Banach spaces $L^p(\Omega)$ and $L^p(\Omega; \mathbb{R}^N)$, respectively, we may select the subsequence $\{f_{n_k}\}_k$ such that $f_{n_k} \rightharpoonup f$ in $L^p(\Omega)$ and $\frac{\partial f_{n_k}}{\partial x_i} \rightharpoonup v_i$ in $L^p(\Omega)$ for all $i = 1, \ldots, N$ and for some functions $f, v_1, \ldots, v_N \in L^p(\Omega)$. It remains to show that $f \in W^{1,p}(\Omega)$. For every $\phi \in C_c^\infty(\Omega)$, $i = 1, \ldots, N$, and $k \in \mathbb{N}$ we have

$$\int_\Omega f_{n_k} \frac{\partial \phi}{\partial x_i}\,d\boldsymbol{x} = -\int_\Omega \frac{\partial f_{n_k}}{\partial x_i} \phi\,d\boldsymbol{x}.$$

Letting $k \to \infty$ in the previous equality yields

$$\int_\Omega f \frac{\partial \phi}{\partial x_i}\, d\boldsymbol{x} = -\int_\Omega v_i \phi \, d\boldsymbol{x},$$

which shows that $\frac{\partial u}{\partial x_i} = v_i$. Hence, $u \in W^{1,p}(\Omega)$. ∎

Lemma 50 *Let $E \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, $1 \leq p < \infty$, and $f \in L^p(E)$. Then for every Lebesgue measurable set $F \subseteq E$,*

$$\int_F |f_\varepsilon(\boldsymbol{x})|^p d\boldsymbol{x} \leq \int_{F_\varepsilon} |f(\boldsymbol{x})|^p d\boldsymbol{x}$$

where $f_\varepsilon = f * \varphi_\varepsilon$ and $F_\varepsilon = \{\boldsymbol{x} \in E : \operatorname{dist}(\boldsymbol{x}, F) < \varepsilon\}$.

Proof. For $\boldsymbol{x} \in F$ we have

$$\begin{aligned} f_\varepsilon(\boldsymbol{x}) &= \int_E \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_E (\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}))^{1/p'} (\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}))^{1/p} f(\boldsymbol{y})\, d\boldsymbol{y}. \end{aligned}$$

Hence, by Hölder's inequality

$$|f_\varepsilon(\boldsymbol{x})| \leq \left(\int_E \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y} \right)^{1/p'} \left(\int_E \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y})|^p d\boldsymbol{y} \right)^{1/p}.$$

In turn,

$$\begin{aligned} |f_\varepsilon(\boldsymbol{x})|^p &\leq \int_E \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y})|^p d\boldsymbol{y} \\ &= \int_{B(\boldsymbol{x},\varepsilon) \cap E} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y})|^p d\boldsymbol{y} \end{aligned}$$

where we used the fact that $\int_E \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y} \leq \int_{\mathbb{R}^N} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y} = 1$. Note that if $\boldsymbol{y} \in B(\boldsymbol{x}, \varepsilon) \cap E$, then $\|\boldsymbol{x} - \boldsymbol{y}\| < \varepsilon$, and so, $\operatorname{dist}(\boldsymbol{x}, F) < \varepsilon$, that is, $\boldsymbol{y} \in F_\varepsilon$. Therefore, $B(\boldsymbol{x}, \varepsilon) \cap E \subseteq F_\varepsilon$, and so,

$$|f_\varepsilon(\boldsymbol{x})|^p \leq \int_{F_\varepsilon} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y})|^p d\boldsymbol{y}$$

Integrating over F and using Fubini's theorem gives

$$\begin{aligned} \int_F |f_\varepsilon(\boldsymbol{x})|^p d\boldsymbol{x} &\leq \int_F \int_{F_\varepsilon} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y})|^p d\boldsymbol{y} d\boldsymbol{x} \\ &= \int_{F_\varepsilon} |f(\boldsymbol{y})|^p \left(\int_F \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{x} \right) d\boldsymbol{y} \\ &\leq \int_{F_\varepsilon} |f(\boldsymbol{y})|^p \left(\int_{\mathbb{R}^N} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{x} \right) d\boldsymbol{y} = \int_{F_\varepsilon} |f(\boldsymbol{y})|^p d\boldsymbol{y}. \end{aligned}$$

■

We turn to the proof of the theorem.

Proof. Step 1: Assume that $f \in C^{\infty}(\Omega)$. Let $U \Subset \Omega$. Then $U \subseteq \Omega_{h,i}$ for all $h > 0$ sufficiently small. For every $\boldsymbol{x} \in U$,

$$\frac{\partial f}{\partial x_i}(\boldsymbol{x}) = \lim_{h \to 0^+} \frac{f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})}{h},$$

and so, by Fatou's lemma,

$$\begin{aligned}\int_U \left|\frac{\partial f}{\partial x_i}(\boldsymbol{x})\right|^p d\boldsymbol{x} &= \int_U \lim_{h \to 0^+} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x} \\ &\leq \liminf_{h \to 0^+} \int_U \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x} \\ &\leq \liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x}.\end{aligned}$$

Letting $U \nearrow \Omega$ and using the Lebesgue monotone convergence theorem, we have

$$\int_\Omega \left|\frac{\partial f}{\partial x_i}(\boldsymbol{x})\right|^p d\boldsymbol{x} \leq \liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x}.$$

Step 2: Let $p > 1$ and $f \in L^p(\Omega)$ be such that

$$\liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x} =: M < \infty.$$

Let $U \Subset \Omega$ and for $0 < \varepsilon < \operatorname{dist}(U, \partial\Omega)$ define $f_\varepsilon := \varphi_\varepsilon * f$, where φ_ε is a standard mollifier. Then for $0 < h < \operatorname{dist}(U, \partial\Omega) - \varepsilon$, by Lemma 50,

$$\begin{aligned}\int_{U_{h,i}} \frac{|f_\varepsilon(\boldsymbol{x} + h\boldsymbol{e}_i) - f_\varepsilon(\boldsymbol{x})|^p}{h^p} dx &= \int_{U_{h,i}} \frac{|((f(\cdot + h e_i) - f) * \varphi_\varepsilon)(\boldsymbol{x})|^p}{h^p} dx \\ &\leq \int_{(U_{h,i})_\varepsilon} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} dx \\ &\leq \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x}.\end{aligned}$$

Letting $h \to 0^+$ and using the previous step applied to f_ε gives

$$\int_U \left|\frac{\partial f_\varepsilon}{\partial x_i}(\boldsymbol{x})\right|^p dx \leq \liminf_{h \to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x} + h\boldsymbol{e}_i) - f(\boldsymbol{x})|^p}{h^p} dx. \tag{25}$$

By compactness, there exist $\varepsilon_n \to 0^+$ and $v \in W^{1,p}(U)$ such that $f_{\varepsilon_n} \rightharpoonup v$ in $W^{1,p}(U)$, but since $f_{\varepsilon_n} \to f$ in $L^p(U)$, necessarily, $f = v$. Thus, $f \in W^{1,p}(U)$. Since this is true for every $U \Subset \Omega$, we have that $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Since $\frac{\partial f_\varepsilon}{\partial x_i} \to \frac{\partial f}{\partial x_i}$

in $L^p(U)$ as $\varepsilon \to 0^+$ by Lemma 14, letting $\varepsilon \to 0^+$ in the previous inequality,we obtain

$$\int_U \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p dx \le \liminf_{h\to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x}+h\boldsymbol{e}_i)-f(\boldsymbol{x})|^p}{h^p} dx. \tag{26}$$

By letting $U \nearrow \Omega$ and using the Lebesgue monotone convergence theorem, we obtain

$$\int_U \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p dx \le \liminf_{h\to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x}+h\boldsymbol{e}_i)-f(\boldsymbol{x})|^p}{h^p} dx. \tag{27}$$

■

Exercise 51 *Prove that for $p = 1$ the last part of the statement of the theorem is false. Hint: It is enough to construct an example for $N = 1$.*

Exercise 52 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $1 \le p < \infty$. Prove that for every $f \in W^{1,p}(\Omega)$,*

$$\int_\Omega \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right|^p d\boldsymbol{x} = \lim_{h\to 0^+} \int_{\Omega_{h,i}} \frac{|f(\boldsymbol{x}+h\boldsymbol{e}_i)-f(\boldsymbol{x})|^p}{h^p} d\boldsymbol{x}.$$

Monday, February 7, 2022

4 Embeddings: $1 \le p < N$

Consider a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that its distributional gradient ∇f belongs to $L^p(\mathbb{R}^N;\mathbb{R}^N)$ for some $1 \le p < \infty$. We are interested in finding an exponent q such that $f \in L^q(\mathbb{R}^N)$, and so we are after an inequality of the type

$$\|f\|_{L^q(\mathbb{R}^N)} \le c \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}, \tag{28}$$

which should hold for all such f.

Assume for simplicity that $f \in C^1_c(\mathbb{R}^N)$ and for $r > 0$ define the rescaled function

$$f_r(\boldsymbol{x}) := f(r\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

Applying the previous inequality to f_r we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f(r\boldsymbol{x})|^q \, d\boldsymbol{x}\right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}^N} |f_r(\boldsymbol{x})|^q \, d\boldsymbol{x}\right)^{\frac{1}{q}} \\ &\le c\left(\int_{\mathbb{R}^N} \|\nabla f_r(\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} = c\left(r^p \int_{\mathbb{R}^N} \|\nabla f(r\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},\end{aligned}$$

or, equivalently, after the change of variables $\boldsymbol{y} := r\boldsymbol{x}$,

$$\left(\frac{1}{r^N}\int_{\mathbb{R}^N} |f(\boldsymbol{y})|^q \, d\boldsymbol{y}\right)^{\frac{1}{q}} \le c\left(\frac{r^p}{r^N}\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p d\boldsymbol{y}\right)^{\frac{1}{p}},$$

that is,

$$\left(\int_{\mathbb{R}^N} |f(\boldsymbol{y})|^q \, d\boldsymbol{y}\right)^{\frac{1}{q}} \le c r^{1-\frac{N}{p}+\frac{N}{q}} \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p d\boldsymbol{y}\right)^{\frac{1}{p}}.$$

If $1-\frac{N}{p}+\frac{N}{q}>0$, let $r\to 0^+$ to conclude that $f\equiv 0$, while if $1-\frac{N}{p}+\frac{N}{q}<0$, let $r\to\infty$ to conclude again that $f\equiv 0$. Hence, the only possible case is when

$$\frac{N}{q}=\frac{N}{p}-1.$$

So in order for q to be positive, we need $p<N$ in which case

$$q=p^* := \frac{Np}{N-p}.$$

The number p^* is called *Sobolev critical exponent.*

Theorem 53 (Sobolev–Gagliardo–Nirenberg Embedding) *Let $1\le p<N$. Then for every $f\in W^{1,p}\left(\mathbb{R}^N\right)$,*

$$\left(\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^{p^*} \, d\boldsymbol{x}\right)^{\frac{1}{p^*}} \le C \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},$$

where $C=C(N,p)>0$. In particular, $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^{p^*}\left(\mathbb{R}^N\right)$.

The proof makes use of the following result, which follows from Hölder's inequality.

Exercise 54 *Let $1\le p_1,\ldots,p_n,p\le\infty$, with $\frac{1}{p_1}+\cdots+\frac{1}{p_n}=\frac{1}{p}$, and $f_i\in L^{p_i}\left(\mathbb{R}^N\right)$, $i=1,\ldots,n$. Prove that*

$$\left\|\prod_{i=1}^n f_i\right\|_{L^p} \le \prod_{i=1}^n \|f_i\|_{L^{p_i}}.$$

Exercise 55 *Prove that if $g:\mathbb{R}\to\mathbb{R}$ is measurable with $\int_{\mathbb{R}} |g(t)|^p \, dt<\infty$ for some $p>0$, then*

$$\liminf_{x\to-\infty} |g(x)|=0, \quad \liminf_{x\to\infty} |g(x)|=0$$

and that in general one cannot replace the limit inferiors with actual limits.

In what follows, we use the notation (18).

Lemma 56 *Let $N\ge 2$ and let $f_i\in L^{N-1}\left(\mathbb{R}^{N-1}\right)$, $i=1,\ldots,N$. Then the function*

$$f(\boldsymbol{x}) := f_1(\boldsymbol{x}_1') \, f_2(\boldsymbol{x}_2') \cdots f_N(\boldsymbol{x}_N'), \quad \boldsymbol{x}\in\mathbb{R}^N,$$

belongs to $L^1\left(\mathbb{R}^N\right)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \le \prod_{i=1}^N \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof. The proof is by induction on N. If $N = 2$, then

$$f(\boldsymbol{x}) := f_1(x_2) f_2(x_1), \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Integrating both sides with respect to $\boldsymbol{x}$ and using Fubini's theorem, we get

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})| \, d\boldsymbol{x} = \int_{\mathbb{R}} |f_1(x_2)| \, dx_2 \int_{\mathbb{R}} |f_2(x_1)| \, dx_1.$$

Assume next that the result is true for N and let's prove it for $N+1$. Let

$$f(\boldsymbol{x}) := f_1(\boldsymbol{x}_1') f_2(\boldsymbol{x}_2') \cdots f_{N+1}\left(\boldsymbol{x}_{N+1}'\right), \quad \boldsymbol{x} \in \mathbb{R}^{N+1},$$

where $f_i \in L^N\left(\mathbb{R}^N\right)$, $i = 1, \ldots, N+1$. Fix $x_{N+1} \in \mathbb{R}$. Integrating both sides with respect to $x_1, \ldots, x_N$ and using Hölder's inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, dx_1 \cdots dx_N \\
&\leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\boldsymbol{x}_i')|^{\frac{N}{N-1}} \, dx_1 \cdots dx_N\right)^{\frac{N-1}{N}}.
\end{aligned}$$

For every $i = 1, \ldots, N$ we denote by $\boldsymbol{x}_i''$ the $N-1$ dimensional vector obtained by removing the last component from $\boldsymbol{x}_i'$ and with an abuse of notation we write $\boldsymbol{x}_i' = (\boldsymbol{x}_i'', x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Since x_{N+1} is fixed, by the induction hypothesis applied to the functions $g_i(\boldsymbol{x}_i'') := |f_i(\boldsymbol{x}_i'', x_{N+1})|^{\frac{N}{N-1}}$, $\boldsymbol{x}_i'' \in \mathbb{R}^{N-1}$, $i = 1, \ldots, N$, we obtain that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\boldsymbol{x}_i')|^{\frac{N}{N-1}} \, dx_1 \cdots dx_N \leq \prod_{i=1}^{N} \|g_i\|_{L^{N-1}(\mathbb{R}^{N-1})},$$

and so

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, dx_1 \cdots dx_N \\
&\leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \prod_{i=1}^{N} \left(\int_{\mathbb{R}^{N-1}} |f_i(\boldsymbol{x}_i'', x_{N+1})|^N \, d\boldsymbol{x}_i''\right)^{\frac{1}{N}}.
\end{aligned}$$

Integrating both sides with respect to x_{N+1} and using Fubini's theorem and the extended Hölder's inequality (see the previous exercise), with

$$1 = \underbrace{\frac{1}{N} + \cdots + \frac{1}{N}}_{N},$$

we get

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})| \, d\boldsymbol{x} \leq \prod_{i=1}^{N+1} \|f_i\|_{L^N(\mathbb{R}^N)},$$

which concludes the proof. ■

Wednesday, February 9, 2022

We now turn to the proof of the Sobolev–Gagliardo–Nirenberg embedding theorem.

Proof. Step 1: Assume first that $p = 1$. By mollification we can assume that $f \in C^1\left(\mathbb{R}^N\right) \cap W^{1,1}(\mathbb{R}^N)$. Fix $i = 1, \ldots, N$. By Fubini's theorem for $\mathcal{L}^{N-1}$ a.e. $\boldsymbol{x}_i' \in \mathbb{R}^{N-1}$ we have that the function $g(t) := f(\boldsymbol{x}_i', t)$, $t \in \mathbb{R}$, belongs to $L^p(\mathbb{R}) \cap C^1(\mathbb{R})$ with $g' \in L^1(\mathbb{R})$. By the previous exercise

$$\liminf_{t \to -\infty} |g(t)| = 0,$$

and so we may find a sequence $t_n \to -\infty$ such that $g(t_n) \to 0$. Hence, for every $t \in \mathbb{R}$ we have that

$$g(t) = g(t_n) + \int_{t_n}^{t} g'(s)\, ds.$$

Letting $n \to \infty$ and using the fact that $g' \in L^1(\mathbb{R})$, by Lebesgue dominated convergence theorem we conclude that for each $i = 1, \ldots, N$ and $\boldsymbol{x} \in \mathbb{R}^N$ we have

$$f(\boldsymbol{x}) = \int_{-\infty}^{x_i} \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i)\, dy_i,$$

and so

$$|f(\boldsymbol{x})| \le \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i) \right| dy_i$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Multiplying these N inequalities and raising to power $\frac{1}{N-1}$, we get

$$|f(\boldsymbol{x})|^{\frac{N}{N-1}} \le \prod_{i=1}^{N} \left(\int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}_i', y_i) \right| dy_i \right)^{\frac{1}{N-1}} =: \prod_{i=1}^{N} w_i(\boldsymbol{x}_i')$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. We now apply the previous lemma to the function

$$w(\boldsymbol{x}) := \prod_{i=1}^{N} w_i(\boldsymbol{x}_i'), \quad \boldsymbol{x} \in \mathbb{R}^N,$$

to obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^{\frac{N}{N-1}}\, d\boldsymbol{x} &\le \int_{\mathbb{R}^N} |w(\boldsymbol{x})|\, d\boldsymbol{x} \le \prod_{i=1}^{N} \|w_i\|_{L^{N-1}(\mathbb{R}^{N-1})} \\ &= \prod_{i=1}^{N} \left(\int_{\mathbb{R}^N} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right| d\boldsymbol{x} \right)^{\frac{1}{N-1}} \le \left(\int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|\, d\boldsymbol{x} \right)^{\frac{N}{N-1}},\end{aligned}$$

where we have used Fubini's theorem. This gives the desired inequality for $p = 1$.

Note that Step 1 continues to hold if we assume that $f \in L^q(\mathbb{R}^N)$ for some $q \ge 1$ and $\nabla f \in L^1(\mathbb{R}^N; \mathbb{R}^N)$.

Step 2: Assume next that $1 < p < N$ and that $f \in L^{p^*}\left(\mathbb{R}^N\right) \cap W^{1,p}(\mathbb{R}^N)$. Again by mollification we can assume that $f \in C^1\left(\mathbb{R}^N\right)$. Define

$$g := |f|^q, \quad q := \frac{p(N-1)}{N-p}.$$

Note that since $q > 1$, we have that $g \in C^1\left(\mathbb{R}^N\right)$. Moreover, $\nabla g \in L^p\left(\mathbb{R}^N; \mathbb{R}^N\right)$ (see below), while $g \in L^{1^*}\left(\mathbb{R}^N\right)$. Applying Step 1 to the function g we get

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} &= \left(\int_{\mathbb{R}^N} |g|^{\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} \\ &\le \int_{\mathbb{R}^N} \|\nabla g\|\, d\boldsymbol{x} \le q \int_{\mathbb{R}^N} |f|^{q-1} \|\nabla f\|\, d\boldsymbol{x} \\ &\le q \left(\int_{\mathbb{R}^N} |f|^{(q-1)p'}\, d\boldsymbol{x}\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Since

$$(q-1)\, p' = p^*,$$

if $f \neq 0$ we obtain

$$\left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N} - \frac{p-1}{p}} = \left(\int_{\mathbb{R}^N} |f|^{\frac{pN}{N-p}}\, d\boldsymbol{x}\right)^{\frac{N-p}{Np}} \le q \left(\int_{\mathbb{R}^N} \|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}},$$

which proves the result. Note that here it was important to know that $f \in L^{p^*}\left(\mathbb{R}^N\right)$, since we divided by $\left(\int_{\mathbb{R}^N} |f|^{(q-1)p'}\, d\boldsymbol{x}\right)^{\frac{1}{p'}}$.

Step 3: Assume that $f \in W^{1,p}\left(\mathbb{R}^N\right)$. For $n \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^N$ define

$$g_n(\boldsymbol{x}) := \begin{cases} |f(\boldsymbol{x})| - \frac{1}{n} & \text{if } \frac{1}{n} \le |f(\boldsymbol{x})| \le n, \\ 0 & \text{if } |f(\boldsymbol{x})| < \frac{1}{n}, \\ n - \frac{1}{n} & \text{if } |f(\boldsymbol{x})| > \frac{1}{n}. \end{cases}$$

By the chain rule (see Exercise 44 (i) and (vi)) for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$

$$\|\nabla g_n(\boldsymbol{x})\| = \begin{cases} \|\nabla f(\boldsymbol{x})\| & \text{if } \frac{1}{n} < |f(\boldsymbol{x})| < n, \\ 0 & \text{otherwise,} \end{cases}$$

and so $\nabla g_n \in L^p\left(\mathbb{R}^N; \mathbb{R}^N\right)$, while for every $s \ge 1$,

$$\begin{aligned}\int_{\mathbb{R}^N} |g_n|^s\, d\boldsymbol{x} &= \int_{\left\{|f| > \frac{1}{n}\right\}} |g_n|^s\, d\boldsymbol{x} \\ &\le \left(n - \frac{1}{n}\right)^s \mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : |f(\boldsymbol{x})| > \frac{1}{n}\right\}\right) < \infty,\end{aligned}$$

since $f \in L^p(\mathbb{R})$. Hence, $g_n \in L^{p^*}(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)$ and so by the previous step

$$\left(\int_{\left\{\frac{1}{n}\leq|f|\leq n\right\}}\left(|f(\boldsymbol{x})|-\frac{1}{n}\right)^{\frac{pN}{N-p}}d\boldsymbol{x}\right)^{\frac{N-p}{Np}} \leq \left(\int_{\mathbb{R}^N}|g_n|^{\frac{pN}{N-p}}\,d\boldsymbol{x}\right)^{\frac{N-p}{Np}}$$

$$\leq q\left(\int_{\mathbb{R}^N}\|\nabla g_n\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} = q\left(\int_{\left\{\frac{1}{n}\leq|f|\leq n\right\}}\|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}} \leq q\left(\int_{\mathbb{R}^N}\|\nabla f\|^p d\boldsymbol{x}\right)^{\frac{1}{p}}.$$

Letting first $n \to \infty$ and using Fatou's lemma we obtain the desired result. ■

Exercise 57 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp < N$. Prove that*

(i) $W^{k+j,p}(\mathbb{R}^N)$ is continuously embedded in $W^{j,q}(\mathbb{R}^N)$ for all $j \in \mathbb{N}$ and for all $p \leq q \leq \frac{Np}{N-kp}$,

(ii) $W^{k,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \leq q \leq \frac{Np}{N-kp}$.

Remark 58 *Note that in the last step of the proof of the previous theorem we only used the fact that f vanishes at infinity and its distributional gradient $\nabla f \in L^p(\mathbb{R}^N;\mathbb{R}^N)$. In particular, it holds if we assume that $f \in L^q(\mathbb{R}^N)$ for some $1 \leq q < \infty$ and the distributional gradiend $\nabla f \in L^p(\mathbb{R}^N;\mathbb{R}^N)$.*

Remark 59 *In view of Theorem 30 in Step 1 and 2 we could have assumed that $f \in C_c^1(\mathbb{R}^N)$ and so avoid Step 3. However, see the previous remark.*

Remark 60 *The previous theorem continues to hold in BV. To be precise, one can show that if $N \geq 2$ and $f \in BV(\mathbb{R}^N)$, then*

$$\left(\int_{\mathbb{R}^N}|f(\boldsymbol{x})|^{1^*}\,d\boldsymbol{x}\right)^{\frac{1}{1^*}} \leq C\|Df\|(\mathbb{R}^N),$$

where $C = C(N) > 0$.

Friday, February 11, 2022

Next we discuss the validity of the Sobolev–Gagliardo–Nirenberg embedding theorem for arbitrary domains.

Exercise 61 (Room and Passages) *Let $\{h_n\}$ and $\{\delta_{2n}\}$ be two sequences of positive numbers such that*

$$\sum_{n=1}^{\infty} h_n = \ell < \infty, \quad 0 < const. \leq \frac{h_{n+1}}{h_n} \leq 1, \quad 0 < \delta_{2n} \leq h_{2n+1},$$

and for $n \in \mathbb{N}$ let

$$c_n := \sum_{i=1}^{n} h_i.$$

Define $\Omega \subset \mathbb{R}^2$ to be the union of all sets of the form

$$R_j := (c_j - h_j, c_j) \times \left(-\frac{1}{2}h_j, \frac{1}{2}h_j\right),$$
$$P_{j+1} := [c_j, c_j + h_{j+1}] \times \left(-\frac{1}{2}\delta_{j+1}, \frac{1}{2}\delta_{j+1}\right),$$

for $j = 1, 3, 5, \ldots,$

(i) Prove that $\partial\Omega$ is a rectifiable curve but Ω is not of class C.

(ii) Let

$$h_n := \frac{1}{n^{\frac{3}{2}}}, \quad \delta_{2n} := \frac{1}{n^{\frac{5}{2}}},$$

and for $j = 1, 3, 5, \ldots,$

$$f(x, y) := \begin{cases} \dfrac{j}{\log 2j} =: K_j & \text{in } R_j, \\ K_j + (K_{j+2} - K_j) \dfrac{x - c_j}{h_{j+1}} & \text{in } P_{j+1}. \end{cases}$$

Prove that $f \in W^{1,2}(\Omega)$ but $f \notin L^q(\Omega)$ for any $q > 2$.

(iii) Let $p > 1$, $q \geq \frac{1}{2}(2p - 1)$,

$$h_{2n-1} = h_{2n} := \frac{1}{n^p}, \quad \delta_{2n} := \frac{1}{3^p n^{2q+p}},$$

and for $n \in \mathbb{N}$,

$$f(x, y) := \frac{1}{n^p} \text{ in } R_{2n-1},$$

and

$$\nabla f(x, y) := \left(\frac{(n+1)^q - n^q}{\frac{1}{n^p}}, 0\right) \text{ in } P_{2n}.$$

Prove that $\nabla f \in L^2(\Omega; \mathbb{R}^{2\times 2})$ but $f \notin L^2(\Omega)$.

Theorem 62 (Rellich-Kondrachov) *Let $1 \leq p < N$ and let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and a function $f \in L^{p^*}(\mathbb{R}^N)$ such that $f_{n_k} \to f$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for all $1 \leq q < p^*$. Moreover, $f \in W^{1,p}(\mathbb{R}^N)$ if $p > 1$.*

The proof makes use of the following auxiliary results.

Lemma 63 *Let $1 \leq p < \infty$ and let $f \in W^{1,p}(\mathbb{R}^N)$. Then for all $\boldsymbol{h} \in \mathbb{R}^N \setminus \{0\}$,*

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} \leq \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Proof. Exercise. ■

Lemma 64 *Let $1 \leq p < \infty$ and let $f \in W^{1,p}\left(\mathbb{R}^N\right)$. For $\varepsilon > 0$ consider standard mollifiers φ_ε. Then*

$$\int_{\mathbb{R}^N} |(f * \varphi_\varepsilon)(\boldsymbol{x}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} \leq C\varepsilon^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p \, d\boldsymbol{x}.$$

Proof. By Hölder's inequality and (5) we have

$$\begin{aligned} |(f * \varphi_\varepsilon)(\boldsymbol{x}) - f(\boldsymbol{x})|^p &\leq \int_{\mathbb{R}^N} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p \, d\boldsymbol{y} \\ &\leq \frac{C}{\varepsilon^N} \int_{B(0,\varepsilon)} |f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})|^p \, d\boldsymbol{h}. \end{aligned}$$

Hence, by Fubini's Theorem,

$$\int_{\mathbb{R}^N} |(f * \varphi_\varepsilon)(\boldsymbol{x}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} \leq \frac{C}{\varepsilon^N} \int_{B(0,\varepsilon)} \int_{\mathbb{R}^N} |f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} d\boldsymbol{h}. \tag{29}$$

In turn, by the previous lemma we get

$$\begin{aligned} \int_{\mathbb{R}^N} |(f * \varphi_\varepsilon)(\boldsymbol{x}) - f(\boldsymbol{x})|^p \, d\boldsymbol{x} &\leq \frac{C}{\varepsilon^N} \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p \, d\boldsymbol{x} \int_{B(0,\varepsilon)} \|\boldsymbol{h}\|^p \, d\boldsymbol{h} \\ &= C\varepsilon^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p \, d\boldsymbol{x}. \end{aligned}$$

■

We now turn to the proof of the Rellich-Kondrachov Theorem.

Proof of Theorem 62. Since the sequence $\{f_n\}_n$ is bounded in $W^{1,p}\left(\mathbb{R}^N\right)$, by the Sobolev–Gagliardo–Nirenberg embedding theorem, $\{f_n\}_n$ is bounded in $L^{p^*}\left(\mathbb{R}^N\right)$. Since $p^* > 1$, by the reflexivity of $L^{p^*}\left(\mathbb{R}^N\right)$ we may find a subsequence $\{f_{n_k}\}_k$ such that

$$f_{n_k} \rightharpoonup f \text{ in } L^{p^*}\left(\mathbb{R}^N\right).$$

We claim that $f_{n_k} \to f$ in $L^p(\Omega)$ for every open set $\Omega \subset \mathbb{R}^N$ with finite measure.

By the previous lemma and the fact that $\{f_{n_k}\}_k$ is bounded in $W^{1,p}\left(\mathbb{R}^N\right)$, we get

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{R}^N} |f_{n_k} * \varphi_\varepsilon - f_{n_k}|^p \, d\boldsymbol{x} \leq C\varepsilon^p \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^N} \|\nabla f_{n_k}\|^p \, d\boldsymbol{x} \leq M\varepsilon^p,$$

and so,

$$\lim_{\varepsilon \to 0^+} \sup_{k \in \mathbb{N}} \int_{\mathbb{R}^N} |f_{n_k} * \varphi_\varepsilon - f_{n_k}|^p \, d\boldsymbol{x} = 0. \tag{30}$$

By Minkowski's inequality

$$\|f_{n_k} - f\|_{L^p(\Omega)} \leq \|f_{n_k} * \varphi_\varepsilon - f_{n_k}\|_{L^p(\Omega)} + \|f_{n_k} * \varphi_\varepsilon - f * \varphi_\varepsilon\|_{L^p(\Omega)} + \|f * \varphi_\varepsilon - f\|_{L^p(\Omega)}.$$

Fix $\epsilon > 0$. By (30) and Theorem 13(iii) there exists $\bar{\varepsilon}$ depending only on ϵ such that for all $0 < \varepsilon \leq \bar{\varepsilon}$ and all $k \in \mathbb{N}$ the first and last term in the previous inequality are both bounded by ϵ, and so

$$\|f_{n_k} - f\|_{L^p(\Omega)} \leq \|f_{n_k} * \varphi_\varepsilon - f * \varphi_\varepsilon\|_{L^p(\Omega)} + 2\epsilon \tag{31}$$

for all $0 < \varepsilon \leq \bar{\varepsilon}$ and all $k \in \mathbb{N}$. Hence, to complete the proof it suffices to show that

$$\lim_{k\to\infty} \|f_{n_k} * \varphi_{\bar{\varepsilon}} - f * \varphi_{\bar{\varepsilon}}\|_{L^p(\Omega)} = 0. \tag{32}$$

Since $f_{n_k} \rightharpoonup f$ in $L^p\left(\mathbb{R}^N\right)$ it follows that for all $\boldsymbol{x} \in \mathbb{R}^N$

$$\begin{aligned}(f_{n_k} * \varphi_{\bar{\varepsilon}})(\boldsymbol{x}) &= \int_{\mathbb{R}^N} \varphi_{\bar{\varepsilon}}(\boldsymbol{x} - \boldsymbol{y})\, f_n(\boldsymbol{y})\, d\boldsymbol{y} \\ &\to \int_{\mathbb{R}^N} \varphi_{\bar{\varepsilon}}(\boldsymbol{x} - \boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} = (f * \varphi_{\bar{\varepsilon}})(\boldsymbol{x})\end{aligned}$$

as $k \to \infty$. Moreover, reasoning as in (29) and since $\{f_n\}_k$ is bounded in $L^p\left(\mathbb{R}^N\right)$, we get

$$\begin{aligned}|(f_{n_k} * \varphi_{\bar{\varepsilon}})(\boldsymbol{x}) - (f * \varphi_{\bar{\varepsilon}})(\boldsymbol{x})|^p &\leq \frac{c}{\bar{\varepsilon}^N} \int_{B(0,\varepsilon)} |f_{n_k}(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x} + \boldsymbol{h})|^p\, d\boldsymbol{h} \\ &\leq \frac{c}{\bar{\varepsilon}^N}\end{aligned}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$ and all $k \in \mathbb{N}$. Since Ω has finite measure, we are in a position to apply the Lebesgue dominated convergence theorem to conclude that (32) holds.

Hence, we have shown that $f_{n_k} \to f$ in $L^p(\Omega)$. Since $\{f_{n_k}\}_k$ is bounded in $L^{p^*}\left(\mathbb{R}^N\right)$, by Vitali's convergence theorem this implies that $f_{n_k} \to f$ in $L^q(\Omega)$ for all $1 \leq q < p^*$. ■

Remark 65 *Note that in the case $p > 1$ we do not need to use the Sobolev–Gagliardo–Nirenberg embedding theorem since $L^p\left(\mathbb{R}^N\right)$ is reflexive.*

Remark 66 *If $p = 1$, one can show that the function f in the previous theorem belongs to $BV(\mathbb{R}^N)$.*

The following exercises show that compactness fails for $q = p^*$ even for nice domains and that for general domains even the embedding

$$\begin{aligned}W^{1,p}(\Omega) &\to L^q(\Omega) \\ f &\mapsto f\end{aligned}$$

may fail to be compact.

