Monday, January 11, 2010

1 Vector Spaces

Definition 1 *A* vector space*, or* linear space*, over* $\mathbb{R}$ *is a nonempty set* X*, whose elements are called* vectors*, together with two operations,* addition *and* multiplication by scalars*,*

$$\begin{array}{ccc} X \times X \to X & & \mathbb{R} \times X \to X \\ (x, y) \mapsto x + y & \text{and} & (t, x) \mapsto tx \end{array}$$

with the properties that

(i) $(X, +)$ *is a commutative group, that is,*

- *(a)* $x + y = y + x$ *for all* $x, y \in X$ *(commutative property),*
- *(b)* $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in X$ *(associative property),*
- *(c) there is a vector* $0 \in X$*, called* zero*, such that* $x + 0 = 0 + x$ *for all* $x \in X$*,*
- *(d) for every* $x \in X$ *there exists a vector in* X*, called the* opposite *of* x *and denoted* $-x$*, such that* $x + (-x) = 0$*,*

(ii) for all $x, y \in X$ *and* $s, t \in \mathbb{R}$*,*

- *(a)* $s\,(tx) = (st)\,x$*,*
- *(b)* $1x = x$*,*
- *(c)* $s\,(x + y) = (sx) + (sy)$*,*
- *(d)* $(s + t)x = (sx) + (tx)$*.*

Remark 2 *Instead of using real numbers, one can use* $\mathbb{C}$ *or a field* F*. For most or our purposes the real numbers will suffice.* From now on, whenever we don't specify, it is understood that a vector space is over $\mathbb{R}$.

Example 3 *Some important examples of vector spaces over* $\mathbb{R}$ *are the following.*

(i) The Euclidean space $\mathbb{R}^N$*.*

(ii) The collection of all polynomials $p : \mathbb{R} \to \mathbb{R}$*.*

(iii) The space of continuous functions $C\,(X)$*, where* (X, τ) *is a topological space.*

Definition 4 *Let* X *be a vector space over* $\mathbb{R}$ *and let* $E \subset X$*. The set* E *is said to be*

(i) convex *if* $tx_1 + (1 - t)\,x_2 \in E$ *for all* $x_1, x_2 \in E$ *and all* $t \in [0, 1]$*,*

(ii) *a* subspace *of* X *if* $sx_1 + tx_2 \in E$ *for all* $x_1, x_2 \in E$ *and all* $s, t \in \mathbb{R}$.

Examples??

Exercise 5 *Let* X *be a vector space over* $\mathbb{R}$ *and let* $E \subset X$. *Prove that* E *is convex if and only if*

$$sE + tE = (s + t) E$$

for all $s, t \geq 0$.

The arbitrary intersection of convex sets is still convex, but in general the union is not (the simplest example is the union of two disjoint closed segments on the real line).

Proposition 6 *Let* X *be a vector space and let* $\{E_\alpha\}_{\alpha \in \Lambda}$ *be an arbitrary family of subspaces (respectively, convex subsets) of* X. *Then*

$$E_- := \bigcap_{\alpha \in \Lambda} E_\alpha$$

is a subspace (respectively, a convex set) of X. *If* $\{E_\alpha\}_{\alpha \in \Lambda}$ *is totally ordered with respect to inclusion (that is, for* $\alpha, \beta \in \Lambda$, *either* $E_\alpha \subset E_\beta$ *or* $E_\beta \subset E_\alpha$*), then*

$$E_+ := \bigcup_{\alpha \in \Lambda} E_\alpha$$

is a subspace (respectively, a convex set) of X.

Proof. We give the proof only for convex sets. Let $x_1, x_2 \in E_-$ and $t \in [0, 1]$. Then $x_1, x_2 \in E_\alpha$ for all $\alpha \in \Lambda$, and since E_α is convex, $tx_1 + (1 - t) x_2 \in E_\alpha$ for all $\alpha \in \Lambda$. Hence $tx_1 + (1 - t) x_2 \in E_-$.

