Contents

Monday, January 13, 2020

1 The Field of Complex Numbers

We define $\mathbb{C}$, the *complex numbers,* to be the set of all ordered pairs $z = (x, y)$ of real numbers x, y with operations of addition and multiplication defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2), \tag{1}$$

$$(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2), \tag{2}$$

for all $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$. It can be checked that with these two operations $\mathbb{C}$ is a field. This means that addition and multiplication are associative and commutative, $(0, 0)$ and $(1, 0)$ are the identities for addition and multiplication, respectively, every complex number has an additive inverse, every complex number different from zero has a multiplicative inverse, and distributivity of multiplication over addition holds. The set of complex numbers of the form $(x, 0)$, $x \in \mathbb{R}$ is a subfield of $\mathbb{C}$, and it is the isomorphic image of $\mathbb{R}$ through the mapping

$$x \mapsto (x, 0).$$

Hence, from now on we will consider $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifying the pair $(x, 0)$ with the real number x. Using this identification, if we define $i := (0, 1)$ then $x + iy = (x, y)$. From now on we will use notation. The real numbers x and y are called the *real* and *imaginary parts of* z, and we write

$$\operatorname{Re} z = x \quad \operatorname{Im} z = y.$$

Complex numbers of the form yi are called *purely imaginary numbers.*

Observe that using (2) we have that $i^2 = -1$ and so the equation $z^2 + 1 = 0$ has a root in $\mathbb{C}$. Indeed, $z^2 + 1 = (z + i)(z - i)$. More generally, if $z, w \in \mathbb{C}$ we have that

$$z^2 + w^2 = (z + iw)(z - iw).$$

Using the previous formula, given $z = x + iy \neq 0$ we have

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2},$$

which is the formula for the multiplicative inverse, or the opposite, of z.

Given a complex number $z = x + iy$, $x, y \in \mathbb{R}$, we define the *absolute value of* or *modulus of* z as

$$|z| = \sqrt{x^2 + y^2}.$$

Note that this is the norm of the vector $(x, y) \in \mathbb{R}^2$. Hence, we have

$$|z| = 0 \quad \text{if and only if } z = 0,$$

$$|z + w| \leq |z| + |w| \quad \text{for all } z, w \in \mathbb{C},$$

$$|tz| = |t||z| \quad \text{for all } z \in \mathbb{C} \text{ and } t \in \mathbb{R}.$$

We leave as an exercise to show that

$$
\begin{aligned}
|zw| &= |z||w| \quad \text{for all } z, w \in \mathbb{C}, \\
\left|\frac{z}{w}\right| &= \frac{|z|}{|w|} \quad \text{for all } z, w \in \mathbb{C}, \text{ with } w \neq 0.
\end{aligned}
$$

Since the absolute value of $z = x + iy$ is the norm in $\mathbb{R}^2$ of (x, y), if we define the open ball centered at $z_0 = x_0 + iy_0 \in \mathbb{C}$ and radius $r > 0$ as

$$
B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\},
$$

this is nothing else than the ball $B((x_0, y_0), r) \subset \mathbb{R}^2$. Hence, the topology in $\mathbb{C}$ coincides with the topology in $\mathbb{R}^2$. So we will have the same open sets, the same closed sets, the same compact sets, the same connected sets, and so on.

Given a complex number $z = x + iy \in \mathbb{C}$, the *complex conjugate* of z is defined as the complex number

$$
\bar{z} := x = iy.
$$

The following properties are left as an exercise:

$$
|z|^2 = z\overline{z}, \quad \operatorname{Re} z = \frac{z + \overline{z}}{2}, \quad \operatorname{Im} z = \frac{z - \overline{z}}{2i} \quad \text{for all } z \in \mathbb{C}, \tag{3}
$$

$$
\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w} \quad \text{for all } z, w \in \mathbb{C}, \tag{4}
$$

$$
\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}} \quad \text{for all } z, w \in \mathbb{C}, \text{ with } w \neq 0. \tag{5}
$$

A complex number $z = x + iy \in \mathbb{C} \setminus \{0\}$ can be written in *polar form* as

$$
z = re^{i\theta},
$$

where $r = |z|$ and (we will justify this later)

$$
e^{i\theta} = \cos\theta + i \sin\theta, \tag{6}
$$

where θ is the angle between the positive real axis and the half-line starting at the origin and passing through z. The number θ is called the *argument* of z and is denoted $\arg z$.

The following properties are left as an exercise:

$$
\begin{aligned}
&\text{if } z = re^{i\theta} \text{ and } w = se^{i\varphi}, \text{ then } zw = rse^{i(\theta+\varphi)}, \\
&\text{if } z = re^{i\theta} \text{ and } n \in \mathbb{N}, \text{ then } z^n = r^n e^{in\theta}.
\end{aligned}
$$

Exercise 1 *Given $n \in \mathbb{N}$, solve the equation $z^n = 1$.*

2 Complex Functions

Definition 2 *Let $E \subseteq \mathbb{C}$, let $z_0 \in \mathbb{C}$ be an accumulation point of E and let $f : E \to \mathbb{C}$. We say that $\ell \in \mathbb{C}$ is the* limit of f as z approaches z_0 *and we write*

$$\lim_{z \to z_0} f(z) = \ell$$

if for every $\varepsilon > 0$ there exists $\delta = \delta(z_0, \varepsilon) > 0$ such that

$$|f(z) - \ell| < \varepsilon$$

for all $z \in E$ with $0 < |z - z_0| < \delta$.

Given $E \subseteq \mathbb{C}$ and a function $f : E \to \mathbb{C}$, since the absolute value in $\mathbb{C}$ is the norm in $\mathbb{R}^2$, the the basic properties of limits (sum, composition, multiplication by a scalar) will not change. The only additional property is the product of limits.

Exercise 3 *Let $E \subseteq \mathbb{C}$, let $z_0 \in \mathbb{C}$ be an accumulation point of E and let $f : E \to \mathbb{C}$ and $g : E \to \mathbb{C}$. Assume that there exist*

$$\lim_{z \to z_0} f(z) = \ell \in \mathbb{C}, \quad \lim_{z \to z_0} g(z) = L \in \mathbb{C}.$$

Prove that

(i) there exist
$$\lim_{z \to z_0} (f + g)(z) = \ell + L;$$

(ii) there exist
$$\lim_{z \to z_0} (fg)(z) = \ell L;$$

(iii) if $L \neq 0$, then z_0 is an accumulation point for $E_0 := \{z \in E : g(z) \neq 0\}$, and if we restrict f/g to E_0, then there exists
$$\lim_{z \to z_0} \left(\frac{f}{g}\right)(z) = \frac{\ell}{L}.$$

Exercise 4 *State and prove a similar result for the limit of compositions.*

Next we discuss differentiation.

Definition 5 *Let $E \subseteq \mathbb{C}$, let $z_0 \in E$ be an accumulation point of E and let $f : E \to \mathbb{C}$. We say that f is* differentiable at z_0 *if there exists the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \ell \in \mathbb{C}.$$

We call the limit ℓ the derivative of f at z_0 *and we denote it by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.*

Definition 6 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$. We say that f is* holomorphic *in U, if f is differentiable in U.*

The following properties are left as an exercise;

Exercise 7 *Let $E \subseteq \mathbb{C}$, let $z_0 \in E$ be an accumulation point of E and let $f : E \to \mathbb{C}$ and $g : E \to \mathbb{C}$ be differentiable at z_0. Prove that*

(i) $f + g$ is differentiable at z_0 and $(f + g)'(z_0) = f'(z_0) + g'(z_0)$,

(ii) fg is differentiable at z_0 and $(fg)'(z_0) = g(z_0)f'(z_0) + f(z_0)g'(z_0)$,

(iii) if $g(z_0) \neq 0$ then $\frac{f}{g} : E_0 \to \mathbb{C}$ is differentiable at z_0 and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2},$$

where $E_0 := \{z \in E : g(z) \neq 0\}$.

Exercise 8 *Let $E, F \subseteq \mathbb{C}$, let $z_0 \in E$ be an accumulation point of E , let $f : E \to F$ be differentiable at z_0, let $f(z_0)$ be an accumulation point of $f(E)$ and let $g : F \to \mathbb{C}$ be differentiable at $f(z_0)$. Prove that $g \circ f$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

Exercise 9 *Let $U, V \subseteq \mathbb{C}$ be open sets, let $f : U \to V$ be continuous and let $g : V \to \mathbb{C}$ be differentiable and such that*

$$g(f(z)) = z \quad \text{for all } z \in U.$$

Let $z_0 \in U$ be such that $g'(f(z_0)) \neq 0$. Prove that f is differentiable at z_0 and

$$f'(z_0) = \frac{1}{g'(f(z_0))}.$$

Let's discuss the relation between complex and real differentiation. Given $E \subseteq \mathbb{C}$ and $f : E \to \mathbb{C}$, let

$$F := \{(x, y) \in \mathbb{R}^2 : x + iy \in E\}$$

and define $u : F \to \mathbb{R}$ and $v : F \to \mathbb{R}$ by

$$u(x, y) := \operatorname{Re} f(x + iy), \quad v(x, y) := \operatorname{Im} f(x + iy). \tag{7}$$

The following example shows that differentiability of u and v does not imply differentiability of f.

Example 10 *Consider the function $f(z) = \overline{z}$. Then $u(x, y) = x$ and $v(x, y) = -y$, which are C^∞ and even analytic functions. However, f is not differentiable at 0, since*

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{\overline{z}}{z}$$

and this limit does not exist, since taking $z = x + i0$ gives

$$\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1,$$

while taking $z = 0 + iy$ gives

$$\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{y\to 0} \frac{-y}{y} = -1.$$

Wednesday, January 15, 2020

We recall that given a set $F \subseteq \mathbb{R}^N$, a point $\boldsymbol{x}_0 \in F \cap \operatorname{acc} F$, and a real-valued function $u : F \to \mathbb{R}$, we say that u is differentiable at $\boldsymbol{x}_0$ if there exists a linear function $L : \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0} \frac{u(\boldsymbol{x}) - u(\boldsymbol{x}_0) - L(\boldsymbol{x} - \boldsymbol{x}_0)}{\|\boldsymbol{x} - \boldsymbol{x}_0\|} = 0.$$

The linear function L is called the *differential of f at $\boldsymbol{x}_0$* and is denoted $df(\boldsymbol{x}_0)$.

Exercise 11 *Let $F \subseteq \mathbb{R}^N$, let $\boldsymbol{x}_0 \in F^\circ$, and let $u : F \to \mathbb{R}$ be differentiable at $\boldsymbol{x}_0$.*

(i) Prove that u is continuous at $\boldsymbol{x}_0$.

(ii) Prove that there exist all partial derivatives $\frac{\partial u}{\partial x_i}(\boldsymbol{x}_0)$, all directional derivatives $\frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}_0)$ and that

$$\nabla u(\boldsymbol{x}_0) \cdot \boldsymbol{\nu} = \frac{\partial u}{\partial \boldsymbol{\nu}}(\boldsymbol{x}_0) \tag{8}$$

for all $\boldsymbol{\nu} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$.

Exercise 12 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$u(x, y) := \begin{cases} x & \text{if } y = x^2,\ x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that u is continuous at $(0, 0)$, all partial and directional derivatives exist at $(0, 0)$ and that (8) holds but that u is not differentiable at $(0, 0)$.

Next we show that differentiability of f implies the differentiability of u and v. In what follows, given a set $E \subseteq \mathbb{C}$, we denote by E° the set of interior points of E.

Theorem 13 (Cauchy–Riemann Equations) *Let $E \subseteq \mathbb{C}$, let $z_0 = x_0 + iy_0 \in E$ be an accumulation point of E and let $f : E \to \mathbb{C}$ be differentiable at z_0. Then the functions u and v defined in (7) are differentiable at (x_0, y_0). Moreover if z_0 is an interior point of E, then*

$$\begin{aligned}\frac{\partial u}{\partial x}(x_0, y_0) &= \frac{\partial v}{\partial y}(x_0, y_0) = \operatorname{Re} f'(x_0 + iy_0), \qquad (9)\\ -\frac{\partial u}{\partial y}(x_0, y_0) &= \frac{\partial v}{\partial x}(x_0, y_0) = \operatorname{Im} f'(x_0 + iy_0).\end{aligned}$$

In particular,

$$\det \begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix} = |f'(z_0)|^2 \qquad (10)$$

The relations

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad -\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0) \qquad (11)$$

are known as the *Cauchy–Riemann equations.*

Proof. We have

$$0 = \lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0}$$

and so (since the product of a bounded function and a function going to zero goes to zero)

$$0 = \lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{|z - z_0|}.$$

In turn,

$$\begin{aligned}\lim_{z \to z_0} \frac{\operatorname{Re}(f(z) - f(z_0) - f'(z_0)(z - z_0))}{|z - z_0|} &= 0, \qquad (12)\\ \lim_{z \to z_0} \frac{\operatorname{Im}(f(z) - f(z_0) - f'(z_0)(z - z_0))}{|z - z_0|} &= 0. \qquad (13)\end{aligned}$$

Now by (2), writing $z = x + iy$ and $z_0 = x_0 + iy_0$,

$$\begin{aligned}f'(z_0)(z - z_0) = &\operatorname{Re} f'(z_0)(x - x_0) - \operatorname{Im} f'(z_0)(y - y_0)\\ &+ i(\operatorname{Im} f'(z_0)(x - x_0) + \operatorname{Re} f'(z_0)(y - y_0)),\end{aligned}$$

and so (12) and (13) become

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\operatorname{Re} f(x + iy) - \operatorname{Re} f(x_0 + iy_0) - \operatorname{Re} f'(x_0 + iy_0)(x - x_0) + \operatorname{Im} f'(x_0 + iy_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\operatorname{Im} f(x + iy) - \operatorname{Im} f(x_0 + iy_0) - \operatorname{Im} f'(x_0 + iy_0)(x - x_0) - \operatorname{Re} f'(x_0 + iy_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

These can be written as

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)-(\operatorname{Re} f'(x_0+iy_0), -\operatorname{Im} f'(x_0+iy_0))\cdot((x-x_0),(y-y_0))}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0,$$

$$\lim_{(x,y)\to(x_0,y_0)} \frac{v(x,y)-v(x_0,y_0)-(\operatorname{Im} f'(x_0+iy_0), \operatorname{Re} f'(x_0+iy_0))\cdot((x-x_0),(y-y_0))}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

These implies that u and v are differentiable at (x_0, y_0) with

$$du(x_0,y_0)(s,t) = \operatorname{Re} f'(x_0+iy_0)s - \operatorname{Im} f'(x_0+iy_0)t, \quad (s,t)\in\mathbb{R}^{2,},$$
$$dv(x_0,y_0)(s,t) = \operatorname{Im} f'(x_0+iy_0)s + \operatorname{Re} f'(x_0+iy_0)t, \quad (s,t)\in\mathbb{R}^2.$$

In particular, if z_0 belongs to the interior of E then $\nabla u(x_0,y_0)$ and $\nabla v(x_0,y_0)$ exist with

$$\frac{\partial u}{\partial x}(x_0,y_0) = \operatorname{Re} f'(x_0+iy_0), \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\operatorname{Im} f'(x_0+iy_0) \tag{14}$$

$$\frac{\partial v}{\partial x}(x_0,y_0) = \operatorname{Im} f'(x_0+iy_0), \quad \frac{\partial v}{\partial y}(x_0,y_0) = \operatorname{Re} f'(x_0+iy_0). \tag{15}$$

Comparing (14) and (15) gives the Cauchy–Riemann equations. In turn, (10) follows by direct computation. ■

Corollary 14 *Let $U \subseteq \mathbb{C}$ be an open and connected set and let $f : U \to \mathbb{C}$ be a differentiable function with $f' = 0$ in U. Then f is constant.*

Proof. By the previous theorem the functions u and v defined in (7) are differentiable in $V = \{(x,y) \in \mathbb{R}^2 : x+iy \in U\}$, with $\nabla u = \nabla v \equiv (0,0)$ in V. Thus, by a result in Analysis, u and v are constant in V. Again by (7), it follows that f is constant. ■

Theorem 15 *Let $F \subseteq \mathbb{R}^2$, let $(x_0,y_0) \in F$ be an interior point of F and let $u, v : E \to \mathbb{R}$ be differentiable at (x_0,y_0). Assume that the Cauchy–Riemann equations (11) hold at (x_0,y_0). Let $E := \{z = x+iy \in \mathbb{C} : (x,y) \in F\}$ and let $f : E \to \mathbb{C}$ be defined by*

$$f(z) = u(x,y) + iv(x,y), \quad z = x+iy \in E. \tag{16}$$

Then f is differentiable at z_0.

Proof. Set

$$f'(z_0) := \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$

Now by (2), writing $z = x + iy$ and $z_0 = x_0 + iy_0$,

$$\begin{aligned}
f'(z_0)(z - z_0) &= \operatorname{Re} f'(z_0)(x - x_0) - \operatorname{Im} f'(z_0)(y - y_0) \\
&\quad + i(\operatorname{Im} f'(z_0)(x - x_0) + \operatorname{Re} f'(z_0)(y - y_0)) \\
&= \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial v}{\partial x}(x_0, y_0)(y - y_0) \\
&\quad + i\frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + i\frac{\partial u}{\partial x}(x_0, y_0)(y - y_0) \\
&= \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0) \\
&\quad + i\frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + i\frac{\partial v}{\partial y}(x_0, y_0)(y - y_0),
\end{aligned}$$

where in the last equality we used (11) ($\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $-\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)$). Hence, also by (16),

$$\begin{aligned}
f(z) - f(z_0) - f'(z_0)(z - z_0) &= u(x, y) - u(x_0, y_0) - \nabla u(x_0, y_0) \cdot (x - x_0, y - y_0) \\
&\quad + i(v(x, y) - v(x_0, y_0) - \nabla v(x_0, y_0) \cdot (x - x_0, y - y_0)).
\end{aligned}$$

Dividing by $|z - z_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ gives

$$\begin{aligned}
\frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{|z - z_0|} &= \frac{u(x, y) - u(x_0, y_0) - \nabla u(x_0, y_0) \cdot (x - x_0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \\
&\quad + i\frac{v(x, y) - v(x_0, y_0) - \nabla v(x_0, y_0) \cdot (x - x_0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}.
\end{aligned}$$

Since u and v are differentiable at (x_0, y_0), it follows that

$$0 = \lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{|z - z_0|},$$

which implies that

$$0 = \lim_{z \to z_0} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0},$$

and the proof is complete. ■

The following example shows that the previous theorem fails without assuming that u and v are differentiable. We refer to Section 3 for the definition of e^z.

Example 16 *Let*

$$f(z) = \begin{cases} \exp(-z^{-4}) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Prove that the Cauchy–Riemann equations are satisfied but that f is not differentiable at the origin.

Note that the previous function is not continuous at $z = 0$. There is a beautiful theorem, due to Looman and Menchoff, which we will not prove, which says the following.

Theorem 17 (Looman–Menchoff) *Let $V \subseteq \mathbb{R}^2$ be an open set, let $u, v : V \to \mathbb{R}$ be continuous functions in V. Assume that $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ exist in V and satisfy the Cauchy–Riemann equations (11) in V. Let $U := \{z = x+iy \in \mathbb{C} : (x, y) \in V\}$ and let $f : U \to \mathbb{C}$ be defined by*

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy \in U.$$

Then f is differentiable in U.

3 Power Series and Some Elementary Functions

Definition 18 *Given a sequence $\{z_n\}_n$ of complex numbers, we call the n-th* partial sum *the number*

$$s_n = z_1 + \cdots + z_n.$$

The sequence $\{s_n\}_n$ of partial sums is called infinite series *or* series *and is denoted*

$$\sum_{n=1}^{\infty} z_n.$$

If there exists $\lim_{n\to\infty} s_n = S \in \mathbb{C}$, we say that the series $\sum_{n=1}^{\infty} z_n$ is convergent*. The number S is called* sum *of the series. If the limit $\lim_{n\to\infty} s_n$ does not exist, we say that the series $\sum_{n=1}^{\infty} z_n$* oscillates*.*

We say that the series $\sum_{n=1}^{\infty} z_n$ converges absolutely *if the series $\sum_{n=1}^{\infty} |z_n|$ converges.*

Remark 19 *There is nothing special about 1, we will also consider series of the type $\sum_{n=0}^{\infty} z_n$ or $\sum_{n=n_0}^{\infty} z_n$, where $n_0 \in \mathbb{N}$. The only change is that in the partial sums, one should consider $s_n = z_0 + \cdots + z_n$ and $s_n = z_{n_0} + \cdots + z_n$, respectively.*

Theorem 20 *If the series $\sum_{n=1}^{\infty} z_n$ converges, then there exists*

$$\lim_{n\to\infty} z_n = 0.$$

Proof. Since the series $\sum_{n=1}^{\infty} z_n$ converges, there exists $\lim_{n\to\infty} s_n = S \in \mathbb{C}$. Hence,

$$z_n = s_{n+1} - s_n \to S - S = 0$$

as $n \to \infty$. Note that here it is important that $S \in \mathbb{C}$. ■

Definition 21 *A* power series *is a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C},$$

where $a_n \in \mathbb{C}$.

Friday, January 17, 2020

We recall that given a sequence $\{x_n\}_n$ of real numbers, the *limit superior* of $\{x_n\}_n$ is defined as

$$\limsup_{n\to\infty} x_n := \inf_n \sup_{k\geq n} x_k.$$

Exercise 22 *Given a sequence $\{x_n\}_n$ of real numbers and $\ell \in \mathbb{R}$, prove that ℓ is the limit superior of the sequence $\{x_n\}_n$ if and only if*

(i) for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$x_n \leq \ell + \varepsilon \quad \textit{for all } n \geq n_\varepsilon$$

(ii) for every $\varepsilon > 0$,

$$x_n \geq \ell - \varepsilon \quad \textit{for infinitely many } n.$$

State and prove a similar result for the case $\ell = \infty$.

Theorem 23 *Given a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C},$$

let $R \in [0, \infty]$ be given by

$$\frac{1}{R} := \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then for $|z| < R$ the series converges absolutely, while for $|z| > R$ the series oscillates.

Proof. If $|z| < R$, then $|z|/R < 1$. Fix $\varepsilon > 0$ so small that $(1/R + \varepsilon)|z| < 1$. By the previous exercise, there exists $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} \leq 1/R + \varepsilon$$

for all $n \geq N$, and so

$$|a_n| \leq (1/R + \varepsilon)^n$$

for all $n \geq N$. In turn,

$$|a_n z^n| = |a_n||z|^n \leq [(1/R + \varepsilon)\,|z|]^n$$

for all $n \geq N$. Since $(1/R + \varepsilon)\,|z| < 1$, the geometric series $\sum_{n=1}[(1/R + \varepsilon)\,|z|]^n$ converges. Hence, so does $\sum_{n=1}^{\infty} |a_n z^n|$ by the comparison test.

On the other hand, if $|z| > R$, fix $\varepsilon > 0$ so small that $(1/R - \varepsilon)|z| > 1$. By the previous exercise,

$$|a_n|^{1/n} \geq 1/R - \varepsilon > 0$$

for infinitely many n, and so

$$|a_n| \geq (1/R - \varepsilon)^n$$

for infinitely many n. In turn,

$$|a_n z^n| = |a_n||z|^n \geq [(1/R - \varepsilon)\,|z|]^n$$

Thus,

$$\limsup_{n\to\infty} |a_n z^n| \geq \limsup_{n\to\infty}[(1/R - \varepsilon)\,|z|]^n = \infty,$$

since $(1/R - \varepsilon)\,|z| > 1$. It follows by Theorem 20, that the series $\sum_{n=1}^{\infty} a_n z^n$ oscillates. ■

The number R is called *radius of convergence* of the power series.

Exercise 24 *Given* $\{a_k\}_{k=1}^n$ *and* $\{b_k\}_{k=1}^n$ *in* $\mathbb{C}$, *let* $B_l := \sum_{k=1}^{l} b_k$, $B_0 := 0$. *Prove that*

$$\sum_{k=m}^{n} a_k b_k = a_n B_n - a_m B_{m-1} - \sum_{k=m}^{n-1} (a_{k+1} - a_k) B_k.$$

Exercise 25 *Assume that the series of complex numbers* $\sum_{n=1}^{\infty} a_n$ *converges. Use the previous exercise to show that*

$$\lim_{r\to 1^-} \sum_{n=1}^{\infty} a_n r^n = \sum_{n=1}^{\infty} a_n.$$

Exercise 26 *Version of Abel's with angles. Ahlfors.*

Example 27 *When* $|z| = R$ *anything can happen as the two power series*

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} z^n$$

show. Note that for $a > 0$,

$$\left(\frac{1}{n^a}\right)^{1/n} = \frac{1}{n^{a/n}} = \frac{1}{e^{\log n^{a/n}}} = \frac{1}{e^{(a/n)\log n}} \to \frac{1}{e^0} = \frac{1}{R}.$$

Exercise 28 *Let* $\{x_n\}_n$ *be a sequence of real numbers, with* $x_n > 0$ *for all* $n \in \mathbb{N}$. *Prove that*

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Show that the inequality can be strict.

Remark 29 *In view of the previous exercise, if there exists*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

then there exists

$$\lim_{n\to\infty} \sqrt[n]{x_n}$$

and the two limits are the same.

Next we show that a power series is differentiable in $B(0, R)$.

Theorem 30 *Given a power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C},$$

let R be its radius of convergence and assume that $R > 0$. Then the function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is differentiable in the open set $U_R := \{z \in \mathbb{C} : |z| < R\}$ and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Moreover, the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R.

Proof. The fact that the two power series have the same radius of convergence follows from the fact that

$$\begin{aligned}\limsup_{n\to\infty} |na_n|^{1/n} &= \limsup_{n\to\infty} n^{1/n}|a_n|^{1/n} = \limsup_{n\to\infty} e^{\log n^{1/n}}|a_n|^{1/n} \\ &= \limsup_{n\to\infty} e^{(\log n)/n}|a_n|^{1/n} = \lim_{n\to\infty} e^{(\log n)/n} \limsup_{n\to\infty} |a_n|^{1/n} = 1\frac{1}{R}.\end{aligned}$$

Let $z_0 \in U_R$ and find $|z_0| < r < R$. Let $h \in \mathbb{C}$ be so small that $|z_0 + h| < r$. Define

$$g(z) := \sum_{n=0}^{\infty} n a_n z^{n-1}$$

and consider

$$\begin{aligned}\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \sum_{n=0}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h} - \sum_{n=0}^{\infty} n a_n z_0^{n-1} \\ &= \sum_{n=0}^{N} a_n \left[\frac{(z_0+h)^n - z_0^n}{h} - n z_0^{n-1}\right] \\ &\quad + \sum_{n=N+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h} - \sum_{n=N+1}^{\infty} n a_n z_0^{n-1} \\ &=: I + II + III.\end{aligned}$$

Using the facts that $a^n - b^n = (b-a)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$, that $|z_0| < r$, and that $|z_0 + h| < r$ we have that

$$|(z_0+h)^n - z_0^n| \le |h| n r^{n-1}.$$

In turn,

$$|II| \le \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

Since g has the same radius of convergence than f and $r < R$ we have that the right-hand side is the tail of a convergent series and thus goes to zero as $N \to \infty$. Hence, given $\varepsilon > 0$ we can find $N_\varepsilon \in \mathbb{N}$ such that $|II| \le \varepsilon$ for all $N \ge N_\varepsilon$.

Similarly, since $|z_0| < R$ and $g(z_0)$ converges, by taking N_ε larger, if necessary, we have that $|III| \le \varepsilon$ for all $N \ge N_\varepsilon$.

Fix $N = N_\varepsilon$. Since I is the difference quotient of a finite number of differentiable functions, we can find $\delta_\varepsilon > 0$ such that $|I| \le \varepsilon$ for all $h \in \mathbb{C}$ with $|h| \le \delta_\varepsilon$. This concludes the proof. ∎

By repeated applications of the previous theorem we obtain the following:

Corollary 31 *A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is infinitely differentiable in $U_R := \{z \in \mathbb{C} : |z| < R\}$, where R is its radius of convergence. Moreover, the higher derivatives $f^{(k)}$ are power series obtained by pointwise differentiation and with the same radius of convergence. To be precise,*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k}, \quad z \in U_R.$$

Moreover,

$$f^{(k)}(0) = \frac{1}{k!} a_k, \quad k \in \mathbb{N}_0.$$

Remark 32 *Similarly, if we consider $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, then f is infinitely differentiable in $U_R := \{z \in \mathbb{C} : |z - z_0| < R\}$, with $f^{(k)}(z_0) = \frac{1}{k!} a_k$, $k \in \mathbb{N}_0$.*

Using power series we can define e^z, $\cos z$, and $\sin z$ as follows

$$e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad \cos z := \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} z^{2n}, \quad \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} z^{2n+1}.$$

Using Remark 29, we compute

$$\begin{aligned}
\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} &= \frac{\frac{1}{n!(n+1)}}{\frac{1}{n!}} = \frac{1}{n+1} \to 0, \\
\frac{(-1)^{n+1}\frac{1}{(2n+2)!}}{(-1)^n \frac{1}{(2n)!}} &= -\frac{\frac{1}{(2n)!(2n+2)(2n+1)}}{\frac{1}{(2n)!}} = -\frac{1}{(2n+2)(2n+1)} \to 0, \\
\frac{(-1)^{n+1}\frac{1}{(2n+3)!}}{(-1)^n \frac{1}{(2n+1)!}} &= -\frac{\frac{1}{(2n+1)!(2n+3)(2n+2)}}{\frac{1}{(2n+1)!}} = -\frac{1}{(2n+3)(2n+2)} \to 0,
\end{aligned}$$

and so all three series have radius of convergence $R = \infty$, so they converge in $\mathbb{C}$. For $\cos z$ we used the fact that $z^{2n} = (z^2)^n$ and for $\sin z$ we pulled out z and used the same trick.

Note that if we differentiate e^z, by Theorem 30,

$$(e^z)' = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{k=1}^{\infty} \frac{1}{k!} z^k = e^z,$$

which is the same property of the real exponential function. Similarly, by Theorem 30,

$$(\cos z)' = -\sin z, \quad (\sin z)' = \cos z.$$

Consider the function $g(z) = e^z e^{a-z}$, where $a \in \mathbb{C}$ is fixed. By the product rule, Exercise 8, and Theorem 30,

$$g'(z) = e^z e^{a-z} - e^z e^{a-z} = 0$$

and so by Corollary 14, g is constant. Since $e^0 = 1$, taking $z = 0$ gives $g(z) \equiv e^a$. Hence,

$$e^z e^{a-z} = e^a \quad \text{for all } z \in \mathbb{C}.$$

Taking $a = z + w$ we get

$$e^z e^w = e^{z+w} \quad \text{for all } z, w \in \mathbb{C}. \tag{17}$$

Taking $w = -z$ gives $e^z e^{-z} = 1$ so $e^z \neq 0$ for all z and

$$\frac{1}{e^z} = e^{-z}. \tag{18}$$

Observe also that by (4), $\overline{\frac{1}{n!} z^n} = \frac{1}{n!} \overline{z^n} = \overline{z}^n$ and so $\overline{e^z} = e^{\overline{z}}$. In turn, by (3) and (17),

$$|e^z|^2 = e^z \overline{e^z} = e^z e^{\overline{z}} = e^{z+\overline{z}} = e^{2 \operatorname{Re} z}. \tag{19}$$

Note that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{20}$$

These are called *Euler formulas for* $\cos z$ and $\sin z$. From these formulas and (17) we obtain

$$\cos^2 z + \sin^2 z = \frac{e^{2iz} + e^{-2iz} + 2e^{iz}e^{-iz}}{4} - \frac{e^{2iz} + e^{-2iz} - 2e^{iz}e^{-iz}}{4} = 1$$

and

$$e^{iz} = \cos z + i \sin z, \tag{21}$$

which is what we used in (6).

Exercise 33 *Let $x + iy \in \mathbb{C}$. Prove that*

$$|\cos z|^2 = \cos^2 x + \cosh^2 y, \quad |\sin z|^2 = \sin^2 x + \sinh^2 y,$$

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MLK Day, no classes.

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Next we study the periodicity of e^z. We say that a function $f : \mathbb{C} \to \mathbb{C}$ is *periodic* with period $w \in \mathbb{C}$ if

$$f(z+w) = f(z) \quad \text{for all } z \in \mathbb{C}.$$

Given $w \in \mathbb{C}$, assume that

$$e^z = e^{z+w}$$

for all $z \in \mathbb{C}$. Then by multiplying both sides by e^{-z} and using (17) we get $1 = e^w$. Taking the modulus on both sides and using 19 we get

$$1 = e^{2\operatorname{Re} w},$$

which implies that $\operatorname{Re} w = 0$. Thus, $w = i\theta$ for some $\theta \in \mathbb{R}$. In turn, by (21),

$$1 = e^{i\theta} = \cos\theta + i\sin\theta,$$

and so

$$\theta = 2\pi k, \quad k \in \mathbb{Z}.$$

This shows that the exponential function is periodic with period $2\pi i$. This is one of the main differences with the real exponential. In particular, this implies that e^z is not one-to-one in $\mathbb{C}$. Thus, we cannot define the complex logarithmic function as the inverse of the complex exponential function.

Definition 34 *Given a connected open set $U \subseteq \mathbb{C}$, a* branch of the logarithm *is a continuous function $f : U \to \mathbb{C}$ such that*

$$z = e^{f(z)} \quad \textit{for all } z \in U.$$

We sometimes write $f = \log_U$.

Remark 35 *Note that since $e^z \neq 0$ for all z, in order for a branch of the logarithm to exist in U, we must have $0 \notin U$.*

Exercise 36 *Let*

$$W := \mathbb{C} \setminus \{z \in \mathbb{C} : \, z = x + 0i, \, x \leq 0\}.$$

For every $z \in W$, write $z = re^{i\theta}$, $r = |z|$, $-\pi < \theta < \pi$, and define

$$f(z) := \log r + i\theta.$$

(i) Prove that f is branch of the logarithm in W.

(ii) Prove that for all $z \in B(0,1)$ with $1 + z \in W$,

$$f(1+z) = -\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}.$$

(iii) Prove that in general

$$f(z_1 z_2) \neq f(z_1) + f(z_2).$$

The branch of the logarithm constructed in the previous exercise is called the *principal branch of the logarithm.*

Proposition 37 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be a branch of of the logarithm. Then f is differentiable in U, with*

$$f'(z) = \frac{1}{z} \quad \textit{for all } z \in U.$$

Moreover, every other branch of the logarithm in U has the form

$$g(z) = f(z) + 2k\pi i$$

for some $k \in \mathbb{Z}$.

Proof. The differentiability of f follows from Exercise 9. By the chain rule (see Exercise 8) and the definition of f,

$$1 = e^{f(z)} f'(z) = z f'(z) \quad \text{for all } z \in U,$$

which implies that $f'(z) = \frac{1}{z}$.

Given $k \in \mathbb{Z}$, consider the function $g(z) := f(z) + 2k\pi i$, $z \in U$. Then the periodicity of the exponential

$$e^{g(z)} = e^{f(z)+2k\pi i} = e^{f(z)} = z,$$

which shows that g is a branch of $\log z$.

Conversely, assume that $g : U \to \mathbb{C}$ is another branch of $\log z$. Then the function

$$h(z) := \frac{1}{2\pi i}(f(z) - g(z)), \quad z \in U,$$

is continuous and since

$$e^{2\pi i h(z)} = e^{f(z)-g(z)} = e^{f(z)} e^{-g(z)} = z \frac{1}{z} = 1,$$

by (18), we have that $2\pi i h(z) = 2k\pi i$ for some $k \in \mathbb{Z}$ (depending on z). This shows that $h(U) \subseteq \mathbb{Z}$, but since h is continuous and U is connected, h must be constant, and thus there is $k_0 \in \mathbb{Z}$ such that $h(z) = k_0$ for *all* $z \in U$, which completes the proof. ■

If $U \subseteq \mathbb{C}$ is an open connected set and $f : U \to \mathbb{C}$ is a branch of the logarithm in U, then for every $a \in \mathbb{C}$ we define a *branch* of z^a as

$$g(z) := e^{a f(z)}, \quad z \in U.$$

In view of the previous theorem, g is differentiable in U, since composition of differentiable functions, and every other branch is given by

$$h(z) = e^{a f(z) + a 2k\pi i} = e^{a f(z)} e^{a 2k\pi i} = g(z) e^{a 2k\pi i}.$$

4 Riemann-Stieltjes integrals

In what follows, given an interval $[a,b] \subseteq \mathbb{R}$, a *partition* of $[a,b]$ is a finite set $P := \{t_0, \dots, t_n\} \subset [a,b]$, where

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Definition 38 *Let $g : [a,b] \to \mathbb{C}$ be a function. The* pointwise variation *of g on the interval $[a,b]$ is*

$$\operatorname{Var} g := \sup \left\{ \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})| \right\},$$

where the supremum is taken over all partitions $P := \{t_0, \dots, t_n\}$ of $[a,b]$, $n \in \mathbb{N}$. A function $g : [a,b] \to \mathbb{C}$ has finite *or* bounded pointwise variation *if* $\operatorname{Var} g < \infty$.

The space of all functions $g : [a,b] \to \mathbb{C}$ of bounded pointwise variation is denoted by $BV([a,b];\mathbb{C})$.

To highlight the dependence on the interval $[a,b]$, we will sometimes write $\operatorname{Var}_{[a,b]} g$.

Given a function $g : [a,b] \to \mathbb{C}$, we say that g is piecewise C^1, if g is continuous, and there exists a partition $P := \{t_0 \dots, t_n\} \subset [a,b]$ such that $g : [t_{k-1}, t_k] \to \mathbb{C}$ is of class C^1 for every $k = 1, \dots, n$.

Exercise 39 *Let $g : [a,b] \to \mathbb{C}$ be piecewise C^1. Prove that*

$$\operatorname{Var} g = \int_a^b |g'(t)|\, dt.$$

Exercise 40 *Let $g : [a,b] \to \mathbb{C}$ be Lipschitz continuous. Prove that $g \in BV([a,b];\mathbb{C})$.*

Exercise 41 *Let $g : [a,b] \to \mathbb{R}$ be a monotone function. Prove that*

$$\operatorname{Var} g = \sup_{[a,b]} g - \inf_{[a,b]} g.$$

Exercise 42 *Let $f, g \in BV([a,b];\mathbb{C})$. Prove the following.*

(i) $f \pm g \in BV([a,b];\mathbb{C})$.

(ii) $fg \in BV([a,b];\mathbb{C})$.

(iii) If $|g(t)| \geq c > 0$ for all $t \in [a,b]$ and for some $c > 0$, then $\frac{f}{g} \in BV([a,b];\mathbb{C})$.

Exercise 43 *Let $g : [a,b] \to \mathbb{C}$, and let $c \in [a,b]$. Prove that*

$$\operatorname{Var}_{[a,c]} g + \operatorname{Var}_{[c,b]} g = \operatorname{Var}_{[a,b]} g.$$

Exercise 44 *Prove that* $g \mapsto \operatorname{Var} g$ *is a seminorm in* $BV([a,b];\mathbb{C})$.

Theorem 45 *Let* $g \in BV([a,b];\mathbb{C})$ *and let* $f : [a,b] \to \mathbb{C}$ *be a continuous function. Then there exists* $\ell \in \mathbb{C}$ *with the property that for every* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that if* $P = \{t_0, \dots, t_n\}$ *is a partition of* $[a,b]$ *with* $t_k - t_{k-1} \leq \delta_\varepsilon$ *for all* $k = 1, \dots, n$, *then*

$$\left| \ell - \sum_{k=1}^{n} f(s_k)(g(t_k) - g(t_{k-1})) \right| \leq \varepsilon,$$

for every $s_k \in [t_{k-1}, t_k]$, $k = 1, \dots, n$.

The number ℓ is called the *Riemann-Stieltjes integral of* f *with respect to* g *over* $[a,b]$ *and is denoted*

$$\ell = \int_a^b f\, dg.$$

Exercise 46 *Let* $g : [a,b] \to \mathbb{C}$ *be piecewise* C^1 *and let* $f : [a,b] \to \mathbb{C}$ *be a continuous function. Prove that*

$$\int_a^b f\, dg = \int_a^b f(t) g'(t)\, dt.$$

Exercise 47 *Let* $g \in BV([a,b];\mathbb{C})$, *let* $f : [a,b] \to \mathbb{C}$ *be a continuous functions, and* $P = \{t_0, \dots, t_n\}$ *be a partition of* $[a,b]$ *with* $a = t_0$ *and* $b = t_n$. *Prove that*

$$\int_a^b f\, dg = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f\, dg.$$

Exercise 48 *Let* $g \in BV([a,b];\mathbb{C})$, *let* $f : [a,b] \to \mathbb{C}$ *be a continuous functions. Prove that*

$$\left| \int_a^b f\, dg \right| \leq \max_{[a,b]} |f| \operatorname{Var} g.$$

Exercise 49 *Let* $g \in BV([a,b];\mathbb{C})$, *let* $f_1, f_2 : [a,b] \to \mathbb{C}$ *be continuous functions, and let* $\alpha, \beta \in \mathbb{C}$. *Prove that*

$$\int_a^b (\alpha f_1 + \beta f_2)\, dg = \alpha \int_a^b f_1\, dg + \beta \int_a^b f_2\, dg.$$

Exercise 50 *Let* $g_1, g_2 \in BV([a,b];\mathbb{C})$, *let* $f : [a,b] \to \mathbb{C}$ *be continuous functions, and let* $\alpha, \beta \in \mathbb{C}$. *Prove that*

$$\int_a^b f\, d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f\, dg_1 + \beta \int_a^b f\, dg_2.$$

Exercise 51 *Let $g \in BV([a,b];\mathbb{C})$, let $f:[a,b] \to \mathbb{C}$ be a continuous functions, and $P = \{t_0, \ldots, t_n\}$ be a partition of $[a,b]$ with $a = t_0$ and $b = t_n$. Prove that*

$$\int_a^b f\,dg = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f\,dg.$$

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We turn to the proof of Theorem 45.