Exercise 67 *Let* $1 \leq p < N$ *and consider the sequence of functions*

$$f_n(\boldsymbol{x}) := \begin{cases} n^{\frac{N-p}{p}}(1 - n\|\boldsymbol{x}\|) & \text{if } \|\boldsymbol{x}\| < \frac{1}{n}, \\ 0 & \text{if } \|\boldsymbol{x}\| \geq \frac{1}{n}. \end{cases}$$

Prove that the sequence $\{f_n\}_n$ *is bounded in* $W^{1,p}(B(0,1))$*, but does not admit any subsequence strongly convergent in* $L^{p^*}(\Omega)$*.*

Remark 68 *In the proof of the Rellich–Kondrachov compactness theorem, we used the fact that if* $1 \leq p < q < \infty$ *and* $\{f_n\}_n$ *is bounded in* $L^p(\mathbb{R}^N)$ *and* $f_n \rightharpoonup f$ *in* $L^q(\mathbb{R}^N)$*, then* $f \in L^p(\mathbb{R}^N)$*. To see this, consider a ball* B *and a function* $g \in L^{p'}(B)$*. Since* $p' > q'$*, we have that* $g \in L^{q'}(B)$*. Therefore,*

$$\int_B f_n g \, d\boldsymbol{x} \to \int_B f g \, d\boldsymbol{x}$$

as $n \to \infty$*. This shows that* $f_n \rightharpoonup f$ *in* $L^p(B)$*. By the lower semicontinuity of the norm with respect to weak convergence,*

$$\|f\|_{L^p(B)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(B)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(\mathbb{R}^N)} \leq M.$$

Taking $B = B(\mathbf{0}, j)$ *and letting* $j \to \infty$*, it follows from the Lebesgue monotone convergence theorem that*

$$\|f\|_{L^p(\mathbb{R}^N)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p(\mathbb{R}^N)} \leq M.$$

Monday, February 14, 2022

5 Embeddings: $p = N$

The argument at the beginning of the previous section shows that when $p \geq N$ we cannot expect an inequality of the form

$$\|f\|_{L^q(\mathbb{R}^N)} \leq c \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

However, we could still have embeddings of the type

$$\begin{aligned} W^{1,p}(\mathbb{R}^N) &\to L^q(\mathbb{R}^N) \\ f &\mapsto f \end{aligned}$$

that is, inequalities of the type

$$\|f\|_{L^q(\mathbb{R}^N)} \leq c \|f\|_{W^{1,p}(\mathbb{R}^N)}.$$

We now show that this is the case when $p = N$. We begin by observing that when $p \nearrow N$, then $p^* \nearrow \infty$, and so one would be tempted to say that if $f \in$

$W^{1,N}(\mathbb{R}^N)$, then $f \in L^\infty(\mathbb{R}^N)$. For $N = 1$ this is true since if $f \in W^{1,1}(\mathbb{R})$, then a representative $\overline{f}$ is absolutely continuous in $\mathbb{R}$ so that

$$\overline{f}(x) = \overline{f}(0) + \int_0^x \overline{f}'(s)\, ds$$

and since $\overline{f}' = f' \in L^1(\mathbb{R})$, we have that $\overline{f}$ is bounded and continuous. For $N > 1$ this is not the case, as the next exercise shows.

Exercise 69 *Let $\Omega = B(0,1) \subset \mathbb{R}^N$, $N > 1$, and show that the function*

$$f(\boldsymbol{x}) := \log\left(\log\left(1 + \frac{1}{\|\boldsymbol{x}\|}\right)\right), \quad \boldsymbol{x} \in B(0,1) \setminus \{0\},$$

belongs to $W^{1,N}(B(0,1))$ but not to $L^\infty(B(0,1))$.

However, we have the following result.

Theorem 70 *The space $W^{1,N}(\mathbb{R}^N)$ is continuously embedded in the space $L^q(\mathbb{R}^N)$ for all $N \le q < \infty$.*

Proof. Let $f \in W^{1,N}(\mathbb{R}^N)$. Define $g := |f|^t$, where $t > 1$ will be determined so that $g \in L^r(\mathbb{R}^N)$ and $\nabla g \in L^1(\mathbb{R}^N;\mathbb{R}^N)$. By the Sobolev–Gagliardo–Nirenberg embedding theorem with $p = 1$ and Remark 58,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{tN}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} &= \left(\int_{\mathbb{R}^N} |g|^{\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}} \\ &\le \int_{\mathbb{R}^N} \|\nabla g\|\, d\boldsymbol{x} \le t\int_{\mathbb{R}^N} |f|^{t-1} \|\nabla f\|\, d\boldsymbol{x} \\ &\le t\left(\int_{\mathbb{R}^N} |f|^{(t-1)N'}\, d\boldsymbol{x}\right)^{\frac{1}{N'}} \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}},\end{aligned}$$

where in the last inequality we have used Hölder's inequality. Hence,

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |f|^{\frac{tN}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{Nt}} &\le C\left(\int_{\mathbb{R}^N} |f|^{(t-1)\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{tN}} \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{Nt}} \\ &\le C\left[\left(\int_{\mathbb{R}^N} |f|^{(t-1)\frac{N}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N}\frac{1}{t-1}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right],\end{aligned} \tag{33}$$

where we have used Young's inequality $ab \le a^t + b^{t'}$ for $a, b \ge 0$. Taking $t = N$ yields

$$\left(\int_{\mathbb{R}^N} |f|^{\frac{N^2}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N^2}} \le C\left[\left(\int_{\mathbb{R}^N} |f|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right],$$

so that $f \in L^{\frac{N^2}{N-1}}\left(\mathbb{R}^N\right)$ with continuous embedding. In turn by Theorem **??**, we conclude that

$$\|f\|_{L^q(\mathbb{R}^N)} \leq C\, \|f\|_{W^{1,N}(\mathbb{R}^N)}$$

for all $N \leq q \leq \frac{N^2}{N-1}$.

Taking $t = N+1 \leq \frac{N^2}{N-1}$ in (33) and using what we just proved gives

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} |f|^{\frac{N(N+1)}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N(N+1)}} \\
&\qquad\qquad \leq C\left[\left(\int_{\mathbb{R}^N} |f|^{\frac{N^2}{N-1}}\, d\boldsymbol{x}\right)^{\frac{N-1}{N^2}} + \left(\int_{\mathbb{R}^N} \|\nabla f\|^N\, d\boldsymbol{x}\right)^{\frac{1}{N}}\right] \\
&\qquad\qquad \leq C\, \|f\|_{W^{1,N}(\mathbb{R}^N)},
\end{aligned}$$

and so the embedding

$$\begin{aligned}
W^{1,p}\left(\mathbb{R}^N\right) &\to L^q\left(\mathbb{R}^N\right) \\
f &\mapsto f
\end{aligned}$$

is continuous for all $N \leq q \leq \frac{N(N+1)}{N-1}$. We proceed in this fashion taking $t = N+2, N+3$, etc. ■

Exercise 71 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 2$ and $kp = N$. Prove that*

(i) $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $W^{j,q}\left(\mathbb{R}^N\right)$ for all $j \in \mathbb{N}$ and for all $p \leq q < \infty$,

(ii) $W^{k,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q < \infty$.

Exercise 72 *Prove that for every function $f \in W^{N,1}\left(\mathbb{R}^N\right)$,*

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq \left\|\frac{\partial^N f}{\partial x_1 \cdots \partial x_N}\right\|_{L^N(\mathbb{R}^N)}.$$

Theorem 73 (Rellich-Kondrachov) *Let $N \geq 2$ and let $\{f_n\}_n$ be a bounded sequence in $W^{1,N}\left(\mathbb{R}^N\right)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and a function $f \in W^{1,N}\left(\mathbb{R}^N\right)$ such that $f_{n_k} \to f$ in $L^q_{\text{loc}}\left(\mathbb{R}^N\right)$ for all $1 \leq q < \infty$.*

Proof. The proof is similar to the case $p < N$ with the only difference that in place of p^* we can consider any exponent $q > 1$. ■

Wednesday, February 16, 2022

6 Embeddings: $p > N$

We recall that, given an open set $\Omega \subseteq \mathbb{R}^N$, a function $f : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq C\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. We define the space $C^{0,\alpha}\left(\overline{\Omega}\right)$ as the space of all bounded functions that are Hölder continuous with exponent α.

Exercise 74 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\alpha > 0$.*

(i) Prove that if $\alpha > 1$ and Ω is connected, then any function that is Hölder continuous with exponent α is constant.

(ii) Prove that the space $C^{0,\alpha}\left(\overline{\Omega}\right)$, $0 < \alpha \leq 1$, is a Banach space with the norm

$$\|f\|_{C^{0,\alpha}\left(\overline{\Omega}\right)} := \sup_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})| + \sup_{\boldsymbol{x},\,\boldsymbol{y}\in\Omega,\,\boldsymbol{x}\neq\boldsymbol{y}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha}.$$

Note that if Ω is bounded, then every function $f : \Omega \to \mathbb{R}$ that is Hölder continuous with exponent $\alpha > 0$ is uniformly continuous and thus it can be uniquely extended to a bounded continuous function on $\mathbb{R}^N$. Thus, in the definition of $C^{0,\alpha}\left(\overline{\Omega}\right)$ one can drop the requirement that the functions are bounded.

The next theorem shows that if $p > N$ a function $f \in W^{1,p}\left(\mathbb{R}^N\right)$ has a representative in the space $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$.

Theorem 75 (Morrey) *Let $N < p < \infty$. Then the space $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$. Moreover, if $f \in W^{1,p}\left(\mathbb{R}^N\right)$ and $\bar{f}$ is its representative in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, then*

$$\lim_{\|\boldsymbol{x}\|\to\infty} \bar{f}(\boldsymbol{x}) = 0.$$

Proof. Let $f \in W^{1,p}\left(\mathbb{R}^N\right) \cap C^\infty\left(\mathbb{R}^N\right)$ and let Q_r be any cube with sides of length r parallel to the axes. Fix $\boldsymbol{x}, \boldsymbol{y} \in Q_r$ and let

$$g(t) := f(t\boldsymbol{x} + (1-t)\boldsymbol{y}), \quad 0 \leq t \leq 1.$$

By the fundamental theorem of calculus

$$\begin{aligned} f(\boldsymbol{x}) - f(\boldsymbol{y}) = g(1) - g(0) &= \int_0^1 g'(t)\, dt \\ &= \int_0^1 \nabla f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\, dt. \end{aligned}$$

Averaging in the $\boldsymbol{x}$ variable over Q_r yields

$$f_{Q_r} - f(\boldsymbol{y}) = \frac{1}{r^N} \int_{Q_r} \int_0^1 \nabla f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})\, dt\, d\boldsymbol{x},$$

where f_{Q_r} is the integral average of f over Q_r, that is,

$$f_{Q_r} := \frac{1}{r^N} \int_{Q_r} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

Hence,

$$\begin{aligned}
|f_{Q_r} - f(\boldsymbol{y})| &\le \sum_{i=1}^N \frac{1}{r^N} \int_{Q_r} \int_0^1 \left| \frac{\partial f}{\partial x_i}(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \right| |x_i - \boldsymbol{y}_i|\, dt\, d\boldsymbol{x} \\
&\le \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \int_{Q_r} \left| \frac{\partial f}{\partial x_i}(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \right| d\boldsymbol{x}\, dt \\
&= \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \frac{1}{t^N} \int_{(1-t)\boldsymbol{y} + Q_{rt}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{z}) \right| d\boldsymbol{z}\, dt,
\end{aligned}$$

where we have used the fact that $|x_i - y_i| \le r$ in Q_r, Tonelli's theorem, and the change of variables $\boldsymbol{z} = t\boldsymbol{x} + (1-t)\boldsymbol{y}$ (so that $d\boldsymbol{z} = t^N d\boldsymbol{x}$). By Hölder's inequality and the fact that $(1-t)\boldsymbol{y} + Q_{rt} \subset Q_r$, we now have

$$\begin{aligned}
|f_{Q_r} - f(\boldsymbol{y})| &\le \sum_{i=1}^N \frac{1}{r^{N-1}} \int_0^1 \frac{(rt)^{\frac{N}{p'}}}{t^N} \left(\int_{(1-t)\boldsymbol{y} + Q_{rt}} \left| \frac{\partial f}{\partial x_i}(\boldsymbol{z}) \right|^p d\boldsymbol{z} \right)^{\frac{1}{p}} dt \\
&\le N \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)} \frac{r^{N-\frac{N}{p}}}{r^{N-1}} \int_0^1 \frac{t^{N-\frac{N}{p}}}{t^N}\, dt \qquad (34) \\
&= \frac{Np}{p-N} r^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)}.
\end{aligned}$$

Since this is true for all $\boldsymbol{y} \in Q_r$, if $\boldsymbol{x}, \boldsymbol{y} \in Q_r$, then

$$\begin{aligned}
|f(\boldsymbol{x}) - f(\boldsymbol{y})| &\le |f(\boldsymbol{x}) - f_{Q_r}| + |f(\boldsymbol{y}) - f_{Q_r}| \\
&\le \frac{2Np}{p-N} r^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)}.
\end{aligned}$$

Now if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, consider a cube Q_r containing $\boldsymbol{x}$ and $\boldsymbol{y}$ and of side length $r := 2\|\boldsymbol{x} - \boldsymbol{y}\|$. Then the previous inequality yields

$$\begin{aligned}
|f(\boldsymbol{x}) - f(\boldsymbol{y})| &\le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(Q_r;\mathbb{R}^N)} \qquad (35) \\
&\le C\|\boldsymbol{x} - \boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.
\end{aligned}$$

Hence, f is Hölder continuous of exponent $1-\frac{N}{p}$. To prove that $f \in C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, it remains to show that f is bounded. Let $\boldsymbol{x} \in \mathbb{R}^N$ and consider a cube Q_1 containing $\boldsymbol{x}$ and of side length one. By (34) we get

$$\begin{aligned} |f(\boldsymbol{x})| \leq |f_{Q_1}| + |f(\boldsymbol{x}) - f_{Q_1}| &\leq \left| \int_{Q_1} f(\boldsymbol{x})\, d\boldsymbol{x} \right| + C \|\nabla f\|_{L^p(Q_1;\mathbb{R}^N)} \qquad (36) \\ &\leq \|f\|_{L^p(Q_1)} + C\|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^N)}, \end{aligned}$$

where we have used Hölder's inequality.

Next we remove the extra hypothesis that $f \in C^\infty\left(\mathbb{R}^N\right)$. Given any $f \in W^{1,p}\left(\mathbb{R}^N\right)$, let $\bar{f}$ be a representative of f and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ be two Lebesgue points of $\bar{f}$ and let $f_\varepsilon := f * \varphi_\varepsilon$, where φ_ε is a standard mollifier. By (35) we have that

$$|f_\varepsilon(\boldsymbol{x}) - f_\varepsilon(\boldsymbol{y})| \leq C\|\boldsymbol{x}-\boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f_\varepsilon\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)}.$$

Since $\{f_\varepsilon\}$ converge at every Lebesgue point by Theorem 13 and $\nabla f_\varepsilon = (\nabla f)_\varepsilon \to \nabla f$ in $L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)$ by Theorems 13, letting $\varepsilon \to 0^+$, we get

$$\left|\bar{f}(\boldsymbol{x}) - \bar{f}(\boldsymbol{y})\right| \leq C\|\boldsymbol{x}-\boldsymbol{y}\|^{1-\frac{N}{p}} \|\nabla f\|_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \qquad (37)$$

for all Lebesgue points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ of $\bar{f}$. This implies that

$$\bar{f} : \{\text{Lebesgue points of } f\} \to \mathbb{R}$$

can be uniquely extended to $\mathbb{R}^N$ as a Hölder continuous function $\bar{f}$ of exponent $1-\frac{N}{p}$ in such a way that (37) holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$.

With a similar argument from (36) we conclude that

$$\left|\bar{f}(\boldsymbol{x})\right| \leq C\|f\|_{W^{1,p}(\mathbb{R}^N)} \qquad (38)$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Hence,

$$\begin{aligned} \left\|\bar{f}\right\|_{C^{0,1-\frac{N}{p}}(\mathbb{R}^N)} &= \sup_{\boldsymbol{x}\in\mathbb{R}^N} \left|\bar{f}(\boldsymbol{x})\right| + \sup_{\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^N,\, \boldsymbol{x}\neq\boldsymbol{y}} \frac{\left|\bar{f}(\boldsymbol{x}) - \bar{f}(\boldsymbol{y})\right|}{\|\boldsymbol{x}-\boldsymbol{y}\|^{1-\frac{N}{p}}} \\ &\leq C\|f\|_{W^{1,p}(\mathbb{R}^N)}. \end{aligned}$$

Finally, we prove that $\bar{f}(\boldsymbol{x}) \to 0$ as $\|\boldsymbol{x}\| \to \infty$. Let $\{f_n\} \subset C_c^\infty\left(\mathbb{R}^N\right)$ be any sequence that converges to f in $W^{1,p}\left(\mathbb{R}^N\right)$. The inequality (38) implies, in particular, that $f \in L^\infty\left(\mathbb{R}^N\right)$, with

$$\|f\|_{L^\infty(\mathbb{R}^N)} \leq C\|f\|_{W^{1,p}(\mathbb{R}^N)}.$$

Replacing f with $f - f_n$ gives

$$\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \leq C\|f - f_n\|_{W^{1,p}(\mathbb{R}^N)},$$

and so $\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$ and find $\bar{n} \in \mathbb{N}$ such that

$$\|f - f_n\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon$$

for all $n \geq \bar{n}$. Since $f_{\bar{n}} \in C_c^\infty\left(\mathbb{R}^N\right)$, there exists $R_{\bar{n}} > 0$ such that $f_{\bar{n}}\left(\boldsymbol{x}\right) = 0$ for all $\|\boldsymbol{x}\| \geq R_{\bar{n}}$. Hence, for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$ with $\|\boldsymbol{x}\| \geq R_{\bar{n}}$ we get

$$\left|\bar{f}\left(\boldsymbol{x}\right)\right| = \left|\bar{f}\left(\boldsymbol{x}\right) - f_{\bar{n}}\left(\boldsymbol{x}\right)\right| \leq \|f - f_n\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon,$$

and, since $\bar{f}$ is continuous, we get that the previous inequality actually holds for all $\boldsymbol{x} \in \mathbb{R}^N$ with $\|\boldsymbol{x}\| \geq R_{\bar{n}}$. ■

Theorem 76 (Rellich-Kondrachov) *Let $p > N$ and let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}\left(\mathbb{R}^N\right)$. Then (up tp precise representatives) there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_k$ and a function $f \in W^{1,p}\left(\mathbb{R}^N\right)$ such that $f_{n_k} \to f$ in $C^{0,\alpha}\left(\overline{\Omega}\right)$ for all $0 < \alpha < 1 - \frac{N}{p}$ and for every bounded open set $\Omega \subset \mathbb{R}^N$.*

Proof. Exercise. ■

Friday, February 18, 2022

7 Extension Domains

We begin with the case in which Ω is the half space $\mathbb{R}^N_+$.

Theorem 77 *For all $1 \leq p \leq \infty$ there exists a continuous linear operator $\mathcal{E} : W^{1,p}(\mathbb{R}^N_+) \to W^{1,p}(\mathbb{R}^N)$ such that for all $f \in W^{1,p}(\mathbb{R}^N_+)$, $\mathcal{E}(f)(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N_+$ and*

$$\|\mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \leq 2\|f\|_{L^p(\mathbb{R}^N_+)}, \quad \left\|\frac{\partial \mathcal{E}(f)}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} \leq 2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(\mathbb{R}^N_+)},$$

for all $i = 1, \ldots, N$.

Proof. We only do the case $p < \infty$. Let $f \in C^1(\mathbb{R}^{N-1} \times [0, \infty))$ and define

$$g(\boldsymbol{x}) := \begin{cases} f(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ f(\boldsymbol{x}) & \text{if } x_N \geq 0, \end{cases}$$

The $g \in C(\mathbb{R}^N)$ and absolutely continuous on every line parallel to the axes with

$$\frac{\partial g}{\partial x_i}(\boldsymbol{x}) = \begin{cases} \frac{\partial f}{\partial x_i}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ \frac{\partial f}{\partial x_i}(\boldsymbol{x}) & \text{if } x_N > 0, \end{cases}$$

if $i = 1, \ldots, N-1$, while

$$\frac{\partial g}{\partial x_N}(\boldsymbol{x}) = \begin{cases} -\frac{\partial f}{\partial x_N}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \\ \frac{\partial f}{\partial x_N}(\boldsymbol{x}) & \text{if } x_N > 0. \end{cases}$$

It follows by the theorem on absolute continuity that $g \in W^{1,p}(\mathbb{R}^N)$. By a change of variables we have that $\|g\|_{L^p(\mathbb{R}^N)} = 2\|f\|_{L^p(\mathbb{R}^N_+)}$, $\left\|\frac{\partial g}{\partial x_i}\right\|_{L^p(\mathbb{R}^N)} =$

$2\left\|\frac{\partial f}{\partial x_i}\right\|_{L^p(\mathbb{R}^N_+)}$. Hence, the mapping $f \mapsto g$ is linear and continuous from $W^{1,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^{N-1} \times [0,\infty))$ to $W^{1,p}(\mathbb{R}^N)$. By Theorem 30 we can extend it uniquely to a bounded linear map $\mathcal{E} : W^{1,p}(\mathbb{R}^N_+) \to W^{1,p}(\mathbb{R}^N)$. ■

Note that $\frac{\partial g}{\partial x_N}$ is discontinuous at $x_N = 0$ and so we cannot use this extension for function $f \in W^{m,p}(\mathbb{R}^N_+)$ for $m \geq 2$.

Exercise 78 *Given $m \in \mathbb{N}$, and $1 \leq p \leq \infty$, let $f \in W^{m,p}(\mathbb{R}^N_+)$. Prove that there exist $c_1, \ldots, c_{m+1} \in \mathbb{R}$ such that the function*

$$g(\boldsymbol{x}) := \begin{cases} \sum_{n=1}^{m+1} c_n f(\boldsymbol{x}', -n x_N) & \text{if } x_N < 0, \\ f(\boldsymbol{x}) & \text{if } x_N > 0, \end{cases}$$

is well-defined and belongs to $W^{m,p}(\mathbb{R}^N)$. Prove also that for every $0 \leq k \leq m$, $\|\nabla^k g\|_{L^p(\mathbb{R}^N)} \leq c \|\nabla^k f\|_{L^p(\mathbb{R}^N_+)}$ for some constant $c = c(m,N,p) > 0$.

Next we consider the important special case in which Ω lies above the graph of a Lipschitz continuous function.

Theorem 79 *Let $h : \mathbb{R}^{N-1} \to \mathbb{R}$ be a Lipschitz continuous function and let*

$$\Omega := \{(\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > h(\boldsymbol{x}')\}. \tag{39}$$

Then for all $1 \leq p \leq \infty$ there exists a continuous linear operator $\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ such that for all $f \in W^{1,p}(\Omega)$, $\mathcal{E}(f)(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$ and

$$\|\mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \leq 2\|f\|_{L^p(\Omega)}, \quad \|\partial_N \mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \leq 2\|\partial_N f\|_{L^p(\Omega)}, \tag{40}$$

$$\|\partial_i \mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \leq 2\|\partial_i f\|_{L^p(\Omega)} + \operatorname{Lip} h \|\partial_N f\|_{L^p(\Omega)} \tag{41}$$

for all $i = 1, \ldots, N$.

Proof. The idea of the proof is to first flatten the boundary to reduce to the case in which $\Omega = \mathbb{R}^N_+$ and then use the previous theorem. We only prove the case $1 \leq p < \infty$ and leave the easier case $p = \infty$ as an exercise. Consider the transformation $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ given by $\Psi(\boldsymbol{y}) := (\boldsymbol{y}', y_N + h(\boldsymbol{y}'))$. Note that Ψ is invertible, with inverse given by $\Psi^{-1}(\boldsymbol{x}) = (\boldsymbol{x}', x_N - h(\boldsymbol{x}'))$. Moreover, for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^N$,

$$\begin{aligned} \|\Psi(\boldsymbol{y}) - \Psi(\boldsymbol{z})\| &= \|(\boldsymbol{y}' - \boldsymbol{z}', h(\boldsymbol{y}') - h(\boldsymbol{z}') + y_N - z_N)\| \\ &\leq \sqrt{\|\boldsymbol{y}' - \boldsymbol{z}'\|^2 + (\operatorname{Lip} h \, \|\boldsymbol{y}' - \boldsymbol{z}'\| + |y_N - z_N|)^2} \\ &\leq \operatorname{Lip} h \|\boldsymbol{y} - \boldsymbol{z}\|, \end{aligned}$$

which shows that Ψ (and similarly Ψ^{-1}) is Lipschitz continuous. Since h is Lipschitz continuous, by Rademacher's theorem it is differentiable for $\mathcal{L}^{N-1}$-a.e. $\boldsymbol{y}' \in \mathbb{R}^{N-1}$, and so for any such $\boldsymbol{y}' \in \mathbb{R}^{N-1}$ and for all $\boldsymbol{y}_N \in \mathbb{R}$ we have

$$J_\Psi(\boldsymbol{y}) = \begin{pmatrix} I_{N-1} & 0 \\ \nabla_{\boldsymbol{y}'} h(\boldsymbol{y}') & 1 \end{pmatrix},$$

which implies that $\det J_\Psi(\boldsymbol{y}) = 1$. Note that $\Psi(\mathbb{R}^N_+) = \Omega$.

Given a function $f \in W^{1,p}(\Omega)$, $1 \le p < \infty$, define the function

$$w(\boldsymbol{y}) := f(\Psi(\boldsymbol{y})) = f(\boldsymbol{y}', y_N + h(\boldsymbol{y}')), \quad \boldsymbol{y} \in \mathbb{R}^N_+.$$

By Exercise 36 the function w belongs to $W^{1,p}(\mathbb{R}^N_+)$ and the usual chain rule formula for the partial derivatives holds. By the previous theorem the function $\hat{w} : \mathbb{R}^N \to \mathbb{R}$, defined by

$$\hat{w}(\boldsymbol{y}) := \begin{cases} w(\boldsymbol{y}) & \text{if } y_N > 0, \\ w(\boldsymbol{y}', -y_N) & \text{if } y_N < 0, \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$ and the usual chain rule formula for the partial derivatives holds.

Define the function $v : \mathbb{R}^N \to \mathbb{R}$ by

$$v(\boldsymbol{x}) := (\hat{w} \circ \Psi^{-1})(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \text{if } x_N > h(\boldsymbol{x}'), \\ f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N) & \text{if } x_N < h(\boldsymbol{x}'). \end{cases} \tag{42}$$

Again by Exercise 36, we have that $v \in W^{1,p}(\mathbb{R}^N)$ and the usual chain rule formula for the partial derivatives holds.

By a change variables and the fact that $\det \nabla\Psi = \det \nabla\Psi^{-1} = 1$, we have that

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} |v(\boldsymbol{x})|^p d\boldsymbol{x} = \int_{\mathbb{R}^N \setminus \overline{\Omega}} |f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)|^p d\boldsymbol{x} = \int_\Omega |f(\boldsymbol{y})|^p d\boldsymbol{y}.$$

Since for all $i = 1, \ldots, N-1$ and for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}$,

$$\partial_i v(\boldsymbol{x}) = \partial_i f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N) + \partial_N f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)\partial_i h(\boldsymbol{x}'), \tag{43}$$

again by a change variables we have that

$$\begin{aligned}\left(\int_{\mathbb{R}^N \setminus \overline{\Omega}} |\partial_i v(\boldsymbol{x})|^p d\boldsymbol{x}\right)^{1/p} &\le \left(\int_{\mathbb{R}^N \setminus \overline{\Omega}} |\partial_i f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)|^p d\boldsymbol{x}\right)^{1/p} \\ &\quad + \operatorname{Lip} h\left(\int_{\mathbb{R}^N \setminus \overline{\Omega}} |\partial_N f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)|^p d\boldsymbol{x}\right)^{1/p} \\ &\le \left(\int_\Omega |\partial_i f(\boldsymbol{y})|^p d\boldsymbol{y}\right)^{1/p} + \operatorname{Lip} h\left(\int_\Omega |\partial_N f(\boldsymbol{y})|^p d\boldsymbol{y}\right)^{1/p}.\end{aligned}$$

Similarly, using the fact that $\partial_N v(\boldsymbol{x}) = -\partial_N f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N \setminus \overline{\Omega}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \overline{\Omega}} |\partial_N v(\boldsymbol{x})|^p d\boldsymbol{x} &= \int_{\mathbb{R}^N \setminus \overline{\Omega}} |\partial_N f(\boldsymbol{x}', 2h(\boldsymbol{x}') - x_N)|^p d\boldsymbol{x} \\ &= \int_\Omega |\partial_N f(\boldsymbol{y})|^p d\boldsymbol{y}.\end{aligned}$$

Hence, the linear extension operator $f \in W^{1,p}(\Omega) \mapsto \mathcal{E}(f) := v \in W^{1,p}(\mathbb{R}^N)$ is continuous and satisfies (40) and (41). ∎

Remark 80 *Note that the operator $\mathcal{E}$ defined in the previous theorem does not depend on p. However, it has the disadvantage that it cannot be used for higher-order Sobolev spaces since in (43) the derivatives of h appear, unless one assumes that h is more regular.*

Theorem 81 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded open set with $\partial\Omega$ Lipschitz continuous. Then for all $1 \le p \le \infty$ there exists a continuous linear operator $\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ such that for all $f \in W^{1,p}(\Omega)$, $\mathcal{E}(f)(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$ and*

$$\|\mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{L^p(\Omega)},$$
$$\|\nabla\mathcal{E}(f)\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}$$

for some constant $C = C(N, p, \Omega) > 0$.

Proof. This follows by using partition of unity. The details are in the book. ■

Corollary 82 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz continuous boundary and let $1 \le p < \infty$. Then*

(i) If $1 \le p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q \le p^*$,

(ii) If $p = N \ge 2$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$,

(iii) If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}\left(\overline{\Omega}\right)$ for all $0 < \alpha \le 1 - \frac{N}{p}$.

Proof. We only prove item (i). Given $f \in W^{1,p}(\Omega)$, by the previous theorem $\mathcal{E}(f) \in W^{1,p}(\mathbb{R}^N)$. Hence, by the Sobolev–Gagliardo–Nirenberg embedding theorem,

$$\|\mathcal{E}(f)\|_{L^{p^*}(\mathbb{R}^N)} \le C\|\mathcal{E}(f)\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

Since $\mathcal{E}(f)(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \Omega$, it follows that

$$\|f\|_{L^{p^*}(\Omega)} = \|\mathcal{E}(f)\|_{L^{p^*}(\Omega)} \le \|\mathcal{E}(f)\|_{L^{p^*}(\mathbb{R}^N)} \le C\|f\|_{W^{1,p}(\Omega)}.$$

■

Monday, February 21, 2022

8 Poincaré Inequalities

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $1 \le p \le \infty$. Poincaré's inequality is the following

$$\int_\Omega |f(\boldsymbol{x}) - f_E|^p \, d\boldsymbol{x} \le C \int_\Omega \|\nabla f\|^p \, d\boldsymbol{x},$$

where $E \subseteq \Omega$ is a measurable set of finite positive measure and

$$f_E := \frac{1}{|E|} \int_E f(\boldsymbol{x}) \, d\boldsymbol{x}. \tag{44}$$

Theorem 83 (Poincaré Inequality) *Let $1 \leq p < \infty$, $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz continuous boundary, and $E \subseteq \Omega$ be a measurable set with positive measure. Then there exists a constant $C = C(p, \Omega, E) > 0$ such that for all $f \in W^{1,p}(\Omega)$,*

$$\int_\Omega |f(\boldsymbol{x}) - f_E|^p \, d\boldsymbol{x} \leq C \int_\Omega \|\nabla f(\boldsymbol{x})\|^p \, d\boldsymbol{x}.$$

Proof. Assume by contradiction that the result is false. Then we may find a sequence $\{f_n\}_n$ in $W^{1,p}(\Omega)$ such that

$$\int_\Omega |f_n(\boldsymbol{x}) - (f_n)_E|^p \, d\boldsymbol{x} \geq n \int_\Omega \|\nabla f_n(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Define

$$g_n := \frac{f_n - (f_n)_E}{\|f_n - (f_n)_E\|_{L^p(\Omega)}}.$$

Then $g_n \in W^{1,p}(\Omega)$ with

$$\|g_n\|_{L^p(\Omega)} = 1, \quad (g_n)_E = 0, \quad \int_\Omega \|\nabla g_n\|^p \, d\boldsymbol{x} \leq \frac{1}{n}.$$

Extend g_n by reflection to a function $G_n \in W^{1,p}(\mathbb{R}^N)$ with $\|G_n\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|g_n\|_{W^{1,p}(\Omega)}$. Then $\{G_n\}_n$ is bounded in $W^{1,p}(\mathbb{R}^N)$. By the Rellich-Kondrachov theorems ($p < N$, $p = N$ and $p > N$) there exist a subsequence $\{G_{n_k}\}_k$ and a function $G \in L^p(\mathbb{R}^N)$ such that $G_{n_k} \to G$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Let g be the restriction of G to Ω. Since Ω is bounded, we have that $g_{n_k} \to g$ in $L^p(\Omega)$. It follows that

$$\|g\|_{L^p(\Omega)} = 1, \quad g_E = 0.$$

Moreover, for every $\psi \in C^1_c(\Omega)$ and $i = 1, \dots, N$, by Hölder's inequality

$$\begin{aligned}\left|\int_\Omega g \frac{\partial \psi}{\partial x_i} \, d\boldsymbol{x}\right| &= \lim_{k\to\infty} \left|\int_\Omega g_{n_k} \frac{\partial \psi}{\partial x_i} \, d\boldsymbol{x}\right| = \lim_{k\to\infty} \left|\int_\Omega \psi \frac{\partial g_{n_k}}{\partial x_i} \, d\boldsymbol{x}\right| \\ &\leq \lim_{k\to\infty} \left(\int_\Omega \|\nabla g_{n_k}\|^p \, d\boldsymbol{x}\right)^{\frac{1}{p}} \left(\int_\Omega |\psi|^{p'} \, d\boldsymbol{x}\right)^{\frac{1}{p'}} = 0\end{aligned}$$

and so $g \in W^{1,p}(\Omega)$ with $\nabla g = 0$. Since Ω is connected, this implies that g is constant (exercise), but since $g_E = 0$, then, necessarily, $g = 0$. This contradicts the fact that $\|g\|_{L^p(\Omega)} = 1$ and completes the proof. ■

Wednesday, February 23, 2022

9 The Trace Operator

Since Sobolev functions are L^p functions, they are equivalence classes of functions, and thus talking about their pointwise value does not make sense in general. One possibility would be to find a good representative. Indeed, in the supercritical case $p > N$ and if Ω is an open bounded set with Lipschitz continuous boundary, we can extend f to $W^{1,p}(\mathbb{R}^N)$ and then apply Morrey's theorem to conclude that f has a Hölder continuous representative $\bar{f}$. Thus, the value of f on the boundary of Ω is well-defined.