Let $x_1, x_2 \in E_+$ and $t \in [0, 1]$. Then $x_1 \in E_\alpha$ and $x_2 \in E_\beta$ for some $\alpha, \beta \in \Lambda$. Then either $E_\alpha \subset E_\beta$ or $E_\beta \subset E_\alpha$, say, $E_\beta \subset E_\alpha$. It follows that $x_1, x_2 \in E_\alpha$, and since E_α is convex, $tx_1 + (1 - t) x_2 \in E_\alpha$. Hence $tx_1 + (1 - t) x_2 \in E_+$. ■

Let X be a vector space. If $x_1, \ldots, x_n \in X$ and $t_1, \ldots, t_n \in \mathbb{R}$, then the vector $x := t_1 x_1 + \ldots + t_n x_n$ is called a *linear combination* of $x_1, \ldots, x_n \in X$. If in addition $t_i \in [0, 1]$ and $t_1 + \ldots + t_n = 1$, then $x := t_1 x_1 + \ldots + t_n x_n$ is called a *convex combination* of $x_1, \ldots, x_n \in X$.

Proposition 7 *Let* X *be a vector space. A set* $E \subset X$ *is a subspace (respectively a convex set) if and only if every linear (respectively convex) combination of elements of* E *belongs to* E.

Proof. We give the proof only for convex sets. If a set contains every convex combination of its elements, then it is convex (take $n = 2$). Conversely, assume that E is convex. The proof is by induction on the number n of elements in the convex combination. If $n = 2$, this is just the definition that E is convex. Assume that the result is true for $n \in \mathbb{N}$ and let's prove it for $n + 1$. Let $x_1, \ldots, x_{n+1} \in E$, $t_1, \ldots, t_{n+1} \in [0, 1]$, with $t_1 + \ldots + t_{n+1} = 1$, and $x :=$

$t_1x_1 + \ldots + t_{n+1}x_{n+1}$. If $t_{n+1} = 1$, then $x = x_{n+1} \in X$ and there is nothing to prove. Thus, assume that $t_{n+1} < 1$. Then $t_1 + \ldots + t_n = 1 - t_{n+1} > 0$, and so we may rewrite x as

$$x = (t_1 + \ldots + t_n)\left(\frac{t_1}{t_1 + \ldots + t_n}x_1 + \ldots + \frac{t_n}{t_1 + \ldots + t_n}x_n\right) + t_{n+1}x_{n+1}.$$

By the inductive hypothesis, the point

$$z := \frac{t_1}{t_1 + \ldots + t_n}x_1 + \ldots + \frac{t_n}{t_1 + \ldots + t_n}x_n$$

belongs to E, and since $x = (t_1 + \ldots + t_n)\, z + t_{n+1}x_{n+1}$, by the convexity of the set E we have that x belongs to E and the proof is complete. ■

Given any set $E \subset X$, the *convex hull* $\operatorname{co}(E)$ is the intersection of all convex sets that contain E.

Analogously, the *span of* E, denote $\operatorname{span} E$ is the intersection of all subspaces that contain E.

Definition 8 *Given a vector space X, a set of vectors $E \subset X$ is called*

(i) linearly independent *if whenever $\sum_{i=1}^{n} t_i x_i = 0$ for some $t_i \in \mathbb{R}$, $x_i \in E$, $i = 1, \dots, n$, then $t_1 = t_2 = \cdots = t_n = 0$,*

(ii) a basis *if it is linearly independent and* $\operatorname{span} E = X$*. The cardinality of E is called the* dimension *of X.*

It may be shown that all bases have the same cardinality.