Proof of Theorem 45. Since f is uniformly continuous, given $\varepsilon = \frac{1}{m}$ we can find $\delta_m > 0$ such that

$$|f(z) - f(w)| \leq \frac{1}{m} \tag{22}$$

for all $z, w \in [a,b]$ with $|z-w| \leq \delta_m$. By an induction argument, we can assume that $\delta_m \geq \delta_{m+1}$ for all $m \in \mathbb{N}$. For each m let $\mathcal{P}_m$ be the family of all partitions $P = \{t_0, \ldots, t_n\}$ of $[a,b]$ with $t_k - t_{k-1} \leq \delta_m$ for all $k = 1, \ldots, n$. Note that $\mathcal{P}_{m+1} \subseteq \mathcal{P}_m$ for every m. Let

$$E_m := \{S(P) :\ P = \{t_0, \ldots, t_n\} \in \mathcal{P}_m,\ s_k \in [t_{k-1}, t_k],\ k = 1, \ldots, n\}$$

where

$$S(P) := \sum_{k=1}^n f(s_k)(g(t_k) - g(t_{k-1})),$$

and let $C_m = \overline{E}_m$. Since $\mathcal{P}_{m+1} \subseteq \mathcal{P}_m$, we have that $E_{m+1} \subseteq E_m$, and so $C_{m+1} \subseteq C_m$.

Next we claim that

$$\operatorname{diam} C_m \leq \frac{2}{m} \operatorname{Var} g. \tag{23}$$

To see this, let $P, Q \in \mathcal{P}_m$. Let $P = \{t_0, \ldots, t_n\}$ and assume first that Q is obtained from P by adding a point c and let k_0 be such that $t_{k_0-1} < c < t_{k_0}$. Then

$$\begin{aligned} S(Q) = \sum_{k \neq k_0} & f(\tau_k)(g(t_k) - g(t_{k-1})) + f(\tau')(g(c) - g(t_{k_0-1})) \\ & + f(\tau'')(g(t_{k_0}) - g(c)), \end{aligned}$$

where $t_{k-1} \leq \tau_k \leq t_k$, $t_{k_0-1} \leq \tau' \leq c$, $c \leq \tau' \leq t_{k_0}$. In turn, by (22),

$$\begin{aligned} |S(Q) - S(P)| &\leq \sum_{k \neq k_0} |f(\tau_k) - f(s_k)||g(t_k) - g(t_{k-1})| + |f(\tau') - f(s_{k_0})||g(c) - g(t_{k_0-1})| \\ &\quad + |f(\tau'') - f(s_{k_0})||g(t_{k_0}) - g(c)| \\ &\leq \frac{1}{m} \sum_{k \neq k_0} |g(t_k) - g(t_{k-1})| + \frac{1}{m}|g(c) - g(t_{k_0-1})| + \frac{1}{m}|g(t_{k_0}) - g(c)| \\ &\leq \frac{1}{m} \operatorname{Var} g. \end{aligned}$$

With a similar proof, we can show that if $P \subseteq Q$, then $|S(Q) - S(P)| \leq \frac{1}{m} \operatorname{Var} g$.

Finally, if $P, Q \in \mathcal{P}_m$, let $R \in \mathcal{P}_m$ be such that $P, Q \subseteq R$, then

$$|S(Q) - S(P)| \leq |S(Q) - S(R)| + |S(R) - S(P)| \leq \frac{1}{m} \operatorname{Var} g + \frac{1}{m} \operatorname{Var} g.$$

By taking the supremum over all such partitions we conclude that $\operatorname{diam} E_m \leq \frac{2}{m} \operatorname{Var} g$, and in turn, (23) follows.

It now follows from Cantor's theorem that there exists a unique $\ell \in \mathbb{C}$ such that

$$\{\ell\} = \bigcap_{m=1}^{\infty} C_m.$$

Given $\varepsilon > 0$ let m be so large that $\frac{2}{\varepsilon} \operatorname{Var} g < m$ and take $\delta_\varepsilon := \delta_m$. Since $\ell \in C_m$, we have that $C_m \subseteq B(\ell, \varepsilon)$, which proves the theorem. ■

5 Line Integrals

Definition 52 *Given two functions $\varphi : [a, b] \to \mathbb{C}$ and $\psi : [c, d] \to \mathbb{C}$, we say that they are* equivalent *if there exists a continuous, strictly increasing, onto function $h : [a, b] \to [c, d]$ such that*

$$\varphi(t) = \psi(h(t))$$

for all $t \in [a, b]$. We write $\varphi \sim \psi$ and we call φ and ψ parametric representations *and the function h a* parameter change*.*

Note that in view of a theorem real analysis, $h^{-1} : [c, d] \to [a, b]$ is also continuous.

Exercise 53 *Prove that $\sim$ is an equivalence relation.*

Definition 54 *An* oriented curve *γ is an equivalence class of parametric representations.*

Remark 55 *The definition of a curve is a restrictive, although it is what we will need it in this course. More generally, given two intervals $I, J \subseteq \mathbb{R}$, and two functions $\varphi : I \to \mathbb{C}$ and $\psi : J \to \mathbb{C}$, we say that they are* equivalent *if there exists a continuous, strictly increasing, onto function $h : I \to J$ such that*

$$\varphi(t) = \psi(h(t))$$

for all $t \in I$. We write $\varphi \sim \psi$ and we call φ and ψ parametric representations *and the function h a* parameter change*.*

Given an oriented curve γ with parametric representation $\varphi : [a, b] \to \mathbb{C}$ the *multiplicity* of a point $z \in \mathbb{C}$ is the (possibly infinite) number of points $t \in [a, b]$ such that $\varphi(t) = z$. Since every parameter change $h : [a, b] \to [c, d]$ is bijective, the multiplicity of a point does not depend on the particular parametric representation. The *range* of γ is the set of points of $\mathbb{C}$ with positive multiplicity, that is, $\varphi([a, b])$.

A point in the range of γ with multiplicity one is called a *simple point.* If every point of the range is simple, then γ is called a *simple arc.*

Given an oriented curve γ with parametric representation $\varphi : [a, b] \to \mathbb{C}$, the oriented curve γ_1 with parametric representation $\varphi_1 : [a, b] \to \mathbb{C}$ given by

$$\varphi_1(t) := \varphi(-t + b + a)$$

is called the *curve opposite to* γ.

Definition 56 *Given two functions $\varphi : [a, b] \to \mathbb{C}$ and $\psi : [c, d] \to \mathbb{C}$ of class C^k, $k \in \mathbb{N}_0$, we say that they are* equivalent *if there exists a strictly increasing, onto function $h : [a, b] \to [c, d]$ with h and h^{-1} of class C^k such that*

$$\varphi\left(t\right) = \psi\left(h\left(t\right)\right)$$

for all $t \in [a, b]$. We write $\varphi \sim_k \psi$ and we call φ and ψ parametric representations of class C^k *and the function h a* parameter change of class C^k*. An* oriented curve γ *of class C^k is an equivalence class of parametric representations of class C^k.*

Similarly we can define C^∞ oriented curves, Lipschitz oriented curves, analytic oriented curves, and so on.

Given a continuous curve, the points $\varphi(a)$ and $\varphi(b)$ are called *endpoints* of the curve. If $\varphi\left(a\right) = \varphi\left(b\right)$, then the oriented curve γ is called a *closed oriented curve.* A closed curve is called *simple* if every point of the range is simple, with the exception of $\varphi\left(a\right)$, which has multiplicity two.

The following theorem will be used in the sequel.

Theorem 57 (Jordan's curve theorem) *Given a continuous closed simple oriented curve γ in $\mathbb{C}$ with range Γ, the set $\mathbb{C} \setminus \Gamma$ consists of two connected components.*

The bounded connected component of $\mathbb{C} \setminus \Gamma$ is called the *interior* of γ.

We are ready to define the notion of length of a curve.

Exercise 58 *Let γ be an oriented curve in $\mathbb{C}$. Let $\varphi : [a, b] \to \mathbb{C}$ and $\psi : [c, d] \to \mathbb{C}$ be two parametric representations of γ. Prove that* $\operatorname{Var}_{[a,b]} \varphi = \operatorname{Var}_{[c,d]} \psi$.

We are now ready to define the length of a curve.

Definition 59 *Let γ be an oriented curve in $\mathbb{C}$ and let $\varphi : [a, b] \to \mathbb{C}$ be a parametric representation of γ. We define the* length *of γ as*

$$L\left(\gamma\right) := \operatorname{Var} \varphi.$$

We say that the curve γ is rectifiable *if $L\left(\gamma\right) < \infty$.*

Theorem 60 *Given a rectifiable oriented curve γ in $\mathbb{C}$ with range Γ and a continuous function $f : \Gamma \to \mathbb{C}$, let $\varphi : [a, b] \to \mathbb{C}$ and $\psi : [c, d] \to \mathbb{C}$ be two parametric representations of γ. Then*

$$\int_a^b f \circ \varphi \, d\varphi = \int_c^d f \circ \psi \, d\psi.$$

Monday, January 27, 2020

No class

Wednesday, January 29, 2020

No class

Friday, January 31, 2020

2 hours

Proof. Since φ and ψ are equivalent, there exists $h : [c, d] \to [a, b]$ continuous, strictly increasing, with $h(c) = a$ and $h(d) = b$, such that

$$\varphi(h(s)) = \psi(s) \quad \text{for all } s \in [c, d]. \tag{24}$$

By Theorem 45 for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $P = \{t_0, \ldots, t_n\}$ is a partition of $[a, b]$ with $t_k - t_{k-1} \le \delta_\varepsilon$ for all $k = 1, \ldots, n$, then

$$\left| \int_a^b f \circ \varphi \, d\varphi - \sum_{k=1}^n f(\varphi(t_k'))(\varphi(t_k) - \varphi(t_{k-1})) \right| \le \varepsilon, \tag{25}$$

for every $t_k' \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$. Similarly, there exists $\rho_\varepsilon > 0$ such that if $Q = \{s_0, \ldots, s_m\}$ is a partition of $[c, d]$ with $s_l - s_{l-1} \le \rho_\varepsilon$ for all $l = 1, \ldots, m$, then

$$\left| \int_c^d f \circ \psi \, d\psi - \sum_{l=1}^m f(\psi(s_l'))(\psi(s_l) - \psi(s_{l-1})) \right| \le \varepsilon, \tag{26}$$

for every $s_l' \in [s_{l-1}, s_l]$, $l = 1, \ldots, m$. Since h is uniformly continuous, there exists $\eta_\varepsilon > 0$ such that

$$|h(s) - h(s')| \le \delta_\varepsilon$$

for all $s, s' \in [c, d]$ with $|s - s'| \le \eta_\varepsilon$. Let $Q = \{s_0, \ldots, s_m\}$ be a partition of $[c, d]$ with $s_l - s_{l-1} \le \min\{\eta_\varepsilon, \rho_\varepsilon\}$ for all $l = 1, \ldots, m$. Then $P = \{h(s_0), \ldots, h(s_m)\}$ is a partition of $[a, b]$ with $h(s_k) - h(s_{k-1}) \le \delta_\varepsilon$. Hence, (25) holds for this partition,On the other hand, by (24), $\varphi(h(s_l)) = \psi(s_l)$ and so

$$\sum_{l=1}^m f(\varphi(h(s_l')))(\varphi(h(s_l)) - \varphi(h(s_{l-1}))) = \sum_{l=1}^m f(\psi(s_l'))(\psi(s_l) - \psi(s_{l-1})).$$

Hence, by (25) and (26),

$$\begin{aligned} \left| \int_a^b f \circ \varphi \, d\varphi - \int_c^d f \circ \psi \, d\psi \right| &\le \left| \int_a^b f \circ \varphi \, d\varphi - \sum_{l=1}^m f(\varphi(h(s_l')))(\varphi(h(s_l)) - \varphi(h(s_{l-1}))) \right| \\ &\quad + \left| \int_c^d f \circ \psi \, d\psi - \sum_{l=1}^m f(\psi(s_l'))(\psi(s_l) - \psi(s_{l-1})) \right| \le 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0^+$ gives the result. ■

Given a rectifiable oriented curve γ in $\mathbb{C}$ parametrized by $\varphi : [a, b] \to \mathbb{C}$ and a continuous function $f : \varphi([a, b]) \to \mathbb{C}$, the *line integral* of f over γ is defined as

$$\int_\gamma f \, dz := \int_a^b f \circ \varphi \, d\varphi.$$

In view of the previous theorem, the integral does not depend on the particular representation of the curve.

Note that all the properties in the exercises in the previous section continue to hold for line integrals.

Definition 61 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$. We say that f has a* primitive *in U if there exists a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$.*

Remark 62 *The function $f(z) = az^n$, where $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$ has a primitive given by $F(z) = \frac{a}{n+1} z^{n+1} + c$.*

Theorem 63 (Fundamental theorem of calculus) *Let $U \subseteq \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a continuous function, which has a primitive F in U. Then for every $z_1, z_2 \in U$ and for every rectifiable continuous oriented curve γ starting at z_1 and ending at z_2 and with range in U,*

$$\int_\gamma f\, dz = F(z_2) - F(z_1).$$

We begin with a preliminary result.

Lemma 64 *Let $U \subseteq \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a continuous function, and let γ be a rectifiable continuous oriented curve γ with range in U. Then for every $\varepsilon > 0$ there exists a polygonal path γ_ε with the same endpoints of γ and range in U such that*

$$\left| \int_\gamma f\, dz - \int_{\gamma_\varepsilon} f\, dz \right| \le \varepsilon$$

Proof. Step 1: Assume first that $U = B(z_0, r)$. Let $\varphi : [a,b] \to \mathbb{C}$ be a parametric representation of γ. Since $\varphi([a,b])$ is compact, we have that $\operatorname{dist}(\varphi([a,b]), \partial U) = \rho > 0$. Hence, $\varphi([a,b]) \subseteq \overline{B(z_0, r-\rho)} =: K$. Since f is uniformly continuous on K, given $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|f(z) - f(w)| \le \varepsilon \tag{27}$$

for all $z, w \in K$ with $|z - w| \le \delta_\varepsilon$.

Since $\varphi : [a,b] \to \mathbb{C}$ is uniformly continuous, there exists $\eta_\varepsilon > 0$ such that

$$|\varphi(t) - \varphi(s)| \le \delta_\varepsilon \tag{28}$$

for all $s, t \in [a,b]$ with $|s - t| \le \eta_\varepsilon$. Moreover, by Theorem 45, there exists $\rho_\varepsilon > 0$ such that if $P = \{t_0, \dots, t_n\}$ is a partition of $[a,b]$ with $t_k - t_{k-1} \le \rho_\varepsilon$ for all $k = 1, \dots, n$, then

$$\left| \int_\gamma f\, dz - \sum_{k=1}^{n} f(\varphi(s_k))(\varphi(t_k) - \varphi(t_{k-1})) \right| \le \varepsilon, \tag{29}$$

for every $s_k \in [t_{k-1}, t_k]$, $k = 1, \dots, n$. Consider a partition $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$ with $t_k - t_{k-1} \le \min\{\rho_\varepsilon, \eta_\varepsilon\}$. Let φ_ε be the polygonal path joining $\varphi(t_0)$, $\dots$, $\varphi(t_n)$. To be precise

$$\varphi_\varepsilon(t) := \frac{1}{t_k - t_{k-1}}[(t-t_{k-1})\varphi(t_k)+(t_k-t)\varphi(t_{k-1})], \quad t \in [t_{k-1}, t_k],\ k = 1, \dots, n.$$

Note that

$$\varphi_\varepsilon'(t) = \frac{\varphi(t_k) - \varphi(t_{k-1})}{t_k - t_{k-1}}, \quad t \in (t_{k-1}, t_k),\ k = 1, \dots, n.$$

Hence, by Exercise 46,

$$\int_{\gamma_\varepsilon} f\,dz = \int_a^b f(\varphi_\varepsilon(t))\varphi_\varepsilon'(t)\,dt = \sum_{k=1}^n \frac{\varphi(t_k) - \varphi(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\varphi_\varepsilon(t))\,dt.$$

Hence, by (29),

$$\begin{aligned}
&\left|\int_\gamma f\,dz - \int_{\gamma_\varepsilon} f\,dz\right| \le \left|\int_\gamma f\,dz - \sum_{k=1}^n f(\varphi(s_k))(\varphi(t_k) - \varphi(t_{k-1}))\right| \\
&\quad + \left|\sum_{k=1}^n f(\varphi(s_k))(\varphi(t_k) - \varphi(t_{k-1})) - \sum_{k=1}^n \frac{\varphi(t_k) - \varphi(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\varphi_\varepsilon(t))\,dt\right| \\
&\le \varepsilon + \left|\sum_{k=1}^n \frac{\varphi(t_k) - \varphi(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\varphi_\varepsilon(s_k))\,dt - \sum_{k=1}^n \frac{\varphi(t_k) - \varphi(t_{k-1})}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f(\varphi_\varepsilon(t))\,dt\right| \\
&\le \varepsilon + \sum_{k=1}^n \frac{|\varphi(t_k) - \varphi(t_{k-1})|}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} |f(\varphi(s_k)) - f(\varphi_\varepsilon(t))|\,dt.
\end{aligned}$$

For $t \in [t_{k-1}, t_k]$ we have

$$\varphi(s_k) - \varphi_\varepsilon(t) = \frac{1}{t_k - t_{k-1}}[(t - t_{k-1})(\varphi(s_k) - \varphi(t_k)) + (t_k - t)(\varphi(s_k) - \varphi(t_{k-1})]$$

and so by (28),

$$|\varphi(s_k)-\varphi_\varepsilon(t)| \le \frac{1}{t_k - t_{k-1}}[(t-t_{k-1})|\varphi(s_k)-\varphi(t_k)|+(t_k-t)|\varphi(s_k)-\varphi(t_{k-1})|] \le \delta_\varepsilon$$

In turn, by (27), $|f(\varphi(s_k)) - f(\varphi_\varepsilon(t))| \le \varepsilon$. Using this inequality we have that

$$\begin{aligned}
\left|\int_\gamma f\,dz - \int_{\gamma_\varepsilon} f\,dz\right| &\le \varepsilon + \sum_{k=1}^n \frac{|\varphi(t_k) - \varphi(t_{k-1})|}{t_k - t_{k-1}} \varepsilon(t_k - t_{k-1}) \le \varepsilon + \varepsilon \sum_{k=1}^n |\varphi(t_k) - \varphi(t_{k-1})| \\
&\le \varepsilon + \varepsilon L(\gamma).
\end{aligned}$$

Step 2: For a generic open set, since $\varphi([a, b])$ is compact, as before $\operatorname{dist}(\varphi([a, b]), \partial U) > 0$. Let $0 < \rho < \operatorname{dist}(\varphi([a, b]), \partial U)$. Since φ is uniformly continuous, there exists $\delta > 0$ such that

$$|\varphi(t) - \varphi(s)| < \rho$$

for all $s, t \in [a, b]$ with $|s - t| \leq \delta$. Consider a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$ with $t_k - t_{k-1} \leq \delta$ for all $k = 1, \dots, n$. It follows that $\varphi([t_{k-1}, t_k]) \subset B(\varphi(t_{k-1}), \rho)$ and so we may apply the previous step to the curve γ_k parametrized by $\varphi : [t_{k-1}, t_k] \to \mathbb{C}$ to find a polygonal path Γ_k with endpoints $\varphi(t_{k-1})$ and $\varphi(t_k)$ such that

$$\left| \int_{\gamma_k} f\, dz - \int_{\Gamma_k} f\, dz \right| \leq \varepsilon/n.$$

By joining $\Gamma_1, \dots, \Gamma_n$ we get a polygonal path joining $\varphi(a)$ and $\varphi(b)$. The result now follows from the previous inequality and Exercise 47. ■

We turn to the proof of the fundamental theorem of calculus.

Proof. In view of the previous lemma, for every $\varepsilon > 0$ there exists a polygonal path γ_ε with endpoints z_1 and z_2 such that

$$\left| \int_{\gamma} f\, dz - \int_{\gamma_\varepsilon} f\, dz \right| \leq \varepsilon.$$

Let $\varphi_\varepsilon : [a, b] \to \mathbb{C}$ be a parametric representation of γ_ε. By Exercise 46,

$$\begin{aligned}\int_{\gamma_\varepsilon} f\, dz &= \int_a^b f(\varphi_\varepsilon(t))\varphi_\varepsilon'(t)\, dt = \int_a^b F'(\varphi_\varepsilon(t))\varphi_\varepsilon'(t)\, dt = \int_a^b (F \circ \varphi_\varepsilon)'(t)\, dt \\ &= F \circ \varphi_\varepsilon(b) - F \circ \varphi_\varepsilon(a) = F(z_2) - F(z_1).\end{aligned}$$

Hence,

$$\left| \int_{\gamma} f\, dz - (F(z_2) - F(z_1)) \right| \leq \varepsilon.$$

Letting $\varepsilon \to 0^+$ completes the proof. ■

Corollary 65 *Let $U \subseteq \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a continuous function, which has a primitive F in U. Then*

$$\int_{\gamma} f\, dz = 0$$

for every closed rectifiable continuous oriented curve with range in U.

Exercise 66 *Given a rectifiable oriented curve γ in $\mathbb{C}$ with range Γ and a continuous function $f : \Gamma \to \mathbb{C}$, let γ_1 be the curve opposite to γ. Prove that*

$$\int_{\gamma_1} f\, dz = -\int_{\gamma} f\, dz.$$

6 Cauchy's Theorem in a Ball

Theorem 67 (Goursat) *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. Then for every closed triangle $T \subset U$,*

$$\int_{\partial T} f\, dz = 0.$$

Proof. Set $T_0 := T$, bisect each side of T_0 and connect the middle points. This creates four triangles $T_{1,1}$, $T_{1,2}$, $T_{1,3}$, and $T_{1,4}$. By choosing an orientation for these triangles consistent with the one of T_0 and by canceling the sides which are integrated in two opposite directions (see Exercises 47 and 66), we get

$$\int_{\partial T_0} f\,dz = \int_{\partial T_{1,1}} f\,dz + \int_{\partial T_{1,2}} f\,dz + \int_{\partial T_{1,3}} f\,dz + \int_{\partial T_{1,4}} f\,dz.$$

Hence, for some $j \in \{1,2,3,4\}$,

$$\left|\int_{\partial T_0} f\,dz\right| \le 4\left|\int_{\partial T_{1,j}} f\,dz\right|.$$

Let $T_1 := T_{1,j}$. Note that $L\left(\partial T_1\right) = \frac{1}{2}L(\partial T_0)$ and $\operatorname{diam} T_1 = \frac{1}{2}\operatorname{diam} T_0$. We now bisect the sides of T_1 and connect the middle points. Inductively we obtain a decreasing sequence of closed triangles T_n such that

$$\left|\int_{\partial T_0} f\,dz\right| \le 4^n\left|\int_{\partial T_n} f\,dz\right|, \tag{30}$$

$L\left(\partial T_n\right) = \frac{1}{2^n}L(\partial T_0)$ and $\operatorname{diam} T_n = \frac{1}{2^n}\operatorname{diam} T_0$. By Cantor's theorem there exists $z_0 \in T_n$ for all n. Since f is differentiable, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z),$$

where

$$\lim_{z\to z_0}\frac{R(z)}{z - z_0} = 0.$$

Since a constant function and a linear function az have a primitive, by the fundamental theorem of calculus,

$$\int_{\partial T_n} f\,dz = \int_{\partial T_n} R\,dz.$$

Since $z_0 \in T_n$ and $z \in \partial T_n$, we have

$$|R(z)| \le \varepsilon_n|z - z_0| \le \varepsilon_n \operatorname{diam} T_n,$$

where $\varepsilon_n \to 0^+$. Hence,

$$\begin{aligned}\left|\int_{\partial T_n} f\,dz\right| = \left|\int_{\partial T_n} R\,dz\right| &\le \varepsilon_n(\operatorname{diam} T_n)L\left(\partial T_n\right)\\ &\le \varepsilon_n\frac{1}{4^n}L(\partial T_0)\operatorname{diam} T_0.\end{aligned}$$

Using (30) we get

$$\left|\int_{\partial T_0} f\,dz\right| \le 4^n\left|\int_{\partial T_n} f\,dz\right| \le \varepsilon_n L(\partial T_0)\operatorname{diam} T_0 \to 0$$

as $n \to \infty$. ■

Exercise 68 *Let $U \subseteq \mathbb{C}$ be an open set, let $z_0 \in U$, and let $f : U \to \mathbb{C}$ be a continuous function, which is holomorphic in $U \setminus \{z_0\}$. Prove that for every closed triangle $T \subset U$,*

$$\int_{\partial T} f \, dz = 0.$$

Hint: Consider first the case in which z_0 is a vertex of T.

Saturday, February 1, 2020

Make-up class.

As a corollary we get

Corollary 69 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. Then for every closed rectangle $R \subset U$,*

$$\int_{\partial R} f \, dz = 0.$$

Proof. Divide R into two triangles and one side of the triangles is in common and are integrated in two opposite directions. ■

Theorem 70 *Let $B \subset \mathbb{C}$ be an open ball and let $f : B \to \mathbb{C}$ be holomorphic. Then f has a primitive in B.*

Proof. Without loss of generality assume that B is centered at the origin. Given $z = x + iy \in B$, with $x, y \in \mathbb{R}$, we connect the origin to $x + 0i$ and then $x + 0i$ to z and let γ_z be this polygonal path in B. We choose the orientation starting at 0 and ending at z. Define

$$F(z) := \int_{\gamma_z} f \, d\zeta.$$

We claim that $F' = f$. Let $z + h \in B$. Then

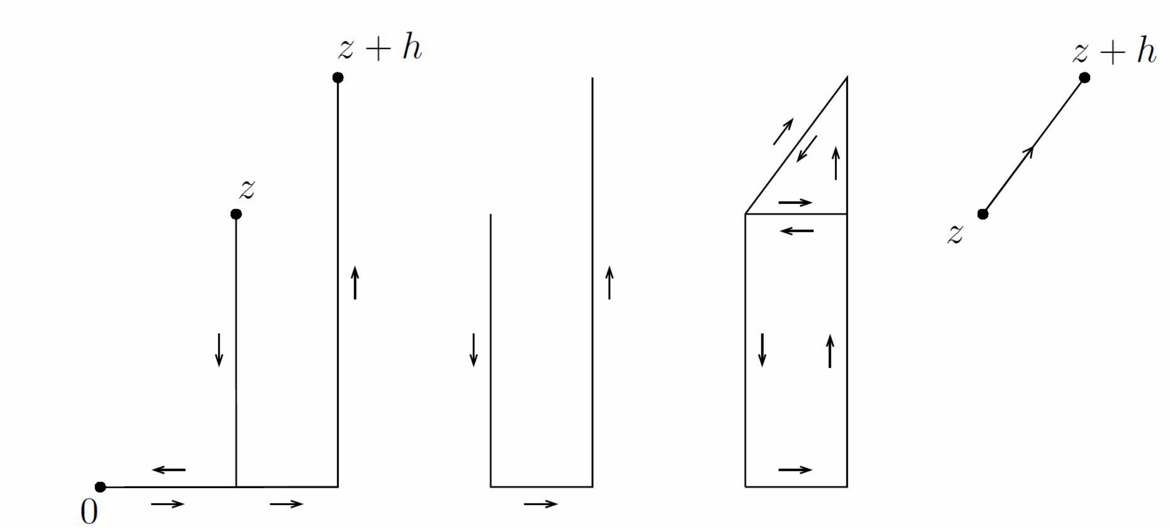
$$F(z + h) - F(z) = \int_{\gamma_{z+h}} f \, d\zeta - \int_{\gamma_z} f \, d\zeta.$$

Using Goursat's theorem for triangles and rectangles we are left with the segment $S_{z,h}$ going from z to $z+h$. Given $\zeta \in S_{z,h}$ write $f(\zeta) = f(z) + r(\zeta)$, where by continuity

$$\lim_{\zeta \to z} r(\zeta) = 0.$$

Then

$$\begin{aligned} F(z + h) - F(z) &= \int_{S_{z,h}} f \, d\zeta = f(z) \int_{S_{z,h}} 1 \, d\zeta + \int_{S_{z,h}} r \, d\zeta \\ &= f(z)h + \int_{S_{z,h}} r \, d\zeta, \end{aligned}$$



where we used the fact that the constant 1 has a primitive. Hence,

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_{S_{z,h}} r \, d\zeta$$

and

$$\left|\frac{1}{h}\int_{S_{z,h}} r \, d\zeta\right| \leq \max_{S_{z,h}} |r| \frac{|h|}{|h|} = \max_{S_{z,h}} |r| \to 0$$

as $h \to 0$. ■

Remark 71 *In the previous proof we only used the fact that f is continuous and for every closed triangle $T \subset B$,*

$$\int_{\partial T} f = 0. \tag{31}$$

Hence, if we assume that $f : B \to \mathbb{C}$ is a continuous function which is holomorphic in $B \setminus \{z_0\}$ for some $z_0 \in B$, then by Exercise 68, (31) holds, and so f has a primitive in B.

Corollary 72 (Cauchy) *Let B be an open ball, let $f : B \to \mathbb{C}$ be holomorphic. Then*

$$\int_\gamma f \, dz = 0$$

for every closed oriented curve γ with range in B.

Proof. This follows from the previous theorem and Corollary 72. ■

Remark 73 *In view of Remark 71, Corollary 72 continues to hold if we assume that $f : B \to \mathbb{C}$ is a continuous function which is holomorphic in $B \setminus \{z_0\}$ for some $z_0 \in B$.*

Exercise 74 *Let $z_0 = x_0 + iy_0 \in B(0,1)$, let $U \subset \mathbb{C}$ be the open set obtained from $B(0,1)$ by removing the segment $\{x_0 + yi : y \geq y_0\}$, and let $f : U \to \mathbb{C}$ be holomorphic. Prove that f has a primitive in U.*

Exercise 75 *Let $U \subseteq \mathbb{C}$ be a star-shaped set and let $f : U \to \mathbb{C}$ be holomorphic. Prove that*

$$\int_\gamma f\, dz = 0$$

for every closed oriented curve γ with range in U.

We are now ready to prove Cauchy's integral formula.

Theorem 76 (Cauchy's integral formula) *Let $U \subseteq \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be holomorphic. Then for every open ball B with $\overline{B} \subset U$ and every $z \in B$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where ∂B is oriented counterclockwise.

Proof. Fix $z \in B$ and consider the closed curve $\Gamma_{\delta,\varepsilon}$ given in the picture below, where ε is the radius of the small circle centered at z and δ is the width of the corridor. Since the function $g(\zeta) := \frac{f(\zeta)}{\zeta - z}$ is holomorphic in $U \setminus \{z\}$, by considering $V := B' \setminus S'$, where B' is a concentric ball contained in U and containing $\overline{B}$, S is the segment obtained when $\varepsilon \to 0$ and $\delta \to 0$, and S' is a slightly larger segment we can apply Exercise 74, to obtain that g has a primitive in V. Since the range of $\Gamma_{\delta,\varepsilon}$ is contained in V, it follows from Corollary 65 that

$$\int_{\Gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

If we let $\delta \to 0^+$ and use the fact that g is continuous, we get that the two segments converge to a segment which is integrated in opposite directions. Hence, we obtain

$$\int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}.$$

Since f is holomorphic, $\frac{f(\zeta)-f(z)}{\zeta - z} \to f'(z)$ as $\zeta \to z$ and so

$$\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \leq M$$

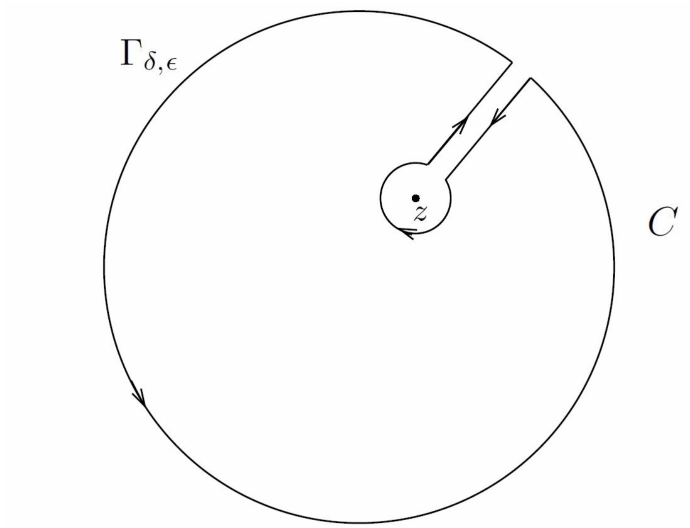


Figure 1: Figure 1: Keyhole contour

for all $\zeta \in \partial B(z,\varepsilon)$. It follows that

$$\int_{\partial B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial B(z,\varepsilon)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + f(z) \int_{\partial B(z,\varepsilon)} \frac{1}{\zeta - z} d\zeta$$
$$= I + II.$$

Then $|I| \le M(2\pi\varepsilon) \to 0$ as $\varepsilon \to 0^+$. On the other hand, if we use the parametrization $\varphi(t) = z + \varepsilon e^{it}$, $t \in [0, 2\pi]$. Then

$$\int_{\partial B(z,\varepsilon)} \frac{1}{\zeta - z} d\zeta = \int_0^{2\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}} dt = 2\pi i.$$

Hence,

$$\int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) 2\pi i,$$

which proves the result. ■

Exercise 77 *Use Remark 73 to give an alternative proof of the previous theorem, which does not make use of* $\Gamma_{\delta,\varepsilon}$.

Exercise 78 *Use contour integration to show that for* $\xi \in \mathbb{R}$,

$$e^{-\pi\xi^2} = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x\xi} dx.$$

Exercise 79 *Use contour integration to show that for* $\xi \in \mathbb{R}$,

$$\frac{\pi}{2} = \int_0^\infty \frac{1 - \cos x}{x^2} dx.$$

Exercise 80 *Use contour integration to show that*

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}\, dx.$$

Monday, February 3, 2020

Corollary 81 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Then f is analytic and for every open ball B with $\overline{B} \subset U$, every $z \in B$, and every $k \in \mathbb{N}$,*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta, \tag{32}$$

where ∂B is oriented counterclockwise.

Proof. Let $B = B(z_0, r)$. Fix $z \in B$. For $\zeta \in \partial B$ write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}.$$

Then

$$\left| \frac{z - z_0}{\zeta - z_0} \right| = \frac{|z - z_0|}{r} =: \delta < 1$$

and so we can use geometric power series to write

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n.$$

Note that the series converges uniformly for all $\zeta \in \partial B$, and so (using Lebesgue dominated convergence theorem or any equivalent theorem for Riemann integration) we can interchange the integral and the series in Cauchy's formula to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n d\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta =: \sum_{n=0}^{\infty} a_n (z - z_0)^n. \end{aligned}$$

The formula for the derivatives now follows by differentiating the power series. To see this, we use Corollary 31 to get

$$\begin{aligned} f^{(k)}(z) &= \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z - z_0)^{n-k} \\ &= \frac{1}{2\pi i} \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)(z - z_0)^{n-k} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \\ &= \frac{1}{2\pi i} \sum_{n=k}^{\infty} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} n(n-1) \cdots (n-k+1) \frac{(z - z_0)^{n-k}}{(\zeta - z_0)^{n-k}} d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial B} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) \frac{(z - z_0)^{n-k}}{(\zeta - z_0)^{n-k}} d\zeta. \end{aligned}$$

Let $w = \frac{z-z_0}{\zeta-z_0}$. Then

$$\begin{aligned}\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)w^{n-k} &= \frac{d^{(k)}}{dw^k}\sum_{n=0}^{\infty} w^n \\ &= \frac{d^{(k)}}{dw^k}(1-w)^{-1} = k!(1-w)^{-k-1}\end{aligned}$$

and so (using again the uniform convergence of the power series and its derivatives)

$$\begin{aligned}f^{(k)}(z) &= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}} \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\frac{(z-z_0)^{n-k}}{(\zeta-z_0)^{n-k}} d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}} k! \frac{1}{\left(1-\frac{z-z_0}{\zeta-z_0}\right)^{k+1}} d\zeta = \frac{k!}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{(\zeta-z)^{k+1}} d\zeta,\end{aligned}$$

which completes the proof. ■

Remark 82 *Note that we have proved that for every open ball $B(z_0, r)$ with $\overline{B(z_0, r)} \subset U$, we can write*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad z \in B(z_0, r),$$

where

$$a_n := \frac{f^{(n)}(z_0)}{n!}.$$

Moreover, if we denote by R the radius of convergence R of the power series, then

$$R \geq \operatorname{dist}(z_0, \partial U) = \sup\{r > 0 : \overline{B(z_0, r)} \subset U\}.$$

Hence, if $U = \mathbb{C}$ then $R = \infty$.

Corollary 83 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Given a closed ball $\overline{B(z_0, r)} \subset U$, let $M \geq \max_{\overline{B(z_0,r)}} |f|$. Then for every $n \in \mathbb{N}$,*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

Proof. In view of (32),

$$\begin{aligned}|f^{(n)}(z_0)| &\leq \frac{n!}{2\pi}\int_{\partial B(z_0,r)} \frac{|f(\zeta)|}{|\zeta-z_0|^{n+1}} d\zeta \leq \frac{n!M}{2\pi}\int_{\partial B(z_0,r)} \frac{1}{|\zeta-z_0|^{n+1}} d\zeta \\ &= \frac{n!M}{2\pi r^{n+1}} 2\pi r = \frac{n!M}{r^n},\end{aligned}$$

which concludes the proof. ■

Definition 84 *Given an open connected set $U \subset \mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$, a point $z_0 \in \partial U$ is called a* regular point *if there exist $r > 0$ and a holomorphic function $g : B(z_0, r) \to \mathbb{C}$ such that $f = g$ on $U \cap B(z_0, r)$. A point $z_0 \in \partial U$ is called a* singular point *if it is not a regular point. We say that ∂U is the* natural boundary of f *if every point on ∂U is a singular point.*

Exercise 85 *Let $f : B(z_0, r) \to \mathbb{C}$ be holomorphic and assume that the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has radius of convergence exactly r. Then there is at least one singular point on $\partial B(z_0, r)$.

Exercise 86 *Given the function*

$$f(z) = \sum_{n=0}^{\infty} z^{2^n},$$

find its natural boundary.

Next we discuss some important consequences of Cauchy's formula.

Corollary 87 (Liouville) *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and bounded. Then f is constant.*

Proof. Let $M > 0$ be such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By the previous corollary,

$$|f'(z)| \leq \frac{M}{r}$$

for every $r > 0$. Hence, letting $r \to \infty$ we get $f'(z) = 0$ for all z. We can now apply Corollary 14. ■

Theorem 88 (Fundamental theorem of algebra) *Every polynomial $P : \mathbb{C} \to \mathbb{C}$ of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$.*

Proof. Step 1: Write

$$P(z) = a_n z^n + \cdots + a_1 z + a_0,$$

where $a_n \neq 0$. We claim that P has at least one root. Assume by contradiction that this is not the case, that is, that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then the function $1/P$ is well-defined and holomorphic. Let's prove that it is bounded. We have

$$\frac{P(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \to a_n + 0$$

as $|z| \to \infty$. Hence, taking $\varepsilon = \frac{1}{2}|a_n| > 0$ we can find $R > 0$ such that

$$\frac{1}{2}|a_n| \leq \left|\frac{P(z)}{z^n}\right| \leq \frac{3}{2}|a_n| \quad \text{for all } |z| \geq R.$$

In particular,

$$\frac{1}{|P(z)|} \leq \frac{2}{|a_n||z|^n} \leq \frac{2}{|a_n|R^n} \quad \text{for all } |z| \geq R.$$

Since $\frac{1}{|P|}$ is continuous on the compact set $\overline{B(0,R)}$, there exists $M \geq 0$ such that $\frac{1}{|P(z)|} \leq M$ for all $|z| \leq R$, which, together with the previous inequality, proves the claim. It now follows from Liouville's theorem that $\frac{1}{P}$ is constant, which is a contradiction since P has degree at least one. ■

Wednesday, February 5, 2020

Proof. Step 2: In view of the previous step there exists $w_1 \in \mathbb{C}$ such that $P(w_1) = 0$. Let $z = (z - w_1) + w_1$. Then

$$P(z) = a_n[(z - w_1) + w_1]^n + \cdots + a_1[(z - w_1) + w_1] + a_0.$$

Using the binomial theorem

$$(a + b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$$

with $a = z - w_1$ and $b = w_1$, we can rewrite $P(z)$ as

$$P(z) = b_n(z - w_1)^n + \cdots + b_1(z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$, we get that $b_0 = 0$. Hence,

$$P(z) = (z - w_1)[b_n(z - w_1)^{n-1} + \cdots + b_1] =: (z - w_1)P_1(z),$$

where P_1 is a polynomial of degree $n - 1$. If $n \geq 2$, we can apply the previous step to P_1 to find a second root w_2.

Inductively, we can find $w_1, \ldots, w_n \in \mathbb{C}$ such that

$$P(z) = a_n(z - w_1) \cdots (z - w_n) \quad \text{for all } z \in \mathbb{C}.$$

This concludes the proof. ■

Another corollary of Cauchy's theorem is the following.

Corollary 89 (Morera) *Let $B \subset \mathbb{C}$ be an open ball, let $f : B \to \mathbb{C}$ a continuous function such that for every closed triangle $T \subset B$,*

$$\int_{\partial T} f = 0.$$

Then f is holomorphic in B.

Proof. In view of Remark 71 we have that f has a primitive $F : B \to \mathbb{C}$. Hence, F is holomorphic. In turn, by the previous corollary, F is infinitely differentiable. Since $F' = f$, it follows that f is also holomorphic. ■

Let's see how to use Morera's theorem. Let $U \subseteq \mathbb{C}$ be an open set. Define

$$\begin{aligned} U^+ &:= \{z = x + iy \in U : y > 0\}, \\ U^- &:= \{z = x + iy \in U : y < 0\}, \\ S &:= \{z = x + i0 \in U\}, \end{aligned}$$

so that $U = U^+ \cup U^- \cup S$.

Theorem 90 *Let $U \subseteq \mathbb{C}$ be an open set, let $f^+ : U^+ \cup S \to \mathbb{C}$ be a continuous function which is holomorphic in U^+ and let $f^- : U^- \cup S \to \mathbb{C}$ be a continuous function which is holomorphic in U^+. Assume that $f^+ = f^-$ in S. Then the function $f : U \to \mathbb{C}$, defined by*

$$f(z) := \begin{cases} f^+(z) & \text{if } z \in U^+ \cup S, \\ f^-(z) & \text{if } z \in U^-, \end{cases}$$

is holomorphic in U.