The situation is quite different in the subcritical and critical cases $p \leq N$ (unless $N = 1$). In this case we will prove that if $\partial\Omega$ is sufficiently regular, say, Lipschitz continuous, we can introduce a linear operator

$$\mathrm{Tr} : W^{1,p}(\Omega) \to L^p_{\mathrm{loc}}(\partial\Omega, \mathcal{H}^{N-1})$$

such that $\mathrm{Tr}(f) = f$ on $\partial\Omega$ for all $f \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and for which integration by parts holds, that is,

$$\int_\Omega f \partial_i \psi \, d\boldsymbol{x} = -\int_\Omega \psi \partial_i f \, d\boldsymbol{x} + \int_{\partial\Omega} \psi \, \mathrm{Tr}(f) \nu_i \, d\mathcal{H}^{N-1} \tag{45}$$

for all $f \in W^{1,p}(\Omega)$, $\psi \in C^1_c(\mathbb{R}^N)$, and $i = 1, \ldots, N$, where ν is the outward unit normal to $\partial\Omega$. We will study the continuity properties of this operator.

In what follows, we will use the abbreviations

$$\begin{aligned} L^p(\partial\Omega) &:= L^p(\partial\Omega, \mathcal{H}^{N-1}), \quad L^p_{\mathrm{loc}}(\partial\Omega) := L^p_{\mathrm{loc}}(\partial\Omega, \mathcal{H}^{N-1}), \\ \|\cdot\|_{L^p(\partial\Omega)} &:= \|\cdot\|_{L^p(\partial\Omega, \mathcal{H}^{N-1})}. \end{aligned} \tag{46}$$

In this section we establish the existence of a trace operator. As usual, we begin with the case $\Omega = \mathbb{R}^N_+$.

Theorem 84 *Let $1 \leq p < \infty$. There exists a unique linear operator*

$$\mathrm{Tr} : W^{1,p}(\mathbb{R}^N_+) \to L^p(\mathbb{R}^{N-1})$$

such that

(i) $\mathrm{Tr}(f)(\boldsymbol{x}') = f(\boldsymbol{x}', 0)$ for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and for all $f \in W^{1,p}(\mathbb{R}^N_+) \cap C(\mathbb{R}^{N-1} \times [0, \infty))$,

(ii) the integration by parts formula

$$\int_{\mathbb{R}^N_+} f(\boldsymbol{x}) \partial_i \psi(\boldsymbol{x}) \, d\boldsymbol{x} = -\int_{\mathbb{R}^N_+} \psi(\boldsymbol{x}) \partial_i f(\boldsymbol{x}) \, d\boldsymbol{x} + \int_{\mathbb{R}^{N-1}} \psi(\boldsymbol{x}', 0) \, \mathrm{Tr}(f)(\boldsymbol{x}') \delta_{i,N} \, d\boldsymbol{x}'$$

holds for all $f \in W^{1,p}(\mathbb{R}^N_+)$, all $\psi \in C^1_c(\mathbb{R}^N)$, and all $i = 1, \ldots, N$,

(iii) for every $0 < \varepsilon < 1$,

$$\int_{\mathbb{R}^{N-1}} |f(\boldsymbol{x}', 0)|^p d\boldsymbol{x}' \leq 2^{p-1} \varepsilon^{-1} \int_{\mathbb{R}^N_+} |f(\boldsymbol{x})|^p d\boldsymbol{x} + 2^{p-1} \varepsilon^{p-1} \int_{\mathbb{R}^N_+} |\partial_N f(\boldsymbol{x})|^p d\boldsymbol{x}. \tag{47}$$

The function $\mathrm{Tr}(f)$ is called the *trace* of f on $\partial\Omega$.

Proof. Step 1: Assume that $f \in W^{1,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^{N-1} \times [0,\infty))$. By the fundamental theorem of calculus, for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and $x_N > 0$, we can write

$$f(\boldsymbol{x}', 0) = f(\boldsymbol{x}', x_N) - \int_0^{x_N} \partial_N f(\boldsymbol{x}', s)\, ds.$$

Hence

$$|f(\boldsymbol{x}', 0)| \le |f(\boldsymbol{x}', x_N)| + \int_0^{x_N} |\partial_N f(\boldsymbol{x}', s)|\, ds.$$

Raising both sides to the power p and using the inequality $(a+b)^p \le 2^{p-1}a^p + 2^{p-1}b^p$ and and Hölder's inequalities (if $p > 1$) gives

$$\begin{aligned} |f(\boldsymbol{x}', 0)|^p &\le 2^{p-1}|f(\boldsymbol{x}', x_N)|^p + 2^{p-1}x_N^{p-1} \int_0^{x_N} |\partial_N f(\boldsymbol{x}', s)|^p ds \qquad (48) \\ &\le 2^{p-1}|f(\boldsymbol{x}', x_N)|^p + 2^{p-1}\varepsilon_N^{p-1} \int_0^{\varepsilon} |\partial_N f(\boldsymbol{x}', s)|^p ds \end{aligned}$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and $0 < x_N < \varepsilon \le 1$. Integrating in $\boldsymbol{x}'$ over $\mathbb{R}^{N-1}$ and in x_N over $(0, \varepsilon)$ gives

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |f(\boldsymbol{x}', 0)|^p d\boldsymbol{x}' \le\ & 2^{p-1}\varepsilon^{-1} \int_{\mathbb{R}^{N-1}} \int_0^{\varepsilon} |f(\boldsymbol{x}', x_N)|^p dx_N d\boldsymbol{x}' \\ &+ 2^{p-1}\varepsilon^{p-1} \int_{\mathbb{R}^{N-1}} \int_0^{\varepsilon} |\partial_N f(\boldsymbol{x}', s)|^p ds d\boldsymbol{x}'. \end{aligned}$$

This shows that (iii) holds for every $f \in W^{1,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^{N-1} \times [0,\infty))$.

Step 2: If now $f \in W^{1,p}(\mathbb{R}^N_+)$, reflect f to find a function $F \in W^{1,p}(\mathbb{R}^N)$ and consider a sequence $F_\varepsilon = \varphi_\varepsilon * F \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. Then $F_\varepsilon \to F$ in $W^{1,p}(\mathbb{R}^N)$ as $\varepsilon \to 0^+$. Let $\varepsilon_n \to 0^+$ and let f_n be the restriction of F_{ε_n} to $\mathbb{R}^N_+$. Then $f_n \to f$ in $W^{1,p}(\mathbb{R}^N_+)$. Applying (47) to $f_n - f_m$, we get

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |(f_n - f_m)(\boldsymbol{x}', 0)|^p d\boldsymbol{x}' \le 2^{p-1}\varepsilon^{-1} \int_{\mathbb{R}^N_+} |(f_n - f_m)(\boldsymbol{x})|^p d\boldsymbol{x} + 2^{p-1}\varepsilon^{p-1} \int_{\mathbb{R}^N_+} |\partial_N (f_n - f_m)(\boldsymbol{x})|^p d\boldsymbol{x} \\ \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Thus, $\{f_n(\cdot, 0)\}_n$ is a Cauchy sequence in $L^p(\mathbb{R}^{N-1})$ and thus it converges to a function $g \in L^p(\mathbb{R}^{N-1})$. Note that if we consider another sequence $\{g_n\}_n$ of functions in $W^{1,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^{N-1} \times [0,\infty))$ such that $g_n \to f$ in $W^{1,p}(\mathbb{R}^N_+)$, then by applying (47) to $f_n - g_n$, we get

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} |(f_n - g_n)(\boldsymbol{x}', 0)|^p d\boldsymbol{x}' \le 2^{p-1}\varepsilon^{-1} \int_{\mathbb{R}^N_+} |(f_n - g_n)(\boldsymbol{x})|^p d\boldsymbol{x} + 2^{p-1}\varepsilon^{p-1} \int_{\mathbb{R}^N_+} |\partial_N (f_n - g_n)(\boldsymbol{x})|^p d\boldsymbol{x} \\ \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $f_n(\cdot, 0) \to g$ in $L^p(\mathbb{R}^{N-1})$, it follows that $g_n(\cdot, 0) \to g$ in $L^p(\mathbb{R}^{N-1})$. This argument proves that the function g does not depend on the particular sequence of smooth functions that converges to f. We define $\operatorname{Tr}(f) := g$. Applying (47) to f_n, we get

$$\int_{\mathbb{R}^{N-1}} |f_n(\boldsymbol{x}',0)|^p d\boldsymbol{x}' \leq 2^{p-1}\varepsilon^{-1} \int_{\mathbb{R}^N_+} |f_n(\boldsymbol{x})|^p d\boldsymbol{x} + 2^{p-1}\varepsilon^{p-1} \int_{\mathbb{R}^N_+} |\partial_N f_n(\boldsymbol{x})|^p d\boldsymbol{x}.$$

Letting $n \to \infty$, we obtain (47).

Since (45) holds for each f_n and $\psi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N_+} f_n(\boldsymbol{x})\partial_i\psi(\boldsymbol{x})\, d\boldsymbol{x} = -\int_{\mathbb{R}^N_+} \psi(\boldsymbol{x})\partial_i f_n(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbb{R}^{N-1}} \psi(\boldsymbol{x}',0) f_n(\boldsymbol{x}',0)\delta_{i,N}\, d\boldsymbol{x}'.$$

Letting $n \to \infty$ and using the fact that $f_n \to f$ in $W^{1,p}(\mathbb{R}^N_+)$ and $f_n(\cdot,0) \to g$ in $L^p(\mathbb{R}^{N-1})$, we obtain

$$\int_{\mathbb{R}^N_+} f(\boldsymbol{x})\partial_i\psi(\boldsymbol{x})\, d\boldsymbol{x} = -\int_{\mathbb{R}^N_+} \psi(\boldsymbol{x})\partial_i f(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbb{R}^{N-1}} \psi(\boldsymbol{x}',0) g(\boldsymbol{x}')\delta_{i,N}\, d\boldsymbol{x}'.$$

■

Remark 85 *If we integrate (48) in $\boldsymbol{x}'$ over a cube Q' and in x_N over $(0,\varepsilon)$ we obtain*

$$\int_{Q'} |f(\boldsymbol{x}',0)|^p d\boldsymbol{x}' \leq 2^{p-1}\varepsilon^{-1} \int_{Q'}\int_0^\varepsilon |f(\boldsymbol{x}',x_N)|^p dx_N d\boldsymbol{x}' + 2^{p-1}\varepsilon^{p-1} \int_{Q'}\int_0^\varepsilon |\partial_N f(\boldsymbol{x}',s)|^p ds d\boldsymbol{x}'$$

for every $f \in W^{1,p}(\mathbb{R}^N_+) \cap C^1(\mathbb{R}^{N-1}\times[0,\infty))$. Reasoning as in Step 2, we obtain

$$\int_{Q'} |\operatorname{Tr}(f)(\boldsymbol{x}')|^p d\boldsymbol{x}' \leq 2^{p-1}\varepsilon^{-1} \int_{Q'\times(0,\varepsilon)} |f(\boldsymbol{x})|^p d\boldsymbol{x} + 2^{p-1}\varepsilon^{p-1} \int_{Q'\times(0,\varepsilon)} |\partial_N f(\boldsymbol{x})|^p d\boldsymbol{x}$$

for every $f \in W^{1,p}(\mathbb{R}^N_+)$.

An important corollary of the previous theorem is compactness of the trace operator for $p > 1$.

Corollary 86 (Compactness of Traces) *Let $1 < p < \infty$, and let $\{f_n\}_n$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N_+)$. Then there exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and a function $f \in W^{1,p}(\mathbb{R}^N_+)$ such that $f_{n_k} \to f$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N_+)$ and $\operatorname{Tr}(f_{n_k}) \to \operatorname{Tr}(f)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{N-1})$.*

Proof. Let $M > 0$ be such that $\|f_n\|_{W^{1,p}(\mathbb{R}^N_+)} \le M$ for every n. Extend f_n by reflection to a function $F_n \in W^{1,p}(\mathbb{R}^N)$ with $\|F_n\|_{W^{1,p}(\mathbb{R}^N)} \le C\|f_n\|_{W^{1,p}(\mathbb{R}^N_+)}$. Then $\{F_n\}_n$ is bounded in $W^{1,p}(\mathbb{R}^N)$. By the Rellich-Kondrachov theorems ($p < N$, $p = N$ and $p > N$) there exist a subsequence $\{F_{n_k}\}_k$ and a function $F \in W^{1,p}(\mathbb{R}^N)$ such that $F_{n_k} \to F$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Let f be the restriction of G to $\mathbb{R}^N_+$.

By the previous remark,

$$\begin{aligned}\int_{Q'} |\operatorname{Tr}(f - f_{n_k})(\boldsymbol{x}')|^p d\boldsymbol{x}' &\le 2^{p-1}\varepsilon^{-1} \int_{Q'\times(0,\varepsilon)} |(f - f_{n_k})(\boldsymbol{x})|^p d\boldsymbol{x} \\ &\quad + 2^{p-1}\varepsilon^{p-1} \int_{Q'\times(0,\varepsilon)} |\partial_N(f - f_{n_k})(\boldsymbol{x})|^p d\boldsymbol{x} \\ &\le 2^{p-1}\varepsilon^{-1} \int_{Q'\times(0,\varepsilon)} |(f - f_{n_k})(\boldsymbol{x})|^p d\boldsymbol{x} + 2^p\varepsilon^{p-1}M,\end{aligned}$$

where in the last inequality we used the fact that $\|f_n\|_{W^{1,p}(\mathbb{R}^N_+)} \le M$. Letting $k \to \infty$ and using the fact that $F_{n_k} \to F$ in $L^p(Q' \times (0,\varepsilon))$ gives

$$\limsup_{k\to\infty} \int_{Q'} |\operatorname{Tr}(f - f_{n_k})(\boldsymbol{x}')|^p d\boldsymbol{x}' \le 2^p\varepsilon^{p-1}M.$$

Letting $\varepsilon \to 0^+$ we have

$$\limsup_{k\to\infty} \int_{Q'} |\operatorname{Tr}(f - f_{n_k})(\boldsymbol{x}')|^p d\boldsymbol{x}' = 0.$$

Since Q' is an arbitrary cube in $\mathbb{R}^{N-1}$, we have shown that $\operatorname{Tr}(f_{n_k}) \to \operatorname{Tr}(f)$ in $L^p_{\text{loc}}(\mathbb{R}^{N-1})$. ■

Example 87 *The previous corollary fails for $p = 1$. Indeed, taking $f_n(x) = (1-nx)^+$ for $x \in (0,1)$, we have that $\operatorname{Tr}(f_n)(0) = f_n(0) = 1$. The sequence $\{f_n\}$ is bounded in $W^{1,1}((0,1))$ and converges to $f = 0$ in $L^1((0,1))$ but $\operatorname{Tr}(f_n)(0) = 1 \nrightarrow \operatorname{Tr}(f)(0) = 0$.*

Exercise 88 *Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 2$, be an open bounded set whose boundary $\partial\Omega$ is Lipschitz continuous. Prove that if $f_n \rightharpoonup f$ in $W^{1,1}(\Omega)$, then $\operatorname{Tr}(f_n)$ converges to $\operatorname{Tr}(f)$ in $L^1(\partial\Omega)$. Hint: Use equi-integrability.*

Exercise 89 *Let $N \ge 2$. Prove that for all functions $f \in W^{1,1}(\mathbb{R}^N_+)$,*

$$\|\operatorname{Tr}(f)(\cdot,0)\|_{L^1(\mathbb{R}^{N-1})} \le \|\partial_N f\|_{L^1(\mathbb{R}^N_+)}.$$

Friday, February 25, 2022

Next we prove that the operator Tr is onto. The following theorem is due to Gagliardo. The proof we present here is due to Mironescu.

Theorem 90 (Gagliardo) *Let $g \in L^1(\mathbb{R}^{N-1})$, $N \geq 2$. Then for every $0 < \varepsilon < 1$ there exists a function $f \in W^{1,1}(\mathbb{R}^N_+)$ such that $\operatorname{Tr}(f) = g$ and*

$$\|f\|_{L^1(\mathbb{R}^N_+)} \leq \varepsilon \|g\|_{L^1(\mathbb{R}^{N-1})}, \quad \|\nabla f\|_{L^1(\mathbb{R}^N_+)} \leq (1+\varepsilon)\|g\|_{L^1(\mathbb{R}^{N-1})}.$$

Proof. Step 1: Assume that $g \in C_c^\infty(\mathbb{R}^{N-1})$, with $g \neq 0$. Fix $\varepsilon > 0$ and let $\varphi \in C_c^\infty([0,\infty))$ be such that $\varphi(0) = 1$ and $\int_{\mathbb{R}^+} |\varphi'(t)|\, dt < 1+\varepsilon$. Note that the Lipschitz continuous function $\varphi_0(t) = (1-t)^+$, $t \geq 0$, satisfies $\int_{\mathbb{R}^+} |\varphi_0'(t)|\, dt = 1$. Hence, to obtain φ it suffices to regularize φ_0. For $n \in \mathbb{N}$ and $\boldsymbol{x} = (\boldsymbol{x}', x_N) \in \mathbb{R}^{N-1} \times [0,\infty)$ define $f_n(\boldsymbol{x}) := g(\boldsymbol{x}')\varphi(nx_N)$. Then $f_n \in C^\infty(\mathbb{R}^{N-1} \times [0,\infty))$, $f_n(\boldsymbol{x}', 0) = g(\boldsymbol{x}')$ for every $\boldsymbol{x}' \in \mathbb{R}^{N-1}$, while

$$\partial_i f_n(\boldsymbol{x}) = \partial_i g(\boldsymbol{x}')\varphi(nx_N), \quad \partial_N f_n(\boldsymbol{x}) = ng(\boldsymbol{x}')\varphi'(nx_N)$$

for $\boldsymbol{x} \in \mathbb{R}^N_+$, $i = 1, \ldots, N-1$. Moreover by Fubini's theorem and the change of variables $t = nx_N$,

$$\int_{\mathbb{R}^N_+} |f_n(\boldsymbol{x})|\, d\boldsymbol{x} = \frac{1}{n}\int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|\, d\boldsymbol{x}' \int_{\mathbb{R}^+} |\varphi(t)|\, dt \to 0$$

as $n \to \infty$. Similarly, for $i = 1, \ldots, N-1$,

$$\int_{\mathbb{R}^N_+} |\partial_i f_n(\boldsymbol{x})|\, d\boldsymbol{x} = \frac{1}{n}\int_{\mathbb{R}^{N-1}} |\partial_i g(\boldsymbol{x}')|\, d\boldsymbol{x}' \int_{\mathbb{R}^+} |\varphi'(t)|\, dt \to 0$$

as $n \to \infty$, while

$$\int_{\mathbb{R}^N_+} |\partial_N f_n(\boldsymbol{x})|\, d\boldsymbol{x} = \int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|\, d\boldsymbol{x}' \int_{\mathbb{R}^+} |\varphi'(t)|\, dt \leq (1+\varepsilon)\int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|\, d\boldsymbol{x}'.$$

Since $g \neq 0$, by taking n large enough we obtain that

$$\int_{\mathbb{R}^N_+} |f_n(\boldsymbol{x})|\, d\boldsymbol{x} \leq \varepsilon \int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|\, d\boldsymbol{x}', \quad \int_{\mathbb{R}^N_+} |\partial_i f_n(\boldsymbol{x})|\, d\boldsymbol{x} \leq \varepsilon \int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|\, d\boldsymbol{x}'$$

$i = 1, \ldots, N-1$, which gives the desired result in the case $g \in C_c^\infty(\mathbb{R}^{N-1})$.
Step 2: Let $g \in L^1(\mathbb{R}^{N-1})$. Find a sequence $g_n \in C_c^\infty(\mathbb{R}^{N-1})$ such that $g = \sum_{n=1}^\infty g_n$ and

$$\sum_{n=1}^\infty \|g_n\|_{L^1(\mathbb{R}^{N-1})} \leq (1+\varepsilon)\|g\|_{L^1(\mathbb{R}^{N-1})}.$$

By Step 1 there exists $f_n \in W^{1,1}(\mathbb{R}^N_+)$ such that $\operatorname{Tr}(f_n) = g_n$ and

$$\|f_n\|_{L^1(\mathbb{R}^N_+)} \leq \varepsilon\|g_n\|_{L^1(\mathbb{R}^{N-1})}, \quad \|\nabla f_n\|_{L^1(\mathbb{R}^N_+)} \leq (1+\varepsilon)\|g_n\|_{L^1(\mathbb{R}^{N-1})}.$$

Then

$$\sum_{n=1}^\infty \|f_n\|_{L^1(\mathbb{R}^N_+)} \leq \varepsilon(1+\varepsilon)\|g\|_{L^1(\mathbb{R}^{N-1})}, \quad \sum_{n=1}^\infty \|\nabla f_n\|_{L^1(\mathbb{R}^N_+)} \leq (1+\varepsilon)^2\|g\|_{L^1(\mathbb{R}^{N-1})}$$

Define

$$f := \sum_{n=1}^{\infty} f_n.$$

Then $f \in W^{1,1}(\mathbb{R}^N_+)$ (as in your homework). Moreover, since Tr is linear,

$$\operatorname{Tr}\left(\sum_{n=1}^{\ell} f_n\right) = \sum_{n=1}^{\ell} \operatorname{Tr}(f_n) = \sum_{n=1}^{\ell} g_n \to g$$

in $L^1(\mathbb{R}^{N-1})$ as $\ell \to \infty$. Since $\sum_{n=1}^{\ell} f_n \to f$ in $W^{1,1}(\mathbb{R}^N_+)$, it follows by the continuity of the trace operator, $\operatorname{Tr}(f) = g$. ■

Exercise 91 *Let X be a Banach space and let Y be a dense subspace. Prove that for every $x \in X$ and every $\varepsilon > 0$ there exists a sequence $\{y_n\}_n$ in Y such that*

$$x = \sum_{n=1}^{\infty} y_n$$

and

$$\sum_{n=1}^{\infty} \|y_n\|_X \leq (1+\varepsilon)\|x\|_X.$$

Remark 92 *Peetre has proved that there does not exist a bounded linear operator*

$$\begin{aligned} L :& L^1(\mathbb{R}^{N-1}) \to W^{1,1}(\mathbb{R}^N_+) \\ & g \mapsto L(g) \end{aligned}$$

with the property that $\operatorname{Tr}(L(g)) = g$.

In the the case $1 < p < \infty$ the operator $\operatorname{Tr} : W^{1,p}(\mathbb{R}^N_+) \to L^p(\mathbb{R}^{N-1})$ is not onto. We will show that

$$\operatorname{Tr}(W^{1,p}(\mathbb{R}^N_+)) = W^{1-1/p,p}(\mathbb{R}^{N-1}).$$

We begin by showing that

$$\operatorname{Tr}(W^{1,p}(\mathbb{R}^N_+)) \supseteq W^{1-1/p,p}(\mathbb{R}^{N-1}).$$

Theorem 93 (Gagliardo) *Let $N \geq 2$, $1 < p < \infty$, and $g \in W^{1-1/p,p}(\mathbb{R}^{N-1})$. Then there exists a function $f \in W^{1,p}(\mathbb{R}^N_+)$ such that $\operatorname{Tr}(f) = g$ and $\|f\|_{W^{1,p}(\mathbb{R}^N_+)} \leq C\|g\|_{W^{1-1/p,p}(\mathbb{R}^{N-1})}$ for some constant $C = C(N,p) > 0$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^{N-1})$ be such that $\operatorname{supp}\varphi \subseteq \overline{B_{N-1}(0,1)}$ and $\int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{x}')\,d\boldsymbol{x}' = 1$. For $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and $x_N > 0$ define

$$v(\boldsymbol{x}) := (\varphi_{x_N} * g)(\boldsymbol{x}') = \frac{1}{x_N^{N-1}} \int_{\mathbb{R}^{N-1}} \varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N) g(\boldsymbol{y}')\,d\boldsymbol{y}'. \tag{49}$$

By standard properties of mollifiers, where x_N plays the role of ε, for any $i = 1, \dots, N-1$ we have that

$$\begin{aligned}\frac{\partial v}{\partial x_i}(\boldsymbol{x}) &= \frac{1}{x_N^N}\int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial x_i}((\boldsymbol{x}'-\boldsymbol{y}')/x_N) g(\boldsymbol{y}')\, d\boldsymbol{y}' \\ &= \frac{1}{x_N^N}\int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial x_i}((\boldsymbol{x}'-\boldsymbol{y}')/x_N)[g(\boldsymbol{y}')-g(\boldsymbol{x}')]\, d\boldsymbol{y}',\end{aligned}$$

where in the second equality we used the fact that

$$\begin{aligned}0 = \frac{\partial}{\partial x_i}(1) &= \frac{\partial}{\partial x_i}\Big(\frac{1}{x_N^{N-1}}\int_{\mathbb{R}^{N-1}} \varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\, d\boldsymbol{y}'\Big) \\ &= \frac{1}{x_N^N}\int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial x_i}((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\, d\boldsymbol{y}'.\end{aligned}$$

Since $\operatorname{supp}\varphi \subseteq \overline{B_{N-1}(0,1)}$, we have that

$$|\partial_i v(\boldsymbol{x})| \le C\frac{1}{x_N^N}\int_{B_{N-1}(\boldsymbol{x}',x_N)} |g(\boldsymbol{y}')-g(\boldsymbol{x}')|\, d\boldsymbol{y}'. \tag{50}$$

■

Monday, February 28, 2022

Proof. Raising both sides to the power p, integrating in $\boldsymbol{x}$ over $\mathbb{R}^N_+$, and using Hölder's inequality, we get

$$\begin{aligned}&\int_{\mathbb{R}^N_+} |\partial_i v(\boldsymbol{x})|^p d\boldsymbol{x} \\ &\quad\le C\int_{\mathbb{R}^N_+} \frac{1}{x_N^{Np}}\Big(\int_{B_{N-1}(\boldsymbol{x}',x_N)} |g(\boldsymbol{y}')-g(\boldsymbol{x}')|\, d\boldsymbol{y}'\Big)^p d\boldsymbol{x} \\ &\quad\le C\int_{\mathbb{R}^N_+} \frac{x_N^{(N-1)(p-1)}}{x_N^{Np}}\int_{B_{N-1}(\boldsymbol{x}',x_N)} |g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p\, d\boldsymbol{y}' d\boldsymbol{x} =: C\mathcal{A}.\end{aligned}$$

By Tonelli's theorem, we get that

$$\begin{aligned}\mathcal{A} &= \int_0^\infty \int_{\mathbb{R}^{N-1}}\int_{B_{N-1}(\boldsymbol{x}',x_N)} \frac{1}{x_N^{p+N-1}}|g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p\, d\boldsymbol{y}' d\boldsymbol{x}' dx_N \\ &= \int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} |g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p \int_{\|\boldsymbol{x}'-\boldsymbol{y}'\|_{N-1}}^\infty \frac{1}{x_N^{p+N-1}}\, dx_N\, d\boldsymbol{y}' d\boldsymbol{x}' \\ &\le C\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p}{\|\boldsymbol{x}'-\boldsymbol{y}'\|_{N-1}^{p+N-2}}\, d\boldsymbol{y}' d\boldsymbol{x}'.\end{aligned}$$

Note that $N-1+sp = N-1+\left(1-\frac{1}{p}\right)p = N+p-2$

Hence, we have shown that

$$\int_{\mathbb{R}^N_+} |\partial_i v(\boldsymbol{x})|^p d\boldsymbol{x} \le C\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p}{\|\boldsymbol{x}'-\boldsymbol{y}'\|_{N-1}^{p+N-2}}\, d\boldsymbol{y}' d\boldsymbol{x}' \tag{51}$$

for all $i = 1, \ldots, N-1$.

Similarly, by differentiating under the integral sign (see [**?**]), we obtain that

$$\begin{aligned}\partial_N v(\boldsymbol{x}) &= \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial x_N}\Big(\frac{1}{x_N^{N-1}}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\Big) g(\boldsymbol{y}')\,d\boldsymbol{y}' \\ &= \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial x_N}\Big(\frac{1}{x_N^{N-1}}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\Big)[g(\boldsymbol{y}')-g(\boldsymbol{x}')]\,d\boldsymbol{y}',\end{aligned}$$

where in the second equality we used the fact that

$$\begin{aligned}0 = \frac{\partial}{\partial x_N}(1) &= \frac{\partial}{\partial x_N}\Big(\int_{\mathbb{R}^{N-1}} \frac{1}{x_N^{N-1}}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\,d\boldsymbol{y}'\Big) \\ &= \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial x_N}\Big(\frac{1}{x_N^{N-1}}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\Big)\,d\boldsymbol{y}'.\end{aligned}$$

Since $\operatorname{supp}\varphi \subseteq \overline{B_{N-1}(0,1)}$, we have that

$$\partial_N v(\boldsymbol{x}) = \int_{B_{N-1}(\boldsymbol{x}',x_N)} \frac{\partial}{\partial x_N}\Big(\frac{1}{x_N^{N-1}}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)\Big)[g(\boldsymbol{y}')-g(\boldsymbol{x}')]\,d\boldsymbol{y}'.$$

Now for $\boldsymbol{y}' \in B_{N-1}(\boldsymbol{x}', x_N)$,

$$\begin{aligned}|\partial_N(x_N^{1-N}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N))| &= \Big|-(N-1)x_N^{-N}\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N) \\ &\qquad - x_N^{1-N}\sum_{i=1}^{N-1}\partial_i\varphi((\boldsymbol{x}'-\boldsymbol{y}')/x_N)(x_i-y_i)x_N^{-2}\Big| \\ &\le C(x_N^{-N} + \|\boldsymbol{x}'-\boldsymbol{y}'\|_{N-1}x_N^{-1-N}) \le Cx_N^{-N}.\end{aligned}$$

In turn,

$$|\partial_N v(\boldsymbol{x})| \le C\frac{1}{x_N^N}\int_{B_{N-1}(\boldsymbol{x}',x_N)} |g(\boldsymbol{y}')-g(\boldsymbol{x}')|\,d\boldsymbol{y}'.$$

We can now continue as before (see (50)) to conclude that

$$\int_{\mathbb{R}^N_+} |\partial_N v(\boldsymbol{x})|^p d\boldsymbol{x} \le C\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{|g(\boldsymbol{y}')-g(\boldsymbol{x}')|^p}{\|\boldsymbol{x}'-\boldsymbol{y}'\|_{N-1}^{p+N-2}}\,d\boldsymbol{y}'d\boldsymbol{x}'. \tag{52}$$

Step 2: Since $v(\cdot, x_N) = (\varphi_{x_N} * g)(\cdot)$, by standard properties of mollifiers (see [**?**]), we have that for all $x_N > 0$,

$$\int_{\mathbb{R}^{N-1}} |v(\boldsymbol{x}', x_N)|^p d\boldsymbol{x}' \le \int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|^p d\boldsymbol{x}'. \tag{53}$$

Integrating in x_N in $(0, \varepsilon)$ gives

$$\int_0^\varepsilon \int_{\mathbb{R}^{N-1}} |v(\boldsymbol{x}', x_N)|^p d\boldsymbol{x}' dx_N \le C\varepsilon\int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|^p d\boldsymbol{x}'. \tag{54}$$

Let $\psi \in C^\infty([0,\infty))$ be a decreasing function such that $\psi = 1$ in $[0, \varepsilon/2]$, $\psi(x_N) = 0$ for $x_N \geq \varepsilon$ and $\|\psi'\|_\infty \leq C\varepsilon^{-1}$. For $x = (\boldsymbol{x}', x_N) \in \mathbb{R}^N_+$ we define $f(x) := \psi(x_N)v(x)$.