Proposition 9 *Let X be a vector space and let $E \subset X$. Then*

$$\operatorname{co}(E) = \left\{ \sum_{i=1}^{n} t_i x_i : n \in \mathbb{N}, \sum_{i=1}^{n} t_i = 1, t_i \geq 0, x_i \in E, i = 1, \dots, n \right\}. \tag{1}$$

Proof. If F is any convex set that contains E, then it must contain all convex combinations of elements of E, and so

$$F \supset \left\{ \sum_{i=1}^{n} t_i x_i : n \in \mathbb{N}, \sum_{i=1}^{n} t_i = 1, t_i \geq 0, x_i \in E, i = 1, \dots, n \right\} =: G.$$

Since this holds for all convex sets containing E, it follows that

$$\operatorname{co}(E) \supset G.$$

To prove the opposite inclusion, it suffices to show that G is convex and contains E. The latter assertion follows from the fact that if $u \in E$, then we can take $n = 1$ and $t_1 = 1$. To show that G is convex, let $0 \leq t \leq 1$ and let $x, y \in X$ be of the form

$$x = \sum_{i=1}^{n} s_i x_i, \quad y = \sum_{j=1}^{l} t_j y_j,$$

where $\sum_{i=1}^{n} s_i = \sum_{j=1}^{l} t_j = 1$, $s_i, t_j \geq 0$, $x_i, y_j \in E$, $i = 1, \dots, n$, $j = 1, \dots, l$. Note that without loss of generality, we may always assume that $n = l$. Indeed, if $n \neq l$, say $n < l$, then it suffices to set $s_{n+1}, \dots, s_l := 0$ and $x_{n+1}, \dots, x_n := x_1$. Then

$$tx + (1-t)\,y = \sum_{i=1}^{n} t s_i x_i + \sum_{i=1}^{n} (1-t)\,t_i y_i,$$

which is still a convex combination of elements of E, and so it belongs to G. ■

Reasoning as in the proof of Proposition 9, it can be shown that

$$\operatorname{span} E = \left\{ \sum_{i=1}^{n} t_i x_i : n \in \mathbb{N},\ t_i \in \mathbb{R},\ x_i \in E,\ i = 1, \dots, n \right\}. \tag{2}$$

Note that without loss of generality, in (1) one may consider only positive coefficients t_i. Carathéodory's theorem improves (1) in that it limits the number of terms in the convex combination to at most $N + 1$.

Theorem 10 (Carathéodory) *Let $E \subset \mathbb{R}^N$. Then*

$$\mathrm{co}E = \left\{ \sum_{i=1}^{N+1} t_i x_i : \sum_{i=1}^{N+1} t_i = 1,\, t_i \geq 0,\, x_i \in E,\, i = 1, \dots, N+1 \right\}.$$

Proof. Fix $x \in \mathrm{co}E$ and let

$$S := \{\ell \in \mathbb{N} : x \text{ is a convex combination of } \ell \text{ vectors of } E\}.$$

Note that by the previous proposition, S is nonempty. Let $k := \min S$. We claim that $k \leq N + 1$. Assume by contradiction that $k > N + 1$ and let

$$x = \sum_{i=1}^{k} t_i x_i,$$

where $\sum_{i=1}^{k} t_i = 1$, $t_i \in (0, 1)$, $x_i \in E$, $i = 1, \dots, k$. Since $k - 1 > N$, the $k - 1$ vectors $x_2 - x_1, \dots, x_k - x_1$ are linearly dependent, and so we may find $s_2, \dots, s_k \in \mathbb{R}$ not all zero such that

$$\sum_{i=2}^{k} s_i (x_i - x_1) = 0.$$

Let $s_1 := -\sum_{i=2}^{k} s_i$. Then $\sum_{i=1}^{k} s_i x_i = 0$ and $\sum_{i=1}^{k} s_i = 0$. Since not all the s_i are zero, there must be positive ones. Define

$$c := \min \left\{ \frac{t_i}{s_i} : s_i > 0,\ i = 1, \dots, k \right\}$$

and let m be such that $c = \frac{t_m}{s_m}$. Then $t_i - cs_i \geq 0$ for all $i = 1, \dots, k$ (if $s_i > 0$, then this follows from the definition of c, while if $s_i \leq 0$, then $-cs_i \geq 0$), $t_m - cs_m = 0$, and

$$\sum_{i=1}^{k} (t_i - cs_i) = \sum_{i=1}^{k} t_i - c \sum_{i=1}^{k} s_i = 1 - 0.$$