Proof. We only need to prove differentiability at points in S. Fix $z_0 \in S$ and let $\overline{B(z_0, r)} \subseteq U$. Since f is continuous, we can use Morera's theorem to prove that f is holomorphic in $B(z_0, r)$. Let $T \subset B(z_0, r)$ be a closed triangle. If T does not intersect S, then it is contained either in U^+ or in U^- and so $\int_{\partial T} f \, dz = 0$ by Exercise 75 since f^+ and f^+ are holomorphic in U^+ and U^-, respectively. If $T^\circ \subseteq U^+$ and one of its sides lies in S, for $\varepsilon > 0$ small consider the triangle $T_\varepsilon := T \cap \{z = x + yi : y \geq \varepsilon\}$. Then again by Exercise 75, $\int_{\partial T_\varepsilon} f^+ \, dz = 0$. Since f is continuous, letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence theorem (or Arzelá's convergence theorem for Riemann's integration) we get $\int_{\partial T} f \, dz = 0$. The case in which $T^\circ \subseteq U^-$ and one of its sides lies in S is similar.

If T has a vertex in S and is contained in U^+ (or U^-) we either raise (lower) T so that it is contained in U^+ (U^-) and reason as above.

If the interior of T intersects S, we split T using S into three triangles whose interior is contained in U^+ or U^- and which have one side or a vertex in S. We then apply the previous cases and Exercise 66 to conclude that $\int_{\partial T} f \, dz = 0$. Hence, the hypotheses of Morera's theorem are satisfied and so f is holomorphic in $B(z_0, r)$. ■

We are ready to prove Schwarz's reflection principle

Theorem 91 (Schwarz reflection principle) *Let $U \subseteq \mathbb{C}$ be an open set which is symmetric with respect to the real line, that is,*

$$z \in U \quad \text{if and only if } \overline{z} \in U.$$

and let $f^+ : U^+ \cup S \to \mathbb{C}$ be a continuous function which is holomorphic in U^+ and real-valued on S. Then the function $f : U \to \mathbb{C}$, defined by

$$f(z) := \begin{cases} f^+(z) & \text{if } z \in U^+ \cup S, \\ \overline{f^+(\overline{z})} & \text{if } z \in U^-, \end{cases}$$

is holomorphic in U.

Proof. Given $z_0 \in U^-$, we have that $\bar{z}_0 \in U^+$. By Corollary 81 we can write

$$f^+(w) = \sum_{n=0}^{\infty} a_n (w - \bar{z}_0)^n$$

for all $w \in B(\bar{z}_0, r) \subset U^+$ and for some $r > 0$. By symmetry $B(z_0, r) \subset U^-$ and for every $z \in B(z_0, r)$ we have that $\bar{z} \in B(\bar{z}_0, r)$ and so

$$f^+(\bar{z}) = \sum_{n=0}^{\infty} a_n (\bar{z} - \bar{z}_0)^n.$$

Taking the conjugate in the partial sums and then passing to the limit we have that

$$f(z) = \overline{f^+(\bar{z})} = \sum_{n=0}^{\infty} \bar{a}_n \overline{(\bar{z} - \bar{z}_0)^n} = \sum_{n=0}^{\infty} \bar{a}_n (z - z_0)^n.$$

Since the radius of convergence of $\sum_{n=0}^{\infty} \bar{a}_n (z - z_0)^n$ is the same as $\sum_{n=0}^{\infty} a_n \xi^n$, we conclude that f is holomorphic in $B(z_0, r)$.

To conclude observe that since f^+ is real-valued on S,

$$\overline{f^+(x)} = f^+(x)$$

for all $x \in S$. Hence, f is continuous at points of S. Thus, by the previous theorem we conclude that f is holomorphic in U. ■

7 Cauchy's Theorem, General Case

In this section we extend Corollary 72 to simply connected domains.

In what follows, given the unit square $Q = [0,1] \times [0,1]$, we consider the oriented closed simple curve obtained by moving along ∂Q counterclockwise starting from $(0,0)$. Denote by $\varphi_0 : [0,4] \to \partial Q$ the parametric representation obtained by using arclength.

Theorem 92 *Let $U \subseteq \mathbb{C}$ be an open set, let $h : Q \to U$ be Lipschitz continuous, let γ be the Lipschitz continuous oriented closed curve parametrized by $h \circ \varphi_0 : [0,4] \to U$, and let $f : U \to \mathbb{C}$ be holomorphic. Then*

$$\int_\gamma f \, dz = 0.$$

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Proof. Assume by contradiction that

$$\int_\gamma f \, dz = c \neq 0.$$

By replacing f with f/c, without loss of generality, we may assume that $c = 1$. Divide Q into four squares $Q_{1,1}$, $Q_{1,2}$, $Q_{1,3}$, $Q_{1,4}$ of side-length $\frac{1}{2}$ and parametrize their boundaries as we did for ∂Q. Let $\varphi_{1,1}$, $\varphi_{1,2}$, $\varphi_{1,3}$, $\varphi_{1,4}$ be the corresponding parametric representations and let $\gamma_{1,1}$, $\gamma_{1,2}$, $\gamma_{1,3}$, $\gamma_{1,4}$ be the oriented closed curve parametrized by $h \circ \varphi_{1,k} : [0, 4/2^1] \to U$, $k = 1, \ldots, 4$, respectively. Using Exercise 66 we have that

$$1 = \int_{\gamma_{1,1}} f\, dz + \int_{\gamma_{1,2}} f\, dz + \int_{\gamma_{1,3}} f\, dz + \int_{\gamma_{1,4}} f\, dz$$

and thus there exists $k_1 \in \{1, \ldots, 4\}$ such that

$$\left| \int_{\gamma_{1,k}} f\, dz \right| \geq \frac{1}{4}.$$

Let $Q_1 := Q_{1,k_1}$ and $\gamma_1 := \gamma_{1,k_1}$. We now divide Q_1 into four squares $Q_{2,1}$, $Q_{2,2}$, $Q_{2,3}$, $Q_{2,4}$ of side-length $\frac{1}{16}$. Proceeding as before we find $k_2 \in \{1, \ldots, 4\}$ such that

$$\left| \int_{\gamma_{2,k_2}} f\, dz \right| \geq \frac{1}{16}.$$

Inductively we obtain a decreasing sequence of closed squares Q_n of side-length $\frac{1}{2^n}$ such that

$$\left| \int_{\gamma_n} f\, dz \right| \geq \frac{1}{4^n}. \tag{33}$$

where γ_n is the oriented closed curve parametrized by $h \circ \varphi_n : [0, \frac{4}{2^n}] \to U$ and $\varphi_n : [0, \frac{4}{2^n}] \to \partial Q_n$. By Cantor's theorem there exists $(x_0, y_0) \in Q_n$ for all n. Let $z_0 = h((x_0, y_0))$. Since f is differentiable, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + R(z),$$

where

$$\lim_{z \to z_0} \frac{R(z)}{z - z_0} = 0. \tag{34}$$

Since a constant function and a linear function az have a primitive, by the fundamental theorem of calculus,

$$\int_{\gamma_n} f\, dz = \int_{\gamma_n} R\, dz.$$

Let Γ_n be the range of γ_n. If $z \in \Gamma_n = h(\varphi_n([0, \frac{4}{2^n}]))$, we can find $(x, y) \in \partial Q_n$ such that $z = h(x, y)$. Hence, if $L > 0$ is the Lipschitz constant of h, we have that

$$|z - z_0| = |h(x, y) - h(x_0, y_0)| \leq L\sqrt{(x - x_0)^2 + (y - y_0)^2} \leq L \operatorname{diam} Q_n = L\frac{\sqrt{2}}{2^n}.$$

In turn, by (34),

$$|R(z)| \leq \varepsilon_n |z - z_0| \leq \varepsilon_n L \frac{\sqrt{2}}{2^n},$$

where $\varepsilon_n \to 0^+$. Hence,

$$\left|\int_{\gamma_n} f\,dz\right| = \left|\int_{\gamma_n} R\,dz\right| \le \varepsilon_n L \frac{\sqrt{2}}{2^n} L(\gamma_n)$$
$$\le \varepsilon_n \frac{4L^2}{4^n},$$

where we used the fact that

$$L(\gamma_n) = \int_0^{\frac{4}{2^n}} |(h\circ\varphi_n)'(s)|\,ds \le L \int_0^{\frac{4}{2^n}} |\varphi_n'(s)|\,ds = L \int_0^{\frac{4}{2^n}} 1\,ds = \frac{4L}{2^n}$$

Using (34) we get

$$\frac{1}{4^n} \le \left|\int_{\gamma_n} f\,dz\right| \le \varepsilon_n \frac{4L^2}{4^n}$$

as $n \to \infty$, which is a contradiction. ■

Next we consider the case in which h is only continuous.

Theorem 93 *Let $U \subseteq \mathbb{C}$ be an open set, let $h : Q \to U$ be continuous, let γ be the oriented closed curve parametrized by $h \circ \varphi_0 : [0,4] \to U$, and let $f : U \to \mathbb{C}$ be holomorphic. If γ is rectifiable, then*

$$\int_\gamma f\,dz = 0.$$

Proof. Since Q is compact and h is continuous, $h(Q)$ is compact. Hence, $d := \operatorname{dist}(h(Q), \partial U) > 0$. For every n consider a partition $t_0 = 0 < t_1 < \cdots < t_n = 1$ with $t_k - t_{k-1} \le \delta_n$ for every $k = 1, \ldots, n$ (for example $\delta_n = \frac{1}{n}$ and $t_k = k/n$, $k = 0, \ldots, n$). We construct $h_n : Q \to U$ by defining $h_n(t_j, t_k) := h(t_j, t_k)$ for each $j, k = 0, \ldots, n$ and by interpolating linearly in each subrectangle

$$h_n(rt_j + (1-r)t_{j-1}, st_k + (1-s)t_{k-1}) := (1-r)(1-s)h_n(t_{j-1}, t_{k-1})$$
$$+ r(1-s)h_n(t_j, t_{k-1}) + (1-r)sh_n(t_{j-1}, t_k) + rsh_n(t_j, t_k)$$

for $r, s \in [0,1]$. Then $h_n : Q \to \mathbb{C}$ is Lipschitz continuous. Using the uniform continuity of h we have that $h_n \to h$ uniformly in Q as $n \to \infty$. In particular, $\operatorname{dist}(h_n(Q), \partial U) \ge d/2$ for all n sufficiently large. Hence, $h_n : Q \to U$ for n large. By the previous theorem

$$\int_{\gamma_n} f\,dz = 0.$$

Since γ_n is parametrized by $h_n \circ \varphi_0 : [0,4] \to U$ we have that $h_n \circ \varphi_0 \to h \circ \varphi_0$ uniformly, and since f is continuous and $h \circ \varphi_0$ has finite length, it follows that (Exercise, see the proof of Lemma 64)

$$0 = \lim_{n\to\infty} \int_{\gamma_n} f\,dz = \int_\gamma f\,dz,$$

which concludes the proof. ■

Corollary 94 *Let $U \subseteq \mathbb{C}$ be an open set, let $h : Q \to U$ be continuous and such that $h(s,0) = h(s,1)$ for all $s \in [0,1]$, let γ be the oriented closed curve parametrized by $h \circ \varphi_0 : [0,4] \to U$, and let $f : U \to \mathbb{C}$ be holomorphic. Assume that the curves γ_1 and γ_2 parametrized by $h \circ \varphi_0 : [1,2] \to U$ and $h \circ \varphi_0 : [3,4] \to U$ are rectifiable, then*

$$\int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz = 0.$$

Proof. Since $h(s,0) = h(s,1)$ for all $s \in [0,1]$, in the previous proof we will have $h_n(s,0) = h_n(s,1)$ for all $s \in [0,1]$. Hence, the Lipschitz curves parametrized by $h \circ \varphi_0 : [0,1] \to U$ and $h \circ \varphi_0 : [2,3] \to U$ are one the opposite of the other and so their corresponding integrals will cancel each other. In turn,

$$\int_{\gamma_{1,n}} f\,dz + \int_{\gamma_2,n} f\,dz = 0.$$

Letting $n \to \infty$ will give the desired result. ■

Definition 95 *Given a set $E \subseteq \mathbb{C}$, two continuous oriented closed curves γ_1 and γ_2 with range in E and parametric representations $\varphi_1 : [a,b] \to \mathbb{C}$ and $\varphi_2 : [a,b] \to \mathbb{C}$, respectively, are* homotopic *in E if there exists a continuous function $h : [0,1] \times [a,b] \to \mathbb{C}$ such that $h([0,1] \times [a,b]) \subseteq E$,*

$$h(0,t) = \varphi_1(t) \text{ for all } t \in [a,b], \quad h(1,t) = \varphi_2(t) \text{ for all } t \in [a,b],$$
$$h(s,a) = h(s,b) \text{ for all } s \in [0,1].$$

The function h is called a homotopy *in E between the two curves.*

Roughly speaking, two curves are homotopic in E if it is possible to deform the first continuously until it becomes the second *without leaving* the set E.

Definition 96 *A set $E \subseteq \mathbb{C}$ is* simply connected *if it is pathwise connected and if every continuous closed curve with range in E is homotopic in E to a point in E (that is, to a curve with parametric representation a constant function).*

Example 97 *A star-shaped set is simply connected. Indeed, let $E \subseteq \mathbb{C}$ be star-shaped with respect to some point $z_0 \in E$ and consider a continuous closed curve $\boldsymbol{\gamma}$ with parametric representation $\varphi : [a,b] \to \mathbb{C}$ such that $\varphi([a,b]) \subseteq E$. Then the function*

$$h(s,t) := s\varphi(t) + (1-s)\,z_0$$

is an homotopy between γ and the point z_0.

Theorem 98 (Cauchy) *Let $U \subseteq \mathbb{C}$ be an open set, let γ_1 and γ_2 be two oriented closed rectifiable curves which are homotopic in U and let $f : U \to \mathbb{C}$ be holomorphic. Then*

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$$

In particular, if U is simply connected, then

$$\int_{\gamma} f\,dz = 0$$

for every rectifiable closed oriented curve γ with range in U.

Proof. Let $\varphi_1 : [0,1] \to U$ and $\varphi_2 : [0,1] \to U$ be parametric representations of γ_1 and γ_2, respectively, and let $h : [0,1] \times [0,1]$ be a corresponding homotopy. Then $h \circ \varphi_0$ is composed of four curves: first $s \in [0,1] \to h(s,0)$ followed by γ_1, then the opposite of $s \in [0,1] \to h(s,1)$ and finally the opposite of γ_2. Since the first and the third of these four curves are the opposite to each other, the corresponding integrals will cancel out. Hence, in view of Corollary 94,

$$\int_{\gamma_1} f\,dz + \int_{-\gamma_2} f\,dz = 0.$$

The result now follows from 66. ■

Exercise 99 *Let $U \subseteq \mathbb{C}$ be a simply connected open set and let $f : U \to \mathbb{C}$ be holomorphic. Prove that f has a primitive in U.*

Using the previous exercise we can show that in a simply connected open set which does not contain the origin there is a branch of the logarithm. More generally, we have the following important result.

Corollary 100 *Let $U \subseteq \mathbb{C}$ be a simply connected open set and let $f : U \to \mathbb{C}$ be a holomorphic function such that $f(z) \neq 0$ for all $z \in U$. Then there exists a holomorphic function $g : U \to \mathbb{C}$ such that*

$$f(z) = e^{g(z)} \quad \textit{for all } z \in U.$$

If $z_0 \in U$ and $f(z_0) = e^{w_0}$ for $w_0 \in \mathbb{C}$, then we can choose g in such a way that $g(z_0) = w_0$.

Proof. Fix $z_0 \in U$ and use polar coordinates to write $f(z_0) = re^{i\theta}$. Taking $w_0 := \log r + i\theta$, we have that $f(z_0) = e^{w_0}$. Since $f(z) \neq 0$ for all $z \in U$, the function f'/f is well-defined and holomorphic in U. By the previous exercise, f'/f has a primitive F_1, that is, $F_1' = f'/f$ in U. By adding a constant, we can assume that $F_1(z_0) = w_0$. Then $h(z) := e^{F_1(z)}$ is holomorphic in U and never vanishes (since the exponential never does). In turn, f/h is holomorphic. Let's compute its derivative

$$\left(\frac{f}{h}\right)'(z) = \frac{f'(z)h(z) - f(z)h'(z)}{h^2(z)} = \frac{f'(z)e^{F_1(z)} - f(z)F_1'(z)e^{F_1(z)}}{e^{2F_1(z)}}$$
$$= \frac{f'(z)e^{F_1(z)} - f(z)\frac{f'(z)}{f(z)}e^{F_1(z)}}{e^{2F_1(z)}} = 0.$$

Since U is connected, it follows from Corollary 14 that f/h is a constant function. Hence, there is $c \in \mathbb{C} \setminus \{0\}$ such that

$$f(z) = ch(z) = ce^{F_1(z)}.$$

Taking $z = z_0$ we get

$$e^{w_0} = f(z_0) = ce^{F_1(z_0)} = ce^{w_0}$$

and so $c = 1$. This completes the proof. ∎

Exercise 101 *Let $U \subset \mathbb{C}$ be a simply connected open set with $0 \notin U$. Prove that in U there exists a branch $\log_U$ of the logarithm. Prove also that if $1 \in U$, then we can assume that $\log_U r = \log r$ whenever r is a real number sufficiently close to 1.*

Exercise 102 *Prove that the previous exercise continues to hold if in place of U simply connected we assume that*

$$\int_\gamma f\, ds = 0$$

for every holomorphic function $f : U \to \mathbb{C}$ and for every closed oriented Lipschitz continuous curve with range contained in U.

Remark 103 *In view of Exercise 101, if $U \subset \mathbb{C}$ is a simply connected open set with $0 \notin U$ and $a \in \mathbb{C}$, then in U there is a branch of z^a, defined as usual by*

$$z^a := e^{a \log_U z}.$$

Definition 104 *Given a set $E \subseteq \mathbb{C}$, two continuous oriented curves, with parametric representations $\varphi : [a, b] \to \mathbb{C}$ and $\psi : [a, b] \to \mathbb{C}$ such that $\varphi([a, b]) \subseteq E$, $\psi([a, b]) \subseteq E$, $\varphi(a) = \psi(a) = \alpha$, $\varphi(b) = \psi(b) = \beta$ are* fixed-endpoint homotopic *in E if there exists a continuous function $h : [0, 1] \times [a, b] \to \mathbb{C}$ such that $h([0, 1] \times [a, b]) \subseteq E$,*

$$h(0, t) = \varphi(t) \text{ for all } t \in [a, b], \quad h(1, t) = \psi(t) \text{ for all } t \in [a, b],$$
$$h(s, a) = \alpha, \quad h(s, b) = \beta \text{ for all } s \in [0, 1].$$

The function h is called a fixed-endpoint homotopy *in E between the two curves.*

Exercise 105 *Let $U \subseteq \mathbb{C}$ be an open set, let γ_1 and γ_1 be two oriented rectifiable continuous curves with the same endpoints and which are fixed-endpoint homotopic in U, and let $f : U \to \mathbb{C}$ be holomorphic. Prove that*

$$\int_{\gamma_1} f\, dz = \int_{\gamma_2} f\, dz.$$

8 Harmonic Functions

Given an open set $\Omega \subseteq \mathbb{R}^N$, a function $u : \Omega \to \mathbb{R}$ of class C^2 is called *harmonic in* Ω if it satisfies

$$\Delta u(\boldsymbol{x}) = 0 \quad \text{for all } \boldsymbol{x} \in \Omega,$$

where we recall that Δ is the Laplace operator defined by

$$\Delta := \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}.$$

As a consequence of Cauchy's integral formula we have the following important result.

Theorem 106 *Let $U \subseteq \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function. Then the real-valued functions*

$$u(x,y) := \operatorname{Re} f(x+iy), \quad v(x,y) := \operatorname{Im} f(x+iy)$$

are harmonic in $\Omega := \{(x,y) \in \mathbb{R}^2 : x+iy \in U\}$.

Proof. In what follows given a function $g : U \to \mathbb{C}$ we define $R_g : \Omega \to \mathbb{R}$ and $I_g : \Omega \to \mathbb{R}$ via

$$R_g(x,y) = \operatorname{Re} g(x+iy), \quad I_g(x,y) := \operatorname{Im} g(x+iy).$$

Recall that by (9),

$$\begin{aligned} \frac{\partial u}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) = \operatorname{Re} f'(x+iy), \\ -\frac{\partial u}{\partial y}(x,y) &= \frac{\partial v}{\partial x}(x,y) = \operatorname{Im} f'(x+iy). \end{aligned}$$

This shows that $R_{f'} = \frac{\partial u}{\partial x}$, $I_{f'} = -\frac{\partial u}{\partial y}$. By Corollary 81, the function f is analytic. In particular, it is of class $C^\infty(U)$. In particular, f' is holomorphic, and so we can apply Theorem 13 to f' to conclude that $R_{f'} = \frac{\partial u}{\partial x}$, $I_{f'} = -\frac{\partial u}{\partial y}$ are differentiable, with

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(x,y) &= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)(x,y) = \operatorname{Re} f''(x+iy), \qquad (35) \\ -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x,y) &= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)(x,y) = \operatorname{Im} f''(x+iy). \end{aligned}$$

This implies that all second order partial derivatives of u exist and since f'' is continuous, so are they. Thus, $u \in C^2(\Omega)$. Moreover, from the first equation in (35) we get that u is harmonic.

We can repeat a similar argument for v since $R_{f'} = \frac{\partial v}{\partial y}$, $I_{f'} = \frac{\partial v}{\partial x}$ or use the Cauchy-Riemann equations, to obtain that $v \in C^2(\Omega)$ and is harmonic. ■

We also have the converse of this theorem.

Theorem 107 *Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let $u, v : \Omega \to \mathbb{R}$ be two harmonic functions satisfying the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \text{in } \Omega. \tag{36}$$

Then the function $f : U \to \mathbb{C}$ defined by

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy \in U,$$

where $U := \{z = x + iy : \ (x, y) \in \Omega\}$, is holomorphic in U.

Proof. This follows from Theorem 15. ■

An interesting problem is, given an open set $\Omega \subseteq \mathbb{R}^2$ and an harmonic function $u : \Omega \to \mathbb{R}$, to find another harmonic function $v : \Omega \to \mathbb{R}$ in such a way that the Cauchy–Riemann equations hold in Ω. If such a function v exists, it is called *complex conjugate of u.*

Exercise 108 *Let $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$. Prove that the function $u(x, y) := \log(x^2 + y^2)$, $(x, y) \in \Omega$, is harmonic but does not have a complex conjugate v.*

Theorem 109 *Let $\Omega \subseteq \mathbb{R}^2$ be simply connected and let $u : \Omega \to \mathbb{R}$ be an harmonic function. Then u admits a complex conjugate $v : \Omega \to \mathbb{R}$.*

Proof. Define

$$g(z) = \frac{\partial u}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y)i, \quad z = x + iy \in U,$$

where as before $U := \{z = x + iy : \ (x, y) \in \Omega\}$. Since u is of class C^2 and harmonic ,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)(x, y) &= \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)(x, y) \quad \text{in } \Omega, \\ -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)(x, y) &= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right)(x, y) \quad \text{in } \Omega, \end{aligned}$$

and so $\frac{\partial u}{\partial x}$ and $-\frac{\partial u}{\partial y}$ satisfy the Cauchy–Riemann equations. In turn, by the previous theorem the function g is holomorphic in U. Since Ω is simply connected, so is U, and so we can apply Exercise 99 to conclude that g has a primitive, that is, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $f' = g$.

Let

$$u_1(x,y) := \operatorname{Re} f(x+iy), \quad v(x,y) := \operatorname{Im} f(x+iy).$$

By (9),

$$\begin{aligned}\frac{\partial u_1}{\partial x}(x,y) &= \frac{\partial v}{\partial y}(x,y) = \operatorname{Re} f'(x+iy) = \operatorname{Re} g(x+iy) = \frac{\partial u}{\partial x}(x,y),\\ -\frac{\partial u_1}{\partial y}(x,y) &= \frac{\partial v}{\partial x}(x,y) = \operatorname{Im} f'(x+iy) = \operatorname{Im} g(x+iy) = -\frac{\partial u}{\partial y}(x,y)\end{aligned}$$

and so

$$\frac{\partial u_1}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,y), \quad \frac{\partial u_1}{\partial y}(x,y) = \frac{\partial u}{\partial y}(x,y).$$

Since U is connected, this implies that $u - u_1$ must be constant. Since v is a complex conjugate of u_1, it follows that it is also a complex conjugate to u, and the proof is complete. ■

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As a corollary of Cauchy's integral form we obtain the mean value theorem.

Theorem 110 (Mean value theorem) *Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let $u : \Omega \to \mathbb{R}$ be an harmonic function. Then for every closed ball $\overline{B((x_0, y_0), r)} \subset \Omega$ we have*

$$u(x_0, y_0) = \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta.$$

Proof. Let $z_0 = x_0 + iy_0$. By applying the previous theorem in a larger open ball B containing z_0 we can find a function v which is conjugate to u in B. In turn, the function

$$f(z) = u(x,y) + iv(x,y), \quad z = x + iy \in B,$$

is holomorphic and so by Cauchy's formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(z)}{z - z_0} dz.$$

Taking as parametric representation of $\partial B(z_0, r)$ the function $\varphi(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$, we get

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$$

where we used the fact that $\varphi'(\theta) = rie^{i\theta}$. In particular, taking the real part on both sides

$$\operatorname{Re} f(z_0) = \frac{1}{2\pi}\operatorname{Re}\left(\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta\right) = \int_0^{2\pi} (\operatorname{Re} f)(z_0 + re^{i\theta})\, d\theta, \tag{37}$$

which gives the result. ■

Using this formula, one can show as in Corollary 81 that u is analytic in Ω. We leave this as an exercise.

9 Zeros and Isolated Singularities

In this section we study zeros and isolated singularities of holomorphic functions. We begin by showing that zeros of holomorphic functions are isolated.

Theorem 111 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be holomorphic. Assume that there exists a sequence $\{z_k\}_k$ in U with $z_k \neq z_m$ for $k \neq m$ such that $z_k \to z_0 \in U$ as $n \to \infty$ and $f(z_k) = 0$ for all k. Then $f = 0$.*

Proof. Since f is analytic by Corollary 81, there exists $r > 0$ such that $B(z_0, r) \subseteq U$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in B(z_0, r)$. If $f \neq 0$ in $B(z_0, r)$, at least one of a_n must be different from 0. Let m be the first integer such that $a_m \neq 0$. Let $m \in \mathbb{N}$ be the smallest integer such that $a_m \neq 0$. Then as in the proof of Theorem 111 we can write

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \\ &= (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k \\ &=: (z - z_0)^m g(z). \end{aligned}$$

Now

$$g(z) = a_m + \sum_{k=1}^{\infty} a_{k+m} (z - z_0)^k,$$

where the power series is convergent. Hence $g(z) \to a_m \neq 0$ as $z \to z_0$. Hence, taking $\varepsilon = \frac{1}{2}|a_m|$, there exists $0 < \delta < r$ such that

$$|g(z) - a_m| \leq \frac{1}{2}|a_m|$$

for all z with $|z - z_0| \leq \delta$, and so $|g(z)| \geq |a_m| - |g(z) - a_m| \geq \frac{1}{2}|a_m|$, and in turn,

$$|f(z)| \geq \frac{1}{2}|a_m||z - z_0|^m$$

for all z with $|z - z_0| \leq \delta$. Since $z_k \to z_0$ we have that $|z_k - z_0| \leq \delta$ for all k large. In particular, there are infinitely many z_k such that $z_k \neq z_0$ and $|z_k - z_0| \leq \delta$. But

$$0 = |f(z_k)| \geq \frac{1}{2}|a_m||z_k - z_0|^m > 0,$$

which is a contradiction. This shows that $f = 0$ in $B(z_0, r)$.

Let

$$V := \{z \in U : f(z) = 0\}^{\circ}.$$

The set V is open by definition and $B(z_0, r) \subseteq V$. The set V is also closed in U, since if $w_k \in V$ and $w_k \to w_0 \in U$, then either $w_k = w_0$ for some k and so $w_0 \in V$ or $w_k \neq w_0$ for all k, in which case the sequence must have infinitely many distinct elements. Hence, by the previous argument we can find a ball centered at w_0 where f is zero. This shows that $w_0 \in V$. Hence, V is closed in U. Hence, $U = V \cup (U \setminus V)$, with $U \setminus V$ open. Since U is connected, it follows that $U \setminus V$ must be empty. ■

Observe that in the previous proof we actually showed that each zero of a holomorphic function f is isolated and has finite multiplicity, unless $f = 0$.

Corollary 112 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be holomorphic and not identically zero. Assume that there exists $z_0 \in U$ such that $f(z_0) = 0$. Then there exists $m \in \mathbb{N}$ such that*

$$f(z) = (z - z_0)^m g(z),$$

where $g : U \to \mathbb{C}$ is holomorphic and $g(z_0) \neq 0$. Moreover, there exists $r > 0$ such that $f(z) \neq 0$ for all $z \in B(z_0, r) \setminus \{z_0\} \subset U$

Proof. Writing f as a power series centered at z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

If $a_n = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$ by Theorem 111. Let $m \in \mathbb{N}$ be the smallest integer such that $a_m \neq 0$. Then as in the proof of Theorem 111 we can write

$$\begin{aligned} f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n &= (z - z_0)^m \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m} \\ &= (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k \\ &=: (z - z_0)^m g(z). \end{aligned}$$

Then $g(z_0) = a_m + 0 + \cdots + 0 = a_m \neq 0$. The function g is holomorphic in $B(z_0, R)$, where R is its radius of convergence. On the other hand, in $U \backslash B(z_0, R)$ the function

$$g(z) := \frac{f(z)}{(z - z_0)^m}$$

is holomorphic, since quotient of two holomorphic functions.

The last statement follows from Theorem 111. ■

The number m is called *multiplicity* of z_0. We say that f has a *zero of order* m or *of multiplicity* m.

Example 113 *Consider the function*

$$f(z) = \cos \frac{1+z}{1-z}, \quad z \in B(0, 1).$$

The function f is holomorphic and has infinitely many zeros when $\frac{1+z}{1-z} = \frac{\pi}{2} + n\pi$, that is, $1 + z = (\frac{\pi}{2} + n\pi)(1 - z)$, or

$$z = \frac{-1 + \frac{\pi}{2} + n\pi}{1 + \frac{\pi}{2} + n\pi} \to 1$$

as $n \to \infty$. Note that $1 \in \partial B(0,1)$, and so this does not contradict Theorem 111.

Corollary 114 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be holomorphic. Assume that there exists $z_0 \in U$ such that $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}_0$. Then $f = 0$.*

Proof. Writing f as a power series centered at z_0 we get that $f = 0$ in $B(z_0, r) \subseteq U$. But then we can apply the previous theorem to conclude that $f = 0$ in U. ■

Corollary 115 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f, g : U \to \mathbb{C}$ be holomorphic. Assume that there exists a sequence $\{z_k\}_k$ in U with $z_k \neq z_m$ for $k \neq m$ such that $z_k \to z_0 \in U$ as $n \to \infty$ and $f(z_k) = g(z_k)$ for all k. Then $f = g$ in U.*

Next we study isolated singularities.

Definition 116 *Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. We say that $z_0 \in \mathbb{C} \setminus U$ is a* point singularity *or* isolated singularity of f *if there exists $r > 0$ such that $B(z_0, r) \setminus \{z_0\} \subseteq U$.*

Example 117 *If we take $U = \mathbb{C} \setminus \{0\}$ then the holomorphic function $f(z) = z$ has an isolated singularity at 0. In this case we can extend f to 0 as a holomorphic function by setting $f(0) := 0$. This is called a* removable singularity. *The functions $f(z) = \frac{1}{z}$ and $g(z) = e^{1/z}$ have an isolated singularity at $z = 0$.*

We will show that isolated singularities are of three types;

1. removable singularities;
2. poles;
3. essential singularities

Definition 118 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be an isolated singularity of f. We say that z_0 is a* removable singularity *if we can define f at z_0 in such a way that the resulting function is homomorphic in $U \cup \{z_0\}$.*

Theorem 119 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be an isolated singularity of f. Then z_0 is a removable singularity if and only if*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0. \tag{38}$$

In particular, if f is bounded near z_0, then z_0 is a removable singularity.

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Proof. If z_0 is a removable singularity for f then f is continuous at z_0 and so

$$\lim_{z\to z_0}(z-z_0)f(z) = 0f(z_0) = 0.$$

Conversely, assume that (38) holds. Define $g : U \cup \{z_0\} \to \mathbb{C}$ via

$$g(z) := \begin{cases} (z-z_0)f(z) & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

In view of (38), the function g is holomorphic in U and continuous at z_0. In view of Remark 71, g has a primitive G in $B(z_0, r) \subseteq U \cup \{z_0\}$, and so G is holomorphic. By Corollary 81, G is analytic. Since $G' = g$, we have that g is holomorphic. Since $g(z_0) = 0$, by Corollary 112, there exists $m \in \mathbb{N}$ such that

$$g(z) = (z-z_0)^m h(z),$$

where $h : U \cup \{z_0\} \to \mathbb{C}$ is holomorphic and $h(z_0) \neq 0$. Set $f_1(z) = (z-z_0)^{m-1}h(z)$. Then f_1 is holomorphic in $U \cup \{z_0\}$. Since $B(z_0, r) \setminus \{z_0\}$ is connected, it follows that f and f_1 must coincide in $B(z_0, r) \setminus \{z_0\}$ by Corollary 115. Thus, f_1 extends f to z_0 as an holomorphic function. ■

Definition 120 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be an isolated singularity of f. We say that z_0 is a* pole *if*

$$\lim_{z\to z_0}|f(z)| = \infty. \tag{39}$$

Theorem 121 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be a pole of f. Then there exist $m \in \mathbb{N}$, $r > 0$, and a holomorphic function $g : B(z_0, r) \to \mathbb{C}$ such that $B(z_0, r) \subseteq U \setminus \{z_0\}$, $g(z) \neq 0$ for all $z \in B(z_0, r)$ and*

$$f(z) = \frac{g(z)}{(z-z_0)^m} \quad \textit{for all } z \in B(z_0, r) \setminus \{z_0\}.$$

Proof. By the definition of limit there exists $r > 0$ such that $B(z_0, r) \subseteq U \setminus \{z_0\}$ and $|f(z)| \geq 1$ for all $z \in B(z_0, r) \setminus \{z_0\}$. Hence, the function $\frac{1}{f}$ is well-defined and holomorphic in $B(z_0, r) \subseteq U \setminus \{z_0\}$. Moreover, by (39),

$$\lim_{z\to z_0}\frac{1}{f(z)} = 0.$$

Thus, if we define

$$h(z) := \begin{cases} \frac{1}{f(z)} & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

Then h is holomorphic in $B(z_0, r)$ by the previous theorem. Since $h(z_0) = 0$, by by Corollary 112, there exists $m \in \mathbb{N}$ such that

$$h(z) = (z-z_0)^m q(z),$$

where $q : B(z_0, r) \to \mathbb{C}$ is holomorphic and $q(z_0) \neq 0$. By continuity and taking r smaller, if necessary, we can assume that $q(z) \neq 0$ for all $z \in B(z_0, r)$. Then

$$\frac{1}{f(z)} = (z - z_0)^m q(z)$$

for all $z \in B(z_0, r) \setminus \{z_0\}$, that is,

$$f(z) = \frac{1}{(z - z_0)^m q(z)} =: \frac{g(z)}{(z - z_0)^m},$$

where $g(z) := 1/q(z)$. ■

The number m is called *multiplicity* of z_0. We say that f has a *pole of order* m or *of multiplicity* m. When $m = 1$, we say that f has a *simple pole* at z_0.

Theorem 122 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be a pole of f or order m. Then there exist b_1, $\ldots$, $b_m \in \mathbb{C}$, $r > 0$, and a holomorphic function $h : B(z_0, r) \to \mathbb{C}$ such that $B(z_0, r) \subseteq U \setminus \{z_0\}$, and*

$$f(z) = \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} + h(z) \quad \text{for all } z \in B(z_0, r) \setminus \{z_0\}. \tag{40}$$

Proof. By the previous theorem, there exist $m \in \mathbb{N}$, $r > 0$, and a holomorphic function $g : B(z_0, r) \to \mathbb{C}$ such that $B(z_0, r) \subseteq U \setminus \{z_0\}$, $g(z) \neq 0$ for all $z \in B(z_0, r)$ and

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad \text{for all } z \in B(z_0, r) \setminus \{z_0\}.$$

Since g is analytic, by taking r smaller, if necessary, we can write

$$g(z) = a_0 + a_1(z - z_0) + \cdots + a_{m-1}(z - z_0)^{m-1} + \sum_{n=m}^{\infty} a_n (z - z_0)^n$$

and so

$$f(z) = \frac{g(z)}{(z - z_0)^m} = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \cdots + \frac{a_{m-1}}{(z - z_0)} + \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m}.$$

It suffices to define

$$h(z) := \sum_{n=m}^{\infty} a_n (z - z_0)^{n-m},$$

which is holomorphic. ■

The sum

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}$$

is called the *principal part of* f at the pole z_0 and the number b_1 is the *residue* of f at z_0. We write

$$\operatorname{res}_{z_0} f = b_1.$$

Theorem 123 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be a pole of f or order m. Then*

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)).$$

In particular, if f has a simple pole at z_0, then

$$\operatorname{res}_{z_0} f = \lim_{z \to z_0} (z-z_0) f(z).$$

Proof. By (40),

$$(z-z_0)^m f(z) = b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \cdots + b_m + (z-z_0)^m h(z).$$

Hence

$$\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)) = b_1(m-1)! + 0 + \cdots + 0 + \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m h(z)).$$

To conclude observe that

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m h(z)) = 0$$

since we are differentiating $m-1$ times and so by the product rule each term in $\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m h(z))$ will have some power of $z - z_0$. ■

Next we prove the *residue formula.* We begin with a simple case.

Theorem 124 (Residue formula) *Let $U \subseteq \mathbb{C}$ be an open set, let $z_0 \in U$, and let $f : U \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function having a pole at z_0. Then for closed ball $\overline{B} \subset U$ having z_0 in its interior,*

$$\int_{\partial B} f \, dz = 2\pi i \operatorname{res}_{z_0} f.$$

Proof. Consider the closed curve $\Gamma_{\delta,\varepsilon}$ given in Figure 1, where ε is the radius of the small circle centered at z_0 and δ is the width of the corridor. Since the function f is holomorphic in $U \setminus \{z_0\}$, by considering $V := B \setminus S$, where S is the segment obtained when $\varepsilon \to 0$ and $\delta \to 0$, we can apply Exercise 74, to obtain that f has a primitive in V. Since the range of $\Gamma_{\delta,\varepsilon}$ is contained in V, it follows from Corollary 65 that

$$\int_{\Gamma_{\delta,\varepsilon}} f \, dz = 0.$$

If we let $\delta \to 0^+$ and use the fact that f is continuous, we get that the two segments converge to a segment which is integrated in opposite directions. Hence, we obtain

$$\int_{\partial B} f \, dz - \int_{\partial B(z_0,\varepsilon)} f \, dz = 0. \tag{41}$$

Thus to prove the theorem it suffices to show that

$$\int_{\partial B(z_0,\varepsilon)} f\,dz = 2\pi i \operatorname{res}_{z_0} f. \tag{42}$$

By Theorem 122 there exist $b_1, \dots, b_m \in \mathbb{C}$, $r > 0$, and a holomorphic function $h : B(z_0, r) \to \mathbb{C}$ such that $B(z_0, r) \subseteq U \setminus \{z_0\}$, and

$$f(z) = \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} + h(z) \quad \text{for all } z \in B(z_0, r) \setminus \{z_0\}.$$

Hence,

$$\int_{\partial B(z_0,\varepsilon)} f\,dz = \int_{\partial B(z_0,\varepsilon)} \frac{b_1}{z - z_0} dz + \cdots + \int_{\partial B(z_0,\varepsilon)} \frac{b_m}{(z - z_0)^m} dz + \int_{\partial B(z_0,\varepsilon)} h\,dz. \tag{43}$$

By Cauchy's integral formula applied to the constant function b_1 we have that

$$b_1 = \frac{1}{2\pi i} \int_{\partial B(z_0,\varepsilon)} \frac{b_1}{z - z_0} dz, \tag{44}$$

while by Corollary 81 applied to the constant functions b_k ,

$$0 = \frac{d^{k-1}}{dz^{k-1}}(b_k) = \frac{(k-1)!}{2\pi i} \int_{\partial B(z_0,\varepsilon)} \frac{b_k}{(z - z_0)^k} dz. \tag{45}$$

Since h is holomorphic in $B(z_0, r)$, taking $\varepsilon < r$ we have that

$$\int_{\partial B(z_0,\varepsilon)} h\,dz = 0 \tag{46}$$

by Corollary 72. Formula (42) follows by combining (43)–(46). ∎

Remark 125 *Note that since ∂B and $\partial B(z_0, \varepsilon)$ are homotopic in U, we could have used Theorem 98 to obtain (41).*

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Exercise 126 *Let $U \subseteq \mathbb{C}$ be an open set, let $z_1, \dots, z_n \in U$, and let $f : U \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ be a holomorphic function having poles at $z_1, \dots, z_n$. Prove that for every closed ball $\overline{B} \subset U$ having $z_1, \dots, z_n$ in its interior,*

$$\int_{\partial B} f \, dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f.$$

Exercise 127 (Residue formula) *Let $U \subseteq \mathbb{C}$ be an open set, let $z_1, \dots, z_n \in U$, and let $f : U \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ be a holomorphic function having poles at $z_1, \dots, z_n$. Prove that for every continuous rectifiable closed simple curve γ homotopic to 0 in U and having $z_1, \dots, z_n$ in its interior,*

$$\int_{\gamma} f \, dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f.$$

Note that in the previous exercise we are using Jordan's curve theorem (see Theorem 57).

The calculus of residues can be used to compute many interesting improper integrals.