By (54), Tonelli's theorem, and the fact that $\psi(x_N) = 0$ for $x_N \geq \varepsilon$ we have

$$\begin{aligned}\int_{\mathbb{R}^N_+} |f(\boldsymbol{x})|^p d\boldsymbol{x} &= \int_0^\varepsilon (\psi(x_N))^p \int_{\mathbb{R}^{N-1}} |v(\boldsymbol{x})|^p d\boldsymbol{x}' dx_N \\ &\leq \int_0^\varepsilon \int_{\mathbb{R}^{N-1}} |v(\boldsymbol{x})|^p d\boldsymbol{x}' dx_N \\ &\leq C\varepsilon \int_{\mathbb{R}^{N-1}} |g(\boldsymbol{x}')|^p d\boldsymbol{x}',\end{aligned} \tag{55}$$

while for $i = 1, \dots, N-1$,

$$|\partial_i f(\boldsymbol{x})| = |\psi(x_N)\partial_i v(\boldsymbol{x})| \leq C|\partial_i v(\boldsymbol{x})|.$$

In turn, by (51), we obtain that

$$\int_{\mathbb{R}^N_+} |\partial_i f(\boldsymbol{x})|^p d\boldsymbol{x} \leq C\varepsilon^{-p} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|g(\boldsymbol{y}') - g(\boldsymbol{x}')|^p}{\|\boldsymbol{x}' - \boldsymbol{y}'\|_{N-1}^{p+N-2}} \, d\boldsymbol{y}' d\boldsymbol{x}'$$

for all $i = 1, \dots, N-1$. On the other hand,

$$\partial_N f(\boldsymbol{x}) = -\psi'(x_N)v(\boldsymbol{x}) + \psi(x_N)\partial_N v(\boldsymbol{x}),$$

and so, by (52), (54), and the facts that $\psi'(x_N) = 0$ for $x_N \geq \varepsilon$, and $\|\psi'\|_\infty \leq C\varepsilon^{-1}$,

$$\begin{aligned}&\int_{\mathbb{R}^N_+} |\partial_N f|^p d\boldsymbol{x} \\ &\quad \leq C\varepsilon^{-p} \int_0^\varepsilon \int_{\mathbb{R}^{N-1}} |v|^p d\boldsymbol{x}' dx_N + \int_{\mathbb{R}^N_+} |\partial_N v|^p d\boldsymbol{x} \\ &\quad \leq C\varepsilon^{-\sigma p} \int_{\mathbb{R}^{N-1}} |g|^p d\boldsymbol{x}' + \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|g(\boldsymbol{y}') - g(\boldsymbol{x}')|^p}{\|\boldsymbol{x}' - \boldsymbol{y}'\|_{N-1}^{p+N-2}} \, d\boldsymbol{y}' d\boldsymbol{x}'.\end{aligned}$$

Step 3: Since $v \in C^1(\mathbb{R}^N_+)$, we have shown that $v \in \dot{W}^{1,p}(\mathbb{R}^N_+)$. To conclude the proof, it remains to prove that $\mathrm{Tr}(v) = g$. Using standard mollifiers (exercise), we may find a sequence $\{g_n\}_n$ in $C^\infty(\mathbb{R}^{N-1})$ with $|g_n|_{W^{\sigma,p}(\mathbb{R}^{N-1})} < \infty$ such that $|g - g_n|_{W^{\sigma,p}(\mathbb{R}^{N-1})} \to 0$. Let v_n be defined as in (49) with g replaced by g_n. Then $v_n \in C(\mathbb{R}^{N-1} \times [0,\infty))$, with $v_n(\boldsymbol{x}', 0) = g_n(\boldsymbol{x}')$. By (52) applied to $v_n - v$ and $g_n - g$, we have that $\partial_i v_n \to \partial_i v$ in $L^p(\mathbb{R}^N_+)$ for every $i = 1, \dots, N$. In turn, by (**??**), we obtain that $\mathrm{Tr}(f) = g$.Reasoning as in the last part of the previous theorem we can show that $\mathrm{Tr}(f) = g$. ■

Wednesday, March 2, 2022

Theorem 94 (Minkowski's inequality for integrals) *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and σ-finite. Let $f : X \times Y \to [0, \infty]$ be an $(\mathfrak{M} \times \mathfrak{N})$-measurable function and let $1 \leq p \leq \infty$. Then*

$$\left\| \int_X |f(x, \cdot)| \, d\mu(x) \right\|_{L^p(Y,\mathfrak{N},\nu)} \leq \int_X \|f(x, \cdot)\|_{L^p(Y,\mathfrak{N},\nu)} \, d\mu(x).$$

Theorem 95 (Gagliardo) *Let $N \geq 2$ and $1 < p < \infty$. Then for all $f \in W^{1,p}(\mathbb{R}^N_+)$,*

$$|\operatorname{Tr}(f)|_{W^{1-1/p,p}(\mathbb{R}^{N-1})} \leq C \|\nabla f\|_{L^p(\mathbb{R}^N_+)} \tag{56}$$

for some constant $C = C(N, p) > 0$.

Proof. Let $f \in W^{1,p}(\mathbb{R}^N_+)$. By Theorem 42, f has a representative $\overline{f}$ that is absolutely continuous on $\mathcal{L}^{N-1}$-a.e. line segments of $\mathbb{R}^N_+$ that are parallel to the coordinate axes. Moreover the first-order (classical) partial derivatives of $\bar{f}$ agree $\mathcal{L}^N$-a.e. with the weak derivatives of f. Also, $\operatorname{Tr}(f)(\boldsymbol{x}') = \bar{f}(\boldsymbol{x}', 0)$ for $\mathcal{L}^{N-1}$-a.e. $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ (exercise). For $\boldsymbol{x}'$, $\boldsymbol{h}' \in \mathbb{R}^{N-1}$, set $r := \|\boldsymbol{h}'\|_{N-1}$. Then

$$\begin{aligned}
|\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', 0) - \bar{f}(\boldsymbol{x}', 0)| &\leq \left| \bar{f}(\boldsymbol{x}', 0) - \frac{1}{r} \int_0^r \bar{f}(\boldsymbol{x}', t) \, dt \right| \\
&\quad + \left| \bar{f}(\boldsymbol{x}' + \boldsymbol{h}', 0) - \frac{1}{r} \int_0^r \bar{f}(\boldsymbol{x}', t) \, dt \right| \\
&\leq \frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}', 0) - \bar{f}(\boldsymbol{x}', t)| \, dt + \frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t)| \, dt \\
&\quad + \frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', 0) - \bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t)| \, dt.
\end{aligned}$$

Hence, by the inequality $(a + b + c)^p \leq 3^{p-1} a^p + 3^{p-1} b^p + 3^{p-1} c^p$ and the change of variables $\boldsymbol{x}' + \boldsymbol{h}' = z'$,

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', 0) - \bar{f}(\boldsymbol{x}', 0)|^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \\
&\quad \leq 3^{p-1} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{\left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}', 0) - \bar{f}(\boldsymbol{x}', t)| \, dt \right)^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \qquad (57) \\
&\quad + 3^{p-1} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{\left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t)| \, dt \right)^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \\
&\quad + 3^{p-1} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{\left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', 0) - \bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t)| \, dt \right)^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \\
&\quad = 3^{p-1} 2 \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{\left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}', 0) - \bar{f}(\boldsymbol{x}', t)| \, dt \right)^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \\
&\quad + r^{p-1} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{\left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t)| \, dt \right)^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' =: 3^{p-1} 2 \mathcal{A} + 3^{p-1} \mathcal{B}.
\end{aligned}$$

Let $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ be such that $\bar{f}(\boldsymbol{x}', \cdot)$ is absolutely continuous in $(0, r)$. By the fundamental theorem of calculus

$$\frac{1}{r}\int_0^r |\bar{f}(\boldsymbol{x}', 0) - \bar{f}(\boldsymbol{x}', t)|\, dt \leq \frac{1}{r}\int_0^r \left|\int_0^t \partial_N f(\boldsymbol{x}', \rho)\, d\rho\right| dt \leq \int_0^r \|\nabla f(\boldsymbol{x}', \rho)\|\, d\rho.$$

Using Minkowski's inequality for integrals,

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \left(\frac{1}{r}\int_0^r |\bar{f}(\boldsymbol{x}', 0) - \bar{f}(\boldsymbol{x}', t)|\, dt\right)^p d\boldsymbol{x}' &\leq \int_{\mathbb{R}^{N-1}} \left(\int_0^r \|\nabla f(\boldsymbol{x}', \rho)\|\, d\rho\right)^p d\boldsymbol{x}' \\ &\leq \left(\int_0^r \|\nabla f(\cdot, \rho)\|_{L^p(\mathbb{R}^{N-1})}\, d\rho\right)^p.\end{aligned} \tag{58}$$

Fix $0 < \varepsilon < 1/p'$. By Hölder's inequality and the identity $1 = \rho^{-\varepsilon}\rho^{\varepsilon}$, the right-hand side of the previous inequality is bounded from above by

$$\begin{aligned}&\leq \left(\int_0^r \rho^{-\varepsilon p'} d\rho\right)^{p/p'} \int_0^r \rho^{\varepsilon p}\, \|\nabla f(\cdot, \rho)\|^p_{L^p(\mathbb{R}^{N-1})}\, d\rho \\ &= \frac{r^{p/p' - \varepsilon p}}{(1 - \varepsilon p')^{p-1}} \int_0^r \rho^{\varepsilon p}\, \|\nabla f(\cdot, \rho)\|^p_{L^p(\mathbb{R}^{N-1})}\, d\rho.\end{aligned}$$

Recalling that $r = \|\boldsymbol{h}'\|_{N-1}$, by Tonelli's theorem we have

$$\begin{aligned}\mathcal{A} &\leq C \int_{\mathbb{R}^{N-1}} \frac{1}{\|\boldsymbol{h}'\|^{p+N-2-p+1+\varepsilon p}_{N-1}} \int_0^{\|\boldsymbol{h}'\|_{N-1}} \rho^{\varepsilon p} \int_{\mathbb{R}^{N-1}} \|\nabla f(\boldsymbol{x}', \rho)\|^p d\boldsymbol{x}' d\rho d\boldsymbol{h}' \\ &\leq C \int_{\mathbb{R}^{N-1}} \int_0^\infty \rho^{\varepsilon p} \|\nabla f(\boldsymbol{x}', \rho)\|^p \left(\int_{\mathbb{R}^{N-1} \setminus B_{N-1}(0,\rho)} \frac{1}{\|\boldsymbol{h}'\|^{N-1+\varepsilon p}_{N-1}} d\boldsymbol{h}'\right) d\rho d\boldsymbol{x}' \\ &\leq C \int_{\mathbb{R}^{N-1}} \int_0^\infty \|\nabla f(\boldsymbol{x}', \rho)\|^p d\rho d\boldsymbol{x}',\end{aligned} \tag{59}$$

where we used the facts that $0 < \varepsilon$ and

$$\int_{\mathbb{R}^{N-1} \setminus B_{N-1}(0,\rho)} \frac{1}{\|\boldsymbol{h}'\|^{N-1+\varepsilon p}_{N-1}} d\boldsymbol{h}' = \frac{\alpha_{N-1}}{\varepsilon p} \frac{1}{\rho^{\varepsilon p}}.$$

To estimate $\mathcal{B}$, define

$$\mathbf{X}'_0 := \boldsymbol{x}', \quad \mathbf{X}'_n := \mathbf{X}'_{n-1} + h_n \boldsymbol{e}_n, \quad n = 1, \ldots, N-1.$$

Then

$$\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t) = \sum_{n=1}^{N-1} \bar{f}(\mathbf{X}'_n, t) - f(\mathbf{X}'_{n-1}, t).$$

By fixing $\boldsymbol{x}'$, $\boldsymbol{h}'$, and t in such a way that $\bar{f}$ is absolutely continuous along the segments $(\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1}, t)$, $\eta \in [0,1]$ for every $n = 1, \ldots, N-1$, by the fundamental theorem of calculus we have

$$\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t) = \sum_{n=1}^{N-1} \int_0^1 \partial_n f(\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1}, t) h_n \, d\eta.$$

Hence,

$$\frac{1}{r}\int_0^r |\bar{f}(\boldsymbol{x}'+\boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t)| \, dt \leq \sum_{n=1}^{N-1} \int_0^r \int_0^1 |\partial_n f(\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1}, t)| \, d\eta dt.$$

Using Minkowski's inequality for integrals and the change of variables $\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1} = z'$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \left(\frac{1}{r} \int_0^r |\bar{f}(\boldsymbol{x}' + \boldsymbol{h}', t) - \bar{f}(\boldsymbol{x}', t)| \, dt \right)^p d\boldsymbol{x}' \\
&\quad \leq C \sum_{n=1}^{N-1} \int_{\mathbb{R}^{N-1}} \left(\int_0^r \int_0^1 |\partial_n f(\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1}, t)| \, d\eta dt \right)^p d\boldsymbol{x}' \\
&\quad \leq C \sum_{n=1}^{N-1} \left(\int_0^r \int_0^1 \left(\int_{\mathbb{R}^{N-1}} |\partial_n f(\eta \boldsymbol{X}'_{n-1} + (1-\eta)\boldsymbol{X}'_{n-1}, t)|^p d\boldsymbol{x}' \right)^{1/p} d\eta dt \right)^p \\
&\quad \leq C \left(\int_0^r \|\nabla f(\cdot, t)\|_{L^p(\mathbb{R}^{N-1})} \, dt \right)^p.
\end{aligned}$$

Since the right-hand side is the same as the right-hand side in (58) we can now continue as in the estimate (59) of $\mathcal{A}$ to obtain

$$\mathcal{B} \leq C \int_{\mathbb{R}^{N-1}} \int_0^\infty \|\nabla f(\boldsymbol{x}', \rho)\|^p d\rho d\boldsymbol{x}'.$$

Combining this inequality with (57) and (59) and using the fact that $\operatorname{Tr}(f)(\boldsymbol{x}') = \bar{f}(\boldsymbol{x}', 0)$ for $\mathcal{L}^{N-1}$-a.e. $\boldsymbol{x}' \in \mathbb{R}^{N-1}$, we have

$$\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|\operatorname{Tr}(f)(\boldsymbol{x}' + \boldsymbol{h}') - \operatorname{Tr}(f)(\boldsymbol{x}')|^p}{\|\boldsymbol{h}'\|_{N-1}^{p+N-2}} d\boldsymbol{x}' d\boldsymbol{h}' \leq C \int_{\mathbb{R}^N_+} \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

■

Monday, March 14, 2022

10 Interpolation Spaces

Definition 96 *A* vector space *over* $\mathbb{R}$ *is a nonempty set* X*, whose elements are called* vectors, *together with two operations,* addition *and* multiplication by scalars*,*

$$\begin{array}{ccc} X \times X \to X & \textit{and} & \mathbb{R} \times X \to X \\ (x, y) \mapsto x + y & & (t, x) \mapsto tx \end{array}$$

with the properties that

(i) $(X, +)$ *is a commutative group, that is,*

(a) $x + y = y + x$ *for all* $x, y \in X$ *(commutative property),*

(b) $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in X$ *(associative property),*

(c) there is a vector $0 \in X$*, called* zero*, such that* $x + 0 = 0 + x$ *for all* $x \in X$*,*

(d) for every $x \in X$ *there exists a vector in* X*, called the* opposite *of* x *and denoted* $-x$*, such that* $x + (-x) = 0$*,*

(ii) for all $x, y \in X$ *and* $s, t \in \mathbb{R}$*,*

(a) $s\,(tx) = (st)\,x$*,*

(b) $1x = x$*,*

(c) $s\,(x + y) = (sx) + (sy)$*,*

(d) $(s + t)x = (sx) + (tx)$*.*

A topological space (X, τ) is a *Hausdorff space* if for any $x, y \in X$ with $x \neq y$ we may find two disjoint open sets U and V containing x and y, respectively.

Definition 97 *Given a topological space* (X, τ)*, we say that a sequence* $\{x_n\}_n$ *in* X converges *to some* $x \in X$ *if for every open set* $U \in \tau$ *with* $x \in U$ *there exists* $n_U \in \mathbb{N}$ *such that* $x_n \in U$ *for all* $n \in \mathbb{N}$ *with* $n \geq n_U$*. We write* $x_n \to x$ *or*

$$\lim_{n \to \infty} x_n = x.$$

Remark 98 *If* (X, τ) *is a Hausdorff space and* $x_n \to x$ *and* $x_n \to y$*, then* $x = y$*.*

Definition 99 *Given two topological vector spaces* (X, τ_X) *and* (Y, τ_Y)*, we say that* X *is* embedded *in* Y *and we write*

$$X \hookrightarrow Y$$

if X *is a subspace of* Y *and the* immersion

$$\begin{aligned} i : (X, \tau_X) &\to (Y, \tau_Y) \\ x &\mapsto x \end{aligned}$$

is continuous, that is, if $i^{-1}(V) \in \tau_X$ *for every* $V \in \tau_Y$ *in* Y*.*

Remark 100 *In particular, if* $X \hookrightarrow Y$ *and if* $x_n \to x$ *in* X*, then* $x_n \to x$ *in* Y*. Indeed, for every open set* V *in* Y *containing* $i(x) = x$ *we have that* $U := i^{-1}(V)$ *is open in* X *and contains* x*. Since* $x_n \to x$*, there exists* n_U *such that* $x_n \in U$ *for all* $n \geq n_U$ *and so* $i(x_n) = x_n \in V$ *for all* $n \geq n_U$*.*

We are given two normed spaces, X_0 and X_1, with $X_0 \supseteq X_1$, (for example $C([0,1])$ and $C^1([0,1])$ or $L^1([0,1])$ and $L^\infty([0,1])$) we want to construct a family of intermediate spaces $X_0 \supseteq X_s \supseteq X_1$, $0 < s < 1$, with the property that

$$\|x\|_{X_s} \le C\|x\|_{X_0}^\theta \|x\|_{X_1}^{1-\theta}$$

for all $x \in X_0 \cap X_1$, where the constants $C > 0$ and $\theta \in (0,1)$ depends on s.

Definition 101 *We say that two normed spaces $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ are an* admissible pair *if they are embedded into a common Hausdorff topological vector space X.*

Theorem 102 *Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ be an admissible pair. Then the vector space $X_0 \cap X_1$ endowed with the norm*

$$\|x\|_{X_0 \cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\} \tag{60}$$

is a normed space. Moreover, if X_0 and X_1 are Banach spaces, then so is $X_0 \cap X_1$.

Proof. Exercise. ■

Theorem 103 *Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ be an admissible pair. Then the vector space*

$$X_0 + X_1 := \{x \in X : x = x_0 + x_1,\, x_0 \in X_0, x_1 \in X_1\}$$

is also a normed space when endowed with the norm

$$\|x\|_{X_0+X_1} := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1}\}, \tag{61}$$

where the infimum is taken over all possible decompositions $x = x_0 + x_1$, $x_0 \in X_0$, $x_1 \in X_1$. Moreover, if X_0 and X_1 are Banach spaces, then so is $X_0 + X_1$.

Proof. Step 1: If $\|x\|_{X_0+X_1} = 0$, then for every n there exist $x_0^n \in X_0$ and $x_1^n \in X_1$ such that $x = x_0^n + x_1^n$ and

$$\|x_0^n\|_{X_0} + \|x_1^n\|_{X_1} \le \|x\|_{X_0+X_1} + \frac{1}{n} = 0 + \frac{1}{n}.$$

It follows that $x_0^n \to 0$ in X_0 and $x_1^n \to 0$ in X_1. In turn, $x_0^n \to 0$ in X and $x_1^n \to 0$ in X by Remark, and so $x_0^n + x_1^n \to 0$ in X, but since $x = x_0^n + x_1^n$, then $x = 0$. Conversely, if $x = 0$, then $\|0\|_{X_0+X_1} = 0$.

If $x \in X_0 + X_1$ and $t \in \mathbb{R}$, let $x_0 \in X_0$, $x_1 \in X_1$ be such that $tx = x_0 + x_1$. Assuming $t \neq 0$, we have that $x = \frac{1}{t}x_0 + \frac{1}{t}x_1$, and $\frac{1}{t}x_0 \in X_0$, $\frac{1}{t}x_1 \in X_2$. Hence,

$$\begin{aligned} \|x\|_{X_0+X_1} &\le \|t^{-1}x_0\|_{X_0} + \|t^{-1}x_1\|_{X_1} \\ &= |t^{-1}|(\|x_0\|_{X_0} + \|x_1\|_{X_1}). \end{aligned}$$

Taking the infimum over all such decompositions gives

$$\|x\|_{X_0+X_1} \leq |t^{-1}| \|tx\|_{X_0+X_1},$$

or, equivalently, $|t|\|x\|_{X_0+X_1} \leq \|tx\|_{X_0+X_1}$. Since this inequality holds for all $x \in X_0 + X_1$ and all $t \in \mathbb{R}$, by applying it to tx and with t^{-1} in place of t we get $|t^{-1}|\|tx\|_{X_0+X_1} \leq \|t^{-1}(tx)\|_{X_0+X_1} = \|x\|_{X_0+X_1}$, that is, $\|tx\|_{X_0+X_1} \leq |t|\|x\|_{X_0+X_1}$.

Finally, given $x, y \in X_0+X_1$, if $x = x_0+x_1$ and $y = y_0+y_1$, with $x_0, y_0 \in X_0$ and $x_1, y_1 \in X_1$, then $x + y = (x_0 + y_0) + (x_1 + y_1)$ with $x_0 + y_0 \in X_0$ and $x_1 + y_1 \in X_1$. Hence,

$$\begin{aligned}\|x + y\|_{X_0+X_1} &\leq \|x_0 + y_0\|_{X_0} + \|x_1 + y_1\|_{X_1} \\ &\leq \|x_0\|_{X_0} + \|x_1\|_{X_1} + \|y_0\|_{X_0} + \|y_1\|_{X_1}.\end{aligned}$$

Taking the infimum over all decompositions of x gives

$$\|x + y\|_{X_0+X_1} \leq \|x\|_{X_0+X_1} + \|y_0\|_{X_0} + \|y_1\|_{X_1}$$

for all decompositions of y. Taking the infimum over all decompositions of y gives

$$\|x + y\|_{X_0+X_1} \leq \|x\|_{X_0+X_1} + \|y\|_{X_0+X_1}.$$

This shows that $\|\cdot\|_{X_0+X_1}$ is a norm.

Step 2: Assume that X_0 and X_1 are Banach spaces. To prove that $X_0 + X_1$ is a Banach space we use Theorem **??**. Let $\{x_n\}_n$ be a sequence in $X_0 + X_1$ such that $\sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1}$ converges in $\mathbb{R}$. For each n find $x_0^n \in X_0$ and $x_1^n \in X_1$ such that $x_n = x_0^n + x_1^n$ and

$$\|x_0^n\|_{X_0} + \|x_1^n\|_{X_1} \leq \|x_n\|_{X_0+X_1} + \frac{1}{2^n}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} \left\|x_n^0\right\|_{X_0} &\leq \sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1} + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty, \\ \sum_{n=1}^{\infty} \left\|x_n^1\right\|_{X_1} &\leq \sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1} + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.\end{aligned}$$

Since X_0 and X_1 are Banach spaces, by Theorem **??** there exist $y_0 \in X_0$ and $y_1 \in X_1$ such that $\sum_{i=1}^{n} x_0^i \to y_0$ in X_0 and $\sum_{i=1}^{n} x_1^i \to y_1$ in X_1. Since $y := y_0 + y_1 \in X_0 + X_1$, we have

$$\left\|y - \sum_{i=1}^{n} x_i\right\|_{X_0+X_1} \leq \left\|y_0 - \sum_{i=1}^{n} x_0^i\right\|_{X_0} + \left\|y_1 - \sum_{i=1}^{n} x_1^i\right\|_{X_1} \to 0$$

as $n \to \infty$. It follows that $\sum_{i=1}^{n} x_i \to y$ in $X_0 + X_1$, and so the series $\sum_{n=1}^{\infty} x_n$ converges in $X_0 + X_1$. By Theorem **??** this implies that $X_0 + X_1$ is a Banach space. ■

Given $t > 0$, in $X_0 + X_1$ we can also consider the equivalent norm

$$x \in X_0 + X_1 \mapsto K(x,t) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\}, \tag{62}$$

where as before the infimum is taken over all possible decompositions $x = x_0 + x_1$, $x_0 \in X_0$, $x_1 \in X_1$. To highlight the dependence of K on X_0 and X_1, when needed, we write

$$K(x,t;X_0,X_1) := K(x,t). \tag{63}$$

Remark 104 *The function $K(\cdot,t)$ can be extended to $X \setminus (X_0 + X_1)$ by setting $K(x,t) := \infty$ if $x \in X \setminus (X_0 + X_1)$, where X is the Hausdorff topological vector space X in Definition 101.*

Proposition 105 *Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ be an admissible pair. Then for every $x \in X_0 + X_1$, the function $t \mapsto K(x,t)$ is an increasing, concave function and such that $t^{-1}K(x,t;X_0,X_1) = K(x,t^{-1};X_1,X_0)$ and*

$$K(x,t) \leq \max\{1, t/\tau\}K(x,\tau)$$

for every $t > 0$ and $\tau > 0$.

Proof. Exercise. ■

For $0 < s < 1$ and $1 \leq q \leq \infty$ we can define the *real interpolation space*

$$(X_0, X_1)_{s,q} := \{x \in X_0 + X_1 : \|x\|_{s,q} < \infty\},$$

where if $1 \leq q < \infty$,

$$\|x\|_{s,q} := \left(\int_0^\infty (K(x,t))^q \frac{dt}{t^{1+sq}}\right)^{1/q}, \tag{64}$$

while if $q = \infty$,

$$\|x\|_{s,\infty} := \sup_{t>0} t^{-s}K(x,t). \tag{65}$$

Theorem 106 *Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ be an admissible pair and let $1 \leq q \leq \infty$ and $0 < s < 1$. Then $\|\cdot\|_{s,q}$ is a norm and the following embeddings hold*

$$X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{s,q} \hookrightarrow X_0 + X_1.$$

Moreover, if X_0 and X_1 are Banach spaces, then so is $(X_0, X_1)_{s,q}$.

Proof. Step 1: If $\|x\|_{s,q} = 0$, then $K(x,t) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$, but since $K(\cdot,t)$ is a norm, necessarily, $x = 0$. On the other hand, since $K(0,t) = 0$ we get $\|0\|_{s,q} = 0$.

Next if $r \in \mathbb{R}$, since $K(\cdot, t)$ is a norm, we have $K(rx, t) = |r|K(x, t)$ and so

$$\begin{aligned}\|rx\|_{s,q} &= \Big(\int_0^\infty (K(rx,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} \\ &= \Big(\int_0^\infty (|r|K(x,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} \\ &= |r|\Big(\int_0^\infty (K(x,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} = |r|\|x\|_{s,q}.\end{aligned}$$

Finally, if $x, y \in (X_0, X_1)_{s,q}$, since $K(\cdot, t)$ is a norm, $K(x + y, t) \le K(x, t) + K(y, t)$. Hence,

$$\begin{aligned}\|x+y\|_{s,q} &= \Big(\int_0^\infty (K(x+y,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} \\ &\le \Big(\int_0^\infty (K(x,t)+K(y,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} \\ &\le \Big(\int_0^\infty (K(x,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q} + \Big(\int_0^\infty (K(y,t))^q \frac{dt}{t^{1+sq}}\Big)^{1/q},\end{aligned}$$

where we used the fact that $\|x\|_{s,q}$ is the $L^q((0,\infty);\mu)$ norm of the function $t \mapsto K(x,t)$ with respect to the measure $\mu = \frac{dt}{t^{1+sq}}$. ■

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Proof. Step 2: In view of (60) and (62) for every $x \in X_0 \cap X_1$,

$$K(x,t) \le \|x\|_{X_0} \le \|x\|_{X_0\cap X_1}, \quad K(x,t) \le t\|x\|_{X_1} \le t\|x\|_{X_0\cap X_1},$$

and so $K(x,t) \le \min\{1,t\}\|x\|_{X_0\cap X_1}$. In turn, if $1 \le q < \infty$, by (64),

$$\|x\|_{s,q} \le \|x\|_{X_0\cap X_1}\Big(\int_0^\infty \min\{1,t^q\}\frac{dt}{t^{1+sq}}\Big)^{1/q} = c_q\|x\|_{X_0\cap X_1},$$

where $c_q = 1/(q(1-s))^{1/q} + 1/(sq)^{1/q}$. If $q = \infty$, then

$$t^{-s}K(x,t) \le \min\{t^{-s}, t^{1-s}\}\|x\|_{X_0\cap X_1} \le \|x\|_{X_0\cap X_1},$$

where we used the fact that $\sup_{t>0}\min\{t^{-s}, t^{1-s}\} = 1$, and so by (65), $\|x\|_{s,\infty} \le \|x\|_{X_0\cap X_1}$. Thus, in both cases, we have shown that $X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{s,q}$.

On the other hand, by (61) and (62), for every $x \in (X_0, X_1)_{s,q}$,

$$\min\{1,t\}\|x\|_{X_0+X_1} \le K(x,t) \le \max\{1,t\}\|x\|_{X_0+X_1} \tag{66}$$

and thus as before, for $1 \le q \le \infty$,

$$c_q\|x\|_{X_0+X_1} \le \|x\|_{s,q}, \tag{67}$$

where $c_\infty := 1$, which proves that $(X_0, X_1)_{s,q} \hookrightarrow X_0 + X_1$.

Step 3: Next we claim that $(X_0, X_1)_{s,q}$ is a Banach space. Let $\{x_n\}_n$ be a Cauchy sequence in $(X_0, X_1)_{s,q}$. In view of (67), $\{x_n\}_n$ is a Cauchy sequence in $X_0 + X_1$, and so, since $X_0 + X_1$ is a Banach space, $x_n \to x$ in $X_0 + X_1$ for some $x \in X_0 + X_1$.

Given $\varepsilon > 0$ we can find $n_\varepsilon \in \mathbb{N}$ such that $\|x_m - x_n\|_{s,q} \le \varepsilon$ for all $m, n \ge n_\varepsilon$. Since $K(\cdot, t)$ is a norm in $X_0 + X_1$, by the triangle inequality and $(66)_2$,

$$t^{-s} K(x - x_n, t) \le t^{-s} K(x_m - x_n, t) + \max\{t^{-s}, t^{1-s}\} \|x - x_m\|_{X_0+X_1}. \quad (68)$$

In turn, if $1 \le q < \infty$, for every $\ell \in \mathbb{N}$,

$$\begin{aligned}\Big(\int_{1/\ell}^{\ell} (K(x - x_n, t))^q \frac{dt}{t^{1+sq}} \Big)^{1/q} &\le \|x_m - x_n\|_{s,q} + c_{\ell,q,s} \|x - x_m\|_{X_0+X_1} \\ &\le \varepsilon + c_{\ell,q,s} \|x - x_m\|_{X_0+X_1},\end{aligned}$$

where

$$c_{\ell,q,s} := \Big(\int_{1/\ell}^{\ell} (\max\{t^{-s}, t^{1-s}\})^q \frac{dt}{t^{1+sq}} \Big)^{1/q} < \infty.$$

Letting first $m \to \infty$ and then $\ell \to \infty$ in the previous inequality gives $\|x - x_n\|_{s,q} \le \varepsilon$ for all $n \ge n_\varepsilon$, which shows that $(X_0, X_1)_{s,q}$ is a Banach space.

If $q = \infty$, then by (65) and (68),

$$t^{-s} K(x - x_n, t) \le \varepsilon + \max\{t^{-s}, t^{1-s}\} \|x - x_m\|_{X_0+X_1}.$$

Letting $m \to \infty$ shows that $t^{-s} K(x - x_n, t) \le \varepsilon$ for all $t > 0$ and for all $n \ge n_\varepsilon$. Hence, $\|x - x_n\|_{s,\infty} \le \varepsilon$ for all $n \ge n_\varepsilon$, and the proof is complete. ■

Remark 107 *If $X_0 = X_1$, then it follows from the previous theorem that $X_0 = X_0 \cap X_1 = (X_0, X_1)_{s,q} = X_0 + X_1 = X_0$.*

On the other hand, if $X_0 \cap X_1 = \{0\}$, then for every $x \in X_0 + X_1$ there exist unique $x_0 \in X_0$ and $x_1 \in X_1$ such that $x = x_0 + x_1$. In turn, by (62), $K(x, t) = \|x_0\|_{X_0} + t\|x_1\|_{X_1}$ and so for $1 \le q < \infty$,

$$\|x\|_{s,q} = \Big(\int_0^\infty (\|x_0\|_{X_0} + t\|x_1\|_{X_1})^q \frac{dt}{t^{1+sq}} \Big)^{1/q} = \infty$$

unless $x_0 = x_1 = 0$. Similarly, $\|x\|_{s,\infty} = \sup_{t>0} t^{-s}(\|x_0\|_{X_0} + t\|x_1\|_{X_1}) = \infty$ unless $x_0 = x_1 = 0$. Thus, $(X_0, X_1)_{s,q} = \{0\}$ for every $1 \le q \le \infty$.

Exercise 108 *Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ be an admissible pair, with $X_1 \hookrightarrow X_0$, and let $1 \le q \le \infty$ and $0 < s < 1$.*

(i) Prove that $\|\cdot\|_{X_0}$ is an equivalent norm in $X_0 + X_1$.

(ii) Prove that for every $T > 0$,

$$x \mapsto \|x\|_{X_0} + \Big(\int_0^T (K(x, t))^q \frac{dt}{t^{1+sq}} \Big)^{1/q}$$

is an equivalent norm in $(X_0, X_1)_{s,q}$ for $1 \le q < \infty$, while

$$x \mapsto \|x\|_{X_0} + \sup_{0<t<T} t^{-s} K(x, t)$$

is an equivalent norm in $(X_0, X_1)_{s,\infty}$.

Exercise 109 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be an admissible pair, let* $1 \le q \le \infty$, *and let* $0 < s < 1$. *Prove that* $x \in X_0 + X_1$ *belongs to* $(X_0, X_1)_{s,q}$ *if and only if the sequence* $\{2^{-ks} K(x, 2^k)\}_{k\in\mathbb{Z}}$ *belongs to* $L^q(\mathbb{Z}, \mathcal{H}^0)$, *where* $\mathcal{H}^0$ *is the counting measure. Prove also that*

$$x \in (X_0, X_1)_{s,q} \mapsto \|\{2^{-ks} K(x, 2^k)\}_{k\in\mathbb{Z}}\|_{L^q(\mathbb{Z},\mathcal{H}^0)}$$

is an equivalent norm in $(X_0, X_1)_{s,q}$.