Since

$$x = \sum_{i=1}^{k} t_i x_i = \sum_{i=1}^{k} t_i x_i - 0 = \sum_{i=1}^{k} (t_i - cs_i)\, x_i,$$

we have written x as a convex combination of less than k elements ($t_m - cs_m = 0$), which contradicts the definition of k. ■

Exercise 11 *Prove that the convex hull of an open set $A \subset \mathbb{R}^N$ is open.*

We now introduce the notion of convex and linear functions.

Definition 12 *Given a vector space X, a function $f : X \to [-\infty, \infty]$ is said to be* convex *if*

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) \tag{3}$$

for all $x_1, x_2 \in X$ and $t \in (0,1)$ for which the right-hand side is well-defined.

The right-hand side is not defined only when $f(x_1) = \pm\infty$ and $f(x_2) = \mp\infty$.

Let X be a vector space. The *effective domain* of a function $f : X \to [-\infty, \infty]$ is the set

$$\mathrm{dom}_e\, f := \{x \in X : f(x) < \infty\}.$$

If E is a subset of the vector space X, then a function $f : E \to [-\infty, \infty]$ is said to be *convex* if the extension

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \infty & \text{if } x \notin E, \end{cases}$$

is a convex function in X. Analogous definitions apply to the concept of strict convexity, concavity, and strict concavity.

Example 13 *Prove that:*

(i) The function $f : \mathbb{R}^N \to \mathbb{R}$ defined by

$$f(x) := Ax \cdot x,$$

where A is a symmetric matrix in $\mathbb{R}^{N\times N}$, is convex if and only if A is positive semidefinite.

(ii) If X is a vector space, the indicator function *of a set $E \subset X$ defined by*

$$f(x) = I_E(x) := \begin{cases} 0 & \text{if } x \in E, \\ \infty & \text{if } x \notin E, \end{cases}$$

is a convex function if and only if the set E is convex.

(iii) The function $f : \mathbb{R} \to [-\infty, \infty]$ defined by

$$f(x) := \begin{cases} -\infty & \text{if } |x| < 1, \\ 0 & \text{if } |x| = 1, \\ \infty & \text{if } |x| > 1, \end{cases}$$

is convex.

Definition 14 *Let X be a vector space.*

(i) A set $F \subset X$ is said to be absorbing, *or* radial, *if for every $x \in X$ there exists $t_0 > 0$ such that $tx \in F$ for all $0 \le t \le t_0$.*

(ii) *Given a set $E \subset X$, a point $x_0 \in E$ is called an* internal point *of E if $-x_0 + E$ is absorbing.*

Remark 15 *Note that an absorbing set always contain the origin.*

Definition 16 *Let X and Y be two vector spaces. A function $T : X \to Y$ is called* linear *if*

(i) $T(x_1 + x_2) = T(x_1) + T(x_2)$ *for all* $x_1, x_2 \in X$,

(ii) $T(tx) = tT(x)$ *for all* $x \in X$ *and* $t \in \mathbb{R}$.

In the infinite dimensional case, many of the properties of convex sets studied in the finite-dimensional setting rely on the Hahn-Banach theorem. The proof makes use of Zorn's lemma.

Given two nonempty sets X, Y, a (binary) *relation* is a subset $\mathcal{R} \subset X \times Y$. Usually, we associate a symbol to it, say $*$, so that $x * y$ means that $(x, y) \in \mathcal{R}$.

A *partial ordering* on a nonempty set is a relation $\mathcal{R} \subset X \times X$, denoted $\leq$, such that

(i) $x \leq x$ for every $x \in X$; that is $(x, x) \in \mathcal{R}$ (reflexivity).

(ii) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$; that is, if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$ (antisymmetry).