Example 128 *Let's prove that for $0 < a < 1$,*

$$\int_{\mathbb{R}} \frac{e^{ax}}{1 + e^x} dx = \frac{\pi}{\sin(\pi a)}.$$

Consider the function

$$f(z) = \frac{e^{az}}{1 + e^z}.$$

Note that $1 + e^z = 0$ for $z = i\pi + 2i\pi k$, $k \in \mathbb{Z}$. Given $\ell > 0$ consider the rectangle $R_\ell = \{z = x + iy : x \in (-\ell, \ell), 0 < y < 2\pi\}$ and let γ_ℓ be the oriented closed curve which parametrizes ∂R_ℓ using arclength and going counterclockwise starting from $-\ell + 0iy$. The only point at which the denominator vanishes in R_ℓ is πi. Note that

$$(z - \pi i) f(z) = e^{az} \frac{z - \pi i}{1 + e^z} = e^{az} \frac{z - \pi i}{e^z - e^{\pi i}}.$$

Since $\frac{d}{dz} e^z = e^z$, we have that

$$\lim_{z \to \pi i} \frac{e^z - e^{\pi i}}{z - \pi i} = e^{\pi i} = -1$$

and so

$$\lim_{z \to \pi i} (z - \pi i) f(z) = -e^{a\pi i}.$$

In turn, by (40),

$$\operatorname{res}_{\pi i} f = -e^{a\pi i}.$$

It follows by the residue formula that

$$\int_{\gamma_\ell} f\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{\pi i} f = -2\pi i e^{a\pi i}. \tag{47}$$

Set

$$I_\ell := \int_{-\ell}^{\ell} f(x)\,dx = \int_{-\ell}^{\ell} \frac{e^{ax}}{1+e^x}dx. \tag{48}$$

On the other hand, to parametrize the top we consider curve $\gamma_{\ell,3}$ parametrized by $\varphi_3(t) = 3\ell + 2\pi - t + 2\pi i$, where $t \in [2\ell + 2\pi, 4\ell + 2\pi]$. Then by the change of variables $s = 3\ell + 2\pi - t$,

$$\begin{aligned}\int_{\gamma_{\ell,3}} f\,dz &= \int_{2\ell+2\pi}^{4\ell++2\pi} f(\varphi_3(t))\varphi_3'(t)\,dt = \int_{\ell}^{-\ell} f(s+2\pi i)\,ds \\ &= -\int_{-\ell}^{\ell} \frac{e^{as}e^{2\pi ia}}{1+e^{s+2\pi i}}\,ds = -\int_{-\ell}^{\ell} \frac{e^{as}e^{2\pi ia}}{1+e^{s}}\,ds = -e^{2\pi ia} I_\ell.\end{aligned} \tag{49}$$

Next to parametrize the right vertical side we consider curve $\gamma_{\ell,2}$ parametrized by $\varphi_2(t) = \ell + i(t - 2\ell)$, where $t \in [2\ell, 2\ell + 2\pi]$. Then by the change of variables $s = t - 2\ell$,

$$\begin{aligned}\int_{\gamma_{\ell,2}} f\,dz &= \int_{2\ell}^{2\ell++2\pi} f(\varphi(t))\varphi'(t)\,dt = \int_0^{2\pi} if(\ell+is)\,ds \\ &= \int_0^{2\pi} \frac{e^{a(\ell+is)}}{1+e^{\ell+is}}\,ds = \frac{e^{a\ell}}{e^\ell}\int_0^{2\pi} \frac{e^{ais}}{e^{-\ell}+e^{is}}\,ds.\end{aligned}$$

Since $|e^{-\ell} + e^{is}| \geq |e^{is}| - e^{-\ell} = 1 - e^{-\ell}$, we have

$$\begin{aligned}\left|\int_{\gamma_{\ell,2}} f\,dz\right| &\leq \frac{1}{e^{\ell(1-a)}}\int_0^{2\pi} \frac{|e^{ais}|}{|e^{-\ell}+e^{is}|}\,ds \\ &\leq \frac{1}{e^{\ell(1-a)}}\frac{2\pi}{1-e^{-\ell}} \to 0\end{aligned} \tag{50}$$

as $\ell \to \infty$. A similar computation holds for the left vertical side, whose integral can be bound in modulus by $ce^{-\ell a}$. It follows from (47)–(50) that

$$-2\pi i e^{a\pi i} = \lim_{\ell\to\infty}\int_{\gamma_\ell} f\,dz = (1-e^{2\pi ia})\int_{\mathbb{R}} \frac{e^{ax}}{1+e^x}dx,$$

that is,

$$\int_{\mathbb{R}} \frac{e^{ax}}{1+e^x}dx = \frac{-2\pi i e^{a\pi i}}{1-e^{2\pi ia}} = \frac{2\pi i e^{a\pi i}}{e^{2\pi ia}-1} = \frac{2\pi i}{e^{\pi ia}-e^{-\pi ia}} = \frac{\pi}{\sin(\pi a)},$$

where we used (6).

Exercise 129 *Use the calculus of residues to prove that*

$$\int_{\mathbb{R}} \frac{1}{1+x^2}dx = \pi.$$

Exercise 130 *Use the calculus of residues to prove that for all $\xi \in \mathbb{R}$,*

$$\int_{\mathbb{R}} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}dx = \frac{1}{\cosh(\pi\xi)}.$$

We now the notion of meromorphic functions. Consider the *extended complex plane* $\mathbb{C}_\infty$ obtained by adding to $\mathbb{C}$ a point not in $\mathbb{C}$ called ∞,

$$\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}.$$

Given an open set $U \subseteq \mathbb{C}$ and $z_0 \in U$, if a holomorphic function $f : U \setminus \{z_0\} \to \mathbb{C}$ has a pole at z_0, we can extend f to z_0 by setting

$$f(z_0) := \infty,$$

so that $f : U \to \mathbb{C}_\infty$.

Definition 131 *Let $U \subseteq \mathbb{C}$ and let $f : U \to \mathbb{C}_\infty$. We say that f is* meromorphic *if there exists a sequence $\{z_n\}_n$ of complex numbers such that the set $\{z_n : n \in \mathbb{N}\}$ has no accumulation points in U, f has poles at z_n for every n, and $f : U \setminus \{z_n : n \in \mathbb{N}\} \to \mathbb{C}$ is holomorphic.*

Let $U \subseteq \mathbb{C}$ be an open set which contains $\mathbb{C} \setminus B(0, R)$ for some $R > 0$ and let $f : U \to \mathbb{C}$ be a holomorphic function. We say that f has a *removable singularity, a pole, or an essential singularity at infinity* if the function $F(z) := f(1/z)$ has a removable singularity, a pole, or an essential singularity at 0, respectively, In the first case we say that f is *holomorphic at infinity.* We say that f is meromorphic in the extended complex plane if it is meromorphic in the complex plane and either has a pole at infinity or is holomorphic at infinity.

Exercise 132 *Prove that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has a removable singularity at infinity iff it is constant.*

Exercise 133 *Prove that a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ has a pole at infinity of order m iff it is polynomial of degree m.*

Exercise 134 *Characterize those rational functions which have a removable singularity at infinity.*

Exercise 135 *Characterize those rational functions which have a pole of order m at infinity.*

Next we prove the argument principle. We have seen that in general for a branch $\log_V$ of the logarithm, the formula

$$\log_V(z_1 z_2) = \log_V z_1 + \log_V z_2.$$

Hence, we cannot expect the formula

$$\log_V(f_1 f_2) = \log_V f_1 + \log_V f_2$$

to holds for holomorphic functions $f_1, f_2 : U \to V$. However, the formula holds for derivatives since

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

More generally,

$$\frac{\left(\prod_{k=1}^{n} f_k\right)'}{\prod_{k=1}^{n} f_k} = \sum_{k=1}^{n} \frac{f_k'}{f_k}. \tag{51}$$

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We will use this observation to prove the argument principle. Given a set E, we denote by $\operatorname{card} E$ its cardinality.

Theorem 136 (Argument principle) *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}_\infty$ be a meromorphic function. Then for every for closed ball $\overline{B} \subset U$ such that f has no poles or zeros on ∂B, we have*

$$\frac{1}{2\pi i}\int_{\partial B} \frac{f'}{f}\,dz = \text{(number of zeros of } f \text{ in } B\text{) minus (number of poles of } f \text{ in } B\text{)},$$

where the zeros and poles are counted with multiplicity.

Proof. Let $z_1, \ldots, z_n$ be the zeros of f inside B and let $p_1, \ldots, p_\ell$ be the poles of f inside B. For every $k = 1, \ldots, n$, let m_k be the order of z_k. By Corollary 112 we can find $r_k > 0$ and a holomorphic function $g_k : B(z_k, r_k) \to \mathbb{C}$ such that $g_k \neq 0$ in $B(z_k, r_k) \subset B$ and

$$f(z) = (z - z_k)^{m_k} g_k(z) \quad \text{for all } z \in B(z_k, r_k).$$

It follows from (51) that

$$\frac{f'(z)}{f(z)} = \frac{m_k}{z - z_k} + \frac{g_k'(z)}{g_k(z)}.$$

The function $\frac{g_k'}{g_k}$ is holomorphic in $B(z_k, r_k)$. This shows that $\frac{f'}{f}$ has a simple pole with residue m_k at z_k, that is, $\operatorname{res}_{z_k} f'/f = m_k$.

Similarly, for every $k = 1, \dots, \ell$, let n_k be the order of p_k. by Theorem 121 we can find $t_k > 0$ and a holomorphic function $h_k : B(p_k, t_k) \to \mathbb{C}$ such that $h_k \neq 0$ in $B(p_k, t_k) \subset B$ and

$$f(z) = \frac{h_k(z)}{(z - p_k)^{n_k}} \quad \text{for all } z \in B(p_k, t_k). \tag{52}$$

Since

$$\frac{d}{dz}\left(\frac{1}{z - p_k}\right) = -\frac{1}{(z - p_k)^2},$$

we have

$$\frac{\frac{d}{dz}\left(\frac{1}{z-p_k}\right)}{\frac{1}{z-p_k}} = \frac{-\frac{1}{(z-p_k)^2}}{\frac{1}{z-p_k}} = -\frac{1}{z - p_k},$$

and so, using (51) and (52) we get

$$\frac{f'(z)}{f(z)} = -\frac{n_k}{z - p_k} + \frac{h_k'(z)}{h_k(z)}.$$

The function $\frac{h_k'}{h_k}$ is holomorphic in $B(p_k, t_k)$. This shows that $\frac{f'}{f}$ has a simple pole with residue $-n_k$ at p_k, that is, $\operatorname{res}_{p_k} f'/f = -n_k$.

The conclusion now follows by applying the residue formula (Theorem 124) to f'/f. ■

Exercise 137 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}_\infty$ be a meromorphic function. Prove that for every continuous rectifiable closed simple curve γ homotopic to 0 in U and whose range contains no zero or pole of f, we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \text{(number of zeros of } f \text{ in the interior of } \gamma\text{) minus}$$
$$\text{(number of poles of } f \text{ in the interior of } \gamma\text{)},$$

where the zeros and poles are counted with multiplicity.

Next we discuss the last type of isolated singularities.

Definition 138 *Let $U \subset \mathbb{C}$ be an open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C} \setminus U$ be an isolated singularity of f. We say that z_0 is an* essential singularity *for f if z_0 is not a removable singularity or a pole.*

Example 139 *The function $f(z) = e^{1/z}$ has an essential singularity at 0. Indeed, if we take $z = iy$ we have that*

$$|f(iy)| = |e^{1/(iy)}| = |e^{-i/y}| = 1,$$

so z is not a pole. On the other hand,

$$\lim_{x \to 0+} xe^{1/x} = \infty$$

and so by Theorem 119, $z = 0$ is not a removable singularity.

10 The Maximum Modulus Principle

In this section we prove some important theorems of holomorphic functions.

Theorem 140 (Rouché) *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be holomorphic functions. Assume that there exists a closed ball $\overline{B} \subset U$ such that*

$$|f(z)| > |g(z)| \quad \text{for all } z \in \partial B. \tag{53}$$

Then f and $f+g$ have the same number of zeros inside B.

Proof. For $t \in [0,1]$ consider the function

$$f_t(z) := f(z) + tg(z), \quad z \in U.$$

Then $f_0 = f$ and $f_1 = f+g$. Moreover f_t is holomorphic in U. Let $n_t \in \mathbb{N}_0$ be the number of zeros of f_t inside B counted with multiplicity. The hypothesis (53) guarantees that f_t has no zeros on ∂B. Hence, by the argument principle

$$n_t = \frac{1}{2\pi i}\int_{\partial B} \frac{f_t'}{f_t}\, dz.$$

Again by (53) we have that the function

$$g(t,z) = \frac{f_t'(z)}{f_t(z)} = \frac{f'(z) + tg'(z)}{f(z) + tg(z)}, \quad t \in [0,1],\, z \in \partial B$$

is continuous in the compact set $[0,1] \times \partial B$. Hence, it is bounded. Using the Lebesgue dominated convergence theorem (or Ascoli's convergence theorem for Riemann integrals), we have that n_t is a continuous function of t. But since it is integer-valued and $[0,1]$ is connected, it follows that n_t must be constant. This concludes the proof. ■

Using Rouché's theorem we can prove that non-constant holomorphic functions are open.

Theorem 141 (Open mapping) *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a non-constant holomorphic function. Then for every $V \subseteq U$ open, $f(V)$ is open.*

Proof. Let $z_0 \in V$ and let $w_0 = f(z_0)$. We must find $\varepsilon > 0$ such that $B(w_0, \varepsilon) \subseteq f(V)$. Since the zeros of $f - w_0$ are isolated by Theorem 111 (or Corollary 112), there exists $\delta > 0$ such that $\overline{B(z_0,\delta)} \subset V$ and $f - w_0 \neq 0$ on $\partial B(z_0, \delta)$. By uniform continuity, we can find $\varepsilon > 0$ such that

$$|f(z) - w_0| > \varepsilon \quad \text{for all } z \in \partial B(z_0, \delta).$$

Let $w \in B(w_0, \varepsilon)$ and define

$$g(z) := f(z) - w = (f(z) - w_0) + (w_0 - w_0) =: F(z) + G(z).$$

By the previous inequality we have that $|F(z)| > |G(z)|$ for all $z \in \partial B(z_0, \delta)$. Hence, by Rouché's theorem F and $F+G=g$ have the same number of zeros in $B(z_0,\delta)$. Since F has one zero in $B(z_0,\delta)$, so must g. Hence, there is $z \in B(z_0,\delta)$ such that $f(z) = w$. This shows that $B(w_0,\varepsilon) \subseteq f(B(z_0,\delta)) \subseteq f(V)$. This concludes the proof. ■

Corollary 142 *Let $U \subseteq \mathbb{C}$ be open and let $f : U \to \mathbb{C}$ be injective and holomorphic. Then $f^{-1} : f(U) \to \mathbb{C}$ is holomorphic and*

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}, \quad w \in f(U).$$

Proof. By the open mapping theorem, f^{-1} is continuous and $f(U)$ is open. Hence, we can apply Exercise 9 to conclude that f^{-1} is differentiable. ■

Theorem 143 (Maximum modulus principle) *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be a nonconstant holomorphic function. Then $|f|$ cannot attains a maximum in U.*

Proof. Assume that $|f|$ assumes a maximum at some point $z_0 \in U$. Let $B(z_0,r) \subseteq U$. By the open mapping theorem, $f(B(z_0,r))$ is open and so there exists $B(f(z_0),\delta) \subseteq f(B(z_0,r))$. This implies that there exists points in U with modulus bigger that $|f(z_0)|$, which is a contradiction. ■

Exercise 144 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f : U \to \mathbb{C}$ be a non-constant holomorphic function such that $f(z) \neq 0$ for all $z \in U$. Prove that $|f|$ cannot attain its minimum on U.*

Corollary 145 *Let $U \subset \mathbb{C}$ be an open bounded set and let $f : \overline{U} \to \mathbb{C}$ be a continuous function which is holomorphic in U. Then*

$$\sup_U |f| \leq \max_{\partial U} |f|.$$

Proof. Since $|f|$ is continuous on the compact set $\overline{U}$, it admits a maximum. By the maximum principle, this maximum must be attained at the boundary of U. ■

The previous corollary fails in general in unbounded domains.

Example 146 *Let $U := \{z = x + iy : x > 0, y > 0\}$ be the first quadrant and let $f(z) = e^{-iz^2}$. Then f is holomorphic in U and continuous on $\overline{U}$. If $z = x \geq 0$, then $|f(x)| = |e^{-ix^2}| = 1$, while if $z = iy$ with $y \geq 0$, then $|f(iy)| = |e^{iy^2}| = 1$. However, f is unbounded. To see this take $z = r\sqrt{i} = re^{i\pi/4}$. Then $f(z) = e^r \to \infty$ as $r \to \infty$.*

Friday, February 21, 2020

11 Essential Singularities

Next we study the behavior of a holomorphic function near an essential singularity.

Theorem 147 (Casorati–Weierstrass) *Let $z_0 \in \mathbb{C}$, $r > 0$, and let $f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function having an essential singularity at z_0. Then $f(B(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$.*

Proof. Assume by contradiction that $f(B(z_0, r) \setminus \{z_0\})$ is not dense in $\mathbb{C}$. Then there exist $w_0 \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w_0| \geq \delta \quad \text{for all } z \in B(z_0, r) \setminus \{z_0\}.$$

It follows that the function

$$g(z) := \frac{1}{f(z) - w_0}, \quad z \in B(z_0, r) \setminus \{z_0\},$$

is well-defined and holomorphic. Moreover, it is bounded by $1/\delta$. Hence, by Theorem 119 it has a removable singularity at z_0. Extend g to z_0 as a holomorphic function. There are now two cases. If $g(z_0) \neq 0$, then $g \neq 0$ in $B(z_0, r)$, and so $f - w_0$ has a removable singularity at z_0, which is a contradiction. If $g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{1}{f(z) - w_0} = 0,$$

which implies that

$$\lim_{z \to z_0} |f(z) - w_0| = \infty,$$

and so f has a pole at z_0, which is again a contradiction. This concludes the proof. ■

There is actually a much stronger result.

Theorem 148 (Picard Big Theorem) *Let $z_0 \in \mathbb{C}$, $r > 0$, and let $f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function having an essential singularity at z_0. Then f takes all possible values of $\mathbb{C}$ with at most a single exception.*

Exercise 149 *Prove that*

$$\pi \cot(\pi z) = \lim_{\ell \to \infty} \sum_{k=-\ell}^{\ell} \frac{1}{z + k} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

The proof relies on several preliminary results. We begin with another important theorem.

Theorem 150 (Bloch) *Let $U \subseteq \mathbb{C}$ be an open set which contains $\overline{B(0, 1)}$ and $f : U \to \mathbb{C}$ be a holomorphic function such that $f'(0) = 1$. Then $f(B(0, 1))$ contains a ball of radius $\frac{3}{2} - \sqrt{2}$.*

We begin with some lemmas.

Exercise 151 *Let $V \subset \mathbb{C}$ be an open bounded set, let $f : \overline{V} \to \mathbb{C}$ be a continuous function such that $f : V \to \mathbb{C}$ is open. Let $w_0 \in V$ be such that*

$$R := \min_{z \in \partial V} |f(z) - f(w_0)| > 0.$$

Prove that $f(V)$ contains $B(f(w_0), R)$.

Lemma 152 *Let $U \subseteq \mathbb{C}$ be an open set which contains $\overline{B(z_0, r)}$ and $f : U \to \mathbb{C}$ be a holomorphic function which is non-constant in $B(z_0, r)$ and such that*

$$|f'(z)| \le 2|f'(z_0)| \quad \text{for all } z \in \overline{B(z_0, r)}. \tag{54}$$

Then $f(B(z_0, r))$ contains $B(f(z_0), r_0)$, where $r_0 = (3 - 2\sqrt{2})|f'(z_0)|r$.

Proof. Without loss of generality we may assume that $z_0 = 0$ and $f(0) = 0$. Define $g(z) = f(z) - f'(0)z$. By the fundamental theorem of calculus,

$$g(z) = \int_{[0,z]} [f'(\zeta) - f'(0)]\, d\zeta.$$

Consider the parametric representation $\varphi(t) = tz$, $t \in [0, 1]$. Then

$$|g(z)| \le |z| \int_0^1 |f'(tz) - f'(0)|\, dt. \tag{55}$$

Let $w \in B(0, r)$. By Cauchy's formula applied to the holomorphic function f',

$$f'(w) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f'(\zeta)}{\zeta - w} d\zeta, \quad f'(0) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f'(\zeta)}{\zeta} d\zeta.$$

Subtracting these identities gives

$$\begin{aligned} f'(w) - f'(0) &= \frac{1}{2\pi i} \int_{\partial B(0,r)} \left[\frac{1}{\zeta - w} - \frac{1}{\zeta} \right] f'(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{w}{\zeta(\zeta - w)} f'(\zeta)\, d\zeta \end{aligned}$$

and so using the the parametric representation $\psi(\theta) = re^{i\theta}$ and the fact that $|\zeta - w| \ge |\zeta| - |w| = r - |w|$, we get

$$|f'(w) - f'(0)| \le |w| \sup_{\partial B(0,r)} |f'| \frac{1}{r - |w|}.$$

Taking $w = tz$ and using this inequality in (55) gives

$$\begin{aligned} |g(z)| &\le |z| \int_0^1 |f'(tz) - f'(0)|\, dt \le |z| \sup_{\partial B(0,r)} |f'| \int_0^1 \frac{t|z|}{r - t|z|}\, dt \\ &\le |z|^2 \sup_{\partial B(0,r)} |f'| \frac{1}{r - |z|} \int_0^1 t\, dt = \frac{1}{2} \frac{|z|^2}{r - |z|} \sup_{\partial B(0,r)} |f'| \qquad (56) \\ &\le \frac{|z|^2}{r - |z|} |f'(0)|, \end{aligned}$$

where in the last inequality we used (54). Now let $0 < \rho < r$ and take z with $|z| = \rho$. Then

$$|g(z)| = |f(z) - f'(0)z| \geq |f'(0)|\rho - |f(z)|$$

Combining this inequality with (56) gives

$$\frac{\rho^2}{r-\rho}|f'(0)| \geq |f'(0)|\rho - |f(z)|,$$

or, equivalently,

$$|f(z)| \geq |f'(0)| \left(\rho - \frac{\rho^2}{r-\rho}\right) =: |f'(0)|h(\rho).$$

We have

$$h'(\rho) = \frac{d}{d\rho}\left(\rho - \frac{\rho^2}{r-\rho}\right) = \frac{r^2 - 4r\rho + 2\rho^2}{(r-\rho)^2} \geq 0$$

for $\rho \geq r\left(\frac{\sqrt{2}}{2} + 1\right)$ and $\rho \leq r\left(1 - \frac{\sqrt{2}}{2}\right)$, so h has a maximum at $\rho_0 = r\left(1 - \frac{\sqrt{2}}{2}\right)$. Hence,

$$|f(z)| \geq |f'(0)|h(\rho_0) = |f'(0)|r\left(3 - 2\sqrt{2}\right) \quad \text{for all } z \in \partial B(0, \rho_0).$$

We now apply the previous exercise with $w_0 = 0$ and $V = B(0, \rho_0)$ to obtain that

$$f(B(0,r)) \supseteq f(B(0,\rho_0)) \supseteq B(0,R),$$

where $R := \min_{\partial B(0,\rho_0)} |f| \geq |f'(0)|r\left(3 - 2\sqrt{2}\right) = r_0$. This concludes the proof. ∎

We now turn to the proof of Bloch's theorem.

Proof. Step 1: Let $U \subseteq \mathbb{C}$ be an open set which contains $\overline{B(0,1)}$ and $f : U \to \mathbb{C}$ be a holomorphic function which is non-constant in $B(0,1)$. Since the function

$$g(z) = |f'(z)|(1 - |z|)$$

is continuous in $\overline{B(0,1)}$, it assumes a maximum at some point z_0. We claim that $f(B(0,1)) \supseteq B(f(z_0), r_0)$, where $r_0 := (\frac{3}{2} - \sqrt{2})g(z_0)$.

To see this, take $t = \frac{1}{2}(1 - |z_0|)$. Then

$$g(z_0) = |f'(z_0)|(1 - |z_0|) = 2t|f'(z_0)|. \tag{57}$$

Moreover, $B(z_0, t) \subseteq B(0,1)$, since if $z \in B(z_0, t)$, then

$$|z| \leq |z - z_0| + |z_0| < t + |z_0| = \frac{1}{2}(1 - |z_0|) + |z_0| = \frac{1}{2} + \frac{1}{2}|z_0| \leq 1.$$

Note that the previous inequality also implies that

$$1 - |z| \geq t. \tag{58}$$

Indeed, the previous inequality can be written $1 \geq t + |z| = \frac{1}{2}(1 - |z_0|) + |z|$,or, equivalently, $\frac{1}{2} + \frac{1}{2}|z_0| \geq |z|$, which is what we just proved.

Using (57) and (58) and the fact that g has a maximum at z_0, we have

$$|f'(z)|(1 - |z|) = g(z) \leq g(z_0) = 2t|f'(z_0)| \leq (1 - |z|)|f'(z_0)|,$$

which gives $|f'(z)| \leq |f'(z_0)|$. It now follows from the previous lemma and the fact that $B(z_0, t) \subseteq B(0, 1)$, that

$$f(B(0, 1)) \supseteq f(B(z_0, t)) \supseteq B(f(z_0), r_0),$$

where $r_0 = (3 - 2\sqrt{2})|f'(z_0)|t = (\frac{3}{2} - \sqrt{2})g(z_0)$, again by (57).

Step 2: To conclude the proof of the theorem, observe that if $f'(0) = 1$, then $g(0) = 1 \leq g(z_0)$ and so $r_0 \geq \frac{3}{2} - \sqrt{2}$. ■

Monday, February 24, 2020

Corollary 153 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. If $z_0 \in U$ is such that $f'(z_0) \neq 0$, then $f(U)$ contains balls of every radius $\frac{1}{12}r|f'(z_0)|$, where $0 < r < \operatorname{dist}(z_0, \partial U)$.*

Proof. Assume that $z_0 = 0$. If $0 < r < \operatorname{dist}(0, \partial U)$, then $\overline{B(0,r)} \subset U$. Consider the function

$$g(z) := \frac{f(rz)}{rf'(0)}, \quad z \in \frac{1}{r}U.$$

Since $\overline{B(0,1)} \subset \frac{1}{r}U$ and $g'(0) = 1$, by Bloch's theorem $g(B(0,1))$ contains a ball of radius $\frac{3}{2} - \sqrt{2} > \frac{1}{12}$. In turn, $f(B(0,r))$ contains a ball of radius $\frac{1}{12}r|f'(0)|$. ■

Corollary 154 *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Then $f(\mathbb{C})$ contains balls of every radius.*

Exercise 155 *Let $f : \mathbb{C} \to \mathbb{C}$. Prove that $f \circ f : \mathbb{C} \to \mathbb{C}$ has a fixed point unless f is of the form $f(z) = z + w$ for all $z \in \mathbb{C}$ and for some $w \in \mathbb{C}$.*

In this subsection we prove the following theorem.

Theorem 156 (Picard Little Theorem) *Every non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ takes every value except at most one.*

We begin with some preliminary results.

Lemma 157 *Let $U \subseteq \mathbb{C}$ be a simply connected open set and let $f : U \to \mathbb{C}$ be a holomorphic function which does not take value -1 and 1. Then there exists a holomorphic function $h : U \to \mathbb{C}$ such that*

$$f(z) = \cos h(z), \quad z \in U.$$

Proof. Since f does not take values -1 and 1, $1 - f^2$ is never equal to 0 and so by by Remark 103 there exists a branch of $\sqrt{1 - f^2}$, that is a holomorphic function $g : U \to \mathbb{C}$ such that $g^2 = 1 - f^2$ in U. Write $1 = f^2 + g^2 = (f + ig)(f - ig)$. Then $f + ig$ has no zeros in U and so by Corollary 100, $f + ig = e^{ih}$ for some holomorphic function $h : U \to \mathbb{C}$. In turn, $1 = (f + ig)(f - ig) = e^{ih}(f - ig)$ and so $f - ig = e^{-ih}$. Using Euler's formula (20) we get

$$f = \frac{e^{ih} + e^{-ih}}{2} = \cos h \quad \text{in } U,$$

which concludes the proof. ■

Lemma 158 *Let $U \subseteq \mathbb{C}$ be a simply connected open set and let $f : U \to \mathbb{C}$ be a holomorphic function which does not take value 0 and 1. Then there exists a holomorphic function $g : U \to \mathbb{C}$ such that*

$$f(z) = \frac{1}{2}[1 + \cos(\pi \cos(\pi g(z)))], \quad z \in U.$$

Moreover, $g(U)$ does not contain any ball of radius 1.

Proof. The function $2f-1$ does not take the values -1 and 1, and so by the previous lemma, there exists a holomorphic function $h:U\to\mathbb{C}$ such that $2f-1=\cos(\pi h)$ in U. Note that by periodicity, the function h does not take any integer values. In particular, it does not take the values -1 and 1. Hence, by the previous lemma again, there exists a holomorphic function $g:U\to\mathbb{C}$ such that we can write $h=\cos(\pi g)$.

To prove the second part of the statement, consider the set

$$E=\{k\pm i\pi^{-1}\log(n+\sqrt{n^2-1}):\ k\in\mathbb{Z},\ n\in\mathbb{N}\}.$$

We claim that $g(U)\cap E=\emptyset$. To see this, let $w\in E$. Then by Euler's formula (20),

$$\begin{aligned}\cos(\pi w)&=\frac{e^{i\pi w}+e^{-i\pi w}}{2}=\frac{1}{2}(e^{i\pi k}e^{\mp\log(n+\sqrt{n^2-1})}+e^{-i\pi k}e^{\pm\log(n+\sqrt{n^2-1})})\\&=\frac{1}{2}(-1)^k\left[\frac{1}{n+\sqrt{n^2-1}}+n+\sqrt{n^2-1}\right]\\&=\frac{1}{2}(-1)^k 2n=(-1)^k n.\end{aligned}$$

Hence, $\cos(\pi\cos(\pi w))=\cos(\pi(-1)^k n)\in\{-1,1\}$. In turn. $\frac{1}{2}[1+\cos(\pi\cos(\pi w))]\in\{0,1\}$. Since f does not take values 0 and 1, g cannot take value w. This proves the claim.

The points in E are the vertices of a rectangular grid. Consider the rectangle of vertices $k+i\pi^{-1}\log(n+\sqrt{n^2-1})$, $k+1+i\pi^{-1}\log(n+\sqrt{n^2-1})$, $k+i\pi^{-1}\log(n+1+\sqrt{(n+1)^2-1})$, and $k+1+i\pi^{-1}\log(n+1+\sqrt{(n+1)^2-1})$. The base has length 1 and the height has length

$$\begin{aligned}&\log(n+1+\sqrt{(n+1)^2-1})-\log(n+\sqrt{n^2-1})\\&=\log\frac{n+1+\sqrt{(n+1)^2-1}}{n+\sqrt{n^2-1}}=\log\frac{1+\frac{1}{n}+\sqrt{1+\frac{2}{n}}}{1+\sqrt{1-\frac{1}{n^2}}}\\&<\log(1+\frac{1}{n}+\sqrt{1+\frac{2}{n}})\le\log(2+\sqrt{3})\sim 1.317<\pi,\end{aligned}$$

where we factor out n and used the monotonicity of the logarithm. Hence, the height of the rectangle is less than 1. Thus for every $w\in\mathbb{C}$ we can find $z\in E$ such that $|\operatorname{Re}w-\operatorname{Re}z|\le\frac{1}{2}$, $|\operatorname{Im}w-\operatorname{Im}z|<\frac{1}{2}$, which implies that $|w-z|<1$. This shows that every ball of radius 1 intersects E. Since $g(U)$ does not intersect E, it cannot intersect any ball of radius 1. ■

Wednesday, February 26, 2020

We are now ready to prove Picard's little theorem.

Proof of Theorem 156. Assume by contradiction that there exist $a,b\in\mathbb{C}$ with $a\neq b$ such that $f:\mathbb{C}\to\mathbb{C}$ does not takes value a and b. The the function

$$h(z)=\frac{f(z)-a}{b-a},\quad z\in\mathbb{C},$$

does not take the values 0 and 1. Hence, by the previous lemma there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$h(z) = \frac{1}{2}[1 + \cos(\pi \cos(\pi g(z)))].$$

Moreover, $g(\mathbb{C})$ does not contain any ball of radius 1. However, since g is not constant, by Corollary 154 we have a contradiction. ■

Another important theorem is the following.

Theorem 159 (Schottky) *Let $U \subseteq \mathbb{C}$ be an open set which contains $\overline{B(0,1)}$, let $\alpha > 0$, $0 < r < 1$, and let $f : U \to \mathbb{C}$ be a holomorphic function which does not take values 0 and 1 and such that $|f(0)| \le \alpha$. Then*

$$|f(z)| \le \exp(\pi \exp(\pi(3 + \alpha + 12r/(1-r)))) \quad \textit{for all } z \in B(0,r). \tag{59}$$

Proof. Since U contains $\overline{B(0,1)}$, we can find $R > 1$ such that contains $\overline{B(0,1)} \subset B(0,R) \subseteq U$. In the remaining of the proof we take $U = B(0,R)$, so that U is simply connected. As in the proof of Lemma 158, since f does not take the values 0 and 1, the function $2f - 1$ does not take the values -1 and 1 and so by Lemma 157 there exists a holomorphic function $h : U \to \mathbb{C}$ such that $2f - 1 = \cos(\pi h)$ in U. By periodicity, we can add to h any integer multiple of 2. Hence, without loss of generality, we may assume that

$$-1 \le \operatorname{Re} h(0) \le 1.$$

By Exercise 33, for every $w = x + iy$ we have that

$$|y| \le \cosh y \le |\cos w| \tag{60}$$

and so

$$\pi|\operatorname{Im} h(0)| \le |\cos(\pi h(0))| = |2f(0) - 1| \le 2|f(0)| + 1.$$

Hence,

$$|h(0)| \le 1 + \frac{2}{\pi}|f(0)| + \frac{1}{\pi} < 2 + |f(0)|. \tag{61}$$

Since $2f - 1$ does not take the values -1 and 1, the function h omits all integer values. In particular, it omits the values -1 and 1 and so by Lemma 158 there exists a holomorphic function $g : U \to \mathbb{C}$ such that $h = \cos(\pi g)$. Moreover, $g(U)$ does not contain any ball of radius 1.

Reasoning as in the first part of the proof, by periodicity we can add to g any integer multiple of 2 and so we can assume that $-1 \le \operatorname{Re} g(0) \le 1$. By (60) and (61),

$$\pi|\operatorname{Im} g(0)| \le |\cos(\pi g(0))| = |h(0)| \le 2 + |f(0)|,$$

and so

$$|g(0)| \le 1 + \frac{2}{\pi}|f(0)| + \frac{2}{\pi} \le 3 + |f(0)|. \tag{62}$$

If $|z| \leq r < 1$, then $\operatorname{dist}(z, \partial B(0,1)) \geq 1 - r$. On one hand, $g(U)$ does not contain any ball of radius 1. On the other hand, by Corollary 153, if $g'(z) \neq 0$, then $g(U)$ contains balls of every radius $\frac{1}{12}(1-r)|g'(z)|$. Hence,

$$\frac{1}{12}(1-r)|g'(z)| < 1$$

for all $z \in B(0,r)$. By the fundamental theorem of calculus,

$$g(z) - g(0) = \int_{[0,z]} g'(\zeta)\, d\zeta$$

and so by the previous inequality, (62), and the fact that $|f(0)| \leq \alpha$,

$$|g(z)| \leq |g(0)| + 12|z|/(1-r) \leq 3 + \alpha + 12r/(1-r). \tag{63}$$

Since $|\cos w| \leq e^{|w|}$ and $\frac{1}{2}|1 + \cos w| \leq e^{|w|}$, it follows that

$$\begin{aligned} |f(z)| &\leq \frac{1}{2}|1 + \cos(\pi \cos(\pi g(z)))| \leq \exp(\pi |\cos(\pi g(z))|) \\ &\leq \exp(\pi \exp(\pi |g(z)|)) \leq \exp(\pi \exp(\pi(3 + \alpha + 12r/(1-r)))), \end{aligned}$$

where in the last inequality we used (63). ■

The beauty of Schottky's theorem is that the right-hand side of (59) depends only on α and r. Hence, we have a universal bound.

12 Sequences of Holomorphic Functions

Theorem 160 *Let $U \subseteq \mathbb{C}$ be an open set and let $f_n : U \to \mathbb{C}$ be holomorphic functions which converge uniformly on compact sets of U to a function $f : U \to \mathbb{C}$. Then f is holomorphic and $\{f_n'\}_n$ converges uniformly to f' on compact sets of U.*

Proof. By Goursat's theorem,

$$\int_{\partial T} f_n = 0$$

for every n and for every closed triangle $T \subset U$. Letting $n \to \infty$ and using uniform convergence we get

$$\int_{\partial T} f = 0$$

and so by the previous corollary f is holomorphic in every open ball contained in U, which implies that f is holomorphic in U.

To prove the second part of the statement, we use (32) to get

$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f_n(\zeta)}{(\zeta - z)^2} d\zeta,$$

for every $\overline{B(z_0, r)} \subset U$ and every $z \in B(z_0, r)$. If $z \in \overline{B(z_0, \rho)}$, where $0 < \rho < r$, since $|\zeta - z| \geq |\zeta - z_0| - |z_0 - z| \geq r - \rho$,

$$|f'(z) - f_n'(z)| = \left| \int_{\partial B(z_0,r)} \frac{f(\zeta) - f_n(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{2\pi r}{(r-\rho)^2} \|f - f_n\|_{C(\partial B(z_0,r))}$$

and so there is uniform convergence in $\overline{B(z_0, \rho)}$. Since any compact set $K \subset U$ can be covered by a finite number of these balls, we have uniform convergence of $\{f_n'\}_n$ on compact sets of U. ■

Definition 161 *A metric space (X, d) is* separable *if there exists a countable subset that is dense in X.*

Definition 162 *Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F}$ of functions $f : X \to Y$ is said to be* equicontinuous at a point $x_0 \in X$ *if for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that*

$$d_Y(f(x), f(x_0)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x \in X$ with $d(x, x_0) \leq \delta$. The family $\mathcal{F}$ of functions $f : X \to Y$ is said to be uniformly equicontinuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \leq \delta$.

Theorem 163 (Ascoli–Arzelà) *Let (X, d) be a separable metric space and let $\mathcal{F} \subseteq C_b(X)$ be a family of functions. Assume that $\mathcal{F}$ is bounded and equicontinuous at every point $x \in X$. Then every sequence in $\mathcal{F}$ has a subsequence that converges pointwise to a function $g \in C_b(X)$ and uniformly on every compact subset of X.*

Friday, February 28, 2020

Theorem 164 (Montel) *Let $U \subseteq \mathbb{C}$ be an open set and let $\mathcal{F}$ be a family of holomorphic functions defined on U. Assume that for every $K \subset U$ there exists a constant $M_K > 0$ such that*

$$|f(z)| \leq M_K$$

for all $f \in \mathcal{F}$ and for all $z \in K$. Then the family $\mathcal{F}$ is equicontinuous on K and for every sequence in $\mathcal{F}$ there is a subsequence which converges uniformly on compact sets to a holomorphic function $f : U \to \mathbb{C}$.

Proof. Fix a compact set $K \subset U$ and let $d_K := \operatorname{dist}(K, \partial U) > 0$ and let $0 < r < \frac{1}{3} d_K$. Then for $z \in K$, $\overline{B(z, 3r)} \subset U$. Hence, for $z, w \in K$ with $|z - w| < r$ we can apply the Cauchy's theorem to get

$$f(z) - f(w) = \frac{1}{2\pi i} \int_{\partial B(w,2r)} \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta - w} \right) d\zeta.$$

For $\zeta \in \partial B(w, 2r)$ we have $|\zeta - w| = 2r$ and $|\zeta - z| \geq |\zeta - w| - |z - w| \geq 2r - r$. Then

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \left| \frac{z - w}{(\zeta - z)(\zeta - w)} \right| \leq \frac{|z - w|}{2r^2}.$$

Hence,

$$|f(z) - f(w)| \leq \frac{2M_K}{2\pi} \frac{|z - w|}{2r^2} (4\pi r)$$

for all $z, w \in K$ with $|z - w| < r$ and for all $f \in \mathcal{F}$. This shows that the family $\mathcal{F}$ is equicontinuous in K. We can now apply the Ascoli–Arzelà to get that for every sequence in $\mathcal{F}$ there a subsequence converging uniformly on compact sets to a continuous function. By the previous theorem, the function is holomorphic. ■

Exercise 165 *Let $U \subseteq \mathbb{R}^N$ be an open connected set and let $f : U \to \mathbb{R}$ be an analytic function such that f is constant in a ball $B \subseteq U$. Prove that f is constant in U.*

Theorem 166 (Hurwitz) *Let $U \subseteq \mathbb{C}$ be an open set, let $f_n : U \to \mathbb{C}$ be a sequence of functions converging uniformly on compact set to a holomorphic function $f : U \to \mathbb{C}$. Assume that there exists $\overline{B(z_0, r)} \subset U$ such that $f(z) \neq 0$ for all $z \in \partial B(z_0, r)$. Then there exists n_1 such that f_n and f have the same number of zeros in $B(z_0, r)$ for all $n \geq n_1$.*

Proof. By continuity

$$\delta := \min_{\partial B(z_0, r)} |f| > 0.$$

In turn, by uniform convergence on compact sets, there is n_* such that $|f_n(z)| \geq \delta/2$ for all $z \in \partial B(z_0, r)$ and all $n \geq n_*$. It follows that

$$\left| \frac{1}{f_n(z)} - \frac{1}{f(z)} \right| = \frac{|f(z) - f_n(z)|}{|f(z)||f_n(z)|} \leq \frac{2}{\delta^2} |f(z) - f_n(z)|,$$

and so $\{1/f_n\}$ converges uniformly to $1/f$ on $\partial B(z_0, r)$. Moreover, since $f_n' \to f'$ uniformly on compact sets by Theorem 160 it follows that $\frac{f_n'}{f_n} \to \frac{f'}{f}$ uniformly on $\partial B(z_0, r)$, and so

$$\lim_{n \to \infty} \int_{\partial B(z_0, r)} \frac{f_n'(z)}{f_n(z)} dz = \int_{\partial B(z_0, r)} \frac{f'(z)}{f(z)} dz.$$

But by the argument principle (see Theorem 136) the integrals $\int_{\partial B(z_0, r)} \frac{f_n'}{f_n} dz$ and $\int_{\partial B(z_0, r)} \frac{f'}{f} dz$ are the numbers of zeros of f_n and f inside $B(z_0, r)$, and these numbers are finite. Since the limit exists, for n large these values must coincide. ■

The following corollary will be useful to prove the Riemann mapping theorem.