Exercise 110 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be an admissible pair, let* $1 \le q \le \infty$, *and let* $0 < s < 1$. *Prove that* $(X_0, X_1)_{s,q} = (X_1, X_0)_{1-s,q}$.

Next we study the inclusions of different interpolation spaces.

Theorem 111 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be an admissible pair and let* $1 \le q_1 < q_2 \le \infty$ *and* $0 < s < 1$. *Then the following embeddings hold*

$$(X_0, X_1)_{s,q_1} \hookrightarrow (X_0, X_1)_{s,q_2} \hookrightarrow (X_0, X_1)_{s,\infty}.$$

Proof. Step 1: We claim that for $x \in (X_0, X_1)_{s,q}$, $1 \le q < \infty$, and $t > 0$,

$$t^{-s} K(x, \tau) \le (sq)^{1/q} \|x\|_{s,q}.$$

To see this note that since $K(x, \cdot)$ is increasing, $K(x, t) \le K(x, r)$ for all $r > t$, and so

$$\begin{aligned}\|x\|_{s,q} &= \Big(\int_0^\infty (K(x,r))^q \frac{dr}{r^{1+sq}}\Big)^{1/q} \ge \Big(\int_t^\infty (K(x,r))^q \frac{dr}{r^{1+sq}}\Big)^{1/q} \\ &\ge K(x,t)\Big(\int_t^\infty \frac{dr}{r^{1+sq}}\Big)^{1/q} = K(x,t)\frac{1}{sq}\frac{1}{t^q}.\end{aligned}$$

Taking the supremum over all $\tau > 0$ gives

$$\|x\|_{s,\infty} \le (sq)^{1/q} \|x\|_{s,q}. \tag{69}$$

This proves the embedding $(X_0, X_1)_{s,q} \hookrightarrow (X_0, X_1)_{s,\infty}$.

Step 2: If now $1 \le q_1 < q_2 \le \infty$, then for $x \in (X_0, X_1)_{s,q_1}$,

$$\begin{aligned}&\Big(\int_0^\infty (K(x,t))^{q_2} \frac{dt}{t^{1+sq_2}}\Big)^{1/q_2} \\ &\qquad \le (\sup_{\tau>0} \tau^{-s} K(x,\tau))^{(q_2-q_1)/q_2} \Big(\int_0^\infty (K(x,t))^{q_1} \frac{dt}{t^{1+sq_1}}\Big)^{1/q_2} \\ &\qquad \le (sq_1)^{(q_2-q_1)/(q_1 q_2)} \|x\|_{s,q_1}^{(q_2-q_1)/q_2 + q_1/q_2} = c\|x\|_{s,q_1},\end{aligned}$$

where in the last inequality we used (69) with q_1 in place of q. ■

Monday, March 21, 2022

An important property of interpolation spaces is given by the following theorem.

Theorem 112 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *and* $(Y_0, \|\cdot\|_{Y_0})$, $(Y_1, \|\cdot\|_{Y_1})$ *be two admissible pairs and let* $T : X_0 + X_1 \to Y_0 + Y_1$ *be a linear operator such that* $T : X_0 \to Y_0$ *and* $T : X_1 \to Y_1$ *are continuous. Then for every* $1 \le q \le \infty$ *and* $0 < \sigma < 1$, $T : (X_0, X_1)_{\sigma,q} \to (Y_0, Y_1)_{\sigma,q}$ *with*

$$\|T\|_{L((X_0,X_1)_{\sigma,q};(Y_0,Y_1)_{\sigma,q})} \le \|T\|^{1-\sigma}_{L(X_0;Y_0)} \|T\|^{\sigma}_{L(X_1;Y_1)}.$$

Proof. Let c_0, $c_1 > 0$ be such that

$$\|T(x_0)\|_{Y_0} \le c_0 \|x_0\|_{X_0}, \quad \|T(x_1)\|_{Y_1} \le c_1 \|x_1\|_{X_1}$$

for all $x_0 \in X_0$ and $x_1 \in X_1$. If $x \in (X_0, X_1)_{\sigma,q}$ and $x = x_0 + x_1$, with $x_0 \in X_0$ and $x_1 \in X_1$, it follows by the linearity of T that $T(x) = T(x_0) + T(x_1)$, with $T(x_0) \in Y_0$ and $T(x_1) \in Y_1$. Hence, by (62) and (63),

$$\begin{aligned} K(T(x), t; Y_0, Y_1) &\le \|T(x_0)\|_{Y_0} + t\|T(x_1)\|_{Y_1} \\ &\le c_0 \|x_0\|_{X_0} + t c_1 \|x_1\|_{X_1} = c_0 (\|x_0\|_{X_0} + t c_1 c_0^{-1} \|x_1\|_{X_1}). \end{aligned}$$

It follows that $K(T(x), t; Y_0, Y_1) \le c_0 K(x, t c_1 c_0^{-1}; X_0, X_1)$, and so, if $1 \le q < \infty$, by (64) and the change of variables $\tau = t c_1 c_0^{-1}$,

$$\begin{aligned} \|T(x)\|_{\sigma,q} &\le c_0 \Big(\int_0^\infty (K(x, t c_1 c_0^{-1}; X_0, X_1))^q \frac{dt}{t^{1+\sigma q}} \Big)^{1/q} \\ &= c_0 (c_1 c_0^{-1})^\sigma \Big(\int_0^\infty (K(x, \tau; X_0, X_1))^q \frac{d\tau}{\tau^{1+\sigma q}} \Big)^{1/q} \\ &= c_0^{1-\sigma} c_1^\sigma \|x\|_{\sigma,q}. \end{aligned}$$

Similarly, if $q = \infty$, $\|T(x)\|_{\sigma,\infty} \le c_0^{1-\sigma} c_1^\sigma \|x\|_{\sigma,\infty}$. It remains to let $c_0 \searrow \|T\|_{L(X_0;Y_0)}$ and $c_1 \searrow \|T\|_{L(X_1;Y_1)}$. ■

Theorem 113 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be an admissible pair, let* $1 \le q \le \infty$ *and* $0 < \sigma < 1$. *Then* $\|x\|_{\sigma,q} \le c \|x\|^{1-\sigma}_{X_0} \|x\|^{\sigma}_{X_1}$ *for every* $x \in X_0 \cap X_1$ *and for some constant* $c = c(q, \sigma) > 0$.

Proof. Let $x \in X_0 \cap X_1 \setminus \{0\}$ and define $T(s) = sx$ for $s \in \mathbb{R}$. Then $\|T\|_{L(\mathbb{R};X_0)} = \|x\|_{X_0}$, $\|T\|_{L(\mathbb{R};X_1)} = \|x\|_{X_1}$, and $\|T\|_{L(\mathbb{R};(Y_0,Y_1)_{\sigma,q})} = \|x\|_{(X_0,X_1)_{\sigma,q}}$. ■

Exercise 114 *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be a admissible pairs of Banach spaces, let* $(Y, \|\cdot\|_Y)$ *be a Banach space and let* $T : X_0 + X_1 \to Y$ *be a linear operator such that* $T : X_0 \to Y$ *is compact and* $T : X_1 \to Y$ *is continuous. Given* $1 \le q \le \infty$ *and* $0 < \sigma < 1$, *prove that* $T : (X_0, X_1)_{\sigma,q} \to Y$ *is compact. Hint: Given* $\varepsilon > 0$ *take* $t > 0$ *so large that* $t^\sigma \le \varepsilon t$ *and use Exercise* **??**.

The following theorem tells us that the interpolation of two interpolation spaces may be realized as an interpolation between the original spaces. This is one of the key results in interpolation theory.

Theorem 115 (Reiteration) *Let* $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ *be an admissible pair, let* $1 \le q \le \infty$, *and let* $0 \le \sigma_0 < \sigma_1 \le 1$. *Then for every* $0 < \sigma < 1$,

$$(X_{\sigma_0}, X_{\sigma_1})_{\sigma,q} = (X_0, X_1)_{\theta,q},$$

where $\theta := (1-\sigma)\sigma_0 + \sigma\sigma_1$, $X_{\sigma_0} := (X_0, X_1)_{\sigma_0,q_0}$ *if* $\sigma_0 > 0$ *for some* $1 \le q_0 \le \infty$, $X_{\sigma_1} := (X_0, X_1)_{\sigma_1,q_1}$ *if* $\sigma_1 < 1$ *for some* $1 \le q_1 \le \infty$.

We now show that $W^{s,p}(\mathbb{R}^N)$ can be obtained as an interpolation space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$. We begin with a preliminary result.

Proposition 116 *For every* $f \in W^{1,p}(\mathbb{R}^N)$ *and every* $\boldsymbol{h} \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p d\boldsymbol{x} \le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Proof. Assume first that $f \in W^{1,p}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$. Then by the fundamental theorem of calculus,

$$f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) = \int_0^1 \nabla f(\boldsymbol{x}+t\boldsymbol{h}) \cdot \boldsymbol{h}\, dt.$$

In turn, by Hölder's inequality

$$|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p \le \|\boldsymbol{h}\|^p \int_0^1 \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\|^p dt.$$

Integrating both sides in $\boldsymbol{x}$ over $\mathbb{R}^N$ and using Tonelli's theorem and the change of variables $\boldsymbol{y} = \boldsymbol{x} + t\boldsymbol{h}$, so that $d\boldsymbol{y} = d\boldsymbol{x}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p d\boldsymbol{x} &\le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \int_0^1 \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\|^p dt d\boldsymbol{x} \\ &= \|\boldsymbol{h}\|^p \int_0^1 \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{x}+t\boldsymbol{h})\|^p d\boldsymbol{x} dt = \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f(\boldsymbol{y})\|^p d\boldsymbol{y}.\end{aligned}$$

In the general case when $f \in W^{1,p}(\mathbb{R}^N)$, we apply the previous inequality to $f_\varepsilon = \varphi_\varepsilon * f$, where φ_ε is a standard mollifier, to obtain

$$\int_{\mathbb{R}^N} |f_\varepsilon(\boldsymbol{x}+\boldsymbol{h}) - f_\varepsilon(\boldsymbol{x})|^p d\boldsymbol{x} \le \|\boldsymbol{h}\|^p \int_{\mathbb{R}^N} \|\nabla f_\varepsilon(\boldsymbol{x})\|^p d\boldsymbol{x}.$$

Using the fact that $f_\varepsilon \to f$ pointwise $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$ and $\nabla f_\varepsilon = \varphi_\varepsilon * \nabla f \to \nabla f$ in $L^p(\mathbb{R}^N; \mathbb{R}^N)$ (see [**?**]), it suffices to let $\varepsilon \to 0^+$ in the previous inequality and use Fatou's lemma on the left. ■

Wednesday, March 23, 2022

Midterm Solutions.

Friday, March 25, 2022

We recall that

$$\|f\|_{W^{s,p}(\Omega)} := \|f\|_{W^{s,p}(\Omega)} + \|f\|_{W^{s,p}(\Omega)},$$

where

$$|f|_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(\boldsymbol{y}) - f(\boldsymbol{x})|^p}{\|\boldsymbol{x} - \boldsymbol{y}\|^{N+sp}} d\boldsymbol{x} d\boldsymbol{y}\right)^{1/p}.$$

Theorem 117 *Let* $1 \le p < \infty$*, and* $0 < s < 1$*. Then*

$$(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p} = W^{s,p}(\mathbb{R}^N).$$

Proof. Step 1: In this step we will show that

$$(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p} \hookrightarrow W^{s,p}(\mathbb{R}^N).$$

Let $f \in (L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p}$ and $v \in L^p(\mathbb{R}^N)$ and $w \in W^{1,p}(\mathbb{R}^N)$ be such that $f = v + w$. By the change of variables $\boldsymbol{x} + \boldsymbol{h} = \boldsymbol{y}$, and the previous proposition,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p}{\|\boldsymbol{h}\|^{N+sp}} d\boldsymbol{x} \\
&\quad \le C \int_{\mathbb{R}^N} \frac{|v(\boldsymbol{x}+\boldsymbol{h}) - v(\boldsymbol{x})|^p}{\|\boldsymbol{h}\|^{N+sp}} d\boldsymbol{x} + \int_{\mathbb{R}^N} \frac{|w(\boldsymbol{x}+\boldsymbol{h}) - w(\boldsymbol{x})|^p}{\|\boldsymbol{h}\|^{N+sp}} d\boldsymbol{x} \\
&\quad \le C \frac{1}{\|\boldsymbol{h}\|^{N+sp}} \int_{\mathbb{R}^N} |v(\boldsymbol{x})|^p d\boldsymbol{x} + \frac{\|\boldsymbol{h}\|^p}{\|\boldsymbol{h}\|^{N+sp}} \int_{\mathbb{R}^N} \|\nabla w(\boldsymbol{y})\|^p d\boldsymbol{y} \qquad (70) \\
&\quad \le C \frac{1}{\|\boldsymbol{h}\|^{N+sp}} \left(\|v\|_{L^p(\mathbb{R}^N)} + \|\boldsymbol{h}\| \|\nabla w\|_{L^p(\mathbb{R}^N)}\right)^p.
\end{aligned}$$

Taking the infimum over all v and w and using the fact that

$$\begin{aligned}
K(f,t) := \inf\{&\|v\|_{L^p(\mathbb{R}^N)} + t\|w\|_{W^{1,p}(\mathbb{R}^N)} : f = v + w, \qquad (71) \\
&v \in L^p(\mathbb{R}^N),\, w \in W^{1,p}(\mathbb{R}^N)\},
\end{aligned}$$

we get

$$\int_{\mathbb{R}^N} \frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p}{\|\boldsymbol{h}\|^{N+sp}} d\boldsymbol{x} \le C \frac{1}{\|\boldsymbol{h}\|^{N+sp}} (K(f, \|\boldsymbol{h}\|))^p.$$

Integrating both sides in $\boldsymbol{h}$ over $\mathbb{R}^N$ and using spherical coordinates gives

$$\begin{aligned}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})|^p}{\|\boldsymbol{h}\|^{N+sp}} d\boldsymbol{x} d\boldsymbol{h} &\le C \int_{\mathbb{R}^N} \frac{1}{\|\boldsymbol{h}\|^{N+sp}} (K(f, \|\boldsymbol{h}\|))^p d\boldsymbol{h} \\
&\le C \int_0^\infty (K(f,t))^p \frac{dt}{t^{1+sp}}.
\end{aligned}$$

Since $W^{1,p}(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$, by Theorem 106 we have that

$$(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p} \hookrightarrow L^p(\mathbb{R}^N) + W^{1,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N).$$

Hence,

$$\|f\|_{L^p(\mathbb{R}^N)} \le C\|f\|_{(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}}.$$

Combining the last two inequalities proves

$$\|f\|_{W^{s,p}(\mathbb{R}^N)} \le C\|f\|_{(L^p(\mathbb{R}^N),W^{1,p}(\mathbb{R}^N))_{s,p}}.$$

■

Monday, March 21, 2022

Proof. Step 2: In this step we will show the other embedding, namely, that

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow (L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p}.$$

Given $f \in W^{s,p}(\mathbb{R}^N)$ and $t > 0$, for $\boldsymbol{x} \in \mathbb{R}^N$ write

$$f(\boldsymbol{x}) = (f(\boldsymbol{x}) - f_t(\boldsymbol{x})) + f_t(\boldsymbol{x}) =: v_t(\boldsymbol{x}) + f_t(\boldsymbol{x}),$$

where $f_t = \varphi_t * t$ and φ_t is a standard mollifier. Since $\int_{\mathbb{R}^N} \varphi_t(\boldsymbol{y})\, d\boldsymbol{y} = 1$,

$$v_t(\boldsymbol{x}) = \int_{\mathbb{R}^n} (f(\boldsymbol{y}) - f(\boldsymbol{x}))\varphi_t(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y}.$$

Writing $\varphi_t = \varphi_t^{1/p}\varphi_t^{1/p'}$, it follows by Hölder's inequality that

$$|v_t(\boldsymbol{x})|^p \le \int_{\mathbb{R}^N} |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p \varphi_t(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} \le \frac{\|\varphi\|_\infty}{t^N} \int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p d\boldsymbol{y},$$

where in the last inequality we used the fact that $\operatorname{supp} \varphi_t(\boldsymbol{x} - \cdot) \subseteq \overline{B(\boldsymbol{x},t)}$. Integrating in $\boldsymbol{x}$ over $\mathbb{R}^N$ gives

$$\int_{\mathbb{R}^N} |v_t(\boldsymbol{x})|^p d\boldsymbol{x} \le \frac{\|\varphi\|_\infty}{t^N} \int_{\mathbb{R}^N} \int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p d\boldsymbol{y} d\boldsymbol{x}.$$

In turn, by Tonelli's theorem

$$\begin{aligned}
\int_0^\infty \|v_t\|^p_{L^p(\mathbb{R}^N)} \frac{dt}{t^{1+sp}} &\le \|\varphi\|_\infty \int_0^\infty \frac{1}{t^{N+1+sp}} \int_{\mathbb{R}^N} \int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p d\boldsymbol{y} d\boldsymbol{x} dt \\
&= \|\varphi\|_\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(\boldsymbol{y}) - f(\boldsymbol{x})|^p \int_{\|\boldsymbol{x}-\boldsymbol{y}\|}^\infty \frac{1}{t^{N+1+sp}} dt d\boldsymbol{x} d\boldsymbol{y} \\
&= \frac{\|\varphi\|_\infty}{N+sp} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(\boldsymbol{y}) - f(\boldsymbol{x})|^p}{\|\boldsymbol{x}-\boldsymbol{y}\|^{N+sp}} d\boldsymbol{x} d\boldsymbol{y}.
\end{aligned} \tag{72}$$

Next we estimate f_t. By standard properties of mollifiers, $f_t \in C^\infty(\mathbb{R}^N)$, with

$$\begin{aligned}
\nabla f_t(\boldsymbol{x}) &= \frac{1}{t^{N+1}} \int_{\mathbb{R}^N} f(\boldsymbol{y}) \nabla\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t}\right) d\boldsymbol{y} \\
&= \frac{1}{t^{N+1}} \int_{\mathbb{R}^N} [f(\boldsymbol{y}) - f(\boldsymbol{x})] \nabla\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t}\right) d\boldsymbol{y},
\end{aligned}$$

where we used the fact that $\int_{\mathbb{R}^N} \nabla\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t}\right) d\boldsymbol{y} = 0$ since $\int_{\mathbb{R}^N} \varphi_t(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = 1$. By the change of variables $\boldsymbol{z} = \frac{\boldsymbol{x}-\boldsymbol{y}}{t}$,

$$\int_{\mathbb{R}^N} \left\|\nabla\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t}\right)\right\|^{p'} d\boldsymbol{y} = t^N \int_{\mathbb{R}^N} \|\nabla\varphi(\boldsymbol{z})\|^{p'} d\boldsymbol{z}.$$

Hence, writing $\frac{1}{t^N}\|\nabla\varphi\| = \frac{1}{t^{N/p}}\|\nabla\varphi\|^{1/p}\frac{1}{t^{N/p'}}\|\nabla\varphi\|^{1/p'}$, it follows by Hölder's inequality that

$$\begin{aligned}\|\nabla f_t(\boldsymbol{x})\|^p &\le C\frac{1}{t^{N+p}}\int_{\mathbb{R}^N} |f(\boldsymbol{y})-f(\boldsymbol{x})|^p \left\|\nabla\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{t}\right)\right\| d\boldsymbol{y} \\ &\le C\frac{1}{t^{N+p}}\int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y})-f(\boldsymbol{x})|^p d\boldsymbol{y},\end{aligned}$$

where in the last inequality we used the fact that $\operatorname{supp}\nabla\varphi((\boldsymbol{x}-\cdot)/t) \subseteq \overline{B(\boldsymbol{x},t)}$. Integrating in $\boldsymbol{x}$ over $\mathbb{R}^N$ gives

$$\int_{\mathbb{R}^N} \|\nabla f_t(\boldsymbol{x})\|^p d\boldsymbol{x} \le C\frac{1}{t^{N+p}}\int_{\mathbb{R}^N}\int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y})-f(\boldsymbol{x})|^p d\boldsymbol{y}d\boldsymbol{x}.$$

In turn, by Tonelli's theorem

$$\begin{aligned}\int_0^\infty t^p\|\nabla f_t\|^p_{L^p(\mathbb{R}^N)}\frac{dt}{t^{1+sp}} &\le C\int_0^\infty \frac{1}{t^{N+1+sp}}\int_{\mathbb{R}^N}\int_{B(\boldsymbol{x},t)} |f(\boldsymbol{y})-f(\boldsymbol{x})|^p d\boldsymbol{y}d\boldsymbol{x}dt \qquad (73)\\ &\le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(\boldsymbol{y})-f(\boldsymbol{x})|^p \int_{\|\boldsymbol{x}-\boldsymbol{y}\|}^\infty \frac{1}{t^{N+1+sp}} dt d\boldsymbol{x}d\boldsymbol{y} \\ &\le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(\boldsymbol{y})-f(\boldsymbol{x})|^p}{\|\boldsymbol{x}-\boldsymbol{y}\|^{N+sp}} d\boldsymbol{x}d\boldsymbol{y}.\end{aligned}$$

Since $\|f_t\|_{L^p(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)}$, we have that

$$\int_0^1 t^p\|f_t\|^p_{L^p(\mathbb{R}^N)}\frac{dt}{t^{1+sp}} \le \|f\|^p_{L^p(\mathbb{R}^N)}\int_0^1 t^p\frac{dt}{t^{1+sp}} = \frac{1}{p(1-s)}\|f\|^p_{L^p(\mathbb{R}^N)}.$$

Combining this inequality with (72) and (73) and using (71) gives

$$\begin{aligned}\int_0^1 (K(f,t))^p\frac{dt}{t^{1+sp}} &\le \int_0^1 (\|v_t\|^p_{L^p(\mathbb{R}^N)} + t^p\|f_t\|^p_{W^{1,p}(\mathbb{R}^N)})\frac{dt}{t^{1+sp}} \\ &\le C\|f\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|f(\boldsymbol{y})-f(\boldsymbol{x})|^p}{\|\boldsymbol{x}-\boldsymbol{y}\|^{N+sp}}d\boldsymbol{x}d\boldsymbol{y}.\end{aligned}$$

Since $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, by Exercise 108 we can endow the space $(L^p(\mathbb{R}^N), W^{1,p}(\mathbb{R}^N))_{s,p}$ with the equivalent norm

$$f \mapsto \|f\|_{L^p(\mathbb{R}^N)} + \left(\int_0^1 (K(f,t))^p\frac{dt}{t^{1+sp}}\right)^{1/p}.$$

Thus the previous inequality completes the proof. ∎

11 Decreasing Rearrangement

Given a measure space $(E, \mathfrak{M}, \mu)$, for every measurable set $F \subseteq E$, we define

$$F^* := [0, \mu(F)) \tag{74}$$

if $\mu(F) > 0$ and $F^* := \emptyset$ if $\mu(F) = 0$. If we now consider the functions χ_F and χ_{F^*}, we see that χ_{F^*} is decreasing in $[0, \infty)$ since $\chi_{F^*}(x) = 1$ if $0 \le x < \mu(F)$, and

$$\mu(\{x \in E : \chi_F(x) > t)\} = \mathcal{L}^1(\{y \in [0, \infty) : \chi_{F^*}(y) > t)\}$$

for every $t \ge 0$. We will see that we can extend this procedure to the case when χ_E is replaced by a qa1 measurable function f, that is, we can construct a function f^*, which is decreasing in $[0, \infty)$, and $\mu(\{|f| > t)\} = \mathcal{L}^1(\{f^* > t)\}$ for every $t \ge 0$.

Definition 118 *Given a measure space $(E, \mathfrak{M}, \mu)$ and a measurable function $f : E \to \mathbb{R}$, the* decreasing rearrangement of f *is the function $f^* : [0, \infty) \to [0, \infty]$, defined by*

$$f^*(y) := \int_0^\infty \chi_{E_t^*}(y)\, dt, \quad y \ge 0, \tag{75}$$

where $E_t := \{x \in E : |f(x)| > t\}$ and

$$E_t^* := (E_t)^* = [0, \mu(E_t)). \tag{76}$$

If $\mu(E_t) = 0$, we set $E_t^* := \emptyset$.

Remark 119 *Let $F \subseteq E$ be a measurable set. Consider the function $\chi_F : E \to \{0, 1\}$. Then $E_t = \{x \in E : \chi_F(x) > t\} = F$ if $0 \le t < 1$ and $E_t = \emptyset$ if $t \ge 1$. In turn,*

$$(\chi_F)^*(y) = \int_0^\infty \chi_{E_t^*}(y)\, dt = \int_0^1 \chi_{F^*}(y)\, dt = \chi_{F^*}(y).$$

Theorem 120 *Let $(E, \mathfrak{M}, \mu)$ be a measurable space, $f : E \to \mathbb{R}$ be a measurable function, and $f^* : [0, \infty) \to [0, \infty]$ be its decreasing rearrangement. Then*

(i) if $0 \le y_1 < y_2$, then $f^*(y_1) \ge f^*(y_2)$,

(ii) for every $t \ge 0$,

$$\{x \in E : |f(x)| > t\}^* = \{y \in [0, \infty) : f^*(y) > t\}$$

and

$$\mu(\{x \in E : |f(x)| > t\}) = \mathcal{L}^1(\{y \in [0, \infty) : f^*(y) > t)\}. \tag{77}$$

Proof. To prove item (i), observe that if $y_2 \in E_t^* = [0, \mu(E_t))$, then, since $0 \le y_1 < y_2$, we have that $0 \le y_1 < y_2 < \mu(E_t)$, and so, $y_1 \in E_t^*$. Thus, if $\chi_{E_t^*}(y_2) = 1$, then $\chi_{E_t^*}(y_1) = 1$, which implies that $\chi_{E_t^*}(y_2) \le \chi_{E_t^*}(y_1)$. Integrating in t and using (75) gives $f^*(y_2) \le f^*(y_1)$.

To prove item (iii), consider $0 \le t_1 < t_2$. Then

$$E_{t_2} = \{x \in E : |f(x)| > t_2\} \subseteq \{x \in E : |f(x)| > t_1\} = E_{t_1}. \tag{78}$$

This proves that the function $g(t) := \mu(E_t)$, $t \ge 0$, is decreasing. We claim that g is right-continuous. Fix $t_0 \ge 0$. Since g is decreasing, there exists

$$\lim_{t \to t_0^+} g(t) \ge g(t_0).$$

To prove equality, consider a decreasing sequence $t_n \to t_0^+$. Then $E_{t_n} \subseteq E_{t_{n+1}}$ and $\bigcup_{n=1}^\infty E_{t_n} = E_{t_0}$, and so, we have that

$$\lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} \mathcal{L}^1(E_{t_n}) = \mathcal{L}^1\Big(\bigcup_{n=1}^\infty E_{t_n}\Big) = g(t_0).$$

This proves the claim.

Let $t \ge 0$ and $y \in E_t^* = [0, \mu(E_t))$. Then $y < \mu(E_t) = g(t)$. Since g is right-continuous, there exists $\delta > 0$ such that $g(r) > y$ for all $r \in [t, t+\delta)$. Thus, we have shown that $y \in [0, \mu(E_r)) = E_r^*$ for all $r \in [t, t+\delta)$. On the other hand, by (78), we have that $y \in E_r^*$ for all $0 \le r \le t$. Hence,

$$f^*(y) = \int_0^\infty \chi_{E_r^*}(y)\,dr = \int_0^{t+\delta} 1\,dr + \int_{t+\delta}^\infty \chi_{E_r^*}(y)\,dr \ge t + \delta > t.$$

This shows that

$$E_t^* \subseteq \{y \in [0,\infty) : f^*(y) > t\}. \tag{79}$$

To prove the other inclusion, assume that $y \notin E_t^* = [0, \mu(E_t))$. Assume $y \ge \mu(E_t)$. Since $\mu(E_r) \le \mu(E_t)$ for all $r \ge t$ by (78), it follows that $y \notin E_r^* = [0, \mu(E_r))$, and so,

$$f^*(y) = \int_0^\infty \chi_{E_r^*}(y)\,dr = \int_0^t \chi_{E_r^*}(y)\,dr \le t,$$

which implies that $y \notin \{z \in [0,\infty) : f^*(z) > t\}$. Together with (79), this proves that

$$E_t^* = \{y \in [0,\infty) : f^*(y) > t\}.$$

Since $\mu(E_t) = \mathcal{L}^1(E_t^*)$ by (76), it follows that

$$\mu(\{x \in E : |f(x)| > t)\} = \mathcal{L}^1(E_t^*) = \mathcal{L}^1(\{y \in [0,\infty) : f^*(y) > t\}),$$

and so item (ii) holds. ■

Wednesday, March 23, 2022

Next we show that the decreasing rearrangement preserves L^p norms. We begin by proving the so-called layer-cake representation.

Theorem 121 *Let $(E,\mathfrak{M},\mu)$ be a measure space, let $0<p<\infty$ and let $f: E\to\mathbb{R}$ be a measurable function. Then*

$$\int_E |f(x)|^p d\mu(x) = p\int_0^\infty t^{p-1}\mu\left(\{x\in E:\,|f(x)|>t\}\right)\,dt.$$

Proof. If $\mu\left(\{x\in E:\,|f(x)|>t_0\}\right)=\infty$ for some $t_0>0$, then $\mu\left(\{x\in E:\,|f(x)|>t\}\right)=\infty$ for all $0\le t<t_0$, and thus both sides of the previous equality are infinite. Thus, assume that $\mu\left(\{x\in E:\,|f(x)|>t\}\right)<\infty$ for all $t>0$. Restrict the measure μ to the set of σ-finite measure

$$E_0 := \{x\in E:\,|f(x)|>0\}.$$

By Tonelli's theorem, which holds since $\mathcal{L}^1$ and μ restricted to E_0 are both σ-finite,

$$\begin{aligned} p\int_0^\infty t^{p-1}\mu\left(\{x\in E_0:\,|f(x)|>t\}\right)\,dt &= p\int_0^\infty t^{p-1}\int_{E_0}\chi_{\{|f|>t\}}(x)\,d\mu(x)dt \\ &= \int_{E_0}\int_0^{|f(x)|} pt^{p-1}dtd\mu(x) \\ &= \int_{E_0}|f(x)|^p d\mu(x) = \int_E |f(x)|^p d\mu(x). \end{aligned}$$

■

Using this result and Theorem 120, we have the following important result.

Theorem 122 *Let $(E,\mathfrak{M},\mu)$ be a measure space and let $f: E\to\mathbb{R}$ be a measurable function. Then for all $0<p<\infty$,*

$$\int_E |f(x)|^p d\mu(x) = \int_0^\infty \left(f^*(y)\right)^p dy, \tag{80}$$

while

$$\operatorname*{esssup}_E |f| = \sup f^* = f^*(0). \tag{81}$$

Proof. It follows from Theorem 121 and Theorem 120 that

$$\begin{aligned} \int_E |f(x)|^p d\mu(x) &= p\int_0^\infty t^{p-1}\mu\left(\{x\in E:\,|f(x)|>t\}\right)\,dt \\ &= p\int_0^\infty t^{p-1}\mathcal{L}^1\left(\{y\ge 0:\,f^*(y)>t\}\right)\,dt \\ &= \int_0^\infty \left(f^*(y)\right)^p dy. \end{aligned}$$

Next we claim that

$$f^*(0) = \operatorname*{esssup}_E |f| := \inf\{t\in[0,\infty):\,|f(x)|\le t \text{ for } \mu \text{ a.e. } x\in E\}. \tag{82}$$

Let $M := \operatorname{esssup}_E |f|$. Let's prove that $f^*(0) \le M$. Assume that $M < \infty$, otherwise there is nothing to prove. If $t > M$, then $E_t = \{x \in E : |f(x)| > t\}$ has measure zero. Thus, $E_t^* = \emptyset$, and by (75),

$$f^*(0) = \int_0^M \chi_{E_t^*}(0)\, dt \le M.$$

On the other hand, if $t < M$, then there exists a measurable set $F \subseteq E$ with $\mu(F) > 0$ such that $|f(x)| > t$ for all $x \in F$. In turn, $E_t \supseteq F$, so $E_t^* = [0, \mu(E_t)) \supseteq [0, \mu(F))$. In particular, $0 \in E_t^*$, so that

$$f^*(0) = \int_0^M \chi_{E_t^*}(0)\, dt = \int_0^M 1\, dt = M.$$

■

Corollary 123 *Let $(E, \mathfrak{M}, \mu)$ be a measure space and let $f : E \to \mathbb{R}$ be a measurable function. Given $t_0 > 0$, let $E_{t_0} := \{x \in E : |f(x)| > t_0\}$. Then*

$$\int_{E_{t_0}} |f(x)|\, d\mu(x) = \int_0^{\mu(E_{t_0})} f^*(y)\, dy.$$

Proof. Exercise. ■

Exercise 124 *Let $(E, \mathfrak{M}, \mu)$ be a measure space and let $f : E \to \mathbb{R}$ be a measurable function. Given $t > 0$, let $E^t := \{x \in E : |f(x)| > f^*(t)\}$. Prove that $\mu(E^t) \le t$ and that f^* is constant on $[\mu(E^t), t]$.*

As an application of the interpolation theory, we prove that the interpolation space between $L^1(E)$ and $L^\infty(E)$ is the Lorentz space $L^{p,q}(E)$,

Theorem 125 *Let $(E, \mathfrak{M}, \mu)$ be a measure space, $1 < p < \infty$, and $1 \le q \le \infty$. Then*

$$(L^1(E), L^\infty(E))_{s,q} = L^{p,q}(E),$$

where $s := 1 - 1/p$. Moreover a norm in $L^{p,q}(E)$ is given by

$$\|f\|_{L^{p,q}} := \Big(\int_0^\infty \Big(\int_0^t f^*(\tau)\, d\tau \Big)^q \frac{dt}{t^{1+sq}} \Big)^{1/q}$$

if $1 \le q < \infty$ and

$$\|f\|_{L^{p,\infty}} := \sup_{t>0} t^{-s} \int_0^t f^*(\tau)\, d\tau$$

if $q = \infty$, where f^* is the decreasing rearrangement of f.

Proof. The fact that $L^1(E)$ and $L^\infty(E)$ are an admissible pair follows from your homework. Given a Lebesgue measurable function $f : E \to \mathbb{R}$ and $t > 0$, we claim that

$$K(f,t) = \int_0^t f^*(\tau)\, d\tau. \tag{83}$$

We begin by proving $\int_0^t f^* d\tau \le K(f,t)$. Without loss of generality we may assume that $K(f,t) < \infty$ (see Remark 104), so that $f \in L^1(E) + L^\infty(E)$. Write $f = g + h$, where $g \in L^1(E)$ and $h \in L^\infty(E)$. Then by Exercise **??**, a change of variables and the fact that h^* is decreasing

$$\begin{aligned}\int_0^t f^*(\tau)\,d\tau &\le \int_0^t g^*((1-\varepsilon)\tau)\,d\tau + \int_0^t h^*(\varepsilon\tau)\,d\tau \\ &\le (1-\varepsilon)^{-1}\int_0^\infty g^*(r)\,dr + th^*(0) \\ &= (1-\varepsilon)^{-1}\|g\|_{L^1(E)} + t\|h\|_{L^\infty(E)},\end{aligned}$$

where the last equality follows from Theorem 122. Letting $\varepsilon \to 0^+$ shows that $\int_0^t f^* d\tau \le \|g\|_{L^1(E)} + t\|h\|_{L^\infty(E)}$. Since this holds for every decomposition of f, it follows that $\int_0^t f^* d\tau \le K(f,t)$. ■

Friday, April 1, 2022

Proof. To prove the converse inequality, assume that $\int_0^t f^* d\tau < \infty$ and define

$$g(x) := \max\{|f(x)| - f^*(t), 0\} \operatorname{sgn} f(x), \quad h(x) := f(x) - g(x).$$

Let $E^t := \{x \in E : |f(x)| > f^*(t)\}$. By Exercise 124, $\mu(E^t) \le t$ and f^* is constant on $[\mu(E^t), t]$. Hence,

$$\begin{aligned}\|g\|_{L^1(E)} &= \int_{E^t} (|f(x)| - f^*(t))\,d\mu = \int_0^{\mu(E^t)} (f^*(\tau) - f^*(t))\,d\tau \\ &\le \int_0^t (f^*(\tau) - f^*(t))\,d\tau,\end{aligned}$$

where in the second equality we used Corollary 123. On the other hand, $|h(x)| = f^*(t)$ in E^t and $|h(x)| \le f^*(t)$ outside E^t. It follows that

$$\begin{aligned}\|g\|_{L^1(E)} + t\|h\|_{L^\infty(E)} &\le \int_0^t (f^*(\tau) - f^*(t))\,d\tau + tf^*(t) \\ &= \int_0^t f^*(\tau)\,d\tau,\end{aligned}$$

which shows that $K(f,t) \le \int_0^t f^* d\tau$. This proves (83). ■

Exercise 126 *Let $(E, \mathfrak{M}, \mu)$ be a measure space, $1 < p < \infty$, and $1 \le q \le \infty$. Prove that $f \in L^1(E) + L^\infty(E)$ belongs to $L^{p,q}(E)$ if and only if*

$$\left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty$$

if $q < \infty$ and

$$\sup_{t>0} t^{1/p} f^*(t) < \infty$$

if $q = \infty$. Deduce that $L^{p,p}(E) = L^p(E)$.

Corollary 127 *Let* $(E, \mathfrak{M}, \mu)$ *be a measure space, let* $1 < p < \infty$*, and let* $1 < p < \infty$ *and let* $1 \le q_1 < q_2 < \infty$*. Then* $L^{p,q_1}(E) \hookrightarrow L^{p,q_2}(E)$*.*

Proof. This follows by applying Theorems 111 and 125. ∎

Corollary 128 *Let* $(E, \mathfrak{M}, \mu)$ *be a measure space, let* $1 < p < \infty$*, and let* $1 \le q \le \infty$*. Then*

$$\|f\|_{L^{p,q}} \le c\|f\|_{L^1}^{1-s}\|f\|_{L^\infty}^s$$

for all $f \in L^1(E) \cap L^\infty(E)$*, where* $s := 1 - 1/p$*.*

Proof. This follows by applying Theorems 113 and 125. ∎

Theorem 129 (Marcinkiewicz) *Let* $(E, \mathfrak{M}, \mu)$ *and* $(F, \mathfrak{N}, \rho)$ *be measure spaces, let*

$$T : L^1(E) + L^\infty(E) \to L^1(F) + L^\infty(F)$$

be a linear operator such that $T : L^1(E) \to L^1(F)$ *with*

$$\|T(f)\|_{L^1(F)} \le c_1 \|f\|_{L^1(E)}$$

for all $f \in L^1(E)$ *and* $T : L^\infty(E) \to L^\infty(F)$ *with*

$$\|T(f)\|_{L^\infty(F)} \le c_2 \|f\|_{L^\infty(E)}$$

for all $f \in L^\infty(E)$*. Then for every* $1 < p < \infty$ *and* $1 \le q \le \infty$*,*

$$T : L^{p,q}(E) \to L^{p,q}(F)$$

and

$$\|T\|_{L(L^{p,q}(E);L^{p,q}(F))} \le \|T\|_{L(L^1(E);L^1(F))}^{1-s}\|T\|_{L(L^\infty(E);L^\infty(F))}^s.$$

Proof. This follows by applying Theorems 112 and 125. ∎

Theorem 130 *Let* $(E, \mathfrak{M}, \mu)$ *be a measure space, let* $1 \le q, q_0, q_1 \le \infty$*, let* $1 \le p_0, p_1 \le \infty$*, with* $p_0 \ne p_1$*, and let* $0 < s < 1$*. Then*

$$(L^{p_0,q_0}(E), L^{p_1,q_1}(E))_{s,q} = L^{p,q}(E), \tag{84}$$

where $\frac{1}{p} = \frac{1-s}{p_0} + \frac{s}{p_1}$*. In particular,*

$$(L^{p_0}(E), L^{p_1}(E))_{s,q} = L^{p,q}(E). \tag{85}$$

Proof. We will only do the case $1 < p_0, p_1 < \infty$ and leave the other cases as an exercise. By Theorem 125,

$$(L^1(E), L^\infty(E))_{1-1/p,q} = L^{p,q}(E),$$
$$(L^1(E), L^\infty(E))_{1-1/p_0,q_0} = L^{p_0,q_0}(E),$$
$$(L^1(E), L^\infty(E))_{1-1/p_1,q_1} = L^{p_1,q_1}(E).$$

Taking $\theta = 1 - \frac{1}{p}$, $s_0 = 1 - \frac{1}{p_0}$, and $s_1 = 1 - \frac{1}{p_1}$ we get

$$\begin{aligned}\theta &= 1 - \frac{1}{p} = 1 - \frac{1-s}{p_0} - \frac{s}{p_1} \\ &= (1-s) + s - \frac{1-s}{p_0} - \frac{s}{p_1} = (1-s)s_0 + ss_1,\end{aligned}$$

and so we are in a position to apply the reiteration theorem (see Theorem 115) to obtain (84). To obtain (85) it suffices to observe that $L^{p_0,p_0}(E) = L^{p_0}(E)$ and $L^{p_1,p_1}(E) = L^{p_1}(E)$ by your homework. ■

Exercise 131 *Let $E \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, let $1 < p < r < s < \infty$ and let $1 \le q \le \infty$. Prove that if $f \in L^{p,\infty}(E) \cap L^{s,\infty}(E)$, then $f \in L^{r,q}(E)$ and estimate its $L^{r,q}(E)$ norm.*

Exercise 132 *Let $E \subseteq \mathbb{R}^N$ be a Lebesgue measurable set, let $1 < p < \infty$ and let $1 \le q < \infty$. Prove that $L^1(E) \cap L^\infty(E)$ is dense in $L^{p,q}(E)$. Deduce that simple functions are dense in $L^{p,q}(E)$.*

Theorem 133 (Marcinkiewicz in Lorentz spaces) *Let $(E, \mathfrak{M}, \mu)$ and $(F, \mathfrak{N}, \rho)$ be measure spaces, let $1 < p_0, p_1, r_0, r_1 < \infty$ with $p_0 \ne p_1$ and $r_0 \ne r_1$, let $1 \le q_0, q_1, s_0, s_1 \le \infty$, and let*

$$T : L^{p_0,q_0}(E) + L^{p_1,q_1}(E) \to L^{p_0,q_0}(F) + L^{p_1,q_1}(F)$$

be a linear operator such that $T : L^{p_0,q_0}(E) \to L^{r_0,s_0}(F)$ and $T : L^{p_1,q_1}(E) \to L^{r_1,s_1}(F)$ with

$$\|T(f)\|_{L^{r_i,s_i}(F)} \le c_i \|f\|_{L^{p_i,q_i}(E)}, \quad i = 0, 1, \tag{86}$$

for all $f \in L^{p_i,q_i}(E)$ and for some positive constants c_0 and $c_1 > 0$. Then for every $\theta \in (0,1)$, there exists a constant $c_\theta > 0$ such that

$$\|T(f)\|_{L^{r,s}(F)} \le c_\theta \|f\|_{L^{p,q}(E)}$$

for all $f \in L^{p,q}(E)$, where $1 \le q \le s \le \infty$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}. \tag{87}$$

Proof. This follows by applying Theorems 112 and 130 and Corollary 127. ■

Monday, April 4, 2022

12 Rapidly Decreasing Functions and Tempered Distributions

Definition 134 *The* space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^N)$ *is the space of all functions $f : \mathbb{R}^N \to \mathbb{C}$ of class C^∞ such that for all multi-indeces $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$,*

$$\|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\boldsymbol{x} \in \mathbb{R}^N} |\boldsymbol{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} f(\boldsymbol{x})| < \infty.$$

Thus $\mathcal{S}\left(\mathbb{R}^N\right)$ consists of all functions that, together with all their derivatives, decay to zero faster than any polynomial. Note that $\|f\|_{\mathbf{0},\mathbf{0}} := \sup_{\boldsymbol{x}\in\mathbb{R}^N} |f(\boldsymbol{x})|$.

Remark 135 *The space $C_c^\infty(\mathbb{R}^N)$ of all C^∞ functions $f : \mathbb{R}^N \to \mathbb{C}$ with compact support is contained in $\mathcal{S}\left(\mathbb{R}^N\right)$. The function $f(\boldsymbol{x}) := e^{-\|\boldsymbol{x}\|^2}$ is an example of a function in $\mathcal{S}\left(\mathbb{R}^N\right)$ without compact support.*

Note that for all multi-indeces $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$, $\|\cdot\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is a seminorm. In $\mathcal{S}\left(\mathbb{R}^N\right)$ we consider the topology τ generated by the family of seminorms $\|\cdot\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$. We recall the following definitions.

Definition 136 *Given a vector space X, a function $p : X \to [0, \infty)$ is a* seminorm *if $p(x + y) \leq p(x) + p(y)$ for every $x, y \in X$ and $p(tx) = |t|p(x)$ for all $t \in \mathbb{R}$ and $x \in X$. Given a seminorm, for every $x \in X$ and $r > 0$ we define $B_p(x, r) := \{y \in X : p(x - y) < r\}$.*

Definition 137 *If X is a vector space and $\mathcal{P}$ is a family of seminorms, the* topology τ generated by $\mathcal{P}$ *is the smallest topology that contains all "balls" $B_p(x, r)$ for all $x \in X$, $r > 0$, and $p \in \mathcal{P}$.*

Theorem 138 *Let $\mathcal{P}$ be a countable family of seminorms on a vector space X with the property that for every $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$ and let τ be the topology generated by $\mathcal{P}$. Then there exists a translation-invariant metric d that generates τ.*

Proof. Let $\mathcal{P} = \{p_n\}_n$ and for $x, y \in X$ define

$$d(x, y) := \sup_n \frac{1}{n} \min\{1, p_n(x - y)\}. \tag{88}$$

We leave as an exercise to prove that d is a metric. Note that $d(x, y) = d(x - y, 0)$, and so d is translation-invariant. Similarly, since every ball

$$B_{p_n}(x, r) = \{y \in X : p_n(x - y) < r\} = x + B_{p_n}(0, r),$$

we have that $U \in \tau$ if and only if $x + U \in \tau$ for any $x \in X$. Thus, τ is translation-invariant.

Step 1: We claim that

$$B_d(0, r) := \{x \in X : d(x, 0) < r\}$$

is open with respect to τ. If $r > 1$, then $B_d(0, r) = X$. To see this, note that $d(x, 0) \leq 1 < r$ for every $x \in X$, which implies that $B_d(0, r) = X \in \tau$.

Next, fix $0 < r \leq 1$. Let $n_1 \in \mathbb{N}$ be so large that $\frac{1}{n} < r$ for all $n > n_1$ and $\frac{1}{n} \geq r$ for $n \leq n_1$. If $d(x, 0) < r$, then

$$\sup_n \frac{1}{n} \min\{1, p_n(x)\} < r \quad \text{if and only if} \quad \max_{1\leq n\leq n_1} \frac{1}{n} \min\{1, p_n(x)\} < r.$$

For $1 \leq n \leq n_1$, since $\frac{1}{n} \geq r$, we have that $\frac{1}{n} \min \{1, p_n(x)\} < r$ if and only if $\frac{1}{n} p_n(x) < r$. Hence,

$$B_d(0, r) = \bigcap_{n=1}^{n_1} B_{p_n}(0, rn) \in \tau.$$

By the translation invariance of d and the seminorms,

$$B_d(x, r) = x + B_d(0, r) = x + \bigcap_{n=1}^{n_1} B_{p_n}(0, rn) = \bigcap_{n=1}^{n_1} B_{p_n}(x, rn) \in \tau.$$

Step 2: Let $k \in \mathbb{N}$ and $r > 0$. We claim that $B_{p_k}(0, r)$ is open with respect to the topology τ_d generated by the metric d. If $x \in B_{p_k}(0, r)$, take

$$R = \frac{\min\{1, r - p_k(x)\}}{k}.$$

We claim that $B_d(x, R) \subseteq B_{p_k}(0, r)$. To see this, let $y \in B_d(x, R)$. Then

$$d(x, y) = \sup_n \frac{1}{n} \min\{1, p_n(x - y)\} < R.$$

In particular, $\frac{1}{k} \min\{1, p_k(x - y)\} < \frac{\min\{1, r - p_k(x)\}}{k}$, which implies that

$$\min\{1, p_k(x - y)\} < \min\{1, r - p_k(x)\} \leq 1.$$

In turn, $p_k(x - y) < \min\{1, r - p_k(x)\} \leq r - p_k(x)$, so that

$$p_k(y) \leq p_k(x - y) + p_k(x) < r.$$

Thus, $y \in B_{p_k}(0, r)$. We have shown that every $x \in B_{p_k}(0, r)$ is an interior point with respect to the metric d. Thus, $B_{p_k}(0, r)$ is open with respect to the metric d. By translation-invariance the same is true for $B_{p_k}(x, r) = x + B_{p_k}(0, r)$. It follows that every open set in τ is open with respect to the metric. ■

Wednesday, April 6, 2022

In view of the previous theorem, the space $\mathcal{S}(\mathbb{R}^N)$ is metrizable. We now show that $\mathcal{S}(\mathbb{R}^N)$ is complete.

Theorem 139 *The space $\mathcal{S}(\mathbb{R}^N)$ endowed with the metric*

$$d(f, g) := \sup_n \frac{1}{n} \min\left\{1, \|f - g\|_{\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n}\right\}$$

is complete. Moreover, $f_n \to f$ in $\mathcal{S}(\mathbb{R}^N)$ if and only if $\|f_n - f\|_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \to 0$ for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^N$.

Proof. Let $\{f_k\}_k$ be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^N)$. Then

$$d(f_k, f_l) = \sup_n \frac{1}{n} \min\left\{1, \|f_k - f_l\|_{\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n}\right\} \to 0$$

as $k,l\to\infty$, which implies that $\frac{1}{n}\min\left\{1,\|f_k-f_l\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}\right\}\to 0$ as $k,l\to\infty$ for every n. In turn, $\|f_k-f_l\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}\to 0$ as $k,l\to\infty$ for every n. Thus, $\left\{\boldsymbol{x}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}f_k\right\}_k$ is a Cauchy sequence in $C_b\left(\mathbb{R}^N\right)$ for every $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}_0^+$ and thus it converges uniformly to a function $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Let $f:=g_{\mathbf{0},\mathbf{0}}$. By the fundamental theorem of calculus

$$f_k(\boldsymbol{x}+t\boldsymbol{e}_i)=f_k(\boldsymbol{x})+\int_0^t\frac{\partial f_k}{\partial x_i}(\boldsymbol{x}+s\boldsymbol{e}_i)\,ds.$$

Letting $k\to\infty$ it follows by uniform convergence that

$$f(\boldsymbol{x}+t\boldsymbol{e}_i)=f(\boldsymbol{x})+\int_0^t g_{\mathbf{0},\boldsymbol{e}_i}(\boldsymbol{x}+s\boldsymbol{e}_i)\,ds.$$

Hence, there exists $\frac{\partial f}{\partial x_i}=g_{\mathbf{0},\boldsymbol{e}_i}$. This proves that f is of class C^1. In a similar way we can show that f is of class C^∞ with $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}=\boldsymbol{x}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}f$. Thus $f\in\mathcal{S}\left(\mathbb{R}^N\right)$. Fix ε and let n_ε be so large that $\frac{1}{n}<\varepsilon$ for all $n>n_\varepsilon$. Since $\|f_k-f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}\to 0$ for all $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}_0^+$ we can find k_ε so large that $\|f_k-f\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}<\varepsilon$ for all $k\ge k_\varepsilon$ and all $n=1,\ldots,n_\varepsilon$. Then

$$d(f_k,f)\le\sup_{1\le n\le n_\varepsilon}\frac{1}{n}\min\left\{1,\|f_k-f_l\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}\right\}+\sup_{n>n_\varepsilon}\frac{1}{n}\le 2\varepsilon$$

for all $k\ge k_\varepsilon$. It follows that $\mathcal{S}\left(\mathbb{R}^N\right)$ is complete. The last part of the statement is a consequence of the previous theorem. ■

The following theorem is important for applications. For $f\in\mathcal{S}\left(\mathbb{R}^N\right)$ and $m,n\in\mathbb{N}_0$ we define

$$\|f\|_{m,n}:=\sum_{|\boldsymbol{\alpha}|\le n}\sum_{|\boldsymbol{\beta}|\le n}\|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,.$$

Theorem 140 *A linear functional $T:\mathcal{S}\left(\mathbb{R}^N\right)\to\mathbb{C}$ is continuous if and only if there exist a constant $C>0$ and some $m,n\in\mathbb{N}_0$ such that*

$$|T(f)|\le C\,\|f\|_{m,n}\,. \tag{89}$$

for every $f\in\mathcal{S}\left(\mathbb{R}^N\right)$.

Proof. Assume that $T:\mathcal{S}\left(\mathbb{R}^N\right)\to\mathbb{C}$ is continuous. Then $T^{-1}(B(0,1))$ is open. Since $0\in T^{-1}(B(0,1))$, we can find $0<r<1$ such that $B_d(0,r)\subseteq T^{-1}(B(0,1))$. We have seen that if $r<1$, there exist multi-indeces $\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\ldots,\boldsymbol{\alpha}_\ell,\boldsymbol{\beta}_\ell$ and $r_1>0,\ldots,r_\ell>0$ such that

$$B_{\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1}(0,r_1)\cap\cdots\cap B_{\boldsymbol{\alpha}_\ell,\boldsymbol{\beta}_\ell}(0,r_\ell)=B_d(0,r)\subseteq T^{-1}(B(0,1)).$$

This means that if $g\in\mathcal{S}\left(\mathbb{R}^N\right)$ is such that $\|g\|_{\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i}<r_i$ for all $i=1,\ldots,\ell$, then $|T(g)|<1$. Let $0<\rho<\min r_i$, $m=\max|\boldsymbol{\alpha}_i|$, $n=\max|\boldsymbol{\beta}_i|$. Given $f\in\mathcal{S}\left(\mathbb{R}^N\right)\setminus\{0\}$, we have that

$$g:=\rho\frac{f}{\|f\|_{m,n}}\in B_{\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1}(0,r_1)\cap\cdots\cap B_{\boldsymbol{\alpha}_\ell,\boldsymbol{\beta}_\ell}(0,r_\ell)\,,$$

and so, by the linearity of T,

$$\frac{\rho}{\|f\|_{m,n}}|T(f)| = |T(g)| < 1,$$

that is,

$$|T(f)| < \frac{1}{\rho}\|f\|_{m,n}.$$

Conversely, assume that there exist a constant $C > 0$ and some $m, n \in \mathbb{N}_0$ such that $|T(f)| \le C\|f\|_{m,n}$ for every $f \in \mathcal{S}(\mathbb{R}^N)$. Let's prove that T is continuous at 0. Consider an open set $V \subseteq \mathbb{C}$ with $T(0) = 0 \in V$. Then we can find $\varepsilon > 0$ such that $B(0,\varepsilon) \subseteq V$. Let $B_{m,n}(0,\delta) = \{f \in \mathcal{S}(\mathbb{R}^N) : \|f\|_{m,n} < \delta\}$. This set is open with respect to τ. If $f \in B_{m,n}(0,\delta)$, then

$$|T(f)| \le C\|f\|_{m,n} < C\delta = \varepsilon,$$

provided $\delta = \varepsilon/C$. Hence, T is continuous at 0. Since the topology is translation invariant, we have continuity at every point. ■

Definition 141 *The dual of* $\mathcal{S}(\mathbb{R}^N)$ *is called the* space of tempered distributions *and is denoted* $\mathcal{S}'(\mathbb{R}^N)$.

In $\mathcal{S}'(\mathbb{R}^N)$ we consider as a topology the topology generated by the basis of neighborhoods of 0 given by

$$B(0,\varepsilon,F) := \{T \in \mathcal{S}'(\mathbb{R}^N) : |T(f)| < \varepsilon \text{ for all } f \in F\},$$

where $\varepsilon > 0$ and $F \subset \mathcal{S}(\mathbb{R}^N)$ is a finite set. With this topology a sequence $\{T_n\}_n$ of tempered distributions converges to T if and only if $T_n(f) \to T(f)$ for every $f \in \mathcal{S}(\mathbb{R}^N)$.

Monday, April 11, 2022

Example 142 *Given a measure* $\mu : \mathcal{B}(\mathbb{R}^N) \to [0,\infty]$ *with the property that*

$$\mu(\overline{B(\mathbf{0},r)}) \le C_0(1+r)^k$$

for some $C_0 > 0$*, some* $k \in \mathbb{N}$*, and for all* $r > 0$*, the linear functional* $T_\mu : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ *defined by*

$$T_\mu(f) := \int_{\mathbb{R}^N} f\,d\mu$$

is well-defined and continuous. Indeed, write

$$\begin{aligned}\int_{\mathbb{R}^N} |f|\,d\mu &= \int_{B(\mathbf{0},1)} |f|\,d\mu + \sum_{n=2}^{\infty}\int_{B(\mathbf{0},n)\setminus B(\mathbf{0},n-1)} |f|\,d\mu \\ &\le \|f\|_\infty 2C_0 + \sum_{n=2}^{\infty}\int_{\overline{B(\mathbf{0},n)}\setminus B(\mathbf{0},n-1)} \frac{(1+\|\boldsymbol{x}\|)^{3k}}{(1+\|\boldsymbol{x}\|)^{3k}}|f|\,d\mu \\ &\le \|f\|_\infty 2C_0 + CC_0\|f\|_{3k,0}\sum_{n=1}^{\infty}\frac{(1+n)^k}{n^{3k}} < \infty.\end{aligned}$$

Hence by (89), $T_\mu \in \mathcal{S}'(\mathbb{R}^N)$.

Example 143 (Principal value of $1/x$) *Let's prove that the linear mapping*

$$T(f) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{x} \, dx, \quad f \in \mathcal{S}(\mathbb{R}),$$

is well-defined and belongs to $\mathcal{S}'(\mathbb{R})$. The functional T is called the principal value *of $\frac{1}{x}$ and is denoted* pv $\frac{1}{x}$. *Write*

$$\begin{aligned} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{x} \, dx &= \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{x} \, dx + \int_{\mathbb{R} \setminus [-1,1]} \frac{f(x)}{x} \, dx \\ &=: I_1 + I_2. \end{aligned}$$

The term I_2 does not give any troubles, since

$$\begin{aligned} \int_{\mathbb{R} \setminus [-1,1]} \left| \frac{f(x)}{x} \right| dx &\leq \int_{\mathbb{R} \setminus [-1,1]} |f(x)| \, dx \\ &\leq 2 \|f\|_{1,0} \int_1^\infty \frac{1}{x^2} \, dx = 2 \|f\|_{0,1}. \end{aligned}$$

Let's study I_1. Since $1/x$ is an odd function,

$$\int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{1}{x} \, dx = 0, \tag{90}$$

we can write

$$I_1 = \int_{[-1,1] \setminus [-\varepsilon, \varepsilon]} \frac{f(x) - f(0)}{x - 0} \, dx.$$

Since $f \in \mathcal{S}(\mathbb{R})$, by the mean value theorem

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = |f'(\theta)| \leq \|f\|_{0,1}$$

for all $x \in [-1, 1]$, with $x \neq 0$, and so by the Lebesgue dominated convergence theorem, there exists

$$\lim_{\varepsilon \to 0+} I_1 = \int_{-1}^1 \frac{f(x) - f(0)}{x - 0} \, dx.$$

Moreover, since $|I_1| \leq 2 \|f\|_{0,1}$, it follows that $|\lim_{\varepsilon \to 0+} I_1| \leq 2 \|f\|_{0,1}$. Thus, we have shown that $T(f)$ is well-defined and

$$|T(f)| \leq 2 \|f\|_{0,1} + 2 \|f\|_{1,0},$$

which, by Theorem 140, implies that pv $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$.

Similarly, for $x_0 \in \mathbb{R}$ we can define the tempered distribution

$$\left(\text{pv} \, \frac{1}{x - x_0} \right)(f) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [x_0 - \varepsilon, x_0 + \varepsilon]} \frac{f(x)}{x - x_0} \, dx, \quad f \in \mathcal{S}(\mathbb{R}).$$

Note that pv $\frac{1}{x}$ *is not of the form 142.*

Remark 144 *The cancellation property (90) will turn out to play a crucial role in the theory of singular operators.*

Next we show that $\mathcal{S}\left(\mathbb{R}^N\right)$ is embedded in L^p for every p.

Theorem 145 *The space $\mathcal{S}\left(\mathbb{R}^N\right)$ is embedded in $L^p\left(\mathbb{R}^N\right)$ for all $1 \le p \le \infty$, while $L^p(\mathbb{R}^N)$ is embedded in $\mathcal{S}'\left(\mathbb{R}^N\right)$ for all $1 \le p \le \infty$.*

Proof. We only need to consider the case $1 \le p < \infty$. Write

$$\begin{aligned}\int_{\mathbb{R}^N} |f|\, d\boldsymbol{x} &= \int_{\mathbb{R}^N} \frac{1 + \|\boldsymbol{x}\|^{N+1}}{1 + \|\boldsymbol{x}\|^{N+1}} |f|\, d\boldsymbol{x} \\ &\le C \|f\|_{N+1,0} \int_{\mathbb{R}^N} \frac{1}{1 + \|\boldsymbol{x}\|^{N+1}}\, d\boldsymbol{x}.\end{aligned}$$

For $1 < p < \infty$ it is enough to observe that

$$\int_{\mathbb{R}^N} |f|^p\, d\boldsymbol{x} \le \|f\|_\infty^{p-1} \int_{\mathbb{R}^N} |f|\, d\boldsymbol{x} \le C \|f\|_{N+1,0}^p .$$

This shows that $\mathcal{S}\left(\mathbb{R}^N\right)$ is embedded in $L^p\left(\mathbb{R}^N\right)$. Given $g \in L^p(\mathbb{R}^N)$, consider the linear functional $T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ defined by

$$T_g(f) := \int_{\mathbb{R}^N} f g\, d\boldsymbol{x}. \tag{91}$$

Then by Hölder's inequality

$$|T_g(f)| \le \|f\|_{L^{p'}} \|g\|_{L^p} \le C \|f\|_{N+1,0} \|g\|_{L^p} .$$

Hence, by (89) the functional T_g belongs to $\mathcal{S}'\left(\mathbb{R}^N\right)$ and the linear mapping $g \in L^p(\mathbb{R}^N) \mapsto T_g$ is a continuous embedding. Indeed, given $\varepsilon > 0$ and a finite set $F \subset \mathcal{S}(\mathbb{R}^N)$,

$$|T_g(f) - T_h(f)| \le C \|f\|_{N+1,0} \|g - h\|_{L^p} < \varepsilon$$

for all $h \in L^p(\mathbb{R}^N)$ with $\|g - h\|_{L^p} < \frac{\varepsilon}{1 + C \max_{f \in F} \|f\|_{N+1,0}}$. ■

Remark 146 *In what follows we identify g with T_g. Hence, $L^p(\mathbb{R}^N)$, and in particular $\mathcal{S}\left(\mathbb{R}^N\right)$, can be thought as contained in $\mathcal{S}'\left(\mathbb{R}^N\right)$.*

We now define the notion of a derivative of a tempered distribution. Let $g \in \mathcal{S}(\mathbb{R}^N)$ and consider the tempered distribution associated to g, that is,

$$T_g(f) := \int_{\mathbb{R}^N} f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x}, \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Given a multi-index $\boldsymbol{\alpha}$, it is natural to ask that the $\boldsymbol{\alpha}$-th *derivative* of T_g should be $T_{\partial^{\boldsymbol{\alpha}} g}$. Using integration by parts it follows that

$$T_{\partial^{\boldsymbol{\alpha}} g}(f) := \int_{\mathbb{R}^N} f(\mathbf{x}) \partial^{\boldsymbol{\alpha}} g(\mathbf{x})\, d\mathbf{x} = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^N} \partial^{\boldsymbol{\alpha}} f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x} = (-1)^{|\boldsymbol{\alpha}|} T_g(\partial^{\boldsymbol{\alpha}} f).$$

This motivates the following definition:

Definition 147 *Given* $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ *and a multi-index* $\boldsymbol{\alpha}$*, we define the* $\boldsymbol{\alpha}$*-th* derivative *of* T *as the linear functional* $\partial^{\boldsymbol{\alpha}} T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ *defined by*

$$\left(\partial^{\boldsymbol{\alpha}} T\right)(f) := (-1)^{|\boldsymbol{\alpha}|} T\left(\partial^{\boldsymbol{\alpha}} f\right), \quad f \in \mathcal{S}\left(\mathbb{R}^N\right).$$

Theorem 148 *For every* $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ *and every multi-index* $\boldsymbol{\alpha}$*, the functional* $\partial^{\boldsymbol{\alpha}} T$ *belongs to* $\mathcal{S}'\left(\mathbb{R}^N\right)$*.*

Proof. Since $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$, by Theorem 140 there exist a constant $C > 0$ and some $m, n \in \mathbb{N}_0$ such that

$$|T(f)| \leq C \|f\|_{m,n} .$$

for every $f \in \mathcal{S}\left(\mathbb{R}^N\right)$. In turn, since for $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, $\partial^{\boldsymbol{\alpha}} f$ still belongs to $\mathcal{S}\left(\mathbb{R}^N\right)$,

$$\left|\left(\partial^{\boldsymbol{\alpha}} T\right)(f)\right| = \left|T\left(\partial^{\boldsymbol{\alpha}} f\right)\right| \leq C \left\|\partial^{\boldsymbol{\alpha}} f\right\|_{m,n} \leq C \|f\|_{m,n+|\boldsymbol{\alpha}|},$$

and so, again by Theorem 140 it follows that $\partial^{\boldsymbol{\alpha}} T$ belongs to $\mathcal{S}'\left(\mathbb{R}^N\right)$. ■

Exercise 149 *The derivative of* $\log|x|$ *is the principal value.*

Exercise 150 *Prove that if* P *is a polynomial,* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$*, and* $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$*, then* PT *and* $fT \in \mathcal{S}'\left(\mathbb{R}^N\right)$*.*

Exercise 151 *Let* $g : \mathbb{R}^N \to \mathbb{C}$ *be a function of class* C^∞ *such that for every multi-index* $\boldsymbol{a}$ *there exist* $C_{\boldsymbol{a}}$ *and* $n_{\boldsymbol{a}} \in \mathbb{N}$ *such that*

$$|\partial^{\boldsymbol{\alpha}} g(\boldsymbol{x})| \leq C_{\boldsymbol{a}} (1 + \|\boldsymbol{x}\|^2)^{n_{\boldsymbol{a}}} \tag{92}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$*.*

- *(i) Prove that if* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ *then* $fg \in \mathcal{S}\left(\mathbb{R}^N\right)$*.*
- *(ii) Prove that if* $h : \mathbb{R}^N \to \mathbb{C}$ *is a measurable function such that* $hf \in \mathcal{S}\left(\mathbb{R}^N\right)$ *for all* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ *and the mapping* $f \in \mathcal{S}\left(\mathbb{R}^N\right) \mapsto hf$ *is continuous, then* h *must be of class* C^∞ *and satisfy (92).*
- *(iii) Given* $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ *prove that the linear functional* $gT : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ *defined by*

  $$(gT)(f) := T(fg), \quad f \in \mathcal{S}\left(\mathbb{R}^N\right),$$

  belongs to $\mathcal{S}'\left(\mathbb{R}^N\right)$*.*

13 Fourier Transforms

Given $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, the *Fourier transform of* f is the function

$$\widehat{f}(\boldsymbol{x}) = \mathcal{F}(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} \tag{93}$$

while the inverse *Fourier transform of* f is the function

$$f^{\vee}(\boldsymbol{x}) := \widehat{f}(-\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y}. \tag{94}$$

Since $\mathcal{S}\left(\mathbb{R}^N\right) \subset L^1\left(\mathbb{R}^N\right)$, the functions $\widehat{f}$ and $f^{\vee}$ are well-defined.