Theorem 167 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $f_n : U \to \mathbb{C}$ be a sequence of injective holomorphic functions converging uniformly on compact set to a holomorphic function $f : U \to \mathbb{C}$. Then either f is injective or constant.*

Proof. Let $z_0 \in U$. Define $g_n(z) = f_n(z) - f_n(z_0)$ and $g(z) := f(z) - f(z_0)$. Assume that there exists $z_1 \neq z_0$ such that $f(z_1) = f(z_0)$. Then g has a zero at z_1. If g is not constant, then since the zeros of g are isolated, we can find $r > 0$ such that $\overline{B(z_1, r)} \subset U$ and $g(z) \neq 0$ for all $z \in \overline{B(z_1, r)} \setminus \{z_1\}$. In particular, we are in a position to apply Hurwitz theorem to conclude that for all n large all functions g_n have a zero in $B(z_1, r)$. But by taking $r > 0$ we can assume that $z_0 \notin \overline{B(z_1, r)}$. Since the functions f_n are injective, they cannot have a zero at z_1, which is a contradiction. ■

An important application of Schottky's theorem is a sharpened version of Montel's theorem. In what follows, given an open set and $f_n : U \to \mathbb{C}$, we say that the sequence $\{f_n\}_n$ *converges uniformly to* ∞ *on compact sets* if for every compact set $K \subset U$ and every $M > 0$ there exists $n_{K,M}$ such that

$$|f_n(z)| \geq M \quad \text{for all } z \in K$$

and all $n \geq n_{K,M}$.

Theorem 168 *Let $U \subseteq \mathbb{C}$ be an open connected set and let $\mathcal{F}$ be the family of holomorphic functions $f : U \to \mathbb{C}$ which do not take the values 0 and 1. Then for every sequence $\{f_n\}_n$ in $\mathcal{F}$ there is a subsequence $\{f_{n_k}\}_k$ such that $\{f_{n_k}\}_k$ converges uniformly on compact sets either to a holomorphic function $f : U \to \mathbb{C}$ or to ∞.*

Proof. Step 1: Let $z_0 \in U$ and $\alpha > 0$ and let

$$\mathcal{F}_{z_0,\delta} := \{f \in \mathcal{F} : |f(z_0)| \leq \alpha\}.$$

We claim that there exist $\delta > 0$ and $M > 0$ such that

$$|f(z)| \leq M$$

for all $z \in B(z_0, \delta)$ and all $f \in \mathcal{F}_{z_0,\delta}$. To see this, let $r > 0$ be so small that $\overline{B(z_0, 2r)} \subset U$. By a dilation and a translation, without loss of generality, we may assume that $z_0 = 0$ and $2r = 1$. Then by Schottky's theorem with $r = 1/2$,

$$|f(z)| \leq \exp(\pi \exp(\pi(3 + \alpha + 12)))$$

for all $z \in B(0, 1/2)$ and all $f \in \mathcal{F}_{z_0,r}$. ■

Proof. Step 2: Fix $z_1 \in U$ and let

$$\mathcal{F}_{z_1,1} := \{f \in \mathcal{F} : |f(z_1)| \leq 1\}.$$

Consider the set $V := \{z \in U : \mathcal{F}_{z_1,1}$ is equibounded in a neighborhood of $z\}$. The set V is open, since if $w \in V$, then there are $B(w,r) \subset U$ and $L > 0$ such that $|f(z)| \leq L$ for all $z \in B(w,r)$ and all $f \in \mathcal{F}_{z_1,1}$. But since $B(z, r - |z-w|) \subset B(w,r)$, it follows that w is an interior point of V, and so V is open. Moreover, V is nonempty in view of Step 1. We claim that $V = U$. If not, then using the previous step there exists $z_2 \in \partial V \cap U$ and a sequence of functions $\{f_n\}_n$ in $\mathcal{F}_{z_1,1}$ such that

$$\lim_{n\to\infty} |f_n(z_2)| = \infty. \tag{64}$$

Define $g_n := 1/f_n$. Then g_n is holomorphic in U and does not take values 0 and 1. Hence, $g_n \in \mathcal{F}$. In view of (64),

$$\lim_{n\to\infty} g_n(z_2) = 0 \tag{65}$$

and so there is $\alpha > 0$ such that $|g_n(z_2)| \leq \alpha$ for all n. In turn, by Step 1, the sequence $\{g_n\}_n$ is equibounded in a neighborhood $B(z_2, r)$ of z_2. It follows by Montel's theorem (Theorem 164) that there exist a subsequence $\{g_{n_k}\}_k$ and a holomorphic function $g : B(z_2, r) \to \mathbb{C}$ such that $g_{n_k} \to g$ uniformly on compact sets of $B(z_2, r)$. In view of (65), $g(z_2) = 0$, but since g_n does not vanish in U, it follows from Hurwitz's theorem (see Theorem 166) that $g \equiv 0$ in $B(z_2, r)$. This implies that $\lim_{n\to\infty} |f_n(z)| = \infty$ for all $z \in B(z_2, r)$. But since $z_2 \in \partial V \cap U$, this implies that there exist points $z \in B(z_2, r) \cap V$ such that $\lim_{n\to\infty} |f_n(z)| = \infty$, which is a contradiction by the definition of V. Hence, the claim holds and so $V = U$.

Step 3: Let $\{f_n\}_n$ be a sequence of functions in $\mathcal{F}$. If there exists countably many n such that $f_n \in \mathcal{F}_{z_1,1}$, say $f_{n_k} \in \mathcal{F}_{z_1,1}$, then by the previous step, the sequence $\{f_{n_k}\}_k$ is locally bounded on compact sets, and thus by Montel's theorem there exists a further subsequence converging uniformly on compact set to a holomorphic function. On the other hand, if only finitely many f_n belong to $\mathcal{F}_{z_1,1}$, then $|f_n(z_1)| > 1$ for all n sufficiently large. In turn, $\frac{1}{f_n} \in \mathcal{F}_{z_1,1}$ for all n sufficiently large. By the previous step and Montel's theorem, there exists a subsequence $\{f_{n_k}\}_k$ and a holomorphic function $g : U \to \mathbb{C}$ such that $\{1/f_{n_k}\}_k$ converges uniformly on compact set to g. If g never vanishes, then $\{f_{n_k}\}_k$ converges uniformly to the holomorphic function $1/g : U \to \mathbb{C}$. If g vanishes at some point, then by Hurwitz's theorem, $g \equiv 0$ (since $1/f_{n_k}$ never vanishes). In turn, $\{f_{n_k}\}_k$ converges uniformly on compact set to ∞. ■

13 Picard's Big Theorem

In this section we prove Picard's big theorem.

Theorem 169 (Picard Big Theorem) *Let $z_0 \in \mathbb{C}$, $r > 0$, and let $f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function having an essential singularity at z_0. Then f takes all possible values of $\mathbb{C}$ with at most a single exception.*

Proof. Without loss of generality we assume that $z_0 = 0$, that $r = 1$. Assume by contradiction that f does not assume two values a and b. By composing f with a linear function, we can assume that f does not take values 0 and 1. Consider the sequences of functions

$$f_n(z) := f(z/n), \quad z \in B(0,1) \setminus \{0\}.$$

In view of the previous theorem, taking $K = \partial B(0, 1/2)$, we can find a subsequence $\{f_{n_k}\}_k$ such that $\{f_{n_k}\}_k$ is equibounded in $\partial B(0, 1/2)$ or $\{1/f_{n_k}\}_k$ is equibounded in $\partial B(0, 1/2)$. In the first case, there exists $M > 0$ such that

$$|f(z/n_k)| \leq M \quad \text{for all } z \in \partial B(0, 1/2)$$

and all k. In turn,

$$|f(w)| \leq M \quad \text{for all } w \in \partial B(0, 1/(2n_k))$$

and all k. It follows by the maximum modulus principle that

$$|f(w)| \leq M \quad \text{for all } 1/(2n_k + 1) < |z| < 1/(2n_k)$$

and for all k. But this implies that f is bounded in a neighborhood of z_0, and so it has a removable singularity at z_0 by Theorem 119, which is a contradiction.

Similarly, if $\{1/f_{n_k}\}_k$ is equibounded in $\partial B(0, 1/2)$, then $1/f$ is bounded in a neighborhood of z_0, which implies that $1/f$ has a removable at z_0, again, by Theorem 119, that is, there exists

$$\lim_{z \to z_0} \frac{1}{f(z)} = \ell \in \mathbb{C}.$$

If $\ell \neq 0$ then f has a removable singularity at z_0, while if $\ell = 0$, then f has a pole at z_0. This is again a contradiction. ∎

Wednesday, March 4, 2020

14 Entire Functions

We begin by reviewing infinite products.

14.1 Infinite Products

Definition 170 *Given a sequence $\{z_n\}_n$ of complex numbers, we say that the* infinite product

$$\prod_{n=1}^{\infty} (1 + z_n)$$

converges *if there exists*

$$\lim_{k\to\infty}\prod_{n=1}^{k}(1+z_n)=\ell\in\mathbb{C}.$$

The following theorem gives a necessary condition for the convergence of an infinite product.

Theorem 171 *Given a sequence* $\{z_n\}_n$ *of complex numbers, if the series* $\sum_{n=1}^{\infty}|z_n|$ *converges, then the infinite product* $\prod_{n=1}^{\infty}(1+z_n)$ *converges. Moreover, the product converges to* 0 *if and only if* $1+a_n=0$ *for some* n.

Proof. By Theorem 20, $\lim_{n\to\infty}z_n=0$, and so there exists $n_1\in\mathbb{N}$ such that $|z_n|<\frac{1}{2}$ for all $n\ge n_1$. By Exercise 36, for $z\in W\cap B(0,1)$,

$$\log_W(1+z)=-\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{n}, \tag{66}$$

where $W=\mathbb{C}\setminus\{z\in\mathbb{C}:\ z=x+0i,\ x\le 0\}$ and $\log_W$ is the principal branch of the logarithm. In particular, if $|z|<\frac{1}{2}$,

$$|\log_W(1+z)|\le\sum_{n=1}^{\infty}\frac{|z|^n}{n}\le\sum_{n=1}^{\infty}|z|^n=\frac{|z|}{1-|z|}\le 2|z|. \tag{67}$$

For $k\ge n_1$ we use (66) to write

$$\prod_{n=n_1}^{k}(1+z_n)=\prod_{n=n_1}^{k}e^{\log_W(1+z_n)}=\exp\left(\sum_{n=n_1}^{n}\log_W(1+z_n)\right).$$

By (67), $|\log_W(1+z_n)|\le 2|z_n|$ and since $\sum_{n=1}^{\infty}|z_n|$ converges, by the comparison test, the series $\sum_{n=n_1}^{\infty}|\log_W(1+z_n)|$ converges. Hence, the series $\sum_{n=n_1}^{\infty}\log_W(1+z_n)$ converges absolutely. In particular, there exists

$$\lim_{k\to\infty}\sum_{n=n_1}^{n}\log_W(1+z_n)=\ell\in\mathbb{C}.$$

By the continuity of the exponential function, there exists

$$\lim_{k\to\infty}\prod_{n=n_1}^{k}(1+z_n)=\lim_{k\to\infty}\exp\left(\sum_{n=n_1}^{n}\log_W(1+z_n)\right)=e^{\ell}.$$

In turn,

$$\prod_{n=1}^{k}(1+z_n)=\prod_{n=1}^{n_1}(1+z_n)\prod_{n=n_1}^{k}(1+z_n)\to\prod_{n=1}^{n_1}(1+z_n)e^{\ell}.$$

This concludes the first part of the proof.

If $1+z_m=0$ for some m, then $\prod_{n=1}^{k}(1+z_n)=0$ for all $k\geq m$ and so the infinite product converges to zero. On the other hand, if $1+z_n\neq 0$ for all n, then by the previous part we have that $\prod_{n=1}^{k}(1+z_n)\to\prod_{n=1}^{n_1}(1+z_n)e^{\ell}=:\ell_1$. Since $e^{\ell}\neq 0$, it follows that $\ell_1\neq 0$. ■

As a corollary of the previous theorem we have the following result.

Theorem 172 *Let $U\subseteq\mathbb{C}$ be an open set and let $f_n:U\to\mathbb{C}$ be holomorphic functions, $n\in\mathbb{N}$. Assume that for each $n\in\mathbb{N}$ there exists $a_n>0$, such that*

$$|f_n(z)-1|\leq a_n \quad \textit{for all } z\in U. \tag{68}$$

If $\sum_{n=1}^{\infty}a_n$ converges, then the infinite product $\prod_{n=1}^{\infty}f_n(z)$ converges uniformly to a holomorphic function $P:U\to\mathbb{C}$. Moreover, if $f_n(z)\neq 0$ for all $z\in U$ and all $n\in\mathbb{N}$, then $P(z)\neq 0$ for all $z\in U$ and

$$\frac{P'(z)}{P(z)}=\sum_{n=1}^{\infty}\frac{f_n'(z)}{f_n(z)} \quad \textit{for all } z\in U.$$

Proof. Let $n_1\in\mathbb{N}$ be such that $a_n<\frac{1}{2}$ for all $n\geq n_1$. In view of (67) and (68),

$$|\log_W f_n(z)|=|\log_W(1+(f_n(z)-1))|\leq 2|f_n(z)-1|\leq 2a_n$$

for all $n\geq n_1$. Taking the supremum over all $z\in U$ gives

$$\sup_U|\log_W f_n(z)|\leq 2a_n$$

and so the series

$$\sum_{n=n_1}^{\infty}\sup_U|\log_W f_n(z)|\leq\sum_{n=n_1}^{\infty}2a_n=R:<\infty.$$

This implies that the series of functions $\sum_{n=n_1}^{\infty}\log_W f_n$ converges uniformly in U and that $\sum_{n=n_1}^{k}\log_W f_n(z)\in B(0,R)$ for all $k\geq n_1$ and all $z\in U$. Since $w\mapsto e^w$ is continuous, it follows that

$$g_k(z):=\prod_{n=n_1}^{k}f_n(z)=\exp\left(\sum_{n=n_1}^{k}\log_W f_n(z)\right)$$

converges uniformly in U to some function $g : U \to \mathbb{C}$, with

$$g(z) = \exp\left(\sum_{n=n_1}^{\infty} \log_W f_n(z)\right). \tag{69}$$

By Theorem 160, g is holomorphic and $g_k' \to g'$ uniformly on compact sets of U.

Define

$$P(z) := g(z)h(z), \quad h(z) := \prod_{n=1}^{n_1} f_n(z),$$

$$P_k(z) := \prod_{n=1}^{k} f_n(z) = g_k(z)h(z)$$

Then

$$\begin{aligned}\sup_U |P_k(z) - P(z)| &= \sup_U |h(z)\,(g_k(z) - g(z))| \\ &= \sup_U |h(z)|\,|g_k(z) - g(z)| \\ &\le L \sup_U |g_k(z) - g(z)| \to 0\end{aligned}$$

as $k \to \infty$, where we used the fact that $|h(z)| \le L$ for all $z \in U$ by (68), with

$$L := \prod_{n=1}^{n_1} (1 + a_n).$$

Next, assume that $f_n(z) \neq 0$ for all $z \in U$ and all $n \in \mathbb{N}$ and fix a compact set $K \subset U$. Since g is the exponential of a holomorphic function $g(z) \neq 0$ for all $z \in U$. In particular, $|g(z)| \ge \delta_0$ for all $z \in K$. Moreover, by assumption $h(z) \neq 0$ for all $z \in U$ and so $|h(z)| \ge \delta_1$ for all $z \in K$. This implies that $|f(z)| \ge \delta_1\delta_0 =: \delta_2$ for all $z \in K$. By uniform convergence we have that

$$|P_k(z)| \ge \frac{1}{2}\delta_1 \quad \text{for all } z \in K \text{ and all } k \ge k_1, \tag{70}$$

where k_1 depends only on K. Since $g_k' \to g'$ uniformly on compact sets and $P_k = hg_k$ then $P_k' = h'g_k + hg_k'$ converges uniformly on compact sets to P'. In turn, by (70), $P_k'/P_k \to P'/P$ uniformly in K. Using (51), we get

$$\frac{P_k'(x)}{P_k(x)} = \sum_{n=1}^{k} \frac{f_n'(x)}{f_n(x)} \to \frac{P'(z)}{P(z)}$$

uniformly in K. In particular,

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \frac{f_n'(x)}{f_n(x)}$$

for all $z \in K$. Since this holds for every compact set $K \subset U$, this concludes the proof. ■

Exercise 173 *Prove that*

$$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Hint: Use Exercise 149.

14.2 Entire Functions of Finite Order

We begin by proving Jensen's formula.

Theorem 174 (Jensen formula) *Let $U \subseteq \mathbb{C}$ be an open set containing 0 and let $f : U \to \mathbb{C}$ be a holomorphic function such that $f(0) \neq 0$. Then for every for closed ball $\overline{B(0,r)} \subset U$ such that f has no zeros on $\partial B(0,r)$, we have*

$$\log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{|z_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta, \tag{71}$$

where $z_1, \ldots, z_n$ are the zeros (if any) of f inside $B(0,r)$ counted with multiplicities. Here, if $n = 0$, we take $\sum_{k=1}^{0} := 0$.

Proof. Step 1: Assume first that f has no zeros inside $B(0,r)$. We claim that

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta. \tag{72}$$

Consider an open ball $B \subseteq U$ containing $\overline{B(0,r)}$. Since B is simply connected, by Corollary 100 there exists a holomorphic function $g : B \to \mathbb{C}$ such that

$$f(z) = e^{g(z)} \quad \text{for all } z \in B.$$

Taking the modulus on both sides we have

$$\begin{aligned}|f(z)| &= |e^{g(z)}| = |e^{\operatorname{Re} g(z) + i \operatorname{Im} g(z)}| = |e^{\operatorname{Re} g(z)} e^{i \operatorname{Im} g(z)}| \\ &= |e^{\operatorname{Re} g(z)}||e^{i \operatorname{Im} g(z)}| = e^{\operatorname{Re} g(z)}\end{aligned}$$

and so $\log|f(z)| = \operatorname{Re} g(z)$. We now apply the mean value formula (37) (see Theorem 110) to $\operatorname{Re} g$, to get (72). ■

Monday, March 16, 2020

No class.

Wednesday, March 16, 2020

Online teaching.

Proof. Step 2: Next assume that $f(z) = z - w_0$ for some $w_0 \in B(0,r) \setminus \{0\}$. We claim that

$$\log|w_0| = \log\left(\frac{|w_0|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|re^{i\theta} - w_0|\,d\theta. \tag{73}$$

Writing $\log\left(\frac{|w_0|}{r}\right) = \log|w_0| - \log r$ and

$$\log|re^{i\theta} - w_0| = \log(r|e^{i\theta} - w_0/r|) = \log r + \log|e^{i\theta} - w_0/r|,$$

we have that formula (73) is equivalent to

$$\begin{aligned} 0 &= \int_0^{2\pi} \log|e^{i\theta} - \zeta_0|\,d\theta = \int_0^{2\pi} \log|e^{-is} - \zeta_0|\,ds \\ &= \int_0^{2\pi} \log|e^{-is} - e^{-is}e^{is}\zeta_0|\,ds = \int_0^{2\pi} \log(|e^{-is}||1 - e^{is}\zeta_0|)\,ds \\ &= \int_0^{2\pi} \log|1 - e^{is}\zeta_0|\,ds, \end{aligned} \tag{74}$$

where $|\zeta_0| < 1$ and we have made the change of variables $\theta = -s$. Since the holomorphic function $h(z) = 1 - z\zeta_0$ does not vanish in $\overline{B(0,1)}$, we can apply Step 1 together with the fact that $h(0) = 1$, to get

$$0 = \log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|1 - e^{is}\zeta_0|\,ds,$$

which proves (73) in view of (74).

Step 3: Let $f_1 : U \to \mathbb{C}$ and $f_2 : U \to \mathbb{C}$ be holomorphic function such that $f_1(0) \neq 0$ and $f_2(0) \neq 0$, and f_1 and f_2 have no zeros on $\partial B(0,r)$. We claim that if f_1 and f_2 satisfy Jensen's formula (71), then so does their product $f_1 f_2$. Let $z_1, \ldots, z_{n_1}$ and $w_1, \ldots, w_{n_2}$ be the zeros of f_1 and f_2 inside $B(0,r)$, respectively. Then $f_1 f_2$ has zeros $z_1, \ldots, z_{n_1}$ and $w_1, \ldots, w_{n_2}$. Moreover,

$$\begin{aligned} \log|(f_1 f_2)(0)| &= \log(|f_1(0)||f_2(0)|) = \log|f_1(0)| + \log|f_2(0)| \\ &= \sum_{k=1}^{n_1} \log\left(\frac{|z_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f_1(re^{i\theta})|\,d\theta \\ &\quad + \sum_{k=1}^{n_2} \log\left(\frac{|w_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f_2(re^{i\theta})|\,d\theta \\ &= \sum_{k=1}^{n_1} \log\left(\frac{|z_k|}{r}\right) + \sum_{k=1}^{n_2} \log\left(\frac{|w_k|}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|(f_1 f_2)(re^{i\theta})|\,d\theta. \end{aligned}$$

Step 4: We are finally ready to prove the general case. Let $f : U \to \mathbb{C}$ be a holomorphic function such that $f(0) \neq 0$ and f has no zeros on $\partial B(0, r)$. Let $z_1, \dots, z_n$ be the zeros of f inside $B(0, r)$ counted with multiplicities. Since the zeros are counted with their multiplicity and are isolated, by Corollary 112 the function

$$q(z) = \frac{f(z)}{(z - z_1) \cdots (z - z_n)}$$

is defined in U, holomorphic, and does not vanish in $\overline{B(0, r)}$. Hence, Jensen's formula (71) holds for q by Step 1. On the other hand, by Step 2 it holds for each function $z \mapsto z - z_k$. Since

$$f(z) = q(z)(z - z_1) \cdots (z - z_n),$$

the conclusion follows from Step 3 and an induction argument. ■

We now define functions of finite order.

Definition 175 *Given an entire function* $f : \mathbb{C} \to \mathbb{C}$ *and* $a > 0$*, we say that* f *has an* order of growth less than or equal a *if there exist constants* $A, B > 0$ *such that*

$$|f(z)| \leq A e^{B|z|^a} \quad \textit{for all } z \in \mathbb{C}. \tag{75}$$

We define the order of growth of f *as* $a_f = \inf a$*, where the infimum is taken over all* $a > 0$ *such that* f *has an order of growth less than or equal to* a*.*

The function $f(z) = e^{z^2}$ has order of growth 2.

Theorem 176 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire function that has an order of growth less than or equal to* $a > 0$*. For every* $r > 0$ *let* $\mathfrak{n}(r)$ *be the number of zeros counted with their multiplicity inside* $B(0, r)$*. Then*

$$\mathfrak{n}(r) \leq C r^a \quad \textit{for all } r \geq 1 \tag{76}$$

and for some constant $C > 0$*. Moreover, if* $\{z_n\}_n$ *are the zeros of* f *different from zero and counted with their multiplicity, then for every* $b > a$*,*

$$\sum_n \frac{1}{|z_n|^b} < \infty. \tag{77}$$

When needed, we write $\mathfrak{n}_f$ for $\mathfrak{n}$ to highlight the dependence on f.

Proof. Step 1: We first show that if $f(0) \neq 0$ and if f does not vanish on $\partial B(0, r)$, then

$$\int_0^r \frac{\mathfrak{n}(s)}{s} \, ds = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta - \log |f(0)|.$$

In view of Jensen's formula, it is enough to show that

$$\int_0^r \frac{\mathfrak{n}(s)}{s} \, ds = \sum_{k=1}^{n} \log \left(\frac{|z_k|}{r} \right),$$

where $z_1, \dots, z_n$ are the zeros of f inside $B(0, r)$ counted with their multiplicity. To see this, observe that

$$\sum_{k=1}^{n} \log\left(\frac{|z_k|}{r}\right) = \sum_{k=1}^{n} \int_{|z_k|}^{r} \frac{1}{s}\, ds.$$

Write

$$\mathfrak{n}(s) = \sum_{k=1}^{n} \chi_{(|z_k|,\infty)}(s).$$

Then

$$\sum_{k=1}^{n} \int_{|z_k|}^{r} \frac{1}{s}\, ds = \sum_{k=1}^{n} \int_{0}^{r} \chi_{(|z_k|,\infty)}(s)\frac{1}{s}\, ds = \int_{0}^{r} \sum_{k=1}^{n} \chi_{(|z_k|,\infty)}(s)\frac{1}{s}\, ds = \int_{0}^{r} \frac{\mathfrak{n}(s)}{s}\, ds,$$

which completes the proof of this step.

Step 2: To prove (76), we first assume that $f(0) \neq 0$. Take $r > 0$ such that f does not vanish on $\partial B(0, 2r)$. Since $\mathfrak{n}$ is increasing,

$$\begin{aligned}\mathfrak{n}(r)\log 2 = \mathfrak{n}(r)\log\frac{2r}{r} = \mathfrak{n}(r)\int_{r}^{2r} \frac{1}{s}\, ds &\le \int_{r}^{2r} \frac{\mathfrak{n}(s)}{s}\, ds. \\ &\le \int_{0}^{2r} \frac{\mathfrak{n}(s)}{s}\, ds = \frac{1}{2\pi}\int_{0}^{2\pi} \log|f(2re^{i\theta})|\, d\theta - \log|f(0)|,\end{aligned}$$

where we used the previous step with r replaced by $2r$. On the other hand by (75),

$$\begin{aligned}\frac{1}{2\pi}\int_{0}^{2\pi} \log|f(2re^{i\theta})|\, d\theta &\le \frac{1}{2\pi}\int_{0}^{2\pi} \log(Ae^{B2^a r^a})\, d\theta \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} [\log A + \log(e^{B2^a r^a})]\, d\theta \\ &= \log A + B2^a r^a.\end{aligned}$$

Combining these inequalities gives

$$\mathfrak{n}(r)\log 2 \le \log A + B2^a r^a.$$

Taking $r \ge 1$ and $C = (\log A + B2^a)/\log 2$, we obtain (76) for all r such that f does not vanish on $\partial B(0, 2r)$. Fix $r \ge 1$. Since the number of zeros in $B(0, 2r+1)$ is finite, we have that f does not vanish on $\partial B(0, 2r+2s)$ for all but finitely many $s \in (0, 1)$. Consider a sequence $s_k \to 0^+$ such f does not vanish on $\partial B(0, 2r + 2s_k)$. By what we just proved and the fact that $\mathfrak{n}$ is increasing,

$$\mathfrak{n}(r) \le \mathfrak{n}(r + s_k) \le C(r + s_k)^a$$

for all k. It suffices to send $k \to \infty$.

Step 3: Next we prove (76), in the case $f(0) = 0$. Assume that ℓ is the multiplicity of 0. Then the function $g(z) := f(z)/z^\ell$ is holomorphic, $\mathfrak{n}_g$ differs from $\mathfrak{n}_f$ by ℓ. Moreover, for $|z| \geq 1$,

$$|g(z)| \leq \frac{|f(z)|}{|z|^\ell} \leq Ae^{B|z|^a}.$$

On the other hand, since g is holomorphic, there exists $A_1 > 0$ such that

$$|g(z)| \leq A_1 \leq A_1 e^{B|z|^a}$$

for all $|z| \leq 1$. Hence, by replacing A with $\max\{A, A_1\}$, we have that g also has an order of growth less than or equal to a. By applying Step 2 to g we get

$$\mathfrak{n}_g(r) \leq Cr^a \quad \text{for all } r \text{ sufficiently large,}$$

say for $r \geq 1$ and for some constant $C \geq 1$. In turn,

$$\mathfrak{n}_f(r) = \mathfrak{n}_g(r) + \ell \leq Cr^a + \ell \leq (C + \ell)r^a.$$

Step 4: We prove (77). If the number of zeros is finite, there is nothing to prove. Thus, we assume that there are infinitely many zeros. Then by (76),

$$\begin{aligned}\sum_{|z_n|\geq 1} \frac{1}{|z_n|^b} &= \sum_{j=0}^\infty \sum_{2^j \leq |z_n| < 2^{j+1}} \frac{1}{|z_n|^b} \leq \sum_{j=0}^\infty \sum_{2^j \leq |z_n| < 2^{j+1}} \frac{1}{2^{jb}} = \sum_{j=0}^\infty \mathfrak{n}_f(2^{j+1}) \frac{1}{2^{jb}} \\ &\leq C \sum_{j=0}^\infty \frac{2^{(j+1)a}}{2^{jb}} = C2^a \sum_{j=0}^\infty \frac{1}{2^{j(b-a)}} < \infty.\end{aligned}$$

Since there are only finitely many zeros in $B(0,1)$, (77) ■

The next example shows that we cannot take b to be the order of growth of f.

Example 177 *Let $f(z) = \sin(\pi z)$. By Euler's identity*

$$f(z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

Hence,

$$|f(z)| \leq e^{\pi |z|},$$

so f has an order of growth less than or equal to 1. Taking $z = -ix$ gives

$$f(ix) = \frac{e^{\pi x} - e^{-\pi x}}{2i},$$

which shows that the order of growth is 1. Note that $f(n) = \sin(\pi n) = 0$ and

$$\sum_{n=1}^\infty \frac{1}{n} = \infty.$$

Friday, March 20, 2020

14.3 Weierstrass Theorem

In this section we show that given a sequence $\{z_n\}_n$ of complex numbers whose moduli converge to infinity, we can construct an entire function which vanishes exactly at each z_n.

Theorem 178 (Weierstrass) *Let $\{z_n\}_n$ be a sequence of complex numbers such that $|z_n| \to \infty$ as $n \to \infty$. Then there exists an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z_n) = 0$ for all n and $f \neq 0$ otherwise. Moreover, any other entire function with the same property is of the form $f(z)e^{g(z)}$, where $g : \mathbb{C} \to \mathbb{C}$ is an entire function.*

The natural choice of f would be

$$f(z) = \prod_{n=1}^{\infty} (1 - z/z_n).$$

However, in general the infinite product will not converge.

Proof. Step 1: Define

$$E_0(z) = 1 - z, \quad E_n(z) = (1 - z)\exp(z + \frac{1}{2}z^2 + \cdots + \frac{1}{n}z^n). \tag{78}$$

We claim that if $|z| \leq 1/2$, then

$$|1 - E_n(z)| \leq 2e|z|^{n+1}.$$

By Exercise 36, for $z \in W \cap B(0, 1)$,

$$\log_W(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k},$$

where $W = \mathbb{C} \setminus \{z \in \mathbb{C} : \ z = x + 0i, \ x \leq 0\}$ and $\log_W$ is the principal branch of the logarithm. Writing $1 - z = e^{\log_W(1-z)}$, we have

$$\begin{aligned} E_n(z) &= \exp\left(\log_W(1 - z) + z + \frac{1}{2}z^2 + \cdots + \frac{1}{n}z^n\right) && (79) \\ &= \exp\left(-\sum_{k=n+1}^{\infty} \frac{z^k}{k}\right) =: e^w. \end{aligned}$$

In particular, if $|z| \leq \frac{1}{2}$,

$$|w| = \left|\sum_{k=n+1}^{\infty} \frac{z^k}{k}\right| \leq |z|^{n+1} \sum_{k=n+1}^{\infty} \frac{|z|^{k-n-1}}{k} \leq |z|^{n+1} \sum_{j=0}^{\infty} |z|^j \leq |z|^{n+1} \sum_{j=0}^{\infty} \frac{1}{2^j} = 2|z|^{n+1} \leq 1. \tag{80}$$

Hence,

$$|1 - E_n(z)| = |1 - e^w| = \left|\sum_{k=1}^{\infty} \frac{w^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|w|^k}{k!} = |w| \sum_{k=1}^{\infty} \frac{|w|^{k-1}}{k!}$$
$$\leq |w| \sum_{k=1}^{\infty} \frac{1}{k!} = |w|(e-1) \leq 2(e-1)|z|^{n+1},$$

which proves the claim for $z \in W \cap \overline{B(0,1/2)}$. For $z \in \overline{B(0,1/2)}$ we can use the fact that E_n and $|z|^{n+1}$ are continuous functions.

Step 2: We are now ready to construct the function f. Since $|z_n| \to \infty$, by relabelling the sequence, we can assume that

$$|z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq |z_{n+1}|$$

for all n. If 0 is one of the numbers z_n with multiplicity ℓ we define

$$f(z) = z^\ell \prod_{n=1}^{\infty} E_n(z/z_n),$$

while if zero is not, we take $\ell = 0$ and set $z^0 := 1$ in the previous definition. Fix $r > 0$ and consider $z \in B(0,r)$. Let $n_1 > 1$ be such that $|z_n| \geq 2r$ for all $n \geq n_1$. Then $|z/z_n| \leq 1/2$ and so by the previous step

$$|1 - E_n(z/z_n)| \leq 2e|z/z_n|^{n+1} \leq 2e/2^{n+1}.$$

Since the series $\sum_{n=n_1}^{\infty} \frac{e}{2^n}$ converges, by Theorem 172, the infinite product $\prod_{n=n_1}^{\infty} E_n(z/z_n)$ converges uniformly to a holomorphic function $P : B(0,r) \to \mathbb{C}$. Moreover, since $E_n(z/z_n)$ vanishes only at z_n, we have that if $E_n(z/z_n) \neq 0$ for all $z \in B(0,r)$ and all $n \geq n_1$. Thus, again by Theorem 172, $P(z) \neq 0$ for all $z \in B(0,r)$. Since

$$f(z) = z^\ell \prod_{n=1}^{n_1 - 1} E_n(z/z_n) P(z),$$

we have that f is holomorphic in $B(0,r)$. Moreover, since $P \neq 0$ in $B(0,R)$, $E_n(z/z_n)$ vanishes only at z_n, we have that f vanishes only at those z_n, $n = 1, \ldots, n_1 - 1$, which are inside $B(0,r)$. By the arbitrariness of $r > 0$ this concludes the first part of the proof.

Step 3: Let $h : \mathbb{C} \to \mathbb{C}$ be an entire function such that $h(z_n) = 0$ for all n and $h(z) \neq 0$ otherwise. If w_k is a zero of h and f with multiplicity m_k, by Corollary 112 applied f and h we can write

$$h(z) = (z - w_k)^{m_k} h_1(z), \quad f(z) = (z - w_k)^{m_k} f_1(z),$$

where h_1 and f_1 are holomorphic functions in some ball $B(w_k, r_k)$ which do not vanish in $B(w_k, r_k)$. Hence,

$$\frac{h(z)}{f(z)} = \frac{h_1(z)}{f_1(z)} \quad \text{for all } z \in B(w_k, r_k) \setminus \{w_k\}.$$

This shows that h/f has a removable singularity at w_k and does not vanishes in $B(w_k, r_k)$. By the arbitrariness of the zero w_k and the fact that the zeros are isolated, we have shown that h/f can be extended to $\mathbb{C}$ as a holomorphic function which vanishes nowhere. We now apply Corollary 100 to write $h/f = e^g$ for some entire function $g : \mathbb{C} \to \mathbb{C}$. This concludes the proof. ■

The functions E_n are called *canonical factors* and n the *degree* of the canonical factor.

Corollary 179 *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then the following hold:*

(i) if $f(z) \neq 0$ for all $z \in \mathbb{C}$, then there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$,

(ii) if f has finitely many zeros $z_1, \dots, z_n$ counted with their multiplicity, then there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = (z - z_1) \cdots (z - z_n) e^{g(z)}$$

for all $z \in \mathbb{C}$,

(iii) if f has infinitely many zeros $\{z_n\}_n$ counted with their multiplicity, then there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = z^\ell \prod_{n=1}^{\infty} E_n(z/z_n) e^{g(z)}$$

for all $z \in \mathbb{C}$.

Proof. Item (i) is Corollary 100. Items (ii) and (iii) follow as in Step 3 of the previous proof. ■

Note that Weierstrass theorem shows that any entire function with infinitely many zeros can be written as the product of the function constructed by Weierstrass and an exponential function. Thus, it provides a way to represent entire functions. This is why this theorem is called *Weierstras representation theorem.*

The next theorem shows that if f has finite order of growth, then the function g in the exponential is a polynomial.

Theorem 180 (Hadamard) *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function which has growth order a and infinitely many zeros z_n. Then*

$$f(z) = e^{p(z)} z^\ell \prod_n E_n(z/z_n),$$

where p is a polynomial of degree less than or equal to $\lfloor a \rfloor$, $\ell \in \mathbb{N}_0$ is the order of the zero of f at $z = 0$.

15 Prime Number Theorem

Throughout this section p denotes a prime number.

Theorem 181 (Prime Number Theorem) *Given $x \in \mathbb{R}$, let $\pi(x)$ be the number of prime numbers which are less than or equal to x. Then*

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

Consider the

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C},\ \operatorname{Re} z > 1. \tag{81}$$

This function is called the *Riemann zeta function.*

Lemma 182 *The function ζ converges absolutely and uniformly on compact sets of $U := \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. Moreover,*

$$\zeta(z) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}}, \quad z \in U.$$

In particular, ζ has no zeros in U.

Proof. We have

$$|n^z| = |e^{z \log n}| = e^{(\operatorname{Re} z) \log n} = n^{\operatorname{Re} z}$$

and so if $\operatorname{Re} z \geq 1 + \varepsilon$, with $\varepsilon > 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{|n^z|} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} < \infty,$$

which implies that there is uniform and absolute convergence in the set $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1 + \varepsilon\}$. In particular, there is absolute convergence in U. Hence, we can rearrange terms in the series. ■

Monday, March 23, 2020

Proof. Let $\{p_n\}_n$ be the ordered sequence of prime numbers. For each $\ell \in \mathbb{N}$ let S_ℓ be the set of all natural numbers which are not divisible by $p_1, \ldots,$ p_ℓ. We claim that

$$\xi(z)\prod_{l=1}^{\ell}\left(1-\frac{1}{p_l^z}\right)=\sum_{n\in S_\ell}\frac{1}{n^z}. \tag{82}$$

For $\ell = 1$ we have $p_1 = 2$ and so

$$\xi(z)\left(1-\frac{1}{2^z}\right)=\sum_{n=1}^{\infty}\frac{1}{n^z}-\sum_{n=1}^{\infty}\frac{1}{(2n)^z}=\sum_{n\in S_1}\frac{1}{n^z},$$

since we removed all the even natural numbers. Hence, the base case $\ell = 1$ is true. Next assume that the claim holds for ℓ and let's prove it for $\ell + 1$. By the induction hypothesis,

$$\xi(z)\prod_{l=1}^{\ell}\left(1-\frac{1}{p_l^z}\right)=\sum_{n\in S_\ell}\frac{1}{n^z}.$$

Multiply both sides by $1-\frac{1}{p_{\ell+1}^z}$ to get

$$\begin{aligned}\xi(z)\prod_{l=1}^{\ell+1}\left(1-\frac{1}{p_l^z}\right)&=\left(1-\frac{1}{p_{\ell+1}^z}\right)\sum_{n\in S_\ell}^{\infty}\frac{1}{n^z}\\&=\sum_{n\in S_\ell}\frac{1}{n^z}-\sum_{n\in S_\ell}\frac{1}{(p_{\ell+1}n^z)}=\sum_{n\in S_{\ell+1}}\frac{1}{n^z},\end{aligned}$$

which proves the claim.

Letting $\ell \to \infty$ in (82) gives

$$\xi(z)\prod_{l=1}^{\infty}\left(1-\frac{1}{p_l^z}\right)=\lim_{\ell\to\infty}\sum_{n\in S_\ell}\frac{1}{n^z}=1,$$

where we used the fact that $S_{\ell+1} \subset S_\ell$ and $\bigcap_{\ell=1}^{\infty} S_\ell = \{1\}$.

The last part of the statement follows from Theorem 172 and the fact that $\frac{1}{1-p^{-z}} \neq 0$ for all $z \in U$. ■

Exercise 183 *Let $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$. Prove that*

$$\int_1^{\infty}\frac{1}{t^z}\,dt=\frac{1}{z-1},\quad \int_n^{x}\frac{z}{t^{z+1}}\,dt=\frac{1}{n^z}-\frac{1}{x^z}$$

for every $n \in \mathbb{N}$.

Lemma 184 *The function $z \mapsto \zeta(z) - \frac{1}{z-1}$ can be extended as an holomorphic function to the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

Proof. By the previous exercise

$$\zeta(z) - \frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{n^z} - \int_1^{\infty} \frac{1}{t^z}\,dt = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^z} - \frac{1}{t^z}\right) dt$$
$$= \sum_{n=1}^{\infty} \int_n^{n+1} \int_n^x \frac{z}{s^{z+1}}\,dsdt.$$

Note that

$$\left|\int_n^{n+1} \int_n^x \frac{z}{s^{z+1}}\,dsdt\right| \le \int_n^{n+1} \int_n^{n+1} \left|\frac{z}{s^{z+1}}\right| dsdt$$
$$\le |z| \max_{s\in[n,n+1]} \frac{1}{|s|^{\operatorname{Re} z+1}} = |z| \frac{1}{n^{\operatorname{Re} z+1}}.$$

Hence, the series $\sum_{n=1}^{\infty} \int_n^{n+1} \int_n^x \frac{z}{s^{z+1}}\,dsdt$ is absolutely convergent for every $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. ■

In view of the previous lemma, the Riemann zeta function can be extended as a meromorphic function to $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ with a simple pole in $z = 1$ and no other poles. Next we study the zeros of ζ. The *Riemann hypothesis* is the conjecture that all zeros of ζ lie on the line $\operatorname{Re} z = \frac{1}{2}$.

The following lemma shows that there are no zeros for $\operatorname{Re} z \ge 1$.

Lemma 185 *The Riemann zeta ζ has no zeros in $\{z \in \mathbb{C} : \operatorname{Re} z = 1\}$.*

Proof. Step 1: Let $U := \{z \in \mathbb{C} : \operatorname{Re} z > 1\}$. Since ζ has no zeros in U, using Lemma 182 and Theorem 172,

$$\frac{\zeta'(z)}{\zeta(z)} = \sum_{p \text{ prime}} \frac{\left(\frac{1}{1-p^{-z}}\right)'}{\frac{1}{1-p^{-z}}} = -\sum_{p \text{ prime}} \frac{\frac{p^{-z}\log p}{(1-p^{-z})^2}}{\frac{1}{1-p^{-z}}} = -\sum_{p \text{ prime}} \frac{p^{-z}\log p}{1-p^{-z}}, \tag{83}$$

where we used the fact that $p^z = e^{z\log p}$ and so $(p^z)' = e^{z\log p}\log p = p^z \log p$. Using the geometric series we have that

$$\frac{1}{1-p^{-z}} = \sum_{k=0}^{\infty} p^{-kz}.$$

Hence,

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{p \text{ prime}} \sum_{k=0}^{\infty} p^{-(k+1)z}\log p = -\sum_{p \text{ prime}} \sum_{n=1}^{\infty} p^{-nz}\log p.$$

Step 2: Assume that $\zeta(1 + iy) = 0$ and consider the function

$$g(z) := (\zeta(z))^3(\zeta(z + iy))^4\zeta(z + 2iy).$$

Note that ζ has a simple pole at $z = 1$, so $(\zeta(z))^3 \sim \frac{c_0}{(z-1)^3}$, while $\zeta_1(z) := \zeta(z + iy)$ has a zero of order $n \in \mathbb{N}$ at $z = 1$ so $(\zeta(z + iy))^4 \sim c_1(z-1)^4$, and $\zeta_2(z) := \zeta(z + 2iy)$ may have a zero of order $m \in \mathbb{N}_0$ at $z = 1$, so $\zeta(z + 2iy) \sim c_2(z-1)^m$. It follows that

$$g(z) \sim \frac{c}{(z-1)^3}(z-1)^{4n}(z-1)^m = c(z-1)^{4n+m-3}$$

as $z \to 1$. Thus g has a zero at $z = 1$ of order $4n + m - 3 \geq 1$. Hence,

$$g(z) = (z-1)^{4n+m-3}h(z),$$

where h is holomorphic near $z = 1$ and $h(1) \neq 0$. In turn, by (51),

$$\begin{aligned}\frac{g'(z)}{g(z)} &= \frac{(4n+m-3)(z-1)^{4n+m-4}}{(z-1)^{4n+m-3}} + \frac{h'(z)}{h(z)} \\ &= \frac{4n+m-3}{z-1} + \frac{h'(z)}{h(z)}\end{aligned}$$

and so

$$\lim_{z\to 1}(z-1)\frac{g'(z)}{g(z)} = 4n + m - 3 > 0. \tag{84}$$

On the other hand, for $z \in \mathbb{C}$ with $\operatorname{Re} z > 1$, by (51) and the previous step,

$$\begin{aligned}\frac{g'(z)}{g(z)} &= 3\frac{(\zeta(z))^2\zeta'(z)}{(\zeta(z))^3} + 4\frac{(\zeta(z+iy))^3\zeta'(z+iy)}{(\zeta(z+iy))^4} + \frac{\zeta'(z+2iy)}{\zeta(z+2iy)} \\ &= 3\frac{\zeta'(z)}{\zeta(z)} + 4\frac{\zeta'(z+iy)}{\zeta(z+iy)} + \frac{\zeta'(z+2iy)}{\zeta(z+2iy)} \\ &= -3\sum_{p \text{ prime}}\sum_{n=1}^{\infty} p^{-nz}\log p - 4\sum_{p \text{ prime}}\sum_{n=1}^{\infty} p^{-nz}p^{-nyi}\log p - \sum_{p \text{ prime}}\sum_{n=1}^{\infty} p^{-kz}p^{-2nyi}\log p \\ &= -\sum_{p \text{ prime}}\sum_{n=1}^{\infty}(3 + 4p^{-nyi} + p^{-2nyi})p^{-nz}\log p.\end{aligned}$$

Taking $z = x > 1$ we have that

$$\begin{aligned}\operatorname{Re}\frac{g'(x)}{g(x)} &= -\sum_{p \text{ prime}}\sum_{n=1}^{\infty}(\operatorname{Re}(3 + 4p^{-nyi} + p^{-2nyi}))p^{-nx}\log p. \\ &= -\sum_{p \text{ prime}}\sum_{n=1}^{\infty}(3 + 4\cos(ny) + \cos(2ny))p^{-nx}\log p.\end{aligned}$$

Since $\cos(2\theta) = 2\cos^2\theta - 1$ we have that

$$3+4\cos\theta+\cos(2\theta) = 3+4\cos\theta+2\cos^2\theta-1 = 2(1+2\cos\theta+\cos^2\theta) = 2(1+\cos\theta)^2.$$

Hence,

$$\lim_{x\to 1^+}(x-1)\operatorname{Re}\frac{g'(x)}{g(x)} \le 0,$$

which contradicts (84). ■

Wednesday, March 25, 2020

The following theorem is of independent interest.

Theorem 186 *Let $f:[0,\infty)\to\mathbb{C}$ be bounded and locally integrable and let*

$$g(z) := \int_0^\infty f(t)e^{-tz}dt, \quad \operatorname{Re} z > 0.$$

Assume that for every $z\in\mathbb{C}$ with $\operatorname{Re} z = 0$ there exists $r_z > 0$ such that g can be extended holomorphically to $B(z,r_z)$. Then the generalized Riemann integral

$$\int_0^\infty f(t)\,dt \tag{85}$$

is well-defined and equals $g(0)$.

Proof. Using Corollary 115 and a compactness argument for every $R>1$ we can find $\delta=\delta(R)\in(0,\frac{1}{2})$ and $M=M(R)>0$ such that g can be extended to a holomorphic function g in an open set U_R containing the set $C_R := \overline{B(0,R)}\cap\{z\in\mathbb{C} : \operatorname{Re} z\ge -\delta\}$ and $|g(z)|\le M$ for every $z\in C_R$. Consider the counterclockwise contour γ given by the intersection of $\partial B(0,R)$ and the segment $\operatorname{Re} z=-\delta$, $|z|\le R$. Also denote by γ_+ and γ_- the parts of γ in the right half-plane $\operatorname{Re} z\ge 0$ and in the left half-plane $\operatorname{Re} z\le 0$, respectively. Let Γ_+ and Γ_- be their ranges. Let $T>0$ and consider the function

$$h_T(z) := g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right), \quad z\in U_R\setminus\{0\}.$$

If $g(0)\neq 0$, the function h_T has only one pole at 0 with residue $\operatorname{res}_0 h_T = g(0)$, while if $g(0)=0$, then h_T is holomorphic in U_R. It follows by the residue's formula

$$\begin{aligned}2\pi i g(0) = 2\pi i \operatorname{res}_0 h_T &= \int_\gamma h_T\,dz = \int_\gamma g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz \\ &= \int_{\gamma_+} g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz + \int_{\gamma_-} g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz.\end{aligned} \tag{86}$$

If z belongs to the range of γ_+, then by (85), we can write

$$g(z) = \int_0^T f(t)e^{-tz}dt + \int_T^\infty f(t)e^{-tz}dt =: S_T(z)+R_T(z). \tag{87}$$

Consider the function

$$q_T(z) := S_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right), \quad z\in B(0,R+1)\setminus\{0\}.$$

Again by the residue's formula

$$
\begin{aligned}
2\pi i S_T(0) = 2\pi i \operatorname{res}_0 q_T &= \int_{\partial B(0,R)} q_T\, dz = \int_{\partial B(0,R)} S_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz \\
&= \int_{\gamma_+} S_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz + \int_{\partial B(0,R)\setminus\Gamma_+} S_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz \\
&= \int_{\gamma_+} S_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz + \int_{\gamma_+} S_T(-w)e^{-wT}\left(\frac{1}{w}+\frac{w}{R^2}\right) dw
\end{aligned}
\tag{88}
$$

where we have made the change of variable $z = -w$. Subtracting (88) from (86), and using (87) gives

$$
\begin{aligned}
2\pi i(g(0) - S_T(0)) = &\int_{\gamma_+} R_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz \\
&- \int_{\gamma_+} S_T(-z)\frac{1}{e^{zT}}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz \\
&+ \int_{\gamma_-} g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R^2}\right) dz =: I + II + III.
\end{aligned}
\tag{89}
$$

We now estimate I, II, and II. Let $z = x + iy$ with $x > 0$. Since f is bounded, say, $|f(t)| \le L$ for all $t \in [0, \infty)$, we have

$$
|R_T(z)| \le \int_T^\infty |f(t)||e^{-tz}|dt \le C\int_T^\infty e^{-tx}dt = \left[-\frac{1}{x}e^{-tx}\right]_{t=T}^{t\to\infty} = \frac{e^{-Tx}}{x}.
$$

On the other hand, for $z \in \partial B(0,R)$, we have that

$$
\begin{aligned}
\frac{1}{z} + \frac{z}{R^2} &= \frac{\bar z}{z\bar z} + \frac{z}{R^2} \\
&= \frac{\bar z}{R^2} + \frac{z}{R^2} = \frac{\operatorname{Re} z}{R^2} = \frac{x}{R^2}
\end{aligned}
\tag{90}
$$

In turn, for $z \in \Gamma_+$,

$$
\left|R_T(z)e^{zT}\right|\left|\frac{1}{z}+\frac{z}{R^2}\right| \le \frac{e^{-Tx}}{x}e^{xT}\frac{x}{R^2} = \frac{1}{R^2}.
$$

Hence,

$$
|I| \le \frac{1}{R^2}\pi R = \frac{\pi}{R}. \tag{91}
$$

Similarly,

$$
|S_T(-z)| \le \int_0^T |f(t)||e^{tz}|dt \le C\int_0^T e^{tx}dt = \left[\frac{1}{x}e^{tx}\right]_{t=0}^{t=T} = \frac{e^{Tx}-1}{x} \le \frac{e^{Tx}}{x}.
$$

In turn, for $z \in \Gamma_+$,

$$\left| S_T(-z)\frac{1}{e^{zT}} \right| \left| \frac{1}{z} + \frac{z}{R^2} \right| \leq \frac{e^{Tx}}{x}\frac{1}{e^{Tx}}\frac{x}{R^2} = \frac{1}{R^2}.$$

It follows that

$$|II| \leq \frac{\pi R}{R^2} = \frac{\pi}{R}. \tag{92}$$

It remains to estimate III. Along the segment Σ given by $\operatorname{Re} z = -\delta$, $|z| \leq R$ we have $z = -\delta + iy$ and so

$$\left| \frac{1}{z} + \frac{z}{R^2} \right| \leq \frac{1}{|z|} + \frac{|z|}{R^2} \leq \frac{1}{\delta} + \frac{1}{R}.$$

Since $|g(z)| \leq M$ for all $z \in C_R$, In turn,

$$\left| g(z)e^{zT} \right| \left| \frac{1}{z} + \frac{z}{R^2} \right| \leq Me^{-\delta T}\left(\frac{1}{\delta} + \frac{1}{R} \right)$$

and so

$$\begin{aligned} \left| \int_\Sigma g(z)e^{zT}\left(\frac{1}{z} + \frac{z}{R^2} \right) dz \right| &\leq Me^{-\delta T}\left(\frac{1}{\delta} + \frac{1}{R} \right)\int_{-R}^{R} 1\, dy \qquad (93) \\ &= Me^{-\delta T}\left(\frac{2R}{\delta} + 2 \right). \end{aligned}$$

■

Friday, March 27, 2020

Proof. On the other hand, on $\gamma_- \setminus \Sigma$, we have $x = \operatorname{Re} z \leq 0$ and $|z| = R$. Using (90) we have

$$\left| g(z)e^{zT} \right| \left| \frac{1}{z} + \frac{z}{R^2} \right| \leq Me^{xT}\frac{|x|}{R^2}.$$

Since $-\delta \leq x \leq 0$ we can parametrize these two arcs by $\varphi(x) = x + \pm i\sqrt{R^2 - x^2}$. Then

$$|\varphi'(x)| = \sqrt{1 + \frac{x^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}} \leq \frac{R}{\sqrt{R^2 - \delta^2}} \leq \frac{1}{2}$$

since $R^2 \geq 1 > \frac{1}{4} \geq \delta^2$. Hence,

$$\begin{aligned} \left| \int_{\gamma_- \setminus \Sigma} g(z)e^{zT}\left(\frac{1}{z} + \frac{z}{R^2} \right) dz \right| &\leq \int_{-\delta}^{0} Me^{xT}\frac{|x|}{R^2}dx \\ &= \frac{M}{R^2}\int_0^\delta e^{-tT}t\, dt = \frac{M}{R^2}\left[-\frac{1}{T^2}e^{-Tt}\left(Tt + 1 \right) \right]_{t=0}^{t=\delta} \\ &\leq \frac{M}{R^2}\left(\frac{1}{T^2} - \frac{1}{T^2}e^{-T\delta}\left(T\delta + 1 \right) \right). \end{aligned}$$

Together with (93) this shows that

$$|III| \leq MT^{-\delta}\left(\frac{2R}{\delta}+2\right)+\frac{M}{R^2T^2}.$$

Combining this inequality with (91) and (92), it follows from (89) that

$$|2\pi i(g(0)-S_T(0))| \leq \frac{\pi}{R}+\frac{\pi}{T}+\frac{\pi}{R}+Me^{-\delta T}\left(\frac{2R}{\delta}+2\right)+\frac{M}{R^2T^2}.$$

We now choose $R=\frac{1}{\varepsilon}$. This determines $\delta=\delta(\varepsilon)$ and $M=M(\varepsilon)$. Since

$$\lim_{T\to\infty}\left[\frac{\pi}{T}+Me^{-\delta T}\left(\frac{2R}{\delta}+2\right)+\frac{M}{R^2T^2}\right]=0,$$

taking T sufficiently large, we have that

$$|2\pi i(g(0)-S_T(0))| \leq 2\pi\varepsilon+\varepsilon,$$

which proves that $S_T(0)\to g(0)$ as $T\to\infty$. ■

Define

$$\theta(x) := \sum_{p \text{ prime}\leq x} \log p, \quad x\in\mathbb{R}.$$

Theorem 187 *The generalized Riemann integral*

$$\int_1^\infty \frac{\theta(x)-x}{x^2}dx$$

converges. In turn,

$$\lim_{x\to\infty}\frac{\theta(x)}{x}=1. \tag{94}$$

Proof. Step 1: We claim that there exists a constant $C>0$ such that

$$|\theta(x)|\leq Cx$$

for all $x>0$ sufficiently large. For $n\in\mathbb{N}$, by the binomial theorem

$$\begin{aligned}
2^{2n} &= (1+1)^{2n} = \binom{2n}{0}+\cdots+\binom{2n}{n} \geq \binom{2n}{n} = \frac{(2n)!}{(n!)^2}\\
&= \prod_{k=0}^{n-1}\frac{(2n-k)}{n!} \geq \prod_{n<p\leq 2n} p = \exp\log\left(\prod_{n<p\leq 2n} p\right) = \exp\log\frac{\prod_{p\leq 2n} p}{\prod_{p\leq n} p}\\
&= \exp\left(\sum_{p\leq 2n}\log p - \sum_{p\leq n}\log p\right) = e^{\theta(2n)-\theta(n)},
\end{aligned}$$

where in the last inequality we used the fact that $\binom{2n}{n}$ is an integer (this can be proved by induction). Taking logarithms on both sides gives

$$2n \log 2 \geq \theta(2n) - \theta(n).$$

Hence for $m \in \mathbb{N}$,

$$\theta(2^m) = \sum_{n=1}^{m} (\theta(2^n) - \theta(2^{n-1})) \leq \log 2 \sum_{n=1}^{m} 2^n = (2^{m+1} - 2) \log 2 < 2^{m+1} \log 2.$$

Given $x > 1$ find $m \in \mathbb{N}$ such that $2^{m-1} \leq x < 2^m$. Since θ is increasing,

$$\theta(x) \leq \theta(2^m) \leq 2^{m+1} \log 2 \leq x 4 \log 2,$$

which proves the claim.

Step 2: Observe that in view of the previous step, for $\operatorname{Re} z > -1$ the integral $\int_0^\infty e^{-(z+1)t} \theta(e^t)\, dt$ is well-defined. Indeed,

$$|e^{-(z+1)t}| = e^{-t(\operatorname{Re} z + 1)}.$$

Let p_n be the n-th prime number. If $p_n < e^t < p_{n+1}$, then

$$\theta(e^t) = \sum_{p \text{ prime} \leq e^t} \log p = \theta(p_n),$$

or equivalently, $\theta(e^t) = \theta(p_n)$ for all $\log p_n < t < \log p_{n+1}$. Also $\theta(e^t) = 0$ for $0 < t < \log 2 = \log p_1$. Hence, for $\operatorname{Re} z > -1$,

$$\begin{aligned}
\int_0^\infty e^{-(z+1)t} \theta(e^t)\, dt &= \sum_{n=1}^{\infty} \int_{\log p_n}^{\log p_{n+1}} e^{-(z+1)t} \theta(e^t)\, dt = \sum_{n=1}^{\infty} \theta(p_n) \int_{\log p_n}^{\log p_{n+1}} e^{-(z+1)t} dt \\
&= \sum_{n=1}^{\infty} \theta(p_n) \left[-\frac{e^{-(z+1)t}}{z+1} \right]_{t=\log p_n}^{t=\log p_{n+1}} = \frac{1}{z+1} \sum_{n=1}^{\infty} \theta(p_n) \left[p_n^{-(z+1)} - p_{n+1}^{-(z+1)} \right] \\
&= \frac{1}{z+1} \sum_{n=1}^{\infty} \theta(p_n) p_n^{-(z+1)} - \frac{1}{z+1} \sum_{k=2}^{\infty} \theta(p_{k-1}) p_k^{-(z+1)} \\
&= \frac{1}{z+1} 2^{-(z+1)} \log 2 + \frac{1}{z+1} \sum_{n=2}^{\infty} (\theta(p_n) - \theta(p_{n-1})) p_n^{-(z+1)} \\
&= \frac{1}{z+1} 2^{-(z+1)} \log 2 + \frac{1}{z+1} \sum_{n=2}^{\infty} p_n^{-(z+1)} \log p_n = \frac{\Phi(z+1)}{z+1},
\end{aligned}$$

where in the second to last equality we used the fact that $\theta(p_n) - \theta(p_{n-1}) = \log p_n$ and we set $k = n+1$ and where

$$\Phi(z) := \sum_{p \text{ prime}} \frac{\log p}{p^z}, \quad z \in \mathbb{C},\ \operatorname{Re} z > 1.$$

■

Proof. Step 3: We prove that the function $z \mapsto \Phi(z) - \frac{1}{z-1}$ can be extended as a meromorphic function to the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 1/2\}$ and is holomorphic for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 1$. Using the identity

$$\frac{1}{p^z - 1} = \frac{1}{p^z} + \frac{1}{p^z(p^z - 1)}$$

by (83) we can write

$$\begin{aligned} -\frac{\zeta'(z)}{\zeta(z)} &= \sum_{p \text{ prime}} \frac{\log p}{p^z - 1} = \sum_{p \text{ prime}} \frac{\log p}{p^z} + \sum_{p \text{ prime}} \frac{\log p}{p^z(p^z - 1)} \\ &= \Phi(z) + \sum_{p \text{ prime}} \frac{\log p}{p^z(p^z - 1)}. \end{aligned}$$

Note that for $\operatorname{Re} z > \frac{1}{2}$, and $p > 4$,

$$|p^z - 1| \geq |p^z| - 1 \geq p^{\operatorname{Re} z} - 1 \geq \frac{1}{2} p^{\operatorname{Re} z}$$

and so

$$\left| \frac{\log p}{p^z(p^z - 1)} \right| \leq \frac{2 \log p}{p^{2 \operatorname{Re} z}}.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{\log n}{n^{2 \operatorname{Re} z}}$$

converges, the series $\sum_{p \text{ prime}} \frac{\log p}{p^z(p^z-1)}$ is absolutely convergent for $\operatorname{Re} z > \frac{1}{2}$. Moreover, by Lemma 184, $\frac{\zeta'(z)}{\zeta(z)}$ is a meromorphic function for $\operatorname{Re} z > 0$. Hence,

$$\Phi(z) = -\frac{\zeta'(z)}{\zeta(z)} - \sum_{p \text{ prime}} \frac{\log p}{p^z(p^z - 1)}$$

can be extended as a meromorphic function to $\operatorname{Re} z > \frac{1}{2}$ with poles at $z = 1$ and at the zeros of ζ.

Step 4: Consider the continuous bounded function

$$f(t) = e^{-t}\theta(e^t) - 1.$$

By Step 2 for $\operatorname{Re} z > 0$, we have that

$$\int_0^{\infty} f(t) e^{-tz} dt = \int_0^{\infty} e^{-t(z+1)} \theta(e^t)\, dt - \int_0^{\infty} e^{-tz}\, dt = \frac{\Phi(z+1)}{z+1} - \frac{1}{z}.$$

It follows from Step 3 that $\frac{\Phi(z+1)}{z+1} - \frac{1}{z}$ can be extended to a meromorphic function g for $\operatorname{Re} z > -\frac{1}{2}$, which is holomorphic for $\operatorname{Re} z \geq 0$. Hence, we are in a position to apply Theorem 186 to conclude that the integral $\int_0^\infty f(t)\,dt$ is well-defined and

$$\int_0^\infty (e^{-t}\theta(e^t)-1)\,dt = \int_0^\infty f(t)dt = g(0).$$

By considering the change of variables $x = e^t$, that is $\log x = t$, so that $\frac{1}{x}dx = dt$ we have that

$$\int_1^\infty \frac{\theta(x)-x}{x^2}dx = \int_0^\infty (e^{-t}\theta(e^t)-1)\,dt = g(0),$$

which proves the first part of the statement.

Step 5: We prove (94). Assume by contradiction that

$$\limsup_{x\to\infty} \frac{\theta(x)}{x} > 1.$$

There there exists an increasing sequence $x_n \to \infty$ such that $\theta(x_n) > (1+\varepsilon)x_n$ for all $n \in \mathbb{N}$ and for some $0 < \varepsilon < 1$. Since θ is increasing, if $x > x_n$, $\theta(x) \geq \theta(x_n) > (1+\varepsilon)x_n$, and so

$$\begin{aligned}\int_{x_n}^{(1+\varepsilon)x_n} \frac{\theta(x)-x}{x^2}dx &\geq \int_{x_n}^{(1+\varepsilon)x_n} \frac{(1+\varepsilon)x_n - x}{x^2}dx \\ &= \int_1^{(1+\varepsilon)} \frac{(1+\varepsilon)-s}{s^2}ds > 0\end{aligned}$$

where we made the change of variables $x = x_n s$ so $dx = x_n ds$.

On the other hand, since

$$\lim_{T\to\infty} \int_1^T \frac{\theta(x)-x}{x^2}dx = \ell \in \mathbb{R}$$

there exists $T_\varepsilon > 0$ such that

$$\left| \int_T^S \frac{\theta(x)-x}{x^2}dx \right| < \int_1^{(1+\varepsilon)} \frac{(1+\varepsilon)-s}{s^2}ds$$

for all $S, T \geq T_\varepsilon$. Hence, by taking n so large that $x_n \geq T_\varepsilon$ we obtain a contradiction.

Similarly, if

$$\liminf_{x\to\infty} \frac{\theta(x)}{x} < 1,$$

There there exists an increasing sequence $y_n \to \infty$ such that $\theta(y_n) < (1-\varepsilon)y_n$ for all $n \in \mathbb{N}$ and for some $0 < \varepsilon < 1$. Since θ is increasing, if $y_n > x$, $\theta(x) \leq \theta(y_n) \leq (1-\varepsilon)y_n$, and so

$$\begin{aligned}\int_{(1-\varepsilon)y_n}^{y_n} \frac{\theta(x)-x}{x^2}dx &\leq \int_{(1-\varepsilon)y_n}^{y_n} \frac{(1-\varepsilon)x_n - x}{x^2}dx \\ &= \int_{1-\varepsilon}^{1} \frac{(1-\varepsilon)-s}{s^2}ds < 0\end{aligned}$$

where we made the change of variables $x = y_n s$. On the other hand, there exists $S_\varepsilon > 0$ such that

$$\left| \int_T^S \frac{\theta(x) - x}{x^2} dx \right| < -\int_{1-\varepsilon}^1 \frac{(1-\varepsilon) - s}{s^2} ds$$

for all $S, T \geq S_\varepsilon$. Hence, by taking n so large that $(1-\varepsilon)y_n \geq S_\varepsilon$ we obtain a contradiction. This shows that

$$\liminf_{x \to \infty} \frac{\theta(x)}{x} \geq 1,$$

which would complete the proof. ■

We turn to the proof of the prime number theorem.

Proof of Theorem 181. For every $\varepsilon \in (0, 1)$ and $x > 1$ we have

$$\theta(x) = \sum_{p \text{ prime} \leq x} \log p \leq \sum_{p \text{ prime} \leq x} \log x = \pi(x) \log x$$

while

$$\begin{aligned} \theta(x) &= \sum_{p \text{ prime} \leq x} \log p \geq \sum_{x^{1-\varepsilon} \leq p \text{ prime} \leq x} \log p \\ &\geq \sum_{x^{1-\varepsilon} \leq p \text{ prime} \leq x} \log x^{1-\varepsilon} = (1-\varepsilon) \sum_{x^{1-\varepsilon} \leq p \text{ prime} \leq x} \log x \\ &= (1-\varepsilon) \log x (\pi(x) - \pi(x^{1-\varepsilon})) \\ &\geq (1-\varepsilon) \log x (\pi(x) - x^{1-\varepsilon}). \end{aligned}$$

Hence,

$$\frac{\pi(x)}{\frac{x}{\log x}} \geq \frac{\theta(x)}{x} \geq (1-\varepsilon) \frac{\pi(x)}{\frac{x}{\log x}} - C \frac{\log x}{x^\varepsilon}.$$

Letting $x \to \infty$ gives

$$\liminf_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} \geq \lim_{x \to \infty} \frac{\theta(x)}{x} \geq (1-\varepsilon) \limsup_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}}.$$

It suffices to let $\varepsilon \to 1^-$. ■

Wednesday, April 1, 2020

16 Conformal Mappings

Definition 188 *Given two open set $U, V \subseteq \mathbb{C}$, a bijective holomorphic function $f : U \to V$ is called a* conformal map. *If such a map exists, the sets U and V are said to be* conformally equivalent.

We have seen in Corollary 142 that the inverse function of a injective holomorphic function is also holomorphic. Hence, the inverse of a conformal mapping is still a conformal mapping.

Exercise 189 *Consider the upper half-plane*

$$H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

and let

$$f(z) = \frac{i - z}{i + z}, \quad z \in H.$$

Prove that $f : H \to B(0,1)$ is a conformal map.

Mappings of the form

$$z \mapsto \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ are called *fractional linear transformations.*

Example 190 *Given $n \in \mathbb{N}$, the function $f(z) = z^n$ is a conformal mapping from the sector $S = \{z \in \mathbb{C} : 0 < \arg z < \pi/n\}$ to the upper half-plane H. Its inverse is $f^{-1}(w) = w^{1/n}$, defined in terms of the principal branch of the logarithm.*

Exercise 191 *Let $0 < \alpha < 2$. Prove that the sector $S = \{z \in \mathbb{C} : 0 < \arg z < \alpha\pi\}$ and the upper half-plane are conformally equivalent.*

The Riemann mapping theorem proves that any simply connected open set which is not the entire space is conformally equivalent to the open unit ball. To prove the Riemann mapping theorem we will need the following auxiliary result.

Theorem 192 (Schwarz's lemma) *Let $f : B(0,1) \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $|f(z)| \le 1$ for all $z \in B(0,1)$. Then $|f(z)| \le |z|$ for all $z \in B(0,1)$ and $|f'(0)| \le 1$. Moreover, if $|f(z_0)| = |z_0|$ for some $z_0 \in B(0,1)$ or $|f'(0)| = 1$, then $f(z) = az$ for all $z \in B(0,1)$ and for some $a \in \mathbb{C}$ with $|a| = 1$.*

Proof. Since $f(0) = 0$, we can write

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in B(0,1).$$

Hence the function

$$h(z) := \sum_{n=1}^{\infty} a_n z^{n-1}, \quad z \in B(0,1)$$

is analytic in $B(0,1)$, since the radius of convergence is the same. In turn,

$$g(z) := \begin{cases} \frac{f(z)}{z} & z \in B(0,1),\, z \neq 0, \\ f'(0) & z = 0, \end{cases}$$

is holomorphic (since $g = h$ near 0). For every $r \in (0,1)$ and every $z \in \partial B(0,r)$, $|g(z)| \leq 1/r$, and so by the maximum modulus principle $|g(z)| \leq 1/r$ for all $z \in \overline{B(0,r)}$. Letting $r \to 1^+$, it follows that $|g(z)| \leq 1$ in $B(0,1)$. Moreover, if $|g(z_0)| = 1$ for some $z_0 \in B(0,1)$, then g must be constant, which shows that $f(z) = az$ for all $z \in B(0,1)$ and for some $a \in \mathbb{C}$ with $|a| = 1$. ■

Exercise 193 *Let $z, \alpha \in \mathbb{C}$ be such that $1 - \overline{\alpha} z \neq 0$.*

(i) Prove that

$$\left| \frac{\alpha - z}{1 - \overline{\alpha} z} \right| < 1$$

if $|z| < 1$ and $|\alpha| < 1$ and that

$$\left| \frac{\alpha - z}{1 - \overline{\alpha} z} \right| = 1$$

if $|z| = 1$ or $|\alpha| < 1$.

(ii) Given $\alpha \in B(0,1)$, the function $\psi_\alpha : B(0,1) \to B(0,1)$ given by

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Prove that ψ_α is a bijection.

We turn to the proof of the Riemann mapping theorem.

Theorem 194 (Riemann mapping) *Let $U \subset \mathbb{C}$ be an open simply connected set. Then U is comformally equivalent to a sphere.*

Proof. Step 1: Since U is strictly contained in $\mathbb{C}$ there exists $\alpha \in \mathbb{C} \setminus U$. Hence the function $z \mapsto z - \alpha$ never vanishes on the simply connected set U and so by Exercise 101 we may define the holomorphic function $f(z) := \log_U(z - \alpha)$. Since $e^{f(z)} = z - \alpha$, we have that f is injective. Fix $z_0 \in U$. We claim that

$$f(z) \neq f(z_0) + 2\pi i \quad \text{for all } z \in U. \tag{95}$$

Indeed, if $f(z) = f(z_0) + 2\pi i$ then by taking the exponential on both sides we get

$$z - \alpha = e^{f(z)} = e^{f(z_0) + 2\pi i} = e^{f(z_0)} e^{2\pi i} = (z_0 - \alpha)1,$$

which implies that $z = z_0$ and in turn that $f(z) = f(z_0)$. This contradicts the fact that $f(z) = f(z_0) + 2\pi i$. Hence, the claim (95) holds.

We claim that there exists $r > 0$ small such that $B(f(z_0) + 2\pi i, r) \cap f(U) = \emptyset$. Indeed, if not then taking $r = \frac{1}{n}$ we could find $z_n \in U$ such that $f(z_n) \to f(z_0) + 2\pi i$. Again by exponentiation

$$z_n - \alpha = e^{f(z_n)} \to e^{f(z_0) + 2\pi i} = e^{f(z_0)} e^{2\pi i} = (z_0 - \alpha)1,$$

which implies that $z_n \to z_0$, and in turn that $f(z_0) = f(z_0) + 2\pi i$, which contradicts (95). It follows that the function

$$F(z) := \frac{1}{f(z) - (f(z_0) + 2\pi i)}, \quad z \in U$$

is holomorphic. Moreover, since $|f(z) - (f(z_0) + 2\pi i)| \geq r > 0$ for all $z \in U$, we have that F is bounded. By a translation and a rescaling we can assume that

$$F : U \to B(0, 1)$$

and that $0 \in F(U)$. By the open mapping theorem the set $F(U)$ is open. Since $F : U \to F(U)$ is a homeomorphism and U is simply connected, it follows that $F(U)$ is also simply connected.

Since U and $F(U)$ are conformally equivalent, it suffices to prove that $F(U)$ and $B(0, 1)$ are conformally equivalent. ■

Friday, April 4, 2020

Proof. Step 2: In view of Step 1, by replacing U with $F(U)$, without loss of generality we may assume that $U \subseteq B(0, 1)$ and that $0 \in U$. Let

$$\mathcal{G} := \{g : U \to B(0, 1) \text{ holomorphic, injective, } g(0) = 0\}.$$

The family $\mathcal{G}$ is nonempty since the identity belongs to $\mathcal{G}$. Since, $|g(z)| \leq 1$ for all $z \in U$ and $0 \in U$, by (32),

$$|g'(0)| \leq \frac{1}{2\pi} \int_{\partial B(0,r)} \frac{|g(\zeta)|}{\zeta^2} d\zeta \leq \frac{2\pi r}{2\pi r^2}$$

for all $g \in \mathcal{G}$ and for $r > 0$ such that $\overline{B(0, r)} \subset U$. Let

$$s := \sup\{|g'(0)| : g \in \mathcal{G}\}.$$

Consider a sequence $\{g_n\}_n$ in $\mathcal{G}$ such that $|g_n'(0)| \to s$. Since the family $\mathcal{G}$ is equibounded, by Montel's theorem there exists a subsequence $\{g_{n_k}\}_k$ which converges uniformly on compact sets to a holomorphic function $g : U \to \mathbb{C}$. By uniform convergence, $g(0) = 0$ and $g : U \to \overline{B(0, 1)}$. By Theorem 70, $|g_n'(0)| \to |g'(0)| = s$. Since $s \geq 1$ (since the identity has derivative with modulus one), the function g cannot be constant and thus by Theorem 167 it must be injective. Since $g(U)$ is open, it follows that $g : U \to B(0, 1)$. It follows that g belongs to $\mathcal{G}$.

Step 3: It remains to show that g is onto. Assume by contradiction that there is $\alpha \in B(0, 1) \setminus g(U)$. Consider the diffeomorphism $\psi_\alpha : B(0, 1) \to B(0, 1)$ given by

$$\psi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Note that ψ_α interchanges 0 with α, since $\psi_\alpha(\alpha) = 0$ and $\psi_\alpha(0) = \alpha$. The set $V := (\psi_\alpha \circ g)(U) \subseteq B(0, 1)$ is open and simply connected and 0 does not belong to V since $\alpha \in B(0,1) \setminus g(U)$ and $\psi_\alpha(\alpha) = 0$. Hence, by Exercise 101 the function $h_1 : V \to \mathbb{C}$ given by

$$h_1(w) := e^{\frac{1}{2}\log_V w} = \sqrt{w}$$

is holomorphic and injective and $h_1 : V \to B(0,1)$. It follows that the function

$$g_1 := \psi_{h_1(\alpha)} \circ h_1 \circ \psi_\alpha \circ g$$

is injective, holomorphic, and

$$\begin{aligned} g_1(0) &= \psi_{h_1(\alpha)}(h_1(\psi_\alpha(g(0)))) = \psi_{h_1(\alpha)}(h_1(\psi_\alpha(0))) \\ &= \psi_{h_1(\alpha)}(h_1(\alpha)) = 0. \end{aligned}$$

Hence, $g_1 \in \mathcal{G}$.

Next consider the function $h_2(w) := w^2$ and $\phi := \psi_\alpha^{-1} \circ h_2 \circ \psi_{h_1(\alpha)}^{-1}$. Then

$$\begin{aligned} \phi \circ g_1 &:= \psi_\alpha^{-1} \circ h_2 \circ \psi_{h_1(\alpha)}^{-1} \circ \psi_{h_1(\alpha)} \circ h_1 \circ \psi_\alpha \circ g \\ &= \psi_\alpha^{-1} \circ h_2 \circ h_1 \circ \psi_\alpha \circ g = \psi_\alpha^{-1} \circ \psi_\alpha \circ g = g \end{aligned}$$

and

$$g'(0) = (\phi \circ g_1)'(0) = \phi'(0) g_1'(0)$$

and so

$$s = |g'(0)| = |\phi'(0)||g_1'(0)|.$$

The function $\phi : B(0,1) \to \mathbb{C}$ satisfies all the hypotheses of Schwarz's lemma, but it is not injective since h_2 is not injective. Hence $|\phi'(0)| < 1$, which implies that $|g_1'(0)| > s$ and contradicts the maximality of s. Hence, g is onto and the proof is complete. ■

Remark 195 *In view of Exercise 102, the Riemann mapping theorem continues to hold is instead of assuming U simply connected, we assume that*

$$\int_\gamma f\, ds = 0$$

for every holomorphic function $f : U \to \mathbb{C}$ and for every closed oriented Lipschitz continuous curve with range contained in U.

An important consequence of the Riemann mapping theorem is the following characterization of simply connected open sets.

Theorem 196 *Let $U \subset \mathbb{C}$ be an open connected set. Then the following are equivalent:*

(i) U is homeomorphic to an open ball,

(ii) U is simply connected,

(iii) $\int_\gamma f\,dz = 0$ *for every holomorphic function* $f : U \to \mathbb{C}$ *and for every rectifiable closed oriented curve* γ *with range contained in* U*.*

Proof. Assume that U is homeomorphic to an open ball, say $B(0,1)$. Then there exists an invertible function $\Psi : U \to B(0,1)$, which is continuous together with its inverse and consider a continuous closed curve, with parametric representation $\varphi : [a,b] \to \mathbb{C}$ such that $\varphi([a,b]) \subseteq U$. Define the function $h : [a,b] \times [0,1] \to \mathbb{C}$ by

$$h(t,s) = \Psi^{-1}(s\Psi(\varphi(t))).$$

Then $h([a,b] \times [0,1]) \subseteq U$,

$$h(t,0) = \Psi^{-1}(0) \text{ for all } t \in [a,b], \quad h(t,1) = \varphi(t) \text{ for all } t \in [a,b],$$
$$h(a,s) = \Psi^{-1}(s\Psi(\varphi(a))) = \Psi^{-1}(s\Psi(\varphi(b))) = h(b,s) \text{ for all } s \in [0,1].$$

Hence, U is simply connected. Hence (ii) holds.

Conversely, assume that U is simply connected. Then by the Riemann mapping theorem U is homeomorphic to a ball. This shows that (i) and (ii) are equivalent.

To show that (ii) and (iii) are equivalent, note that if U is simply connected, then (iii) holds in view of Theorem 98. Conversely, if (iii) holds then by Remark 195, U is homeomorphic to a ball and so it is simply connected by the equivalence between (i) and (ii). ■

Next we study the behavior of conformal mappings at the boundary.

Definition 197 *A set* $E \subseteq \mathbb{C}$ *is* locally connected *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $z, w \in E$ *with* $0 < |z - w| < \delta$ *there exists a compact connected set* $F \subseteq E$ *such that* $z, w \in F$ *and* $\operatorname{diam} F < \varepsilon$*.*

The range of a continuous curve is locally connected.

Monday, April 6, 2020

Exercise 198 *Let $E_1, \dots, E_n$ be locally connected. Prove that their union is locally connected.*

Exercise 199 *Let*

$$E = \{x + iy : |x| < 1, 0 < y < 1\} \setminus \bigcup_{n=1}^{\infty} \left[\frac{i}{n}, \frac{i}{n} + 1 \right].$$

Prove that ∂E is not locally connected.

Theorem 200 *Let $U \subset \mathbb{C}$ be an open bounded simply connected set and and let f map conformally $B(0,1)$ onto U. Then the following conditions are equivalent*

(i) f can be extended continuously to $\overline{B(0,1)}$,

(ii) ∂U is the range of an oriented closed curve,

(iii) ∂U is locally connected,

(iv) $\mathbb{C} \setminus U$ is locally connected.

In general the extension of f to $\partial B(0,1)$ will not be injective.

Example 201 *An example of a simply connected domain whose boundary is not the range of an oriented simple closed curve is $U = B(0,1) \setminus \{x : 0 \leq x < 1\}$.*

Indeed, we have the following result:

Theorem 202 (Carathéodory) *Let $U \subset \mathbb{C}$ be an open bounded simply connected set and let f map conformally $B(0,1)$ onto U. Then f has a continuous and injective extension to $\overline{B(0,1)}$ if and only if ∂U is the range of an oriented simple closed curve.*

17 Runge's Theorem

Next we proof another important theorem. There is a more general statement but we will prove first a simpler version.

Theorem 203 (Runge) *Let $U \subseteq \mathbb{C}$ be an open set, let $K \subset U$ be a compact set with $\mathbb{C} \setminus K$ connected, and let $f : U \to \mathbb{C}$ be a holomorphic function. Then there exists a sequence of polynomials $p_n : \mathbb{C} \to \mathbb{C}$ such that $p_n \to f$ uniformly in K.*

Exercise 204 *Let $K \subset \mathbb{C}$ be a compact set.*

(i) Let B be an open ball such that $K \subset B$ and let $z_1 \in \mathbb{C} \setminus B$. Let $f(z) := \frac{1}{z - z_1}$. Prove that there exists a sequence of polynomials which converges to f uniformly in K.

(ii) Assume that $\mathbb{C} \setminus K$ is connected and let $z_0 \in \mathbb{C} \setminus K$. Let $g(z) := \frac{1}{z-z_0}$. Prove that there exists a sequence of polynomials which converges to g uniformly in K.

Lemma 205 *Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be a holomorphic function. Then there exist finitely many oriented segments $\gamma_1, \ldots, \gamma_n$ with range in $U \setminus K$ such that*

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta$$

for all $z \in K$.