Wednesday, April 13, 2022

Theorem 152 *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}\left(\mathbb{R}^N\right)$ into $\mathcal{S}\left(\mathbb{R}^N\right)$. Moreover, for every $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$,*

$$\widehat{\partial^{\boldsymbol{\alpha}} f}(\boldsymbol{x}) = (2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\alpha}} \widehat{f}(\boldsymbol{x}), \quad \partial^{\boldsymbol{\beta}} \widehat{f}(\boldsymbol{x}) = \widehat{g_{\boldsymbol{\beta}}}(\boldsymbol{x}) \tag{95}$$

where $g_{\boldsymbol{\alpha}}(\boldsymbol{x}) := (-2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\beta}} f(\boldsymbol{x})$.

Proof. By (93),

$$\widehat{\partial^{\boldsymbol{\alpha}} f}(\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}}(\boldsymbol{y})\, d\boldsymbol{y}.$$

By integrating by parts and using the fact that f and its derivatives decay to zero at infinity we get

$$\widehat{\partial^{\boldsymbol{\alpha}} f}(\boldsymbol{x}) = (-1)^{|\boldsymbol{\alpha}|} \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\alpha}} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} = (2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\alpha}} \widehat{f}(\boldsymbol{x}).$$

This proves the first formula in (95).

To prove the second we differentiate under the integral sign to get

$$\begin{aligned} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} \frac{\partial^{\boldsymbol{\beta}}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}}) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} = \widehat{g_{\boldsymbol{\beta}}}(\boldsymbol{x}). \end{aligned}$$

Next we estimate $\|\widehat{f}\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. By (95) we have

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) &= \frac{1}{(-2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} (-2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\alpha}} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \frac{1}{(-2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y}) \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}}(e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}})\, d\boldsymbol{y}. \end{aligned}$$

By integrating by parts and using the fact that f and its derivatives decay to zero at infinity we get

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) = \frac{1}{(2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right) d\boldsymbol{y}.$$

It follows from Leibnitz rule that

$$\begin{aligned}
\|\widehat{f}\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} &\le C \int_{\mathbb{R}^N} \left| \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right) \right| d\boldsymbol{y} \\
&= C \int_{\mathbb{R}^N} \frac{1+\|\boldsymbol{y}\|^{N+1}}{1+\|\boldsymbol{y}\|^{N+1}} \left| \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right) \right| d\boldsymbol{y} \\
&\le C \|f\|_{N+1+|\boldsymbol{\beta}|,|\boldsymbol{\alpha}|},
\end{aligned}$$

which shows that $\widehat{f} \in \mathcal{S}(\mathbb{R}^N)$ and that the linear operator $\mathcal{F} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is continuous. ■

Example 153 *We compute the Fourier transform of the function* $f(\boldsymbol{x}) = e^{-\pi\|\boldsymbol{x}\|^2}$. *By Fubini's theorem and by completing the square we have*

$$\begin{aligned}
\widehat{f}(\boldsymbol{x}) &= \prod_{k=1}^{N} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i} x_k y_k - \pi y_k^2} dy_k \\
&= \prod_{k=1}^{N} e^{\pi(\boldsymbol{i}x_k)^2} \int_{\mathbb{R}} e^{-\pi(\boldsymbol{i}x_k+y_k)^2} dy_k.
\end{aligned}$$

Next observe that the function

$$g(x) := \int_{\mathbb{R}} e^{-\pi(\boldsymbol{i}x+y)^2} dy$$

is constant since

$$\begin{aligned}
g'(x) &= \int_{\mathbb{R}} -2\pi \boldsymbol{i}(\boldsymbol{i}x+y) e^{-\pi(\boldsymbol{i}x+y)^2} dy \\
&= \int_{\mathbb{R}} \boldsymbol{i} \frac{d}{dy}(e^{-\pi(\boldsymbol{i}x+y)^2})\, dy = 0.
\end{aligned}$$

Hence,

$$g(x) = g(0) = \int_{\mathbb{R}} e^{-\pi y^2} dy = 1.$$

If follows that $\widehat{f}(\boldsymbol{x}) = \prod_{k=1}^{N} e^{\pi(\boldsymbol{i}x_k)^2} = f(\boldsymbol{x})$.

Friday, April 15, 2022

Example 154 *Similarly, by taking*

$$f_\varepsilon(\boldsymbol{x}) = e^{2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2\|\boldsymbol{x}\|^2},$$

where $\varepsilon > 0$ *and* $\boldsymbol{x}_0 \in \mathbb{R}^N$ *we get*

$$\begin{aligned}
\widehat{f_\varepsilon}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} e^{2\pi \boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2\|\boldsymbol{y}\|^2}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{y}} e^{-\pi\varepsilon^2\|\boldsymbol{y}\|^2}\, d\boldsymbol{y} \\
&= \frac{1}{\varepsilon^N} \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\varepsilon^{-1}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{z}} e^{-\pi\|\boldsymbol{z}\|^2}\, d\boldsymbol{z} \\
&= \frac{1}{\varepsilon^N} \widehat{f}((\boldsymbol{x}-\boldsymbol{x}_0)/\varepsilon) = \frac{1}{\varepsilon^N} e^{-\pi\|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}\|^2}
\end{aligned}$$

where we have made the change of variables $\boldsymbol{z} := \varepsilon\boldsymbol{y}$*.*

Next we prove that $\mathcal{F}$ is invertible with inverse given by $\mathcal{F}^{-1}(f) = f^\vee$.

Proposition 155 *For every* f*,* $g \in \mathcal{S}\left(\mathbb{R}^N\right)$*, we have*

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}. \tag{96}$$

Proof. By Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\, d\boldsymbol{x} &= \int_{\mathbb{R}^N} f(\boldsymbol{x}) \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} g(\boldsymbol{y})\, d\boldsymbol{y} d\boldsymbol{x} \\
&= \int_{\mathbb{R}^N} g(\boldsymbol{y}) \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{x})\, d\boldsymbol{x} d\boldsymbol{y} \\
&= \int_{\mathbb{R}^N} g(\boldsymbol{y}) \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y},
\end{aligned}$$

which shows (96). ■

Theorem 156 (Fourier inversion theorem) *For every* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$*,*

$$(\widehat{f})^\vee = \widehat{(f^\vee)} = f.$$

In particular, the Fourier transform $\mathcal{F}$ *is an isomorphism from* $\mathcal{S}\left(\mathbb{R}^N\right)$ *to* $\mathcal{S}\left(\mathbb{R}^N\right)$ *with inverse* $\mathcal{F}^{-1}$ *given by* $\mathcal{F}^{-1}(f) = f^\vee$ *for every* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$*.*

Proof. Fix $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ and define $g_\varepsilon(\boldsymbol{x}) := e^{2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2\|\boldsymbol{x}\|^2}$. By Example 153 we have that $\widehat{g_\varepsilon}(\boldsymbol{x}) = \frac{1}{\varepsilon^N} e^{-\pi\|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}\|^2}$ and so, taking $g = g_\varepsilon$ in (96), we get

$$\int_{\mathbb{R}^N} f(\boldsymbol{x}) \frac{1}{\varepsilon^N} e^{-\pi\|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}\|^2}\, d\boldsymbol{x} = \int_{\mathbb{R}^N} e^{2\pi \boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2\|\boldsymbol{y}\|^2} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y}.$$

Note that $\widehat{g_\varepsilon}$ is a mollifier. Hence, the left-hand side converges to $f(\boldsymbol{x}_0)$. On the other hand, by the Lebesgue dominated convergence theorem the right-hand side converges to $(\widehat{f})^\vee(\boldsymbol{x}_0)$. Hence,

$$f(\boldsymbol{x}_0) = (\widehat{f})^\vee(\boldsymbol{x}_0)$$

which shows that $(\widehat{f})^\vee = f$. Similarly we can show that, $\widehat{(f^\vee)} = f$.

Next observe that if $\widehat{f} = 0$, then $f = (\widehat{f})^\vee = 0^\vee = 0$, and so $\mathcal{F}$ is one-to-one. Since $\widehat{(f^\vee)} = f$, it follows that $\mathcal{F}$ is onto and that the inverse of $\mathcal{F}$ is $\mathcal{F}^{-1}(f) = f^\vee$. ■

We recall that for a complex number $z = \operatorname{Re} z + \boldsymbol{i} \operatorname{Im} z$, the complex conjugate of z is the number $\overline{z} := \operatorname{Re} z - \boldsymbol{i} \operatorname{Im} z$.

Corollary 157 *For every $f,\, h \in \mathcal{S}\left(\mathbb{R}^N\right)$,*

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\overline{h(\boldsymbol{x})}\, d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})\overline{\widehat{h}(\boldsymbol{x})}\, d\boldsymbol{x} \qquad \text{Parseval identity}$$

and

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^N} |f^\vee(\boldsymbol{x})|^2 d\boldsymbol{x}. \qquad \text{Plancherel identity}$$

In particular, $\mathcal{F}$ extends uniquely to an isomorphism of $L^2\left(\mathbb{R}^N\right)$ onto itself.

Proof. Let $g := \overline{\widehat{h}}$. Then, using the facts that cos is even and sin is odd, we have

$$\begin{aligned}
\widehat{g}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}}\overline{\widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} = \int_{\mathbb{R}^N} \overline{e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}}}\,\overline{\widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^N} \overline{e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}}\widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} \\
&= \overline{\int_{\mathbb{R}^N} e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}}\widehat{h}(\boldsymbol{y})\, d\boldsymbol{y}} = \overline{(\widehat{h})^\vee(\boldsymbol{x})} = \overline{h(\boldsymbol{x})},
\end{aligned}$$

where in the last equality we have used the inversion theorem. Hence, Parseval's identity follows by (96). Taking $h = f$ and using the fact that $f(\boldsymbol{x})\overline{f(\boldsymbol{x})} = |f(\boldsymbol{x})|^2$ gives the first equality Plancherel's identity. The second equality follows by replacing f with $f^\vee$ and using the inversion theorem.

Since $\mathcal{S}\left(\mathbb{R}^N\right)$ is dense in $L^2\left(\mathbb{R}^N\right)$, if $\{f_n\} \subset \mathcal{S}\left(\mathbb{R}^N\right)$ converges to f in $L^2\left(\mathbb{R}^N\right)$, then by Plancherel's identity the sequence $\widehat{\{f_n\}}$ is a Cauchy sequence in $L^2\left(\mathbb{R}^N\right)$ and so it converges to a function $g \in L^2\left(\mathbb{R}^N\right)$. Again by Plancherel's identity, the function g does not depend on the particular sequence $\{f_n\}$. We define $\widehat{f} := g$. Similar we can extend uniquely the inverse Fourier transform to $L^2\left(\mathbb{R}^N\right)$ and reasoning as in the last part of the proof of the inversion theorem we have that the Fourier transform $\mathcal{F} : L^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right)$ is an isomorphism with inverse given by the extension of $\mathcal{F}^{-1}$ to $L^2\left(\mathbb{R}^N\right)$. ■

Remark 158 (Important) *Note that the Fourier transform of a function f in $L^2(\mathbb{R}^N)$ is obtained as a limit in $L^2(\mathbb{R}^N)$ of functions of the type (93), but in general we cannot say that $\widehat{f}$ has the form (93), since the integral in (93) is well-defined for functions in $L^1(\mathbb{R}^N)$ but not for functions in $L^2(\mathbb{R}^N)$. On the other hand, if $f \in L^1(\mathbb{R}^N)$, then (93) is well-defined. Hence, the Fourier transform of a function in $L^1(\mathbb{R}^N)$ is defined pointwise by (93), while the Fourier transform of a function in $L^2(\mathbb{R}^N)$ is defined as a limit in $L^2(\mathbb{R}^N)$.*

Another consequence of the L^2 theory is that it allows us to define the Fourier transform for functions in $L^p(\mathbb{R}^N)$ for $1 < p < 2$. More generally, given $f \in L^1(\mathbb{R}^N) + L^2(\mathbb{R}^N)$, we can write $f = g + h$, where $g \in L^1(\mathbb{R}^N)$ and $h \in L^2(\mathbb{R}^N)$. We define *the Fourier transform of f* as $\widehat{f} := \widehat{g} + \widehat{h}$. To see that this is a good definition, let $f = g_1 + h_1 = g_2 + h_2$, where $g_i \in L^1(\mathbb{R}^N)$ and $h_i \in L^2(\mathbb{R}^N)$, $i = 1, 2$. Then $g_1 - g_2 = h_1 - h_2 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Since the two definitions of Fourier tranforms coincide for functions in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, we have that $\widehat{g_1} - \widehat{g_2} = \widehat{h_1} - \widehat{h_2}$, that is, $\widehat{g_1} + \widehat{h_1} = \widehat{g_2} + \widehat{h_2}$, which shows that the definition of $\widehat{f}$ is independent of the decomposition of f. In particular, since $L^p(\mathbb{R}^N) \subseteq L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ for all $1 < p < 2$, we have defined the Fourier transform of every function $f \in L^p(\mathbb{R}^N)$ for $1 \le p \le 2$. Next we will show that $\widehat{f} \in L^{p'}(\mathbb{R}^N)$.

Monday, April 18, 2022

Theorem 159 *Let $f \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then the L^2 Fourier transform $\widehat{f}$ of f satisfies*

$$\widehat{f}(\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y}$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$.

Proof. Define $f_n := f\chi_{B(\mathbf{0},n)}$. Since $|f_n| \le |f|$, by the Lebesgue dominated convergence theorem, $f_n \to f$ in $L^1(\mathbb{R}^N)$ and $f_n \to f$ in $L^2(\mathbb{R}^N)$. Consider $f_n * \varphi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$. By the properties of mollifiers, $f_n * \varphi_\varepsilon \to f_n$ in $L^1(\mathbb{R}^N)$ and $f_n * \varphi_\varepsilon \to f_n$ in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0^+$. Hence,

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0^+} \left(\|f_n * \varphi_\varepsilon - f\|_{L^1(\mathbb{R}^N)} + \|f_n * \varphi_\varepsilon - f\|_{L^2(\mathbb{R}^N)} \right) = 0.$$

Hence, by a diagonal argument, we can construct $g_k \in C_c^\infty(\mathbb{R}^N)$ such that

$$\lim_{k\to\infty} \left(\|g_k - f\|_{L^1(\mathbb{R}^N)} + \|g_k - f\|_{L^2(\mathbb{R}^N)} \right) = 0.$$

By selecting a further subsequence, we can assume that $g_k \to f$ pointwise $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$ and that $|g_k(\boldsymbol{x})| \le h(\boldsymbol{x})$ for all k and for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$, where h is a Lebesgue integrable function. Since

$$\widehat{g_k}(\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} g_k(\boldsymbol{y})\, d\boldsymbol{y}$$

and $|e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} g_k(\boldsymbol{y})| \leq |g_k(\boldsymbol{y})| \leq h(\boldsymbol{y})$ for all k and for $\mathcal{L}^N$-a.e. $\boldsymbol{y} \in \mathbb{R}^N$, it follows by the Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} g_k(\boldsymbol{y})\, d\boldsymbol{y} \to \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y}.$$

On the other hand, since $g_k \to f$ in $L^2(\mathbb{R}^N)$, we have that $\widehat{g_k} \to \widehat{f}$ in $L^2(\mathbb{R}^N)$. By selecting another subsequence, we have that $\widehat{g_k} \to \widehat{f}$ pointwise $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. ■

Given an open set $\Omega \subseteq \mathbb{R}^N$, the space $C_0(\Omega)$ is defined as the space of all continuous functions f such that for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|f(\boldsymbol{x})| < \varepsilon$ for all $\boldsymbol{x} \in \Omega \setminus K$.

Theorem 160 (Riemann–Lebesgue lemma) $\mathcal{F} : L^1(\mathbb{R}^N) \to C_0(\mathbb{R}^N)$, *with*

$$\sup_{\boldsymbol{x}\in\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})| \leq \|f\|_{L^1(\mathbb{R}^N)} \tag{97}$$

In particular,

$$\lim_{|\boldsymbol{x}|\to\infty} |\widehat{f}(\boldsymbol{x})| = 0.$$

Proof. By (93), for every $f \in L^1(\mathbb{R}^N)$,

$$|\widehat{f}(\boldsymbol{x})| \leq \|f\|_{L^1(\mathbb{R}^N)}$$

for every $\boldsymbol{x} \in \mathbb{R}^N$. Since $\mathcal{S}\left(\mathbb{R}^N\right)$ is dense in $L^1\left(\mathbb{R}^N\right)$, let $\{f_n\}_n$ in $\mathcal{S}\left(\mathbb{R}^N\right)$ converge to f in $L^1\left(\mathbb{R}^N\right)$. By the previous inequality

$$\sup_{\boldsymbol{x}\in\mathbb{R}^N} |\widehat{f_n}(\boldsymbol{x}) - \widehat{f}(\boldsymbol{x})| \leq \|f_n - f\|_{L^1(\mathbb{R}^N)}.$$

Hence, the sequence $\{\widehat{f_n}\}_n$ converges uniformly to $\widehat{f}$. On the other hand, by Theorem 152 we have that $\widehat{f_n} \in \mathcal{S}\left(\mathbb{R}^N\right) \subset C_0(\mathbb{R}^N)$ and hence, since $C_0(\mathbb{R}^N)$ is a closed under uniform convergence, it follows that $\widehat{f} \in C_0(\mathbb{R}^N)$. ■

Using the previous theorems we can show that Fourier transform maps $L^p(\mathbb{R}^N)$ for $1 < p < 2$ into $L^{p'}(\mathbb{R}^N)$.

Exercise 161 *Let* $1 \leq p \leq \infty$ *and* $1 \leq r < q \leq \infty$. *Prove that*

$$\|f\|_{L^{p,q}(\mathbb{R}^N)} \leq C\, \|f\|_{L^{p,r}(\mathbb{R}^N)}$$

for all $f \in L^{p,r}(\mathbb{R}^N)$.

Corollary 162 (Hausdorff–Young inequality) *Let* $1 < p < 2$. *Then*

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^N)} \leq C\, \|f\|_{L^p(\mathbb{R}^N)} \tag{98}$$

for every $f \in L^p\left(\mathbb{R}^N\right)$.

Proof. By (97), we have that $\mathcal{F} : L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$, with

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^N)} \le \|f\|_{L^1(\mathbb{R}^N)},$$

while by the Plancherel identity $\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$,

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^N)} = \|f\|_{L^2(\mathbb{R}^N)}.$$

Moreover, we have defined the Fourier transform for functions in $L^1(\mathbb{R}^N) + L^2(\mathbb{R}^N)$, by $\widehat{f} := \widehat{g} + \widehat{h}$. Hence,

$$\mathcal{F} : L^1(\mathbb{R}^N) + L^2(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N) + L^2(\mathbb{R}^N).$$

Hence, by Theorem 112, for every $\sigma \in (0,1)$ and $1 \le q \le \infty$,

$$\mathcal{F} : (L^1(\mathbb{R}^N), L^2(\mathbb{R}^N))_{\sigma,q} \to (L^\infty(\mathbb{R}^N), L^2(\mathbb{R}^N))_{\sigma,q},$$

with

$$\|\mathcal{F}\|_{L((L^1(\mathbb{R}^N),L^2(\mathbb{R}^N))_{\sigma,p};(L^\infty(\mathbb{R}^N),L^2(\mathbb{R}^N))_{\sigma,p})} \le \|\mathcal{F}\|^{1-\sigma}_{L(L^1(\mathbb{R}^N);L^\infty(\mathbb{R}^N))} \|\mathcal{F}\|^{\sigma}_{L(L^2(\mathbb{R}^N);L^2(\mathbb{R}^N))} \le 1.$$

By Theorem 130,

$$(L^1(\mathbb{R}^N), L^2(\mathbb{R}^N))_{\sigma,p} = L^{p,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N),$$

where $\frac{1}{p} = \frac{1-\sigma}{1} + \frac{\sigma}{2}$ and

$$(L^\infty(\mathbb{R}^N), L^2(\mathbb{R}^N))_{\sigma,p} = L^{r,p}(\mathbb{R}^N),$$

where $\frac{1}{r} = \frac{1-\sigma}{\infty} + \frac{\sigma}{2}$. Note that $p = \frac{2}{2-\theta} \in (0,1)$ and $r = p' = \frac{2}{2-\theta}/(\frac{2}{2-\theta} - 1) = 2/\theta$. Thus,

$$\|\mathcal{F}(f)\|_{L^{p',p}(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}$$

for every $f \in L^p(\mathbb{R}^N)$.

Since $p < p'$, by the previous exercise,

$$\|\mathcal{F}(f)\|_{L^{p',p'}(\mathbb{R}^N)} \le C\,\|\mathcal{F}(f)\|_{L^{p',p}(\mathbb{R}^N)} \le C\|f\|_{L^p(\mathbb{R}^N)}.$$

■

We can also define the Fourier transform of tempered distributions. Given $g \in \mathcal{S}(\mathbb{R}^N)$, consider the linear functional $T : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ defined by

$$T_g(f) := \int_{\mathbb{R}^N} fg\,d\boldsymbol{x}. \tag{99}$$

By (96), for every $f \in \mathcal{S}(\mathbb{R}^N)$, we have

$$T_{\widehat{g}}(f) = \int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\,d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})g(\boldsymbol{x})\,d\boldsymbol{x} = T_g(\widehat{f}) = \widehat{T_g}(f).$$

This motivates the following definition.

Given $T \in \mathcal{S}'(\mathbb{R}^N)$, the *Fourier transform* $\widehat{T}$ *of* T is the tempered distribution given by

$$\widehat{T}(f) := T(\widehat{f}), \quad f \in \mathcal{S}(\mathbb{R}^N). \tag{100}$$

Similarly, given $T \in \mathcal{S}'(\mathbb{R}^N)$, the *inverse Fourier transform of* T is the tempered distribution given by

$$T^{\vee}(f) := T(f^{\vee}), \quad f \in \mathcal{S}(\mathbb{R}^N). \tag{101}$$

Since we are identifying g with T_g in $\mathcal{S}'\left(\mathbb{R}^N\right)$, this shows that the Fourier transform defined on $\mathcal{S}'\left(\mathbb{R}^N\right)$ extends the Fourier transform defined in $\mathcal{S}\left(\mathbb{R}^N\right)$. In view of Theorem 145, for every function $g \in L^p(\mathbb{R}^N)$ with $p > 2$, the Fourier transform $\widehat{g}$ of g is the Fourier transform $\widehat{T_g}$ of the tempered distribution T_g. Hence, $\widehat{g}$ belongs to $\mathcal{S}'(\mathbb{R}^N)$ but in general $\widehat{g}$ cannot be identified with a function. A simple example is given by $g = 1 \in L^\infty(\mathbb{R}^N)$. In this case

$$T_1(f) = \int_{\mathbb{R}^N} f 1 \, dx, \quad f \in \mathcal{S}(\mathbb{R}^N), \tag{102}$$

and so by inverse Fourier theorem,

$$\widehat{1}(f) = \widehat{T_1}(f) = \int_{\mathbb{R}^N} \widehat{f}(x)\, dx = \int_{\mathbb{R}^N} e^{2\pi i 0 \cdot y} \widehat{f}(x)\, dx = (\widehat{f})^{\vee}(0) = f(0),$$

which shows that $\widehat{1}$ is δ_0.

Exercise 163 *Let* $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$.

(i) Prove that $\widehat{T} \in \mathcal{S}'\left(\mathbb{R}^N\right)$.

(ii) Prove that if $\{T_n\} \subset \mathcal{S}'\left(\mathbb{R}^N\right)$ *is such that* $T_n \overset{*}{\rightharpoonup} T$ *in* $\mathcal{S}'\left(\mathbb{R}^N\right)$, *then* $\widehat{T_n} \overset{*}{\rightharpoonup} \widehat{T}$.

(iii) Prove that $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^N\right) \to \mathcal{S}'\left(\mathbb{R}^N\right)$ *is a bijection.*

As another application of Corollary 157, we can give a characterization of $H^1(\mathbb{R}^N)$ in terms of Fourier transforms.

Theorem 164 *A function* $f \in L^2(\mathbb{R}^N)$ *belongs to the space* $H^1(\mathbb{R}^N) := W^{1,2}(\mathbb{R}^N)$ *if and only if*

$$\int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty.$$

Moreover, for every $f \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x},$$

$$\sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i f(\boldsymbol{x})|^2 d\boldsymbol{x} = 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

In particular,

$$f \mapsto \left(\int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|^2) |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} \right)^{1/2}$$

is an equivalent norm in $H^1(\mathbb{R}^N)$*.*

We begin with a preliminary lemma.

Lemma 165 *Let* $s > 0$ *and* $g \in L^2(\mathbb{R}^N)$ *be such that*

$$\int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |g(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty.$$

Then there exists $g_n \in C_c^\infty(\mathbb{R}^N)$ *such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g - g_n)(\boldsymbol{x})|^2 d\boldsymbol{x} = 0.$$

Proof. Step 1: Let $g \in L^2(\mathbb{R}^N)$ be such that $g = 0$ for $\|\boldsymbol{x}\| > R$ for some $R > 0$. Consider $g * \varphi_\varepsilon$, where φ is a standard mollifier. Then $g * \varphi_\varepsilon \to g$ in $L^2(\mathbb{R}^N)$. Moreover, if $0 < \varepsilon < 1$ and $\|\boldsymbol{x}\| > R + 1$, then

$$(g * \varphi_\varepsilon)(\boldsymbol{x}) = \int_{\mathbb{R}^N} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y} = \int_{B(\boldsymbol{x},\varepsilon)\cap B(\boldsymbol{0},R)} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y} = 0.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g - g * \varphi_\varepsilon)(\boldsymbol{x})|^2 d\boldsymbol{x} &= \int_{B(\boldsymbol{0},R+1)} (1 + \|\boldsymbol{x}\|^{2s}) |(g - g * \varphi_\varepsilon)(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &\le (1 + (R+1)^{2s}) \int_{B(\boldsymbol{0},R+1)} |(g - g * \varphi_\varepsilon)(\boldsymbol{x})|^2 d\boldsymbol{x} \to 0 \end{aligned}$$

as $\varepsilon \to 0^+$.

Step 2: Let g be as in the statement. Given $n \in \mathbb{N}$ consider the function $h_n := g\chi_{B(\boldsymbol{0},n)}$. Since $|g - h_n| \le 2|g|$, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g - h_n)(\boldsymbol{x})|^2 d\boldsymbol{x} = 0.$$

Each function g_k satisfies the hypotheses of Step 1. Hence, we can find $g_n \in C_c^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g_n - h_n)(\boldsymbol{x})|^2 d\boldsymbol{x} \le \frac{1}{n}.$$

In turn,

$$\begin{aligned} \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g - g_n)(\boldsymbol{x})|^2 d\boldsymbol{x} &\le 2 \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g - h_n)(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &\quad + 2 \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|)^{2s} |(g_n - h_n)(\boldsymbol{x})|^2 d\boldsymbol{x} \to 0 \end{aligned}$$

as $n \to \infty$. ■

Remark 166 *If in the lemma we require*

$$\int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|^2)^s |g(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty,$$

then there is a simpler proof (suggested by Spencer). Since the function $h(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{s/2} g(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^N$, *belongs to* $L^2(\mathbb{R}^N)$, *there exists a sequence* $h_n \in C_c^\infty(\mathbb{R}^N)$ *such that* $h_n \to h$ *in* $L^2(\mathbb{R}^N)$. *Define* $g_n(\boldsymbol{x}) := (1 + \|\boldsymbol{x}\|^2)^{-s/2} h_n(\boldsymbol{x})$. *Then* $g_n \in C_c^\infty(\mathbb{R}^N)$ *and satisfy the thesis of the lemma.*

We turn to the proof of Theorem 164.

Proof. Step 1: Given $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, by Theorem 152, for every $i = 1, \ldots, N$ and every $\boldsymbol{x} \in \mathbb{R}^N$,

$$\widehat{\partial_i f}(\boldsymbol{x}) = 2\pi \boldsymbol{i} x_i \widehat{f}(\boldsymbol{x}).$$

Hence, by the previous corollary

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x}$$

and

$$\begin{aligned}\sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i f(\boldsymbol{x})|^2 d\boldsymbol{x} &= \sum_{i=1}^N \int_{\mathbb{R}^N} |2\pi \boldsymbol{i} x_i \widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} = 4\pi^2 \int_{\mathbb{R}^N} \sum_{i=1}^N x_i^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &= 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x}.\end{aligned}$$

Step 2: Let $f \in L^2(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty.$$

Then by the previous lemma there exists $g_n \in C_c^\infty(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (1 + \|\boldsymbol{x}\|^{2s}) |(\widehat{f} - g_n)(\boldsymbol{x})|^2 d\boldsymbol{x} = 0.$$

Define $f_n := g_n^\vee$. By Theorem 152, $f_n \in \mathcal{S}\left(\mathbb{R}^N\right)$, so that, using the previous step and the Fourier inversion theorem

$$\int_{\mathbb{R}^N} |(f_n - f_k)(\boldsymbol{x})|^2 d\boldsymbol{x} = \int_{\mathbb{R}^N} |(g_n - g_k)(\boldsymbol{x})|^2 d\boldsymbol{x} \to 0$$

and

$$\sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i (f_n - f_k)(\boldsymbol{x})|^2 d\boldsymbol{x} = 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |(g_n - g_k)(\boldsymbol{x})|^2 d\boldsymbol{x} \to 0$$

as $n, k \to \infty$. Thus, $\{f_n\}_n$ is a Cauchy sequence in $H^1(\mathbb{R}^N)$, so $f_n \to h$ in in $H^1(\mathbb{R}^N)$ for some function $h \in H^1(\mathbb{R}^N)$. But since $g_n \to \widehat{f}$ in $L^2(\mathbb{R}^N)$, we have that $f_n = g_n^\vee \to f$ in in $L^2(\mathbb{R}^N)$. Hence, $f = h \in H^1(\mathbb{R}^N)$.

Step 3: Let $f \in H^1(\mathbb{R}^N)$. Then by the density of smooth functions we can find $f_n \in C_c^\infty(\mathbb{R}^N)$ such that $f_n \to f$ in $H^1(\mathbb{R}^N)$. In turn, $\widehat{f_n} \to \widehat{f}$ in $L^2(\mathbb{R}^N)$. By extracting a subsequence, not relabeled, we can assume that $\widehat{f_n} \to \widehat{f}$ pointwise $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. By Step 1,

$$4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |(\widehat{f_n} - \widehat{f}_k)(\boldsymbol{x})|^2 d\boldsymbol{x} = \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i (f_n - f_k)(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

Letting $k \to \infty$, it follows by Fatou's lemma that

$$\begin{aligned} 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |(\widehat{f_n} - \widehat{f})(\boldsymbol{x})|^2 d\boldsymbol{x} &\le \liminf_{k\to\infty} 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |(\widehat{f_n} - \widehat{f}_k)(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &= \liminf_{k\to\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i (f_n - f_k)(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &= \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i (f_n - f)(\boldsymbol{x})|^2 d\boldsymbol{x}. \end{aligned}$$

Letting $n \to \infty$ shows that

$$\lim_{n\to\infty} 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |(\widehat{f_n} - \widehat{f})(\boldsymbol{x})|^2 d\boldsymbol{x} = \lim_{n\to\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i (f_n - f)(\boldsymbol{x})|^2 d\boldsymbol{x} = 0.$$

In particular,

$$\begin{aligned} 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} &= \lim_{n\to\infty} 4\pi^2 \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^2 |\widehat{f_n}(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &= \lim_{n\to\infty} \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i f_n(\boldsymbol{x})|^2 d\boldsymbol{x} = \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i f(\boldsymbol{x})|^2 d\boldsymbol{x}. \end{aligned}$$

■

Friday, April 22, 2022

In your homework you will show the following result.

Theorem 167 *Let $0 < s < 1$ and let $f \in L^2(\mathbb{R}^N)$. Then $f \in W^{s,2}(\mathbb{R}^N)$ if and only*

$$\int_{\mathbb{R}^N} \|\boldsymbol{x}\|^{2s} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty.$$

Moreover, there exists a constant $C = C(N, s) > 0$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x})|^2}{\|\boldsymbol{h}\|^{N+2s}} d\boldsymbol{x} d\boldsymbol{h} = C \int_{\mathbb{R}^N} \|\boldsymbol{x}\|^{2s} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

Definition 168 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \le p < \infty$, and $s > 1$, with $s \notin \mathbb{N}$. A function $f \in L^p(\Omega)$ belongs to the fractional Sobolev space $W^{s,p}(\Omega)$*

if $f \in W^{\lfloor s\rfloor,p}(\Omega)$ *and for every multi-index* $\alpha \in \mathbb{N}_0^N$*, with* $|\alpha| = \lfloor s\rfloor$*,* $\partial^\alpha f \in W^{s-\lfloor s\rfloor,p}(\Omega)$*. We endow* $W^{s,p}(\Omega)$ *with the norm*

$$\|f\|_{W^{s,p}(\Omega)} := \|f\|_{W^{\lfloor s\rfloor,p}(\Omega)} + \sum_{|\alpha|=\lfloor s\rfloor} \|\partial^\alpha f\|_{W^{s-\lfloor s\rfloor,p}(\Omega)}.$$

When $p = 2$*, we write* $W^{s,2}(\Omega) =: H^s(\Omega)$*.*

Exercise 169 *Let* $s > 1$ *and* $f \in L^2(\mathbb{R}^N)$*. Prove that* $f \in H^s(\mathbb{R}^N)$ *if and only*

$$\int_{\mathbb{R}^N} \|\boldsymbol{x}\|^{2s} |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty.$$

Let's use Fourier transforms to prove Morrey's embedding theorem.