Proof. Let $d := \operatorname{dist}(K, \partial U)$ and partition $\mathbb{C}$ into squares of side-length less than $\frac{1}{\sqrt{2}} d$. Let $Q_1, \ldots, Q_\ell$ be the closed cubes which intersects K with ∂Q_k oriented counterclockwise. Since $K \cap Q_k \neq \emptyset$ and Q_k has diameter less than d, each Q_k is contained in U. Let $\gamma_1, \ldots, \gamma_n$ be the oriented sides of these cubes which do not belong to two adjacent squares. Then each γ_k does not intersect K since otherwise γ_k would belong to two adjacent cubes intersecting K. Let $z \in K$ and assume that z is not on the boundary of one of the cubes. Then there exists a unique j such that $z \in Q_j$. It follows by Cauchy's theorem and Theorem 98 that

$$f(z) = \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

On the other hand for all $k \neq j$,

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Hence, if we sum these equalities we get

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial Q_k} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where in the second equality we used the fact that integrals over the sides of adjacent cubes cancel out. This proves the result for all $z \in K$ not on the boundary of a cube Q_k. Now if $z \in K$ and z belongs to the boundary of a cube, then z does not belong to any of the segments γ_k and so by continuity we have that the formula

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for all $z \in K$. ■

Lemma 206 *Let γ be a Lipschitz continuous oriented curve in $\mathbb{C}$ parametrized by $\varphi : [a, b] \to \mathbb{C}$, let $f : \varphi([a, b]) \to \mathbb{C}$ be a continuous function, and let $K \subset \mathbb{C}$*

be a compact set with $K \cap \varphi([a,b]) = \emptyset$. Then for every $\varepsilon > 0$ there exists a rational function $R : \mathbb{C} \setminus \bigcup_{j=1}^{n} \{z_j\} \to \mathbb{C}$, where $z_j \in \varphi([a,b])$ such that

$$\left| \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - R(z) \right| \leq \varepsilon \quad \textit{for all } z \in K.$$

Proof. The function

$$g(z,t) := \frac{f(\varphi(t))}{\varphi(t) - z}, \quad (t,z) \in [a,b] \times K$$

is uniformly continuous, therefore we can find a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that

$$\left| \frac{f(\varphi(t))}{\varphi(t) - z} - \frac{f(\varphi(t_j))}{\varphi(t_j) - z} \right| \leq \frac{\varepsilon}{M(b-a)} \quad \text{for all } (t,z) \in [t_{j-1}, t_j] \times K,$$

where $\|\varphi'\|_\infty \leq M$. Define

$$R(z) := \sum_{j=1}^{n} \frac{f(\varphi(t_j))}{\varphi(t_j) - z} (\varphi(t_j) - \varphi(t_{j-1})).$$

Then

$$\begin{aligned} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - R(z) &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{f(\varphi(t))}{\varphi(t) - z} \varphi'(t)\, dt - \sum_{j=1}^{n} \frac{f(\varphi(t_j))}{\varphi(t_j) - z} \int_{t_{j-1}}^{t_j} \varphi'(t)\, dt \\ &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(\frac{f(\varphi(t))}{\varphi(t) - z} - \frac{f(\varphi(t_j))}{\varphi(t_j) - z} \right) \varphi'(t)\, dt \end{aligned}$$

and so

$$\begin{aligned} \left| \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - R(z) \right| &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| \frac{f(\varphi(t))}{\varphi(t) - z} - \frac{f(\varphi(t_j))}{\varphi(t_j) - z} \right| |\varphi'(t)|\, dt \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{\varepsilon}{M(b-a)} M\, dt = \varepsilon. \end{aligned}$$

This concludes the proof. ■

Wednesday, April 8, 2020

Lemma 207 *Let $G \subseteq \mathbb{C}$ be a set and let $\mathcal{F}(G)$ be the family of functions $f : G \to \mathbb{C}$ for which there exists a sequence of polynomials p_n such that $p_n \to f$ uniformly in G as $n \to \infty$. If $f_k \in \mathcal{F}(G)$ and $f_k \to f$ uniformly in G, then $f \in \mathcal{F}(G)$.*

Proof. The proof uses a diagonal argument. Since $f_k \in \mathcal{F}(G)$ there exists a sequence of polynomials $p_{n,k}$ such that $p_{n,k} \to f_k$ uniformly in G as $n \to \infty$. Hence we can find $n_k \geq k$ such that

$$\sup_{z \in K} |p_{n,k}(z) - f_k(k)| \leq \frac{1}{k}$$

for all $n \geq n_k$. Define

$$q_k(z) := p_{n_k,k}(z).$$

Then

$$\begin{aligned}|f(z) - q_k(z)| = |f(z) - p_{n_k,k}(z)| &\leq |f(z) - f_k(z)| + |f_k(z) - p_{n_k,k}(z)| \\ &\leq |f(z) - f_k(z)| + \frac{1}{k}.\end{aligned}$$

Taking the supremum over all $z \in G$, we have that the right-hand side converges uniformly to zero in G as $k \to \infty$. ■

Lemma 208 *Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Given $z_0 \in \mathbb{C} \setminus K$, let $g_{z_0}(z) := \frac{1}{z-z_0}$. Then there exists a sequence of polynomials which converges to g_{z_0} uniformly in K.*

Proof. Let $\mathcal{F}(K)$ be the space of all functions $f : K \to \mathbb{C}$ such that there exists a sequence of polynomials $p_n : \mathbb{C} \to \mathbb{C}$ such that $p_n \to f$ uniformly in K. Note that if $f, g \in \mathcal{F}(K)$, then fg and $f + g \in \mathcal{F}(K)$. Moreover, if $f_k \in \mathcal{F}(K)$ and $f_k \to f$ uniformly in K, then by the previous lemma, $f \in \mathcal{F}(K)$.

Step 1: Let $R > 0$ be so large that $K \subset B(0, R)$, let $z_1 \in \mathbb{C} \setminus B(0, R)$, and let $g_{z_1}(z) := \frac{1}{z-z_1}$. We claim that $g_{z_1} \in \mathcal{F}(K)$. Find $0 < r < R$ such that $K \subset B(0, r)$. For $z \in K$, write

$$\frac{1}{z - z_1} = -\frac{1}{z_1}\frac{1}{1 - \frac{z}{z_1}}.$$

Then

$$\left|\frac{z}{z_1}\right| \leq \frac{r}{R} =: \delta < 1$$

and so we can use geometric power series to write

$$\frac{1}{z - z_1} = -\frac{1}{z_1}\frac{1}{1 - \frac{z}{z_1}} = -\frac{1}{z_1}\sum_{k=0}^{\infty}\left(1 - \frac{z}{z_1}\right)^k.$$

Since this geometric series converges uniformly in K (since the number δ is independent of z), we have that and the polynomials $-\frac{1}{z_1}\sum_{k=0}^{\ell}\left(1 - \frac{z}{z_1}\right)^k$ converge uniformly to g_{z_1} in K.

Step 2: Let $w_1 \in \mathbb{C} \setminus K$ and assume that $g_{w_1} \in \mathcal{F}(K)$. Let $0 < \delta < \frac{1}{4}\operatorname{dist}(w_1, K)$. We claim that for every $w_2 \in \mathbb{C}$ with $|w_1 - w_2| < \delta$ we have that $g_{w_2} \in \mathcal{F}(K)$ in K. To see this we proceed as in the previous step to write for $z \in K$,

$$g_{w_2}(z) = \frac{1}{z - w_2} = \frac{1}{z - w_1 - (w_1 - w_2)} = \frac{1}{z - w_1}\frac{1}{1 - \frac{w_1 - w_2}{z - w_1}}.$$

Then $|z - w_1| \geq 4\delta$ and so

$$\left|\frac{w_1 - w_2}{z - w_1}\right| \leq \frac{\delta}{4\delta} = \frac{1}{4} < 1$$

and so we can use geometric power series to write

$$g_{w_2}(z) = \frac{1}{z-w_1}\frac{1}{1-\frac{w_1-w_2}{z-w_1}} = \frac{1}{z-w_1}\sum_{k=0}^{\infty}\left(\frac{w_1-w_2}{z-w_1}\right)^k,$$

where this geometric series converges uniformly in K. Hence, the sequence of functions

$$\sum_{k=0}^{\ell}\left(\frac{w_1-w_2}{z-w_1}\right)^k$$

converges uniformly in K as $\ell \to \infty$. Since $g_{w_1} \in \mathcal{F}(K)$ we have that $g_{w_1}^k \in \mathcal{F}(K)$. In turn, $(w_1-w_2)^k g_{w_1}^k \in \mathcal{F}(K)$ and so

$$\sum_{k=0}^{\ell}(w_1-w_2)^k g_{w_1}^k \in \mathcal{F}(K).$$

Hence, $\sum_{k=0}^{\infty}\left(\frac{w_1-w_2}{z-w_1}\right)^k \in \mathcal{F}(K)$ since the series converges uniformly in K. It follows that $g_{w_2} \in \mathcal{F}(K)$, since it is the product of g_{w_1} and this series.

Step 3: Let $R > 0$ be so large that $K \subset B(0,R)$. Let $z_1 \in \mathbb{C}\setminus B(0,R)$. Given $z_0 \in \mathbb{C}\setminus K$, since $\mathbb{C}\setminus K$ is connected, we can find a polygonal path γ that joins z_0 and z_1 with range Γ in $\mathbb{C}\setminus K$. Let $0 < \delta < \frac{1}{4}\operatorname{dist}(\Gamma, K)$. Without loss of generality we can assume that the endpoints of the segments of γ have distance less than δ. Hence, we can apply Step 2 starting from z_1 until we reach z_0. ■

We turn to the proof of the theorem.

Proof of Runge's theorem. By Lemma 205 there exist finitely many oriented segments $\gamma_1, \ldots, \gamma_n$ with range in $U\setminus K$ such that

$$f(z) = \sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(\zeta)}{\zeta - z}d\zeta$$

for all $z \in K$. By Lemma 205 for each $\varepsilon > 0$ there exists a rational function R_k such that

$$\left|\frac{1}{2\pi i}\int_{\gamma_k}\frac{f(\zeta)}{\zeta - z}d\zeta - R_k(z)\right| \le \varepsilon/n \quad \text{for all } z \in K.$$

Hence,

$$\left|f(z) - \sum_{k=1}^{n}R_k(z)\right| \le \varepsilon \quad \text{for all } z \in K.$$

Now each R_k is a sum of rational functions whose denominator has the form $\frac{1}{z-z_0}$ for some $z_0 \in U\setminus K$. We now apply Lemma 208. ■

Friday, April 10, 2020

We now present a more general version. Let $\mathbb{S}^2 := \partial B((0,0,0),1)$ be the unit sphere in $\mathbb{R}^3$. We can view the complex plane as the plane the plane $\{(x,y,0) : x,y \in \mathbb{R}\}$ inside $\mathbb{R}^3$. Let $N = (0,0,1) \in \mathbb{S}^2$ be the north pole. Given a point $z = x + iy$ there is a unique line passing through N and $(x, y, 0)$ which intersects $\mathbb{S}^2$ at a point $S(z) \in \mathbb{S}^2 \setminus \{N\}$. The map S gives a bijection between $\mathbb{C}$ and $\mathbb{S}^2 \setminus \{N\}$. Indeed, given $(X, Y, Z) \in \mathbb{S}^2 \setminus \{N\}$ consider

$$x = \frac{X}{1 - Z}, \quad y = \frac{Y}{1 - Z}.$$

Conversely, given $z = x + iy \in \mathbb{C}$ we have that

$$\begin{aligned} S(z) &= \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) \\ &= \frac{1}{1 + |z|^2} (2 \operatorname{Re} z, 2 \operatorname{Im} z, |z|^2 - 1). \end{aligned}$$

If we set $S(\infty) := N$ we have a bijection between $\mathbb{C}_\infty$ and $\mathbb{S}^2$. Note that $S(z) \to N$ in $\mathbb{R}^3$ if and only if $|z| \to \infty$ in $\mathbb{C}$.

Hence, we can regard $\mathbb{C}_\infty$ as a subset of $\mathbb{R}^3$. In turn, the metric in $\mathbb{R}^3$ induces a metric on $\mathbb{C}_\infty$. We leave as an exercise to show that this metric is given by

$$d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}$$

for $z, w \in \mathbb{C}$ and that this metric induces the same topology in $\mathbb{C}$. Note that since $\mathbb{S}^2$ is compact, so is $\mathbb{C}_\infty$.

Theorem 209 (Runge) *Let $U \subseteq \mathbb{C}$ be an open set, let $K \subset U$ be a compact set, let $E \subseteq \mathbb{C}_\infty \setminus U$ be such that E contains at least one point in each component of $\mathbb{C}_\infty \setminus K$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Then there exists a sequence of rational functions $r_n : \mathbb{C} \setminus E \to \mathbb{C}$ with poles in E such that $r_n \to f$ uniformly in K.*

We will need two more lemmas.

Lemma 210 *Let $V, W \subseteq \mathbb{C}$ be two open sets with $V \subseteq W$ and $\partial V \cap W = \emptyset$. If H is any component of W and $H \cap V \neq \emptyset$, then $H \subseteq V$.*

Proof. Let H be as in the statement and let $z_0 \in H \cap V$. Then there exists a connected component G of V such that $z_0 \in G$. To conclude the proof, it is enough to show that $H = G$.

We have that $G \subseteq H$, since G is a connected subset of V (and so of W) containing z_0 and H is the union of all connected subsets of W containing z_0. Write

$$H = G \cup (H \setminus G) = G \cup ((H \cap \partial G) \cup (H \setminus \overline{G})).$$

But $H \cap \partial G \subseteq W \cap \partial G \subseteq W \cap \partial V = \emptyset$. Hence, the connected set H is the union of two disjoint open sets. Since G is nonempty, it follows that $H \setminus \overline{G} = \emptyset$, which shows that $H = G$. ■

Lemma 211 *Let $K \subset \mathbb{C}$ be a compact set, let $z_0 \in \mathbb{C} \setminus K$, let $g(z) := \frac{1}{z-z_0}$, and let $E \subseteq \mathbb{C}_\infty \setminus K$ be such that E contains at least one point in each component of $\mathbb{C}_\infty \setminus K$. Then there exists a sequence of rational functions $R_n : \mathbb{C} \setminus E \to \mathbb{C}$ with poles in E such that $R_n \to g$ uniformly in K.*

Proof. Step 1: Let $B(E)$ be the space of all functions $f : K \to \mathbb{C}$ such that there exists a sequence of rational functions $R_n : \mathbb{C} \setminus E \to \mathbb{C}$ with poles in E such that $R_n \to f$ uniformly in K. Note that if $f, g \in B(E)$, then fg and $f + g \in B(E)$. Moreover, if $f_k \in B(E)$ and $f_k \to f$ uniformly in K, then by Lemma 207 (which continues to hold if we replace polynomials with rational functions), $f \in B(E)$.

Step 2: Assume that $E \subset \mathbb{C} \setminus K$. Let $W := \mathbb{C} \setminus K$ and let V be the set of all $w \in W$ such that $g_w \in B(E)$, where $g_w(z) = \frac{1}{z-w}$, $z \in K$. We claim that V is an open set. To see this, let $w_0 \in V$ and $w \in B(w_0, r)$, where $r := \operatorname{dist}(w_0, K) > 0$. For $z \in K$, write

$$\frac{1}{z-w} = \frac{1}{z - w_0 - (w - w_0)} = \frac{1}{z - w_0} \frac{1}{1 - \frac{w-w_0}{z-w_0}}.$$

Then $|z - w_0| \geq r$ and so

$$\left| \frac{w - w_0}{z - w_0} \right| \leq \frac{|w - w_0|}{r} =: \delta < 1$$

and so we can use geometric power series to write

$$\frac{1}{1 - \frac{w-w_0}{z-w_0}} = \sum_{k=0}^{\infty} \left(\frac{w - w_0}{z - w_0} \right)^k.$$

Since this geometric series converges uniformly in K (since the number δ is independent of z), and $\sum_{k=0}^{\ell} \left(\frac{w-w_0}{z-w_0} \right)^k$ belongs to $B(E)$, because is it given by products and sums of functions in $B(E)$, by Step 1, $\frac{1}{1-\frac{w-w_0}{z-w_0}} \in B(E)$, and so also $g_w \in B(E)$. This shows that $B(w_0, r) \subseteq V$. Thus, V is open.

Next we claim that $\partial V \cap W = \emptyset$. Let $w \in \partial V$ and find $w_n \in V$ such that $w_n \to w$. By what we just proved, if $|w_n - w| < \operatorname{dist}(w_n, K)$, then $w \in V$. Since $w \notin V$, it must be that

$$|w_n - w| \geq \operatorname{dist}(w_n, K) \geq \operatorname{dist}(w, K) - |w_n - w|.$$

Letting $n \to \infty$ gives $\operatorname{dist}(w, K) = 0$, which implies that $w \in K$, since K is compact. Recalling that $W := \mathbb{C} \setminus K$, it follows that $w \notin W$.

This proves that all the hypotheses of the previous lemma are satisfied. Let H be any component of $W = \mathbb{C} \setminus K$. By hypothesis there exists $w \in E \cap H$. Moreover g_w is a rational function itself with pole in E. Hence, w belongs to V. By the previous lemma, it follows that $H \subseteq V$. This shows that $V = \mathbb{C} \setminus K$, that is, that for every $w \in \mathbb{C} \setminus K$ there exists a sequence of rational functions $R_n : \mathbb{C} \setminus E \to \mathbb{C}$ with poles in E such that $R_n \to g_w$ uniformly in K. ■

Proof. Step 3: Assume that $\infty \in E \subset \mathbb{C}_\infty \setminus K$. Since K is bounded, there exists a unique unbounded connected component H of $\mathbb{C} \setminus K$. If $w_0 \in H$ and $|w_0|$ is very large, then the Taylor series of g_{w_0} converges uniformly in K (see Lemma 208). Thus, $w_0 \in B(S)$.

By applying Step 2 to $(E \cup \{w_0\}) \setminus \{\infty\}$, we conclude that for every $w \in \mathbb{C} \setminus K$ there exists a sequence of rational functions $R_n : \mathbb{C} \setminus ((E \cup \{w_0\}) \setminus \{\infty\}) \to \mathbb{C}$ with poles in $(E \cup \{w_0\}) \setminus \{\infty\}$ such that $R_n \to g_w$ uniformly in K. Write

$$R_n = Q_n + S_n,$$

where the poles of Q_n are in $E \setminus \{\infty\}$ and S_n is either zero or has only a pole in w_0. Since S_n can be approximated uniformly in K by polynomials, by a diagonal argument, we can find a sequence of rational functions with poles in $E \setminus \{\infty\}$ converging uniformly to g_{w_0} in K. This concludes the proof. ■

We turn to the proof of Runge's theorem.

Proof. We proceed as in the proof of Theorem 203 with the only difference that in place of Lemma 208 we apply the previous lemma. ■

17.1 Mittag-Leffler Theorem

This is the analog of Weierstrass representation theorem for meromorphic functions. In the statement we will use the fact that if $U \subseteq \mathbb{C}$ is an open set and $E \subset U$ is a set with no accumulation points in U, then E is countable.

Theorem 212 *Let $U \subseteq \mathbb{C}$ be an open set, let $E = \{w_n : n \in I\} \subset U$ be a set with no accumulation points in U, where $I \subseteq \mathbb{N}$ and let*

$$S_n(z) = \frac{a_{n,1}}{z - w_n} + \cdots + \frac{a_{n,\ell_k}}{(z - w_n)^{\ell_k}}.$$

Then there exists a meromorphic function $f : U \setminus E \to \mathbb{C}$ whose only poles are at E and whose principal part at w_n is S_n.

Proof. Step 1: Let $K_0 := \emptyset$ and

$$K_j := \overline{B(0,j)} \cap \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{C} \setminus U) \geq 1/j\}.$$

Then $K_j \subset K_{j+1}^\circ$ and $\bigcup_{j=1}^\infty K_j = U$.

Note that

$$\mathbb{C}_\infty \setminus K_j = (\mathbb{C}_\infty \setminus \overline{B(0,j)}) \cup (\overline{B(0,j)} \setminus U) \cup \{z \in U \cap \overline{B(0,j)} : \operatorname{dist}(z, \mathbb{C} \setminus U) < 1/j\}. \tag{96}$$

We claim that each component of $\mathbb{C}_\infty \setminus K_j$ contains a component of $\mathbb{C}_\infty \setminus U$. Indeed, since $\mathbb{C}_\infty \setminus U \subset \mathbb{C}_\infty \setminus K_j$, if we consider the component G of $\mathbb{C}_\infty \setminus K_j$ which contains ∞, it must contain the component H of $\mathbb{C}_\infty \setminus U$ which contains ∞ (since H is connected, $\infty \in H$ and $H \subseteq \mathbb{C}_\infty \setminus K_j$). On the other hand, since $K_j \subseteq \overline{B(0,j)}$, we have that G contains $\mathbb{C}_\infty \setminus \overline{B(0,j)}$, since the latter is a connected set contained in $\mathbb{C}_\infty \setminus K_j$. It follows that if D is a component of $\mathbb{C}_\infty \setminus K_j$ which does not contain ∞, then $D \subseteq \overline{B(0,j)}$ and so by (96), D contains a point $z_0 \in \mathbb{C}$ with $\operatorname{dist}(z_0, \mathbb{C}\setminus U) < 1/j$. It follows from the definition of distance that there exists $w_0 \in \mathbb{C}\setminus U \subset \mathbb{C}_\infty \setminus K_j$ with $|z_0 - w_0| < 1/j$. Hence, $z_0 \in B(w_0, 1/j)$. But $B(w_0, 1/j) \subseteq \mathbb{C}_\infty \setminus K_j$. Indeed, let $w \in B(w_0, 1/j)$. If $w \in (\mathbb{C}_\infty \setminus \overline{B(0,j)}) \cup (\overline{B(0,j)} \setminus U)$ there is nothing to prove, so assume that $w \in U$ and $|w| \leq j$. Since $w_0 \in \mathbb{C} \setminus U$,

$$\operatorname{dist}(w, \mathbb{C}\setminus U) \leq |w - w_0| < 1/j,$$

and so by the definition of K_j, $w \notin K_j$.

Thus, $z_0 \in B(w_0, 1/j) \subseteq \mathbb{C}_\infty \setminus K_j$. Since D and $B(w_0, 1/j)$ are connected and contain z_0, $D \cup B(w_0, 1/j)$ is connected. But D is maximal, so $B(w_0, 1/j) \subseteq D$. Let D_1 be the component of $\mathbb{C} \setminus U$ which contains w_0. Then $D \subseteq D_1$ again because $D \subseteq \mathbb{C} \setminus U \subset \mathbb{C}_\infty \setminus K_j$ and $w_0 \in D$. This proves the claim.

Step 2: Let

$$I_j := \{n \in I : \, w_n \in K_j \setminus K_{j-1}\}.$$

The sets I_j are disjoint and each I_j has only finitely many elements, since E has no accumulation points in U. Define

$$Q_j := \sum_{n \in I_j} S_n,$$

if I_j is nonempty and $Q_j = 0$ otherwise. Then Q_j is a rational functions with poles in $K_j \setminus K_{j-1}$. By Runge's theorem with $E = \mathbb{C}\setminus U$, there exists a rational functions R_j with poles in $\mathbb{C} \setminus U$ such that

$$|Q_j(z) - R_j(z)| \leq 1/2^j \quad \text{for all } z \in K_{j-1}.$$

We claim that the function

$$f(z) = Q_1(z) + \sum_{j=2}^{\infty} (Q_j(z) - R_j(z))$$

is well-defined and has all the desired property of the theorem. To see this let we beging by showing that f is holomorphic in $U \setminus E$. Note that since each w_n is isolated and don't accumulate at points of U, $U \setminus E$ is open. Let $K \subset U \setminus E$ be a compact set. Then there exists m such that $K \subset K_m$. If $j \geq m+1$, then $K \subset K_{j-1}$ and so

$$|Q_j(z) - R_j(z)| \leq 1/2^j \quad \text{for all } z \in K.$$

It follows that the series $\sum_{j=m+1}^{\infty} (Q_j(z) - R_j(z))$ is uniformly convergent in K. Since $Q_1(z) + \sum_{j=2}^{m} (Q_j(z) - R_j(z))$ have poles in E or in $\mathbb{C} \setminus U$, we have that f is holomorphic in K°. By considering an increasing sequence of compact sets T_l, with $T_l \subset T_{l+1}^\circ$ and $\bigcup_j T_l = U \setminus E$, we have that f is holomorphic in $U \setminus E$. ■

Wednesday, April 15, 2020

Proof. It remains to show that f has poles at each w_n and that its principal part is Q_n. Since w_n is isolated, there exists $r > 0$ such that $|w_n - w_j| > r$ for all $j \neq n$. For $z \in U \cap B(w_n, r) \setminus \{w_n\}$ we can write

$$f(z) = S_n(z) + f(z) - S_n(z),$$

and the function $f - S_n$ is holomorphic in $U \cap B(w_n, r)$ since the poles of R_j are in $\mathbb{C} \setminus U$ for all j and Q_j has poles in $w_j \notin B(w_n, r)$ for all $j \neq n$. Thus, S_n is the principal part of f at w_n. ■

18 Simply Connected Domains

Using Runge's theorem we can give another characterization of simply connected sets. Given $z \in \mathbb{C}$ and a Lipschitz continuous closed oriented curve γ with range not containing z the *winding number* of γ around z is defined as

$$\operatorname{ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}. \tag{97}$$

It is also called the index of z with respect to γ.

Theorem 213 *Let γ be a rectifiable closed oriented curve in $\mathbb{C}$ with range Γ. Then*

(i) for every $z \in \mathbb{C} \setminus \Gamma$, $\operatorname{ind}_\gamma(z)$ is an integer,

(ii) if z, w belong to the same connected component of $\mathbb{C} \setminus \Gamma$, then $\operatorname{ind}_\gamma(z) = \operatorname{ind}_\gamma(w)$,

(iii) $\operatorname{ind}_\gamma(z) = 0$ for all z in the unbounded connected component of $\mathbb{C} \setminus \Gamma$.

Proof. (i) Fiz $z \in \mathbb{C} \setminus \Gamma$. Assume that γ is a polygonal path. Let $\varphi : [0, 1] \to \mathbb{C}$ be a parametrization of γ and consider the function

$$g(t) := \int_0^t \frac{\varphi'(r)}{\varphi(r) - z} \, dr.$$

Then g is absolutely continuous and $g'(t) = \frac{\varphi'(t)}{\varphi(t) - z}$ for $\mathcal{L}^1$-a.e. $t \in [0, 1]$. Define

$$h(t) = (\varphi(t) - z)e^{-g(t)}.$$

By the chain rule,

$$\begin{aligned} h'(t) &= \varphi'(t)e^{-g(t)} - (\varphi(t) - z)e^{-g(t)}g'(t) \\ &= \varphi'(t)e^{-g(t)} - (\varphi(t) - z)e^{-g(t)}\frac{\varphi'(t)}{\varphi(t) - z} = 0 \end{aligned}$$

for $\mathcal{L}^1$-a.e. $t \in [0,1]$ and since h is absolutely continuous, it follows that h is constant,say $h \equiv \frac{1}{c}$. Since $\varphi(0) = \varphi(1)$ we get

$$1 = e^{g(0)} = c(\varphi(0) - z) = c(\varphi(1) - z) = e^{g(1)}$$

and so

$$1 = e^{\int_\gamma \frac{d\zeta}{\zeta - z}},$$

which implies that $\int_\gamma \frac{d\zeta}{\zeta - z}$ is a multiple of $2\pi i$. Hence, $\operatorname{ind}_\gamma(z)$ is an integer.

On the other hand, if γ is only rectifiable, by Lemma 64 for every $0 < \varepsilon < \frac{1}{2}$ there exists a polygonal path γ_ε with the same endpoints of γ such that

$$|\operatorname{ind}_\gamma(z) - \operatorname{ind}_{\gamma_\varepsilon}(z)| \leq \varepsilon.$$

Since $\operatorname{ind}_{\gamma_\varepsilon}(z)$ is an integer, letting $\varepsilon \to 0^+$ we conclude that $\operatorname{ind}_\gamma(z)$ is also an integer.

(ii) Since the function $\operatorname{ind}_\gamma : \mathbb{C} \setminus \Gamma \to \mathbb{Z}$ is continuous and it is integer-valued, it must be constant in any connected component of $\mathbb{C} \setminus \Gamma$.

(iii) Let $C > 0$ be such that $|\varphi(t)| \leq C$ for all $t \in [0,1]$. Hence, for $|z| > R > C$, we have that

$$|\varphi(t) - z| \geq |z| - |\varphi(t)| \geq |z| - C > 0,$$

and so

$$\left| \frac{\varphi'(t)}{\varphi(t) - z} \right| \leq \frac{M}{|\varphi(t) - z|} \leq \frac{M}{|z| - C} < \pi$$

provided R is sufficiently large. It follows that for $|z| > R$,

$$|\operatorname{ind}_{\boldsymbol{\gamma}}(z)| \leq \frac{1}{2},$$

and since $\operatorname{ind}_{\boldsymbol{\gamma}}$ takes only integer values, $\operatorname{ind}_{\boldsymbol{\gamma}}(z) = 0$. The result now follows from part (ii). ■

Another important application of Theorem **??** is the following.

Theorem 214 *Let $U \subseteq \mathbb{C}$ be an open set and let γ_1 and γ_2 be two continuous, closed, oriented curves that are homotopic in U. Then*

$$\operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z)$$

for all $z \in \mathbb{C} \setminus U$. In particular, if U is simply connected, then $\operatorname{ind}_{\boldsymbol{\gamma}}(z) = 0$ for every continuous closed oriented curve γ with range contained in U and for every $z \in \mathbb{C} \setminus U$.

Proof. Fix $z_0 \in \mathbb{C} \setminus U$ and let γ_1 and γ_2 be as in the statement. Since the the function $f(z) = \frac{1}{z - z_0}$ is holomorphic in U, it follows by Theorem 98, that $\int_{\gamma_1} \frac{d\zeta}{\zeta - z_0} = \int_{\gamma_2} \frac{d\zeta}{\zeta - z_0}$, and so $\operatorname{ind}_{\gamma_1}(z_0) = \operatorname{ind}_{\gamma_2}(z_0)$. On the other hand, if U is simply connected, then every continuous closed oriented curve g_1 is homotopic to a point. But for a curve γ_2 with constant parametric representation we have that $\int_{\gamma_2} \frac{d\zeta}{\zeta - z_0} = 0$, and so by the first part of the theorem, $\operatorname{ind}_{\gamma_1}(z_0) = 0$. ■

Given n closed continuous oriented curves $\gamma_1, \ldots, \gamma_n$, the family $\Xi := \{\gamma_1, \ldots, \gamma_n\}$ is called a *cycle*. The *range* of Ξ is given by the union of the ranges of $\gamma_1, \ldots, \gamma_n$. Given a point $z \in \mathbb{C}$ not contained in the range of Ξ, we define the *winding number* of Ξ around z to be the integer

$$\operatorname{ind}_{\Xi}(z) := \sum_{k=1}^{n} \operatorname{ind}_{\gamma_k}(z).$$

Theorem 215 *Let $U \subseteq \mathbb{C}$ be an open set and let $K \subset U$ be a compact set. Then there exists a cycle Ξ with range contained in $U \setminus K$ such that*

$$\operatorname{ind}_{\Xi}(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \in \mathbb{C} \setminus U. \end{cases}$$

Proof. Let $0 < \delta < \frac{1}{2} \operatorname{dist}(K, \partial U)$ and consider a grid of squares of diameter less than δ. Since K is compact, only finitely many closed squares $Q_1, \ldots, Q_n$ intersect K. If $z \in Q_j$ for some j, then $\operatorname{dist}(z, K) < \delta$. Hence, $Q_j \subset U$. Also if Q_j and Q_k have a side S in common, then if we consider the closed curves ∂Q_j and ∂Q_k oriented counterclockwise, then S will be traversed in both directions and so the integrals of any continuous function over S^+ and S^- will cancel out.

Let $S_1, \ldots, S_n$ be the segments which are the sides of only one the rectangles. Note that if one of these segments S_j intersects K then necessarily there must be two rectangles which intersect K, which contradicts the definition of S_j. It follows that $S_k \subseteq U \setminus K$.

If $z \in K$, then there exists $j \in \{1, \ldots, m\}$ such that $z \in R_j$. If $z \in R_j^\circ$, then

$$\operatorname{ind}_{\partial R_j}(z) = \frac{1}{2\pi i} \int_{\partial R_j} \frac{d\zeta}{\zeta - z} = 1,$$
$$\operatorname{ind}_{\partial R_k}(z) = \frac{1}{2\pi i} \int_{\partial R_k} \frac{d\zeta}{\zeta - z} = 0, \quad k \neq j.$$

Hence, summing these two identities

$$\operatorname{ind}_{\Xi}(z) = \sum_{k=1}^{n} \operatorname{ind}_{\partial R_k}(z).$$

If z belongs to ∂R_j, then either z is a vertex, in which case it belongs to four rectangles, say R_{j_1}, R_{j_2}, R_{j_3}, R_{j_4}. Then, setting $R = \bigcup_{l=1}^{4} R_{j_l}$,

$$\sum_{l=1}^{4} \operatorname{ind}_{\partial R_{j_l}}(z) = \sum_{l=1}^{4} \frac{1}{2\pi i} \int_{\partial R_{j_l}} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\partial R} \frac{d\zeta}{\zeta - z} = 1$$

since all the integral along common edges cancel out. On the other hand,

$$\operatorname{ind}_{\partial R_k}(z) = \frac{1}{2\pi i} \int_{\partial R_k} \frac{d\zeta}{\zeta - z} = 0, \quad k \notin \{j_1, j_2, j_3, j_4\}.$$

Hence, as before $\text{ind}_{\Xi}(z) = 1$. Finally if z belongs to ∂R_j but it is not a vertex, in which case it belongs to two rectangles, say R_{j_1}, R_{j_2} Then, setting $R = \bigcup_{l=1}^{2} R_{j_l}$, as before

$$\sum_{l=1}^{2} \text{ind}_{\partial R_{j_l}}(z) = \sum_{l=1}^{2} \frac{1}{2\pi i} \int_{\partial R_{j_l}} \frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\partial R} \frac{d\zeta}{\zeta - z} = 1.$$

Also, On the other hand,

$$\text{ind}_{\partial R_k}(z) = \frac{1}{2\pi i} \int_{\partial R_k} \frac{d\zeta}{\zeta - z} = 0, \quad k \notin \{j_1, j_2\}.$$

This shows that $\text{ind}_{\Xi}(z) = 1$.

If $z \in \mathbb{C} \setminus U$, then $z \notin R_k$ for any k and since z belongs to the unbounded component of $\mathbb{C} \setminus \partial R_k$, $\text{ind}_{\partial R_k}(z) = 0$ for all k, which shows that $\text{ind}_{\Xi}(z) = 0$. This completes the proof. ■

Friday, April 17, 2020

Theorem 216 *Let $U \subset \mathbb{C}$ be an open connected set. Then the following are equivalent:*

(i) $\mathbb{C}_\infty \setminus U$ *is connected,*

(ii) U *is simply connected,*

(iii) $\text{ind}_\gamma(z) = 0$ *for every continuous closed oriented curve* γ *with range contained in* U *and for every* $z \in \mathbb{C} \setminus U$*.*

Proof. Step 1: We prove that (i) implies (ii). Assume that $\mathbb{C}_\infty \setminus U$ is connected. Fix an holomorphic function $f : U \to \mathbb{C}$ and a rectifiable closed oriented curve γ with range Γ contained in U. Taking $E = \{\infty\}$ in Runge's theorem there exists a sequence of rational functions $r_n : \mathbb{C} \to \mathbb{C}$ with poles in ∞ such that $r_n \to f$ uniformly in Γ. But this implies that these rational functions are polynomials. Since each polynomial has a primitive, by Remark **??**,

$$\int_\gamma r_n \, dz = 0.$$

Letting $n \to \infty$ and using uniform convergence in Γ, it follows that $\int_\gamma f \, ds = 0$. Thus (ii) holds. In view of Theorem 196 it follows that U is simply connected.

Step 2: That (ii) implies (iii) follows from Theorem 214.

Step 3: Assume that (iii) holds but that $\mathbb{C}_\infty \setminus U$ is not connected. Since $\mathbb{C}_\infty \setminus U$ is closed, its connected components are also closed. Moreover, since $\mathbb{C}_\infty$ is compact, so is any closed subset of $\mathbb{C}_\infty$. Hence, we can find two disjoint nonempty compact sets C and K (with respect to the metric in $\mathbb{C}_\infty$) such that

$$\mathbb{C}_\infty \setminus U = C \cup K.$$

Moreover, since $U \subseteq \mathbb{C}$ we have that $\infty \in \mathbb{C}_\infty \setminus U$, so $\infty \in C \cup K$. Assume that $\infty \in C$. Then $\infty \notin K$ and so K must be bounded, since otherwise we could find a sequence $\{z_n\}_n$ in K such that $|z_n| \to \infty$. This would imply that ∞ is an accumulation point of K and so it would belong to K since K is closed. Thus K is compact in $\mathbb{C}$.

Let $V := \mathbb{C} \setminus C$. Then V is open and contains K. By Theorem 215 there exists a cycle Ξ with range contained in $V \setminus K$ such that

$$\operatorname{ind}_\Xi(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \in \mathbb{C} \setminus V. \end{cases}$$

But $V \setminus K = (\mathbb{C} \setminus C) \setminus K = \mathbb{C} \setminus (C \cup K) = \mathbb{C} \setminus (\mathbb{C} \setminus U) = U$. Hence, the range of Ξ is contained in U but $\operatorname{ind}_\Xi(z) = 1$ for all $z \in K \subset \mathbb{C} \setminus U$, which contradicts hypothesis (iii), since the winding number of each closed curve in the cycle should be zero. ■

Remark 217 *Note that saying that $\mathbb{C}_\infty \setminus U$ is connected is not equivalent to saying that $\mathbb{C} \setminus U$ is connected. Indeed, consider the set $E = \{z = x + iy : y \in (0,1)\}$. Then its complement is not connected in $\mathbb{C} \setminus U$ but it is connected in $\mathbb{C}_\infty \setminus U$.*

Corollary 218 *Let $U \subset \mathbb{C}$ be an open bounded connected set. Then U is connected if and only if $\mathbb{C} \setminus U$ is connected.*

Exercise 219 *Let $U \subseteq \mathbb{C}$ be an open set. Prove that $\mathbb{C}_\infty \setminus U$ is connected if and only if every component of $\mathbb{C} \setminus U$ is unbounded.*

19 Proof of Caratheodory's Theorem

Given an open set $U \subseteq \mathbb{C}$ an *oriented continuous half-open curve* γ in U is an equivalence class of continuous equivalent functions $\varphi : [a, b) \to U$. We define the length of γ as

$$L(\gamma) := \lim_{r \to b^-} \operatorname*{Var}_{[a,r]} \varphi.$$

We say that the curve γ *ends at b* if there exists

$$\lim_{t \to b^-} \varphi(t) = b \in \overline{U}.$$

Exercise 220 *Let γ be an oriented continuous half-curve with range in some open set $U \subseteq \mathbb{C}$. Prove that if γ has finite length, then it ends at some point $b \in \overline{U}$.*

We begin with a preliminary result.

Lemma 221 *Let $V \subseteq \mathbb{C}$ be an open set and assume that $f : V \to f(V)$ be a conformal map with $f(V) \subseteq B(0, R)$ for some $R > 0$. If $z_0 \in \mathbb{C}$ and*

$$C(r) := V \cap \partial B(z_0, r),$$

then

$$\inf_{\rho<r<\sqrt{\rho}} L\left(f(C(r))\right) \leq \frac{2\pi R}{\sqrt{\log(1/\rho)}}, \quad 0<\rho<1.$$

In particular, there exists $r_n \searrow 0^+$ *such that* $L\left(f(C(r_n))\right) \to 0$ *as* $n \to \infty$.

Proof. Let $D_r := \{t \in [0,2\pi] : z_0 + re^{it} \in V\}$ and define $\varphi(t) = z_0 + re^{it}$, $t \in D_r$. The set D_r is the union of disjoint intervals, Let I be one of these intervals and consider $[a,b] \subseteq I$. Then $f \circ \varphi : [a,b] \to \mathbb{C}$ is a curve of class C^∞ and so

$$L(f(\varphi([a,b])) = \int_a^b |f'(\varphi(t))||\varphi'(t)|\,dt.$$

Letting $[a,b] \nearrow I$ if needed, we get

$$L(f(\varphi(I)) = \int_I |f'(\varphi(t))||\varphi'(t)|\,dt.$$

Summing over all disjoint intervals in D_r we obtain

$$g(r) := L\left(f(C(r))\right) = L(f(\varphi(D_r)) = \int_{D_r} |f'(\varphi(t))||\varphi'(t)|\,dt.$$

In turn, by Hölder's inequality

$$\begin{aligned}(g(r))^2 &= \left(\int_D |f'(\varphi(t))||\varphi'(t)|^{1/2}|\varphi'(t)|^{1/2}\,dt\right)^2 \leq \int_{D_r} |\varphi'(t)|\,dt \int_{D_r} |f'(\varphi(t))|^2|\varphi'(t)|\,dt \\ &\leq 2\pi r \int_{D_r} |f'(\varphi(t))|^2|\varphi'(t)|\,dt = 2\pi r \int_{D_r} |f'(z_0+re^{it})|^2 r\,dt.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^\infty (g(r))^2 \frac{dr}{r} &\leq 2\pi \int_0^\infty \int_{D_r} |f'(z_0+re^{it})|^2 r\,dtdr \\ &= 2\pi \int_U |f'(x+iy)|^2 dxdy\end{aligned}$$

where we used polar coordinates. Recalling that

$$|f'(x+iy)|^2 = \det \begin{pmatrix} \frac{\partial u}{\partial x}(x,y) & \frac{\partial u}{\partial y}(x,y) \\ \frac{\partial v}{\partial x}(x,y) & \frac{\partial v}{\partial y}(x,y) \end{pmatrix}$$

(see (10)), using the theorem on change of variables for Lebesgue (or Riemann) integration we get

$$\int_0^\infty (g(r))^2 \frac{dr}{r} \leq 2\pi \int_V |f'(x+iy)|^2 dxdy = 2\pi \mathcal{L}^2(f(V)).$$

Since $f(V) \subseteq B(0,R)$ we obtain

$$\frac{1}{2}\log\frac{1}{\rho} \inf_{\rho<r<\sqrt{\rho}} (g(r))^2 \leq \int_\rho^{\sqrt{\rho}} (g(r))^2 \frac{dr}{r} \leq 2\pi^2 R^2.$$

Dividing by $\log \frac{1}{\rho}$ proves the first part of the theorem, while to prove the second part of the statement it suffice to observe that $\frac{1}{\log \frac{1}{\rho}} \to 0$ as $\rho \to 0^+$. ∎

Exercise 222 *Let $U, V \subseteq \mathbb{C}$ be open sets and let $f : U \to V$ be continuous, one-to-one, onto, with $f^{-1} : V \to U$ continuous.*

(i) *Let $\{z_n\}_n$ be a sequence of points in U such that $z_n \to z_0 \in \partial U$. Assume that there exists*

$$\lim_{n\to\infty} f(z_n) = w_0 \in \mathbb{C}.$$

Prove that $w_0 \in \partial V$.