Theorem 170 (Morrey) *Let* $s = N/2 + \alpha$*, where* $0 < \alpha < 1$*, and let* $f \in H^s(\mathbb{R}^N)$*, then* f *admits a representative that is Hölder continuous of exponent* α*.*

Proof. Step 1: Let's prove first that f has a representative in $C_0(\mathbb{R}^N)$. By Hölder's inequality

$$\begin{aligned}\int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})|\, d\boldsymbol{x} &= \int_{\mathbb{R}^N} (1+\|\boldsymbol{x}\|^2)^{-s/2}(1+\|\boldsymbol{x}\|^2)^{s/2}|\widehat{f}(\boldsymbol{x})|\, d\boldsymbol{x} \\ &\le \left(\int_{\mathbb{R}^N} \frac{1}{(1+\|\boldsymbol{x}\|^2)^s} d\boldsymbol{x}\right)^{1/2} \left(\int_{\mathbb{R}^N} (1+\|\boldsymbol{x}\|^2)^s |\widehat{f}(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{1/2} \\ &= C\|f\|_{H^s(\mathbb{R}^N)}.\end{aligned}$$

Hence, $\widehat{f} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. In particular, by Theorem 159 (which continues to hold for the inverse Fourier transform) the inverse Fourier tranform of $\widehat{f}$ is given by

$$(\widehat{f})^\vee(\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y} \quad \text{for } \mathcal{L}^N\text{-a.e. } \boldsymbol{x} \in \mathbb{R}^N,$$

while by the Fourier inversion theorem the right-hand side is given by $f(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$. Since $\widehat{f} \in L^1(\mathbb{R}^N)$, it follows by the Riemann-Lebesgue lemma that the function

$$g(\boldsymbol{x}) := \int_{\mathbb{R}^N} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^N,$$

belongs to $C_0(\mathbb{R}^N)$, with

$$\|g\|_{C_0(\mathbb{R}^N)} \le \|\widehat{f}\|_{L^1(\mathbb{R}^N)} \le C\|f\|_{H^s(\mathbb{R}^N)}$$

Step 2: Let $\boldsymbol{h}, \boldsymbol{x} \in \mathbb{R}^N$ with $0 < \|\boldsymbol{h}\| \le 1/2$. Then by Hölder's inequality,

$$\begin{aligned}|g(\boldsymbol{x}+\boldsymbol{h}) - g(\boldsymbol{x})| &= \left|\int_{\mathbb{R}^N} [e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1] e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y}\right| \\ &= \left|\int_{\mathbb{R}^N} (1+\|\boldsymbol{y}\|^2)^{-s/2}(1+\|\boldsymbol{y}\|^2)^{s/2}[e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1] e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y}\right| \\ &\le \left(\int_{\mathbb{R}^N} \frac{|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y}\right)^{1/2} \left(\int_{\mathbb{R}^N} (1+\|\boldsymbol{y}\|^2)^s |\widehat{f}(\boldsymbol{y})|^2 d\boldsymbol{y}\right)^{1/2}.\end{aligned}$$

Write

$$\int_{\mathbb{R}^N} \frac{|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y} = \int_{B(\boldsymbol{0},1/\|\boldsymbol{h}\|)} \frac{|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y} + \int_{\mathbb{R}^N\setminus B(\boldsymbol{0},1/\|\boldsymbol{h}\|)} \frac{|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y}$$

Since $\sin t = t + o(t)$ and $1 - \cos t = \frac{t^2}{2} + o(t^2)$, for $\boldsymbol{y} \in B(\boldsymbol{0}, 1/\|\boldsymbol{h}\|)$ we have

$$\begin{aligned}|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2 &= (1-\cos(2\pi \boldsymbol{h}\cdot\boldsymbol{y}))^2 + \sin^2(2\pi \boldsymbol{h}\cdot\boldsymbol{y}) \\ &\le C\|\boldsymbol{h}\|^2\|\boldsymbol{y}\|^2,\end{aligned}$$

and so

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{|e^{2\pi i \boldsymbol{h}\cdot\boldsymbol{y}} - 1|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y} \le C\|\boldsymbol{h}\|^2 \int_{B(\boldsymbol{0},1/\|\boldsymbol{h}\|)} \frac{\|\boldsymbol{y}\|^2}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y} \\ &\quad + C\int_{\mathbb{R}^N\setminus B(\boldsymbol{0},1/\|\boldsymbol{h}\|)} \frac{1}{(1+\|\boldsymbol{y}\|^2)^s} d\boldsymbol{y} = C\|\boldsymbol{h}\|^2 \int_0^{1/\|\boldsymbol{h}\|} \frac{r^{N-1}r^2}{(1+r^2)^s} dr \\ &\quad + C\int_{1/\|\boldsymbol{h}\|}^{\infty} \frac{r^{N-1}}{(1+r^2)^s} dr \le C\|\boldsymbol{h}\|^2 \int_0^{1/\|\boldsymbol{h}\|} r^{N+1-2s} dr + C\int_{1/\|\boldsymbol{h}\|}^{\infty} \frac{1}{r^{2s-N+1}} dr \\ &\quad = C\frac{\|\boldsymbol{h}\|^2}{N+2-2s}\left[r^{N+2-2s}\right]_0^{1/\|\boldsymbol{h}\|} + \frac{C}{2s-N}\left[\frac{1}{r^{2s-N}}\right]_0^{1/\|\boldsymbol{h}\|} \int_{1/\|\boldsymbol{h}\|}^{\infty} \frac{1}{r^{2s-N+1}} dr \\ &\quad = C\frac{\|\boldsymbol{h}\|^2}{2(1-\alpha)}\frac{1}{\|\boldsymbol{h}\|^{2(1-\alpha)}} + \frac{C}{2\alpha}\|\boldsymbol{h}\|^{2\alpha} = C\|\boldsymbol{h}\|^{2\alpha},\end{aligned}$$

where we used the fact that $N + 2 - 2s = N + 2 - N - 2\alpha = 2(1-\alpha) > 0$ and $2s - N = 2\alpha > 0$. Hence,

$$\begin{aligned}|g(\boldsymbol{x}+\boldsymbol{h}) - g(\boldsymbol{x})| &\le C\|\boldsymbol{h}\|^\alpha \left(\int_{\mathbb{R}^N} (1+\|\boldsymbol{y}\|^2)^s |\widehat{f}(\boldsymbol{y})|^2 d\boldsymbol{y}\right)^{1/2} \\ &= C\|\boldsymbol{h}\|^\alpha \|f\|_{H^s(\mathbb{R}^N)}.\end{aligned}$$

On the other hand, if $\|\boldsymbol{h}\| > 1/2$, then

$$\begin{aligned}|g(\boldsymbol{x}+\boldsymbol{h}) - g(\boldsymbol{x})| &\le 2\|g\|_{C_0(\mathbb{R}^N)} = 2\frac{\|\boldsymbol{h}\|^\alpha}{\|\boldsymbol{h}\|^\alpha}\|g\|_{C_0(\mathbb{R}^N)} \\ &\le 2^{1+\alpha}\|\boldsymbol{h}\|^\alpha \|f\|_{H^s(\mathbb{R}^N)}.\end{aligned}$$

■

14 Convolutions

Given two measurable functions $f : \mathbb{R}^N \to \mathbb{C}$ and $g : \mathbb{R}^N \to \mathbb{C}$, the *convolution* of f and g is the function $f * g$ defined by

$$(f * g)(\boldsymbol{x}) := \int_{\mathbb{R}^N} f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y} \tag{103}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$ for which the right-hand side is well-defined.

Theorem 171 *Given $f, g \in \mathcal{S}(\mathbb{R}^N)$, the function $f * g$ belongs to $\mathcal{S}(\mathbb{R}^N)$.*

Proof. Fix $\boldsymbol{x} \in \mathbb{R}^N$. For $m \in \mathbb{N}$ with $m > N$, we can write

$$\begin{aligned} |(f * g)(\boldsymbol{x})| &\le \int_{\mathbb{R}^N} |f(\boldsymbol{x} - \boldsymbol{y})||g(\boldsymbol{y})|\, d\boldsymbol{y} \\ &\le C \|g\|_{0,m} \|f\|_{0,m} \int_{\mathbb{R}^N} \frac{1}{(1 + \|\boldsymbol{y}\|)^m} \frac{1}{(1 + \|\boldsymbol{x} - \boldsymbol{y}\|)^m}\, d\boldsymbol{y}. \end{aligned}$$

We now split $\mathbb{R}^N$ in the sets $E := \{\boldsymbol{y} \in \mathbb{R}^N : \frac{1}{2}\|\boldsymbol{x}\| \le \|\boldsymbol{x} - \boldsymbol{y}\|\}$ and $\mathbb{R}^N \setminus E$. Then we have

$$\begin{aligned} &\int_E \frac{1}{(1 + \|\boldsymbol{y}\|)^m} \frac{1}{(1 + \|\boldsymbol{x} - \boldsymbol{y}\|)^m}\, d\boldsymbol{y} \\ &\le \frac{2^m}{(2 + \|\boldsymbol{x}\|)^m} \int_{\mathbb{R}^N} \frac{1}{(1 + \|\boldsymbol{y}\|)^m}\, d\boldsymbol{y} \le \frac{C(m, N)}{(2 + \|\boldsymbol{x}\|)^m}, \end{aligned}$$

while in $\mathbb{R}^N \setminus E$, $\|\boldsymbol{y}\| \ge \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{y}\| \ge \|\boldsymbol{x}\| - \frac{1}{2}\|\boldsymbol{x}\| = \frac{1}{2}\|\boldsymbol{x}\|$, and so

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus E} \frac{1}{(1 + \|\boldsymbol{y}\|)^m} \frac{1}{(1 + \|\boldsymbol{x} - \boldsymbol{y}\|)^m}\, d\boldsymbol{y} \\ &\le \frac{2^m}{(2 + \|\boldsymbol{x}\|)^m} \int_{\mathbb{R}^N} \frac{1}{(1 + \|\boldsymbol{x} - \boldsymbol{y}\|)^m}\, d\boldsymbol{y} \le \frac{C(m, N)}{(2 + \|\boldsymbol{x}\|)^m}. \end{aligned}$$

Hence,

$$(2 + \|\boldsymbol{x}\|)^m |(f * g)(\boldsymbol{x})| \le C \|g\|_{0,m} \|f\|_{0,m}.$$

This shows that f decays to zero faster than any power of $\|\boldsymbol{x}\|$.

On the other hand, by differentiating under the integral sign, for every multi-index $\boldsymbol{\alpha}$,

$$\begin{aligned} \frac{\partial^{\boldsymbol{\alpha}}(f * g)}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \left(\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} * g \right)(\boldsymbol{x}), \end{aligned}$$

and so by repeating the same calculations above with f replaced by $\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}$, we get that all derivatives of $f * g$ decay to zero faster than any power of $\|\boldsymbol{x}\|$, which shows that $f * g \in \mathcal{S}(\mathbb{R}^N)$. ■

Exercise 172 *Prove that for every* $f, g, h \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$(f * g) * h = f * (g * h).$$

Theorem 173 *For every* $f, g \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

Proof. For $\boldsymbol{x} \in \mathbb{R}^N$ by Fubini's theorem we have

$$\begin{aligned}
\widehat{(f * g)}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}}(f * g)(\boldsymbol{y})\, d\boldsymbol{y} \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f\left(\boldsymbol{y}-\boldsymbol{\xi}\right) g\left(\boldsymbol{\xi}\right)\, d\boldsymbol{\xi} d\boldsymbol{y} \\
&= \int_{\mathbb{R}^N} g\left(\boldsymbol{\xi}\right)\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f\left(\boldsymbol{y}-\boldsymbol{\xi}\right)\, d\boldsymbol{y} d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} g\left(\boldsymbol{\xi}\right)\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot(\boldsymbol{y}-\boldsymbol{\xi})} f\left(\boldsymbol{y}-\boldsymbol{\xi}\right)\, d\boldsymbol{y} d\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} g\left(\boldsymbol{\xi}\right)\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\eta}} f\left(\boldsymbol{\eta}\right)\, d\boldsymbol{\eta} d\boldsymbol{\xi} \\
&= \widehat{g}(\boldsymbol{x})\widehat{f}(\boldsymbol{x}),
\end{aligned}$$

where we have made the change of variables $\boldsymbol{\eta} := \boldsymbol{y} - \boldsymbol{\xi}$. ■

Remark 174 *The previous theorem continues to hold for* $f \in L^1\left(\mathbb{R}^N\right)$ *and* $g \in \mathcal{S}\left(\mathbb{R}^N\right)$.

Given two measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$,

Theorem 175 *Let* $f \in L^p\left(\mathbb{R}^N\right)$, $1 \le p \le \infty$, *and* $g \in L^1\left(\mathbb{R}^N\right)$. *Then* $(f * g)(\boldsymbol{x})$ *exists for* $\mathcal{L}^N$*-a.e.* $\boldsymbol{x} \in \mathbb{R}^N$ *and*

$$\|f * g\|_{L^p(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^1(\mathbb{R}^N)}.$$

Proof. Consider two Borel functions f_0 and g_0 such that $f_0(\boldsymbol{x}) = f(\boldsymbol{x})$ and $g_0(\boldsymbol{x}) = g(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$. Since the integral in (103) is unchanged if we replace f and g with f_0 and g_0, respectively, in what follows, without loss of generality we may assume that f and g are Borel functions.

Let $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$h(\boldsymbol{x}, \boldsymbol{y}) := f(\boldsymbol{x} - \boldsymbol{y}), \quad (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then h is a Borel function, since it is the composition of the Borel function f with the continuous function $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ given by $g(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} - \boldsymbol{y}$. In turn, the function

$$(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})$$

is Borel measurable. We are now in a position to apply Minkowski's inequality for integrals and Tonelli's theorem to conclude that

$$\begin{aligned}\|f * g\|_{L^p(\mathbb{R}^N)} &= \left\| \int_{\mathbb{R}^N} |f(\cdot - \boldsymbol{y})\, g(\boldsymbol{y})|\; d\boldsymbol{y} \right\|_{L^p(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} \|f(\cdot - \boldsymbol{y})\, g(\boldsymbol{y})\|_{L^p(\mathbb{R}^N)}\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} |g(\boldsymbol{y})|\, \|f(\cdot - \boldsymbol{y})\|_{L^p(\mathbb{R}^N)}\, d\boldsymbol{y} = \|f\|_{L^p(\mathbb{R}^N)} \int_{\mathbb{R}^N} |g(\boldsymbol{y})|\; d\boldsymbol{y},\end{aligned}$$

where in the last equality we have used the fact that the Lebesgue measure is translation invariant. Hence, $f * g$ belongs to $L^p\left(\mathbb{R}^N\right)$, and so it is finite $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. ■

Wednesday, April 27, 2022

The following is the generalized form of the previous inequality.

Theorem 176 (Young's inequality) *Let $1 \le p < q' \le \infty$ and let $f \in L^p\left(\mathbb{R}^N\right)$ and $g \in L^q\left(\mathbb{R}^N\right)$. Then $(f * g)(\boldsymbol{x})$ exists for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$ and*

$$\|f * g\|_{L^r(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{104}$$

Proof. If $p = 1$, then $r = q$ and the result follows from the previous theorem. Thus assume that $p > 1$. Fix $g \in L^q\left(\mathbb{R}^N\right)$ and consider the linear operator $L_g(h) := g * h$. By the previous theorem we have that $L_g : L^1(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is linear and continuous, with

$$\|L_g(h)\|_{L^q(\mathbb{R}^N)} \le \|g\|_{L^q(\mathbb{R}^N)} \|h\|_{L^1(\mathbb{R}^N)}.$$

Hence,

$$\|L_g\|_{L(L^1(\mathbb{R}^N);L^q(\mathbb{R}^N))} \le \|g\|_{L^q(\mathbb{R}^N)}.$$

Moreover, by Hölder's inequality for every $h \in L^{q'}\left(\mathbb{R}^N\right)$,

$$\begin{aligned}|L_g(h)(\boldsymbol{x})| &= \left| \int_{\mathbb{R}^N} h(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})\; d\boldsymbol{y} \right| \le \|g\|_{L^q(\mathbb{R}^N)} \|h(\boldsymbol{x} - \cdot)\|_{L^{q'}(\mathbb{R}^N)} \\ &= \|g\|_{L^q(\mathbb{R}^N)} \|h\|_{L^{q'}(\mathbb{R}^N)},\end{aligned}$$

where in the last equality we used the translation invariance of the Lebesgue measure. This shows that $L_g : L^{q'}(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ is linear and continuous, with

$$\|L_g(h)\|_{L^\infty(\mathbb{R}^N)} \le \|g\|_{L^q(\mathbb{R}^N)} \|h\|_{L^{q'}(\mathbb{R}^N)}.$$

Hence,

$$\|L_g\|_{L(L^{q'}(\mathbb{R}^N);L^\infty(\mathbb{R}^N))} \le \|g\|_{L^q(\mathbb{R}^N)}.$$

Next we observe that

$$L_g : L^1(\mathbb{R}^N) + L^{q'}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$

is well-defined. Hence, by Theorem 112, for every $\sigma \in (0,1)$,

$$L_g : (L^1(\mathbb{R}^N), L^{q'}(\mathbb{R}^N))_{\sigma,p} \to (L^q(\mathbb{R}^N), L^\infty(\mathbb{R}^N))_{\sigma,p},$$

with

$$\begin{aligned}\|L_g\|_{(L^q(\mathbb{R}^N),L^\infty(\mathbb{R}^N))_{\sigma,p}} &\le \|L_g\|^{1-\sigma}_{L(L^1(\mathbb{R}^N);L^{q'}(\mathbb{R}^N))}\|L_g\|^{\sigma}_{L(L^q(\mathbb{R}^N);L^\infty(\mathbb{R}^N))} \\ &\le \|g\|^{1-\sigma}_{L^q(\mathbb{R}^N)}\|g\|^{\sigma}_{L^q(\mathbb{R}^N)} = \|g\|_{L^q(\mathbb{R}^N)}.\end{aligned}$$

By Theorem 130,

$$(L^1(\mathbb{R}^N), L^2(\mathbb{R}^N))_{\sigma,p} = L^{p_3,p}(\mathbb{R}^N),$$

where $\frac{1}{p_3} = \frac{1-\sigma}{1} + \frac{\sigma}{q'} = \frac{1-\sigma}{1} + \frac{\sigma(q-1)}{q} = 1 - \frac{\sigma}{q}$, while

$$(L^q(\mathbb{R}^N), L^\infty(\mathbb{R}^N))_{\sigma,p} = L^{p_4,p}(\mathbb{R}^N),$$

where $\frac{1}{p_4} = \frac{1-\sigma}{q} + \frac{\sigma}{\infty}$. Since $p > 1$, it follows from (104) that $r > q$ and so we can find $0 < \sigma < 1$ such that $q = (1-\sigma)r$. Hence $p_4 = r$. In turn

$$\frac{1}{p} = 1 - \frac{1}{q} + \frac{1}{r} = 1 - \frac{1}{q} + \frac{1-\sigma}{q} = 1 - \frac{\sigma}{q},$$

which shows that $p_3 = p$.

$$\|L_g(h)\|_{L^{r,p}(\mathbb{R}^N)} \le \|L_g\|_{(L^q(\mathbb{R}^N),L^\infty(\mathbb{R}^N))_{\sigma,p}} \|h\|_{L^{p,p}(\mathbb{R}^N)} \le \|g\|_{L^q(\mathbb{R}^N)} \|h\|_{L^{p,p}(\mathbb{R}^N)}$$

Since $r > p$, it follows that

$$\|L_g(h)\|_{L^{r,r}(\mathbb{R}^N)} \le C\|L_g(h)\|_{L^{r,p}(\mathbb{R}^N)} \le C\,\|g\|_{L^q(\mathbb{R}^N)}\,\|h\|_{L^{p,p}(\mathbb{R}^N)}.$$

To conclude the proof it remains to show that the convolution is defined pointwise. This is left as an exercise. ■

15 Convolution of Tempered Distributions

In this section we define the convolution of a tempered distribution T and a function φ. We begin with the case in which $T = T_f$ for some function $f \in \mathcal{S}(\mathbb{R}^N)$, where we recall that $T_f \in \mathcal{S}'(\mathbb{R}^N)$ is defined by

$$T_f(\phi) := \int_{\mathbb{R}^N} \psi(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}, \quad \phi \in \mathcal{S}(\mathbb{R}^N).$$

By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^N} (f*\varphi)(\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x} &= \int_{\mathbb{R}^N} \phi(\boldsymbol{x}) \int_{\mathbb{R}^N} f(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,d\boldsymbol{y}d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} f(\boldsymbol{\xi}) \int_{\mathbb{R}^N} \varphi(\boldsymbol{x}-\boldsymbol{\xi})\phi(\boldsymbol{x})\,d\boldsymbol{x}d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^N} f(\boldsymbol{\xi}) \int_{\mathbb{R}^N} \widetilde{\varphi}(\boldsymbol{\xi}-\boldsymbol{x})\phi(\boldsymbol{x})\,d\boldsymbol{x}d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^N} f(\boldsymbol{\xi})(\widetilde{\varphi}*\phi)(\boldsymbol{\xi})\,d\boldsymbol{\xi},\end{aligned}$$

where $\boldsymbol{\xi} := \boldsymbol{x} - \boldsymbol{y}$ and $\widetilde{\varphi}(\boldsymbol{x}) := \varphi(-\boldsymbol{x})$. Hence, we have shown that $T_{f*\varphi}(\phi) = T_f(\widetilde{\varphi} * \phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^N)$. Motivated by this formula we define:

Definition 177 *If $T \in \mathcal{S}'(\mathbb{R}^N)$ and $\varphi \in \mathcal{S}(\mathbb{R}^N)$ the* convolution *of T and φ is the linear functional $T * \varphi : \mathcal{S}(\mathbb{R}^N) \to \mathbb{R}$ defined by $(T * \varphi)(\phi) := T(\widetilde{\varphi} * \phi)$, where*

$$\widetilde{\varphi}(\boldsymbol{x}) := \varphi(-\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^N. \tag{105}$$

It turns out that $T * \varphi$ can be identified with a function. Given $\boldsymbol{x} \in \mathbb{R}^N$ and a function $\varphi : \mathbb{R}^N \to \mathbb{R}$ we define the function

$$\varphi^{\boldsymbol{x}}(\boldsymbol{y}) := \varphi(\boldsymbol{x} - \boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^N. \tag{106}$$

Theorem 178 *Let $T \in \mathcal{S}'(\mathbb{R}^N)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^N)$. Then $T * \varphi = T_{f_\varphi}$, where f_φ is the function given by $f_\varphi(\boldsymbol{x}) := T(\varphi^{\boldsymbol{x}})$, $\boldsymbol{x} \in \mathbb{R}^N$. Moreover*

(i) $f_\varphi \in C^\infty(\mathbb{R}^N)$ and for every multi-index $\boldsymbol{\alpha}$ there exist $c_{\boldsymbol{\alpha}} > 0$ and $n_{\boldsymbol{\alpha}} \in \mathbb{N}$ such that

$$\|\partial^{\boldsymbol{\alpha}} f_\varphi(\boldsymbol{x})\| \le c_{\boldsymbol{\alpha}}(1 + \|\boldsymbol{x}\|^2)^{n_{\boldsymbol{\alpha}}} \tag{107}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$.

(ii) $\partial^{\boldsymbol{\alpha}} f_\varphi(\boldsymbol{x}) = f_{\partial^{\boldsymbol{\alpha}}\varphi}(\boldsymbol{x}) = \partial^{\boldsymbol{\alpha}} T(\varphi^{\boldsymbol{x}})$ for all $\boldsymbol{x} \in \mathbb{R}^N$ and for every multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^N$,

(iii) $(T * \varphi) * \psi = T * (\varphi * \psi)$.

Friday, April 29, 2022

Proof. Step 1: If $\boldsymbol{x}_n \to \boldsymbol{x}$ in $\mathbb{R}^N$, then by your homework, $\varphi^{\boldsymbol{x}_n} \to \varphi^{\boldsymbol{x}}$ in $\mathcal{S}(\mathbb{R}^N)$, and so by the continuity of T,

$$f_\varphi(\boldsymbol{x}_n) = T(\varphi^{\boldsymbol{x}_n}) \to T(\varphi^{\boldsymbol{x}}) = f_\varphi(\boldsymbol{x}),$$

which proves that $T * \varphi$ is a continuous function. Let $\boldsymbol{e}_i$ be an element of the canonical basis of $\mathbb{R}^N$ and for every $\boldsymbol{x} \in \mathbb{R}^N$ and $h \neq 0$ consider the function

$$\varphi^{\boldsymbol{x},h,i}(\boldsymbol{y}) := \frac{\varphi(\boldsymbol{x} + h\boldsymbol{e}_i - \boldsymbol{y}) - \varphi(\boldsymbol{x} - \boldsymbol{y})}{h}, \quad \boldsymbol{y} \in \mathbb{R}^N.$$

Again by your homework, as $h \to 0$, we have that $\varphi^{\boldsymbol{x},h,i} \to \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x} - \cdot)$ in $\mathcal{S}(\mathbb{R}^N)$. Hence, by the linearity and continuity of T,

$$\frac{f_\varphi(\boldsymbol{x} + h\boldsymbol{e}_i) - f_\varphi(\boldsymbol{x})}{h} = T(\varphi^{\boldsymbol{x},h,i}) \to T((\partial_i \varphi)^{\boldsymbol{x}})$$

as $h \to 0$, which proves that $\partial_i f_\varphi = f_{\partial_i \varphi}$.

Moreover, since for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$,

$$\left(\frac{\partial \varphi}{\partial x_i}\right)^{\boldsymbol{x}}(\boldsymbol{y}) = \frac{\partial \varphi}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) = -\frac{\partial \varphi}{\partial y_i}(\boldsymbol{x} - \boldsymbol{y}) = -\frac{\partial \varphi^{\boldsymbol{x}}}{\partial y_i}(\boldsymbol{y}),$$

for all $\boldsymbol{x} \in \mathbb{R}^N$ we have

$$\frac{\partial T}{\partial y_i}(\varphi^{\boldsymbol{x}}) = -T\left(\frac{\partial \varphi^{\boldsymbol{x}}}{\partial y_i}\right)$$
$$= T\left(-\left(\frac{\partial \varphi}{\partial y_i}\right)^{\boldsymbol{x}}\right) = f_{\frac{\partial \varphi}{\partial y_i}}(\boldsymbol{x}),$$

which, together with an induction argument, gives (ii).
Step 2: Since $\partial^{\boldsymbol{\alpha}}\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $\partial^{\boldsymbol{\alpha}} f_\varphi(\boldsymbol{x}) = f_{\partial^{\boldsymbol{\alpha}}\varphi}(\boldsymbol{x})$ by part (ii), it suffices to prove the bound (107) for $\boldsymbol{\alpha} = \boldsymbol{0}$. Since T is continuous, there exist a constant $C > 0$ and some $m, n \in \mathbb{N}_0$ such that

$$|T(g)| \leq C \|g\|_{m,n}.$$

for every $g \in \mathcal{S}\left(\mathbb{R}^N\right)$. In particular,

$$|f_\varphi(\boldsymbol{x})| = |T(\varphi^{\boldsymbol{x}})| \leq C\|\varphi^{\boldsymbol{x}}\|_{m,n}.$$

Now

$$\|\varphi^{\boldsymbol{x}}\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\boldsymbol{y}\in\mathbb{R}^N}\left|\boldsymbol{y}^{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\beta}}\varphi}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}}(\boldsymbol{x}-\boldsymbol{y})\right| = \sup_{\boldsymbol{z}\in\mathbb{R}^N}\left|(\boldsymbol{x}-\boldsymbol{z})^{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\beta}}\varphi}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}}(\boldsymbol{z})\right|$$
$$\leq C(1 + \|\boldsymbol{x}\|^{|\boldsymbol{\alpha}|})\|\varphi\|_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|}.$$

Step 3: Fix $\phi \in \mathcal{S}(\mathbb{R}^N)$. For $h > 0$ define

$$f_h(\boldsymbol{x}) := h^N \sum_{\boldsymbol{y}\in\mathbb{Z}^N} \widetilde{\varphi}(\boldsymbol{x} - h\boldsymbol{y})\phi(h\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^N,$$

where $h > 0$. By your homework $f_h \to \widetilde{\varphi} * \phi$ in $\mathcal{S}(\mathbb{R}^N)$. Hence, by the continuity and linearity of T and by Theorem 140 we have that

$$(T * \varphi)(\phi) = T(\widetilde{\varphi} * \phi) = \lim_{h\to 0^+} T(f_h) = \lim_{h\to 0^+} h^N \sum_{\boldsymbol{y}\in\mathbb{Z}^N} T(\widetilde{\varphi}(\cdot - h\boldsymbol{y}))\phi(h\boldsymbol{y})$$
$$= \lim_{h\to 0^+} h^N \sum_{\boldsymbol{y}\in\mathbb{Z}^N} T(\varphi(h\boldsymbol{y} - \cdot))\phi(h\boldsymbol{y}) = \lim_{h\to 0^+} h^N \sum_{\boldsymbol{y}\in\mathbb{Z}^N} f_\varphi(h\boldsymbol{y})\phi(h\boldsymbol{y}) = \int_{\mathbb{R}^N} f_\varphi(\boldsymbol{y})\phi(\boldsymbol{y})\, d\boldsymbol{y}$$
$$= T_{f_\varphi}(\phi),$$

where in the second to last equality we used Riemann sums and the fact that $f_\varphi\phi \in \mathcal{S}(\mathbb{R}^N)$ (which follows from the previous two steps). This shows that $T * \varphi = T_{f_\varphi}$.
Step 4: Finally, to prove (iii), we define

$$f_h(\boldsymbol{x}) := h^N \sum_{\boldsymbol{y}\in\mathbb{Z}^N} \varphi(\boldsymbol{x} - h\boldsymbol{y})\psi(h\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^N,$$

where $h > 0$. As before we have that $f_h \to \varphi * \psi$ in $\mathcal{S}(\mathbb{R}^N)$ as $h \to 0^+$. It follows that for every $\boldsymbol{x}_0 \in \mathbb{R}^N$, $(f_h)^{\boldsymbol{x}_0} \to (\varphi * \psi)^{\boldsymbol{x}_0}$ in $\mathcal{S}(\mathbb{R}^N)$ as $h \to 0$, where

$$(f_h)^{\boldsymbol{x}_0}(\boldsymbol{x}) = h^N \sum_{\boldsymbol{y} \in \mathbb{Z}^N} \varphi(\boldsymbol{x}_0 - \boldsymbol{x} + h\boldsymbol{y})\psi(h\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{R}^N$$
$$(\varphi * \psi)^{\boldsymbol{x}_0}(\boldsymbol{x}) = (\varphi * \psi)(\boldsymbol{x}_0 - \boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

By the linearity and continuity of T and we have that

$$\begin{aligned} f_{\varphi * \psi}(\boldsymbol{x}_0) &= T((\varphi * \psi)^{\boldsymbol{x}_0}) = \lim_{h \to 0} T((f_h)^{\boldsymbol{x}_0}) \\ &= \lim_{h \to 0} h^N \sum_{\boldsymbol{y} \in \mathbb{Z}^N} T(\varphi(\boldsymbol{x}_0 - \cdot + h\boldsymbol{y})\psi(h\boldsymbol{y})) \\ &= \lim_{h \to 0} h^N \sum_{\boldsymbol{y} \in \mathbb{Z}^N} T(\varphi(\boldsymbol{x}_0 - \cdot + h\boldsymbol{y}))\psi(h\boldsymbol{y}) \\ &= \lim_{h \to 0} h^N \sum_{\boldsymbol{y} \in \mathbb{Z}^N} f_\varphi(\boldsymbol{x}_0 + h\boldsymbol{y})\psi(h\boldsymbol{y}) \\ &= \int_{\mathbb{R}^N} f_\varphi(\boldsymbol{x}_0 - \boldsymbol{y})\psi(\boldsymbol{y})\, d\boldsymbol{y} = (f_\varphi * \psi)(\boldsymbol{x}_0). \end{aligned}$$

This completes the proof. ■

As a consequence of the previous theorem, we can approximate distributions with C^∞ functions.

Exercise 179 *Let $T \in \mathcal{S}'(\mathbb{R}^N)$ and let $\{\varphi_\varepsilon\}_\varepsilon$, $\varepsilon > 0$, be a family of standard mollifiers. Prove that $T * \varphi_\varepsilon \to T$ in $\mathcal{S}'(\mathbb{R}^N)$ as $\varepsilon \to 0^+$.*

Given a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^N)$ and a C^∞ function $g : \mathbb{R}^N \to \mathbb{R}$ such that for every multi-index $\boldsymbol{\alpha}$ there exist $c_{\boldsymbol{\alpha}} > 0$ and $n_{\boldsymbol{\alpha}} \in \mathbb{N}$ such that

$$\|\partial^{\boldsymbol{\alpha}} g(\boldsymbol{x})\| \le c_{\boldsymbol{\alpha}}(1 + \|\boldsymbol{x}\|^2)^{n_{\boldsymbol{\alpha}}}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$, we define

$$(gT)(f) := T(gf), \quad f \in \mathcal{S}(\mathbb{R}^N).$$

We leave as an exercise to check that $gT \in \mathcal{S}'(\mathbb{R}^N)$.

Let $s < 0$ and $T \in \mathcal{S}'(\mathbb{R}^N)$. We say that T belongs to the *fractional Sobolev space* $H^s(\mathbb{R}^N)$ if there exists a function $g \in L^2(\mathbb{R}^N)$ such that $(1 + \|\boldsymbol{x}\|^2)^{s/2}\widehat{T} = T_g$.