(ii) *Assume that $U = B(0,1)$ and that f can be extended continuously to $\overline{B(0,1)}$. Prove that $f(\partial U) = \partial V$.*

Monday, April 20, 2020

Given a closed set $C \subset X$, where X is a metric space and $x, y \in X \backslash C$. We say that x, y are *separated by* C if they belong to different connected components of $X \setminus C$. We say that are *not separated by* C if they belong to the same connected component of $X \setminus C$.

Lemma 223 (Janiszweski) *Let $C_1, C_2 \subset \mathbb{C}_\infty$ be two closed sets such that $C_1 \cap C_2$ is connected. If the points $a, b \in \mathbb{C}_\infty \setminus (C_1 \cup C_2)$ are not separated by either C_1 or C_2, then they are not separated by $C_1 \cup C_2$.*

Proof. Assume that $a = 0$ and $b = \infty$ (the other cases are similar). Since $\infty \notin C_k$, we have that C_k is bounded, since otherwise we could find a sequence $\{z_n\}_n$ in C_k such that $|z_n| \to \infty$. This would imply that ∞ is an accumulation point of C_k and so it would belong to C_k since C_k is closed. Hence, C_k is compact. Note that 0 and ∞ belong to the same connected component U of $\mathbb{C}_\infty \setminus C_k$ which is open and connected. Since C_k is bounded, with $C_k \subseteq B(0, R_k)$ we have that the connected set $\mathbb{C} \setminus B(0, R_k)$ is contained in U. Thus, $U \setminus \{\infty\}$ is open and connected in $\mathbb{C}$ and so pathwise connected. Thus we can find a simple infinite polygonal path γ_k joining 0 with ∞ (we can take it to be the union of a half line and a simple polygonal path of finite length). Since the range of Γ_k is connected and $\mathbb{C} \setminus \Gamma_k$ is connected, by Theorem 216, $\mathbb{C} \setminus \Gamma_k$ is simply connected and does not contain 0 and ∞. Hence, by Theorem 100 we can define a branch f_k of the logarithm in $\mathbb{C} \setminus \Gamma_k$. The connected set $C_1 \cap C_2$ lies in one connected component F of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. If $C_1 \cap C_2$ is empty we take F to be any connected component of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_2)$. In the first case, by adding a constant we can assume that $f_1 = f_2$ in F. Since the compact sets $C_1 \setminus F$ and $C_2 \setminus F$ are disjoint, we can find disjoint open sets V_1 and V_2 such that $C_k \setminus F \subset V_k \subset \mathbb{C} \setminus \Gamma_k$, $k = 1, 2$. Define

$$f(z) := \begin{cases} f_k(z) & z \in V_k,\ k = 1, 2, \\ f_1(z) = f_2(z) & z \in F. \end{cases}$$

Then f is holomorphic in the open set $V := V_1 \cup V_2 \cup F$ which contains $C_1 \cup C_2$ and $e^{f(z)} = z$ for all $z \in V$.

Assume by contradiction that $C_1 \cup C_2$ separates 0 and ∞. Then the connected component G of $\mathbb{C}_\infty \setminus (C_1 \cup C_2)$ which contains 0 is bounded. Note that $\partial G \subseteq \partial (C_1 \cup C_2)$ and since V contains $C_1 \cup C_2$ we have that $\partial V \cap \partial G = \emptyset$. Let $0 < \delta < \frac{1}{2} \operatorname{dist}(\partial V, \partial G)$ and consider a grid of closed squares with diameter less than δ and such that 0 lies in the interior of one of these squares, say $0 \in Q_1^\circ$. Note that Q_1 is contained in G. Let $Q_1, \ldots, Q_n$ be the closed squares contained in G. Since $\partial G \subset V$ and $0 < \delta < \frac{1}{2} \operatorname{dist}(\partial V, \partial G)$, we have that the sides of $Q_1, \ldots, Q_n$ which are not counted twice are contained in V. Since $f'(z) = \frac{1}{z}$ for $z \in V$, we have that

$$\operatorname{ind}_\Xi(0) = \sum_{k=1}^n \operatorname{ind}_{\partial Q_k}(0) = \sum_{k=1}^n \frac{1}{2\pi i} \int_{\partial Q_k} \frac{d\zeta}{\zeta} = \int_{\partial \Xi} f'(\zeta)\, d\zeta = 0.$$

On the other hand,

$$\operatorname{ind}_{\partial Q_1}(0) = \frac{1}{2\pi i}\int_{\partial Q_1}\frac{d\zeta}{\zeta} = 1,$$
$$\operatorname{ind}_{\partial Q_k}(0) = \frac{1}{2\pi i}\int_{\partial Q_k}\frac{d\zeta}{\zeta} = 0, \quad k \geq 2.$$

Hence, summing these two identities $\operatorname{ind}_{\Xi}(0) = 1$, which gives a contradiction. ■

We turn to the proof of Theorem 200.

Proof. (i) $\Longrightarrow$ (ii). Assume that f can be extended continuously to $\overline{B(0,1)}$ and still denote by f the extension. Then by the previous exercise, $f(\partial B(0,1)) = \partial U$. It follows that we can parametrize ∂U as

$$\varphi(t) = f(e^{it}), \quad t \in [0, 2\pi],$$

and so ∂U is the range of an oriented closed curve.

(ii) $\Longrightarrow$ (iii) This implication follows from that fact that the range of a continuous curve is locally connected.

(iii) $\Longrightarrow$ (iv) Assume that ∂U is locally connected. For every $\varepsilon > 0$ let $0 < \delta < \varepsilon$ be such that if $z, w \in \partial U$ with $0 < |z - w| < \delta$ there exists a compact connected set $F \subseteq \partial U$ such that $z, w \in F$ and $\operatorname{diam} F < \varepsilon$. Let $z, w \in \mathbb{C} \setminus U$ with $|z - w| < \delta$. If the closed segment $[z, w]$ does not intersect ∂U, then we take $F = [z, w]$. If $[z, w] \cap \partial U \neq \emptyset$, let z' and w' be the first and last points of $[z, w]$ where $[z, w]$ intersects ∂U. Since $|z' - w'| < \delta$ and $z', w' \in \partial U$, there exists a compact connected set $F \subseteq \partial U$ such that $z', w' \in F$ and $\operatorname{diam} F < \varepsilon$. But then $[z, z'] \cup F \cup [w', w]$ is a compact connected set in $\mathbb{C} \setminus U$ with diameter less than 3ε which contains z, w. Hence, $\mathbb{C} \setminus U$ is locally connected. ■

Wednesday, April 22, 2020

Proof. (iv) $\Longrightarrow$ (i) Assume that $\mathbb{C} \setminus U$ is locally connected. Without loss of generality we may assume that $f(0) = 0$. Since U is bounded, there exist $R_0 < R$ such that

$$B(0, R_0) \subseteq U \subseteq B(0, R). \tag{98}$$

We claim that f is uniformly continuous in $B(0,1) \setminus \overline{B(0,1/2)}$. Fix $0 < \varepsilon < R_0$. Since $\mathbb{C} \setminus U$ is locally connected we can find $0 < \delta < \varepsilon$ such that if $z_1, z_2 \in \mathbb{C} \setminus U$ with $0 < |z_1 - z_2| < \delta$ there exists a compact connected set $F \subseteq \mathbb{C} \setminus U$ such that $z_1, z_2 \in F$ and $\operatorname{diam} F < \varepsilon$. Let $0 < \rho < 1/4$ be such that $2\pi R(\log(1/\rho))^{-1/2} < \delta$.

Let $z, w \in B(0,1) \setminus \overline{B(0,1/2)}$ with $|z - w| < \rho$. We claim that

$$|f(z) - f(w)| < 2\varepsilon. \tag{99}$$

Assume by contradiction that

$$|f(z) - f(w)| \geq 2\varepsilon.$$

By applying Lemma 221 with $V = B(0,1)$ and $z_0 = z$ we can find $r \in (\rho, \sqrt{\rho})$ such that

$$L\left(f(C(r))\right) < \delta < \varepsilon, \tag{100}$$

where $C(r) := B(0,1) \cap \partial B(z,r)$. There are two cases. If $\overline{B(z,r)} \subset B(0,1)$. Then $C(r) = \partial B(z,r)$ and $f(\partial B(z,r))$ is the boundary of the simply connected open set $f(B(z,r))$ which contains $f(z)$ and $f(w)$. Since $|z-w| < \rho < r$, we have that $z, w \in B(z,\rho) \subset B(z,r)$, and so $f(z)$ with $f(w)$ belong to the interior of the closed curve $f(\partial B(z,r))$. Consider the segment S joining $f(z)$ with $f(w)$ and extend it on both sides until it meets $f(\partial B(z,r))$. The resulting segment has length bigger than 2ε, which contradicts the fact that $L\left(f(C(r))\right) < \delta < \varepsilon$ (the length is the supremum of the length of all polygonal paths made of segments with endpoints on $f(C(r))$).

Assume next that $\overline{B(z,r)} \cap \partial B(0,1) \neq \emptyset$. In view of (100) and Exercise 220, the continuous rectifiable curve $f(C(r))$ has endpoints a and $b \in \partial U \subset \mathbb{C} \setminus U$. In view of (100), $|b-a| \leq L\left(f(C(r))\right) < \delta$, and so, since $\mathbb{C} \setminus U$ is locally connected there exists a compact connected set $F \subseteq \mathbb{C} \setminus U$ such that $a, b \in F$ and $\operatorname{diam} F < \varepsilon$. Then $F \cup f(C(r))$ is a connected set and

$$F \cup f(C(r)) \subseteq B(a,\varepsilon). \tag{101}$$

On the other hand, by (98), the fact that $a \in \partial U$ and $\varepsilon < R_0$, we have that $0 \notin B(a,\varepsilon)$. Since $|f(z) - f(w)| \geq 2\varepsilon$, it follows that either $f(z)$ or $f(w)$ does not belong to $B(a,\varepsilon)$. Denote this point by c, so $c \notin B(a,\varepsilon)$. Using the fact that $0 \notin B(a,\varepsilon)$ in view of (101) we have that c and 0 are not separated by the connected set $F \cup f(C(r))$. On the other hand $c \in U$ and $f(0) = 0 \in U$ and so c and 0 are also not separated by $\mathbb{C} \setminus U$. Note that $(F \cup f(C(r))) \cap (\mathbb{C} \setminus U) = F$, which is connected. Hence, by Janiszweski's theorem c and 0 are not separated by $F \cup f(C(r)) \cup (\mathbb{C} \setminus U)$. Since the $\mathbb{C} \setminus (F \cup f(C(r)) \cup (\mathbb{C} \setminus U)) = U \setminus (F \cup f(C(r))$ is open, its connected components are open, and so pathwise connected. Hence, there exists a polygonal path in $U \setminus (F \cup f(C(r))$ which joins c and 0. Let $\gamma = [\varphi]$. Since f is a conformal map, $f^{-1} \circ \varphi$ is a curve joining $f^{-1}(c) \in \{z, w\}$ and 0. Moreover, its range its contained in $B(0,1) \setminus C(r) = B(0,1) \setminus \partial B(z,r)$.

Since $z, w \in B(0,1) \setminus \overline{B(0,1/2)}$ with $|z-w| < \rho < r$, we have that $z, w \in B(z,\rho) \subset B(z,r)$, while $\operatorname{dist}(0, B(0,1) \setminus \overline{B(0,1/2)}) = \frac{1}{2} > \sqrt{\rho} > r$. Hence, $0 \notin \overline{B(z,r)}$. In turn, any curve joining 0 and either z or w would intersect $\partial B(z,r)$, and so we have a contradiction. ■

Let $E \subset \mathbb{C}$ be a connected set and let $z \in E$. We say that z is a *cut point of* E is $E \setminus \{z\}$ is no longer connected. If we have a continuous simple arc, then every point except the endpoints is a cut point. If we have a closed simple curve then no point is a cut point.

Theorem 224 *Let $U \subset \mathbb{C}$ be an open bounded simply connected set and let f map conformally $B(0,1)$ onto U. Assume that ∂U is a closed oriented curve and denote by f the continuous extension of f to $\overline{B(0,1)}$ given by Theorem 200. Then $z \in \partial U$ is a cut point of ∂U if and only if the set $f^{-1}(\{z\})$ has more than one element and the components of $\partial U \setminus \{z\}$ are $f(I_k)$, where I_k are the components of $\partial B(0,1) \setminus f^{-1}(\{z\})$.*

Proof. Let $m = \operatorname{card} f^{-1}(\{z\}) \in \mathbb{N} \cup \{\infty\}$. Since $f : \overline{B(0,1)} \to \overline{U}$ is continuous, the set $f^{-1}(\{z\})$ is closed and so $\partial B(0,1) \setminus f^{-1}(\{z\})$ is relatively open and thus it can be written as a countable union of disjoint open maximal arcs I_k. In turn, we may write

$$\partial U \setminus \{z\} = f(\partial B(0,1) \setminus f^{-1}(\{z\})) = f\left(\bigcup_{k=1}^{m} I_k\right) = \bigcup_{k=1}^{m} f(I_k).$$

Since f is continuous and the sets I_k are connected we have that the sets $f(I_k)$ are connected. Note that if $f^{-1}(\{z\})$ is a singleton, then $\partial U \setminus \{z\} = f(I_1)$, which is connected, and so z is not a cut point of ∂U.

Conversely, assume that $m \geq 2$. Then the endpoints a and b of I_1 are distinct. Consider the oriented closed segment $\overrightarrow{ab}$ and let $\varphi(t) = tb + (1-t)a$, $t \in [0,1]$. Consider the continuous curve γ parametrized by $f \circ \varphi$. Since f is injective in $B(0,1)$ and $f(a) = f(b) = z$, we have that γ is a continuous simple closed curve with range in $U \cup \{z\}$. Let $\Sigma = f(\varphi([0,1))$ be its range. By the Jordan's curve theorem, $\mathbb{C} \setminus \Sigma$ has two connected components V_b and V_u, with V_b bounded and V_u unbounded, and with $\partial V_b = \partial V_u = \Sigma = f(\varphi([0,1))$.

Note that $\overline{B(0,1)} \setminus (\overrightarrow{ab} \cup f^{-1}(\{z\}))$ has two connected components E_1 and E_2. Since f is continuous and $f(\overline{B(0,1)} \setminus \overrightarrow{ab}) \subseteq \mathbb{C} \setminus \Sigma = V_b \cup V_u$, and since f maps connected sets into connected sets, we must have that $f(E_1)$ and $f(E_2)$ are contained in V_b or in V_u. But since $f : B(0,1) \to U$ is open, if we take $z_0 \in \overrightarrow{ab} \setminus \{a,b\}$, we can find a small ball $B(z_0, r)$ such that $f(B(z_{0,}r))$ is open and so there exists $B(f(z_0), \delta) \subseteq f(B(z_0, r))$. Since $f(z_0) \in \Sigma = \partial V_b = \partial V_u$, there must be points of $B(z_0, r)$ which end up in V_b and points which end up in V_u. Thus $f(E_1)$ and $f(E_2)$ are contained one in V_b and the other in V_u. Thus $f(I_1)$ and $\bigcup_{k=2}^{m} f(I_k)$ are not connected. In turn, z is a cut point of ∂U. ■

There are examples in which $f^{-1}(\{z\})$ has countably many elements.

We are now ready to prove Carathéodory's theorem.

Proof. Let $U \subset \mathbb{C}$ be an open bounded simply connected set and let f map conformally $B(0,1)$ onto U. If f has a continuous and injective extension to $\overline{B(0,1)}$ then ∂U is parametrized by $f(e^{it})$, $t \in [0, 2\pi]$, which is an oriented simple closed curve. Conversely assume that ∂U is the range of an oriented simple closed curve. In particular, ∂U is locally connected and it has no cut points. Then by Theorem 200, f can be extended continuously to $\overline{B(0,1)}$. By the previous theorem the set $f^{-1}(\{z\})$ is a singleton for every $z \in \partial U$, which implies that f is injective on $\partial B(0,1)$. This concludes the proof. ■

Remark 225 *Note that we actually proved that f has a continuous and injective extension to $\overline{B(0,1)}$ if and only if ∂U is locally connected and it has no cut points.*

20 Elliptic Functions

We are interested in meromorphic functions $f : \mathbb{C} \to \mathbb{C}_\infty$ which have two periods, that is, there exist $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ such that

$$f(z + \omega_1) = f(z), \quad f(z + \omega_2) = f(z)$$

for all $z \in \mathbb{C}$. A function with these properties is called *doubly periodic.*

Exercise 226 *Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be a doubly periodic meromorphic function with periods $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$. Assume that $\tau := \omega_1/\omega_2 \in \mathbb{R}$. Prove that f is either periodic with simple period or constant.,*

In view of the previous exercise, we can assume that $\operatorname{Im} \tau \neq 0$. Since τ and $\frac{1}{\tau}$ have imaginary parts of opposite sign, by interchanging ω_1 and ω_2, in what follows we can assume that $\operatorname{Im} \tau > 0$.

Consider the function

$$g(z) := f(\omega_1 z), \quad z \in \mathbb{C}.$$

Then

$$\begin{aligned} g(z+1) &= f(\omega_1 z + \omega_1) = f(\omega_1 z) = g(z), \\ g(z+\tau) &= f(\omega_1 z + \omega_1 \tau) = f(\omega_1 z + \omega_2) = f(\omega_1 z) = g(z). \end{aligned}$$

Moreover, g is meromorphic if and only if f is and it has the same number of zeros and of poles. Any other property of f can be deduced by the analogous property of g. Thus, in what follows we assume that f has periods 1 and τ, where $\operatorname{Im} \tau > 0$. By induction we have that

$$f(z + j + k\tau) = f(z) \quad \text{for all } z \in \mathbb{C} \text{ and } j, k \in \mathbb{Z}. \tag{102}$$

Consider the lattice

$$\Lambda := \{j + k\tau : \, j, k \in \mathbb{Z}\}. \tag{103}$$

We will show that Λ partitions $\mathbb{C}$ into pairwise disjoint parallelograms congruent to

$$P_0 := \{z \in \mathbb{C} : \, z = x + y\tau, \, 0 \le x < 1, \, 0 \le y < 1\}. \tag{104}$$

To be precise,

$$\mathbb{C} = \bigcup_{j,k \in \mathbb{Z}} (j + k\tau + P_0).$$

We say that 1 and τ *generate* the lattice Λ and we call P_0 the *fundamental parallelogram* of f.

We say that $z, w \in \mathbb{C}$ are *congruent modulo* Λ if

$$z = w + j + k\tau$$

for some $j, k \in \mathbb{Z}$ and we write $z \sim w$. Note that $z - w \in \Lambda$.

Remark 227 *If* $f : \mathbb{C} \to \mathbb{C}_\infty$ *is be a doubly periodic meromorphic function with periods* $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ *such that* $\omega_1/\omega_2 \notin \mathbb{R}$*, then we define*

$$P_0 = \{z \in \mathbb{C} : \; z = x\omega_1 + y\omega_2, \; 0 \le x < 1, \; 0 \le y < 1\}$$

the fundamental parallelogram *of* f*.*

Theorem 228 *Let* $f : \mathbb{C} \to \mathbb{C}_\infty$ *be a doubly periodic meromorphic function with periods* 1 *and* τ*, where* $\operatorname{Im}\tau > 0$*. Then*

(i) every point in $\mathbb{C}$ *is congruent modulo* Λ *to a unique point in the fundamental parallelogram* P_0*,*

(ii) given $j, k \in \mathbb{Z}$*, every point in* $\mathbb{C}$ *is congruent modulo* Λ *to a unique point in the parallelogram* $j + k\tau + P_0$*,*

(iii) we have

$$\mathbb{C} = \bigcup_{j,k\in\mathbb{Z}} (j + k\tau + P_0),$$

where the interiors of the parallelograms are parwise disjoint,

(iv) the function f *is completely determined by its values in* P_0*.*

Proof. (i) Since the vectors 1 and τ form a basis over the reals of the two-dimensional vector space $\mathbb{C}$, given $z \in \mathbb{C}$, we can write $z = x + \tau y$, for some $x, y \in \mathbb{R}$. Let $j, k \in \mathbb{Z}$ be such that $j \le x < j + 1$ and $k \le y < k + 1$. Then

$$w := z - j - k\tau = (x - j) + (y - k)\tau$$

is congruent to z modulo Λ. Moreover, $0 \le x - j < 1$ and $0 \le y - k < 1$, and so $w \in P_0$.

To prove uniqueness, let $w_1, w_2 \in P_0$ be congruent modulo Λ. Then $w_l = x_l + y_j\tau$, where $0 \le x_l < 1$ and $0 \le y_l < 1$, $l = 1, 2$. Since $w_1 \sim w_2$ we have that

$$x_1 + y_1\tau - x_2 - y_2\tau = w_1 - w_2 = j + k\tau$$

for some $j, k \in \mathbb{Z}$. But since $0 \le x_1, x_2 < 1$, we have that $-1 < x_1 - x_2 < 1$ and so $j = x_1 - x_2 = 0$. Similarly, $k = y_1 - y_2 \in (-1, 1)$ and so $k = 0$. Thus $w_1 = w_2$.

(ii) Let $P := j_0 + k_0\tau + P_0$, where $j_0, k_0 \in \mathbb{Z}$. Given $z \in \mathbb{C}$ by item (i) there exists a unique $w \in P_0$ with $z \sim w$. In turn, $j_0 + k_0\tau + w \in P$ and $z \sim j_0 + k_0\tau + w$. By the uniqueness in part (i), it follows that $j_0 + k_0\tau + w$ is the unique point in P which is congruent to z modulo Λ.

(iii) By part (i) each $z \in \mathbb{C}$ is congruent to some $w \in P_0$ modulo Λ, which means that $z = j + k\tau + w$ for some $w \in P_0$. Hence, $z \in j + k\tau + P_0$.

On the other hand, if $P_1 = j_1 + k_1\tau + P_0$ and $P_2 = j_2 + k_2\tau + P_0$, and $z \in P_1 \cap P_2$, then

$$z = j_1 + k_1\tau + w_1 = j_2 + k_2\tau + w_2$$

with $w_1, w_2 \in P_0$. This means that $z \sim w_1$ and $z \sim w_2$. Again by the uniqueness in item (i), $w_1 = w_2$. In turn, $j_1 + k_1\tau = j_2 + k_2\tau$, which implies that $j_1 = j_2$ and $k_1 = k_2$.

(iv) In view of (102),

$$f(z) = f(w) \quad \text{if } z \sim w.$$

The result now follows from item (i). ■

Next we show why we are taking meromorphic functions instead of holomorphic functions.

Corollary 229 *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and doubly periodic with periods 1 and τ, where $\operatorname{Im} \tau > 0$. Then f is constant.*

Proof. Let $M := \max_{\overline{P_0}} |f|$. By item (iv) of the previous theorem for every $z \in \mathbb{C}$ there exists $w \in P_0$ such that $f(z) = f(w)$. Hence, $|f(z)| = |f(w)| \leq M$. It follows by Liouville's theorem that f is constant. ■

Definition 230 *An* elliptic function *is a meromorphic function which is doubly periodic with periods $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ such that $w_1/w_2 \notin \mathbb{R}$.*

We begin by showing that an elliptic function must have more than one pole.

Theorem 231 *Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be an elliptic function. Then f must have at least two poles.*

Proof. Without loss of generality we may assume that the periods are 1 and τ with $\operatorname{Im} \tau > 0$.

Step 1: Assume that f has no poles on ∂P_0. Then by the residue theorem

$$\int_{\partial P_0} f\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f,$$

where $z_1, \ldots, z_n$ are the poles of f inside P_0. Note that there must be at least one in view of the previous two theorems. With a slight abuse of notation we write

$$\int_{\partial P_0} f\, dz = \int_0^1 f\, dz + \int_1^{1+\tau} f\, dz + \int_{1+\tau}^{\tau} f\, dz + \int_\tau^0 f\, dz.$$

Note that by (102),

$$\begin{aligned}\int_0^1 f\, dz + \int_{1+\tau}^{\tau} f\, dz &= \int_0^1 f\, dz + \int_1^0 f(\tau + w)\, dw \\ &= \int_0^1 f\, dz + \int_1^0 f(w)\, dw = \int_0^1 f\, dz - \int_0^1 f(z)\, dz = 0,\end{aligned}$$

and similarly,

$$\int_1^{1+\tau} f\, dz + \int_\tau^0 f\, dz = 0.$$

Hence,

$$0 = \int_{\partial P_0} f\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f.$$

If n were 1, we would have $0 = \operatorname{res}_{z_1} f$, which would impliy that f has a removable singularity at z_1 by Theorems and . This would contradict the previous corollary. Hence, $n \geq 2$. ■

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Proof. Step 2: Since poles do no accumulate in the interior, it follows that f has a finite number of poles in $\overline{P_0}$. Hence, if $B(0,R)$ contains $\overline{P_0}$ then by periodicity f has a finite number of poles in $B(0,R)$. In turn, for $\varepsilon > 0$ the function $f_\varepsilon(z) := f(z + \varepsilon(1+\tau))$ has no poles on ∂P_0. By the previous step we find that f_ε has at least two poles in P_0 for every ε small. Letting $\varepsilon \to 0$ we conclude that f has at least two poles. ■

The number of poles of an elliptic function in its fundamental parallelogram counted with their multiplicity is called its *order*. Next we show that the number of zeros of an elliptic function equals the number of poles.

Theorem 232 *Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be an elliptic function of order ℓ. Then f has ℓ zeros in its fundamental parallelogram counted with their multiplicity.*

Proof. Without loss of generality we may assume that the periods are 1 and τ with $\operatorname{Im} \tau > 0$. Since zeros and poles do no accumulate in the interior, it follows that f has a finite number of poles and zeros in $\overline{P_0}$.

Step 1: Assume that f has no poles and no zeros on ∂P_0. By the argument principle,

$$\frac{1}{2\pi i}\int_{\partial P_0} \frac{f'}{f}\,dz = (\text{number of zeros of } f \text{ in } P_0) \text{ minus } (\text{number of poles of } f \text{ in } P_0)$$
$$=: n_z - \ell.$$

Since $\frac{f'}{f}$ is doubly periodic with periods 1 and τ, reasoning as in the previous theorem, we can show that $\frac{1}{2\pi i}\int_{\partial P_0} \frac{f'}{f}\,dz = 0$. Hence, $n_z = \ell$.

Step 2: Since poles and zeros do no accumulate in the interior, it follows that f has a finite number of poles and zeros in $\overline{P_0}$. Hence, if $B(0,R)$ contains $\overline{P_0}$ then by periodicity f has a finite number of poles and zeros in $B(0,R)$. In turn, for $\varepsilon > 0$ the function $f_\varepsilon(z) := f(z+\varepsilon(1+\tau))$ has no poles or zeros on ∂P_0. By previous step we find that the number of zeros of f_ε in P_0 is the same as the number of poles of f_ε in P_0 for every ε small. Letting $\varepsilon \to 0$ we conclude that the number of zeros of f in P_0 is the same as the number of poles of f in P_0. ■

The next natural question is the existence of elliptic functions. We will construct an elliptic function of order two. The idea is to consider the function

$$\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2}$$

but the problem is that this double series does not converge absolutely. Indeed we will see below that for a double series to converge we need the exponent to be bigger than 2. To fix this problem, we follow the approach in your homework for cot and we define the function

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda_*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right],$$

where $\Lambda_* := \Lambda \setminus \{0\}$. This function is called Weierstrass $\wp$ function. Note that

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2 - z^2 - 2z\omega - \omega^2}{\omega^2(z+\omega)^2} = \frac{-z^2 - 2z\omega}{\omega^2(z+\omega)^2} \sim -\frac{2z}{\omega^3}$$

as $|\omega| \to \infty$.

Theorem 233 *The Weierstrass $\wp$ function is an elliptic function of order two.*

We begin with a preliminary result.

Lemma 234 *The double series*

$$\sum_{(j,k)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(|j|+|k|)^r}, \qquad \sum_{j+k\tau\in\Lambda_*} \frac{1}{|j+k\tau|^r}$$

converge if and only if $r > 2$.

Proof. Step 1: Assume that $r > 2$. For every $j \neq 0$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{1}{(|j|+|k|)^r} &= \frac{1}{|j|^r} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(|j|+|k|)^r} = \frac{1}{|j|^r} + 2\sum_{k\in\mathbb{N}} \frac{1}{(|j|+k)^r} \\ &= \frac{1}{|j|^r} + 2\sum_{n=|j|+1} \frac{1}{n^r} \le \frac{1}{|j|^r} + 2\int_{|j|}^{\infty} \frac{dx}{x^r} = \frac{1}{|j|^r} + \frac{2}{r-1}\frac{1}{|j|^{r-1}}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{(j,k)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(|j|+|k|)^r} &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(0+|k|)^r} + \sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{k\in\mathbb{Z}} \frac{1}{(|j|+|k|)^r} \\ &\le 2\sum_{k\in\mathbb{N}} \frac{1}{k^r} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{|j|^r} + \frac{2}{r-1}\frac{1}{|j|^{r-1}}\right) < \infty\end{aligned}$$

since $r > 2$.

To prove that the second series converges, it suffices to show that there exists a constant $c > 0$ such that

$$|j + k\tau| \ge c(|j| + |k|)$$

for all $(j,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Write $\tau = x + iy$, where $x \in \mathbb{R}$ and $y > 0$. Then

$$|j + k\tau| = \sqrt{(j+kx)^2 + k^2y^2} \ge \frac{1}{2}(|j+kx| + |ky|).$$

If $x = 0$, then

$$\frac{1}{2}(|j| + |ky|) \ge \frac{\min\{1,y\}}{2}(|j| + |k|).$$

Assume that $x \neq 0$. If $|j| \leq 2|kx|$, then

$$
\begin{aligned}
|j + kx| + |ky| \geq |ky| &= \frac{1}{2}|kx|\frac{|y|}{|x|} + \frac{1}{2}|ky| \geq \frac{1}{4}\frac{|y|}{|x|}|j| + \frac{1}{2}|ky| \\
&\geq \frac{|y| \min\{1/|x|, 1\}}{4}(|j| + |k|).
\end{aligned}
$$

If $|j| \geq 2|kx|$, then

$$
|j + kx| + |ky| \geq |j| - |kx| + |ky| \geq \frac{1}{2}|j| + |ky| \geq \frac{\min\{1, y\}}{2}(|j| + |k|).
$$

This concludes the proof of the case $r > 2$.

Step 2: Assume that $r \leq 2$. If $1 \leq k \leq j$ then $j + k \leq 2j$ and so $\frac{1}{j+k} \geq \frac{1}{2j}$. Then

$$
\sum_{(j,k)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(|j| + |k|)^r} \geq \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{1}{(j + k)^r} \geq \sum_{j=1}^{\infty}\sum_{k=1}^{j} \frac{1}{(2j)^r} = \sum_{j=1}^{\infty} \frac{j}{(2j)^r} = \infty.
$$

To prove that the second series diverges, it suffices to show that there exists a constant $c > 0$ such that

$$
|j + k\tau| \leq c(|j| + |k|)
$$

for all $(j, k) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. We have

$$
|j + k\tau| \leq |j| + |k\tau| = |j| + |k||\tau| \leq \max\{1, |\tau|\}(|j| + |k|),
$$

which concludes the proof. ■

We turn to the proof of Weierstrass theorem.

Proof. Let $R > 0$ and let $|z| < R$. Write

$$
\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{|\omega|\leq 2R}\left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}\right] + \sum_{|\omega|>2R}\left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}\right] \\
&=: I + II + III.
\end{aligned}
$$

To estimate III observe that for $|z| < R$ and $|\omega| > 2R$,

$$
|z + \omega| \geq |\omega| - |z| \geq \frac{1}{2}|\omega| + R - |z| \geq \frac{1}{2}|\omega|,
$$

and so

$$
\begin{aligned}
\left|\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{-z^2 - 2z\omega}{\omega^2(z + \omega)^2}\right| &\leq 2\frac{R^2 + 2R|\omega|}{|\omega|^4} \\
&\leq 2\frac{4R|\omega|}{|\omega|^4} = 8R\frac{1}{|\omega|^3}.
\end{aligned}
$$

Hence

$$|III| \leq 8R \sum_{j+k\tau\in\Lambda_*} \frac{1}{(|j+k\tau|)^3}$$

which converges by the previous lemma.

The term II is a finite sum and so it is a meromorphic function in $B(0,R)$ with double poles at those $\omega \in \Lambda_*$ inside $B(0,R)$.

This shows that $\wp$ is well-defined and meromorphic with double poles at each point of the lattice Λ. To prove that $\wp$ is doubly periodic with periods 1 and τ we compute the derivative of $\wp$. We have

$$\wp'(z) = -\frac{2}{z^3} - \sum_{\omega\in\Lambda_*} \frac{2}{(z+\omega)^3} = -\sum_{\omega\in\Lambda} \frac{2}{(z+\omega)^3}.$$

Note that by the previous lemma the series converges absolutely whenever $z \notin \Lambda$. Let's prove that $\wp'$ has periods 1 and τ. Since $\omega+1 \in \Lambda$ and $\omega+\tau \in \Lambda$ whenever $\omega \in \Lambda$, we have

$$\wp'(z+1) = -\sum_{\omega\in\Lambda} \frac{2}{(z+1+\omega)^3} = -\sum_{\zeta\in\Lambda} \frac{2}{(z+\xi)^3} = \wp'(z),$$

$$\wp'(z+\tau) = -\sum_{\omega\in\Lambda} \frac{2}{(z+\tau+\omega)^3} = -\sum_{\zeta\in\Lambda} \frac{2}{(z+\xi)^3} = \wp'(z).$$

Hence, there exist $a, b \in \mathbb{C}$ such that

$$\wp(z+1) = \wp(z) + a, \quad \wp(z+\tau) = \wp(z) + b \tag{105}$$

for all $z \in \mathbb{C} \setminus \Lambda$.

Using the fact that $\omega \in \Lambda$ if and only if $-\omega \in \Lambda$ we have that

$$\begin{aligned}\wp(-z) &= \frac{1}{(-z)^2} + \sum_{\omega\in\Lambda_*} \left[\frac{1}{(-z+\omega)^2} - \frac{1}{\omega^2}\right] \\ &= \frac{1}{z^2} + \sum_{\omega\in\Lambda_*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{(-\omega)^2}\right] = \wp(z).\end{aligned}$$

This shows that $\wp$ is even. Taking $z = -\frac{1}{2}$ and $z = -\frac{\tau}{2}$ in (105) gives $a = 0$ and $b = 0$. We have proved that $\wp$ is doubly periodic with periods 1 and τ. Since the only element of Λ inside the fundamental parallelogram is 0, $\wp$ has order 2. ■

Wednesday, April 29, 2020

Next we show some important properties of the function $\wp$.

Theorem 235 *The function $\wp$ satisfies the equality*

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),$$

where

$$e_1 := \wp(1/2), \quad e_2 := \wp(\tau/2), \quad e_3 := \wp((1+\tau)/2). \tag{106}$$

Proof. Since $\wp$ is even, $\wp'$ is odd, and so using also the fact that $\wp'$ is periodic of period 1,

$$\wp'(1/2) = -\wp'(-1/2) = -\wp'(-1/2+1) = -\wp'(1/2),$$

which implies that $\wp'(1/2) = 0$. Similalrly,

$$\wp'(\tau/2) = -\wp'(-\tau/2) = -\wp'(-\tau/2+\tau) = -\wp'(\tau/2),$$

and so $\wp'(\tau/2) = 0$. Finally,

$$\wp'((1+\tau)/2) = -\wp'(-(1+\tau)/2) = -\wp'(-(1+\tau)/2+1+\tau) = -\wp'((1+\tau)/2),$$

which implies that $\wp'((1+\tau)/2)$. Since $\wp'$ is an elliptic function of order 3, it follows from Theorem 232, that it has three zeros in the fundamental parallelogram P_0 (already counted with their multiplicity). Hence, $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$ are simple zeros of $\wp'$ and they are the only ones in P_0.

Since the function $\wp - e_1$ is elliptic of order two, and it has a double zero at $\frac{1}{2}$ (since its derivative has a simple zero), it follows from Theorem 232 that $\wp - e_1$ has no other zeros in P_0. Similarly, $\wp - e_2$ and $\wp - e_3$ have a double zero at $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$, respectively, and no other zeros in P_0.

Consider the function

$$g(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

The only zeros of g in P_0 are at $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$ and they have multiplicity 2. Moreover, g is an elliptic function of with poles at Λ. Since 0 is the only pole in P_0, it has multiplicity 6 by Theorem 232. Thus, every pole in Λ has multiplicity 6.

On the other hand, since $\wp'$ has poles of multiplicity 3 at Λ, $(\wp')^2$ has poles of multiplicity 6 at Λ. Also, by what we did before it only has zeros of multiplicity 2 at $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$. Thus, if we consider the function $(\wp')^2/g$, we have that it has removable singularities at each point of Λ and at $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$ (and their periodic translates). Hence, $(\wp')^2/g$ can be extended to an entire function. Since it is doubly periodic with periods 1 and τ, by Corollary 229, $(\wp')^2/g$ is constant.

We have seen in the proof of Theorem 233 that if we take $R > 0$ so small that $B(0, 2R) \cap \Lambda = \{0\}$, then the function $\sum_{|\omega|>2R} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = \sum_{\omega \in \Lambda_*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$ is holomorphic in $B(0, R)$. Hence,

$$\lim_{z \to 0} z^2 \wp(z) = 1.$$

Similarly,

$$\lim_{z \to 0} z^3 \wp'(z) = -2.$$

It follows that

$$c = \lim_{z \to 0} \frac{z^6 (\wp')^2}{z^6 g(z)} = \frac{4}{1}.$$

This completes the proof. ■

Remark 236 *The numbers $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$ are called* half-periods. *It follows from the previous proof that $\wp'$ restricted to P_0 has three simple zeros at the half-periods and no other zeros. Hence, for every $a \in P_0$, the function $\wp - \wp(a)$ has a double zero at a if a is a half-period and otherwise a simple zero at a and $-a$ since $\wp$ is even.*

We now demonstrate the importance of the function $\wp$.

Theorem 237 *Every elliptic function $f : \mathbb{C} \to \mathbb{C}_\infty$ with periods 1 and τ, where $\operatorname{Im} \tau > 0$, is a rational function of $\wp$ and $\wp'$.*

Proof. We want to construct a doubly periodic elliptic function g using $\wp$ which has the same zeros and poles of f.

Step 1: Assume that f is even. Then if f has a zero or a pole at some $a \in P_0 \setminus \{0\}$, then it also has a zero or a pole at $-a$. Note that $-a$ is congruent to a modulo Λ if and only if a is a half-period. Indeed, if

$$a = -a + j + k\tau$$

for some $j, k \in \mathbb{Z}$, then $a = \frac{1}{2}j + \frac{1}{2}k\tau \in P_0$, which can happen only if $j, k \in \{0, 1\}$.

Substep 1: Assume that f has no zeros or poles at the origin and at the half-periods. We recall that by Theorem 232, if f has order ℓ, then it has ℓ zeros. Let $a_1, \ldots, a_\ell \in P_0 \setminus \{0\}$ be the zeros of f in P_0 counted with their multiplicity and let $b_1, \ldots, b_\ell \in P_0 \setminus \{0\}$ be the poles of f in P_0 counted with their multiplicity. We claim that

$$f(z) = f(0) \prod_{n=1}^{\ell} \frac{\wp(z) - \wp(a_n)}{\wp(z) - \wp(b_n)}.$$

Indeed, let g denote the function on the right-hand side. In view of the previous remark $\wp - \wp(a_n)$ has a simple zero at a_n while the function $\frac{1}{\wp - \wp(b_n)}$ has a simple pole at b_n. Thus, the function g has the same zeros and poles in P_0 as f. It follows that f/g has removable singularities at a_n and at b_n, $n = 1, \ldots, \ell$. Thus f/g can be extended to a doubly periodic entire function, and so it must be constant in view of Corollary 229. Using the fact that

$$\lim_{z \to 0} z^2 \wp(z) = 1,$$

we obtain that the constant must be $f(0)$.

Substep 2: If f has a zero at at a half-period a, then the zero must have even multiplicity. Indeed $f^{(2n+1)}$ is odd and we can reason as in the proof of Theorem 235 to show that $f^{(2n+1)}$ vanishes at all the half-periods. Similarly, if f has a pole at a half-period a, then $\frac{1}{f}$ is still an even elliptic function with the same periods and so the pole must have even multiplicity. Recalling that $\wp - \wp(a)$ has a double zero if a is a half-period and a pole of multiplicity two at the origin we can find integers $k_0, \ldots, k_3 \in \mathbb{Z}$ such that $\wp^{k_0}$ behaves like f near $z = 0$ and $(\wp(z) - e_j)^{k_j}$, $j = 1, 2, 3$, behaves like f near $\frac{1}{2}$, $\frac{\tau}{2}$, and $\frac{1+\tau}{2}$, respectively. Let $a_1, \ldots, a_n \in P_0 \setminus \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$ be the other zeros of f in P_0 counted with their multiplicity and let $b_1, \ldots, b_m \in P_0 \setminus \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$ be the other poles of f in P_0 counted with their multiplicity. Consider the function

$$g(z) := \wp^{k_0}(z) \prod_{j=1}^{3} (\wp(z) - e_j)^{k_j} \prod_{j=1}^{n} (\wp(z) - \wp(a_j)) \prod_{j=1}^{m} \frac{1}{\wp(z) - \wp(b_j)},$$

where

$$2(k_0 + k_1 + k_2 + k_2) + n - m = 0$$

by Theorem 232. The function g has the same zeros and poles in P_0 as f. It follows that f/g has removable singularities at 0, 0, $\frac{1}{2}$, $\frac{\tau}{2}$, $\frac{1+\tau}{2}$, and at all the a_j and b_k. Thus f/g can be extended to a doubly periodic entire function, and so it must be constant in view of Corollary 229.

Step 2: If f is odd, then $f/\wp'$ is an even elliptic function and so by the previous step it can be written as a rational function of $\wp$. Finally, in the general case we can write f as the sum of an even function and an odd function, to be precise,

$$f(z) = \frac{1}{2}[f(z) + f(-z)] + \frac{1}{2}[f(z) - f(-z)].$$

This concludes the proof. ■

Friday, May 1, 2020

No class.