(iii) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; that is, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$ (transitivity).

The word "partial" means that given $x, y \in X$, in general we cannot always say that $x \leq y$ or $y \leq x$.

Example 17 *Let $X = \mathcal{P}(\mathbb{R}) = \{$all subsets of $\mathbb{R}\}$. Given $E, F \in X$, we say that $E \leq F$ if $E \subset F$. Then $\leq$ is a partial ordering, but given the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$, one is not contained into the other.*

Given a partially ordered set $(X, \leq)$, a *totally ordered set*, or *chain*, $E \subset X$ is a set with the property that for all $x, y \in X$, either $x \leq y$ or $y \leq x$ (or both).

In the previous example $E = \{\{1, 2, 3\}, \{1, 2\}, \{2\}\}$ is a chain.

Given a partially ordered set $(X, \leq)$, and a set $E \subset X$, an *upper bound of* E is an element $x \in X$ such that $y \leq x$ for all $y \in E$. A set E may not have any upper bounds. A *maximal element* of E is an element $x \in E$ such that if $x \leq y$ for some $y \in E$, then $x = y$. A set E may not have maximal elements or it may have maximal elements that are not upper bounds (it can happen that a maximal element cannot be compared with all the elements of E).

Proposition 18 (Zorn's lemma) *Given a partially ordered set $(X, \leq)$, if every totally ordered subset of X has an upper bound, then X has a maximal element.*

Friday, January 15, 2010

Theorem 19 (Hahn-Banach) *Let X be a vector space and let $f : X \to (-\infty, \infty]$ be convex. Let $X_1 \subset X$ be a subspace of X such that X_1 contains an internal point of* $\operatorname{dom}_e f$*, and let $L_1 : X_1 \to \mathbb{R}$ be a linear function such that*

$$f(x) \geq L_1(x)$$

for all $x \in X_1$. Then there exists a linear extension $L : X \to \mathbb{R}$ of L_1 such that

$$f(x) \geq L(x) \quad \textit{for all } x \in X.$$

Proof. Step 1: If $X_1 = X$, then there is nothing to prove. Thus let $y_0 \in X \setminus X_1$, with $y_0 \neq 0$ and consider the subspace Y given by the linear span of $X_1 \cup \{y_0\}$. If $y \in Y$, then $y = x + ty_0$ where $x \in X_1$ and $t \in \mathbb{R}$ (this decomposition is unique). Define

$$\hat{L}(x + ty_0) := L_1(x) + tc,$$

where $c \in \mathbb{R}$ has to be chosen appropriately. We claim that

$$\inf_{x \in X_1,\, t>0} \frac{f(x + ty_0) - L_1(x)}{t} \geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x - sy_0)}{s}, \tag{4}$$

or, equivalently,

$$\frac{f(x_1 + ty_0) - L_1(x_1)}{t} \geq \frac{L_1(x_2) - f(x_2 - sy_0)}{s}$$

for all $x_1, x_2 \in X_1$ and for all $s, t > 0$. This inequality may be rewritten as

$$sf(x_1 + ty_0) + tf(x_2 - sy_0) \geq L_1(tx_2 + sx_1).$$

Since f is convex and L_1 linear,

$$\begin{aligned}
&sf(x_1 + ty_0) + tf(x_2 - sy_0) \\
&\quad = (s+t)\left[\frac{s}{s+t} f(x_1 + ty_0) + \frac{t}{s+t} f(x_2 - sy_0)\right] \\
&\quad \geq (s+t) f\left(\frac{s}{s+t}(x_1 + ty_0) + \frac{t}{s+t}(x_2 - sy_0)\right) \\
&\quad = (s+t) f\left(\frac{s}{s+t}x_1 + \frac{t}{s+t}x_2\right) \geq (s+t) L_1\left(\frac{s}{s+t}x_1 + \frac{t}{s+t}x_2\right) \\
&\quad = L_1(tx_2 + sx_1).
\end{aligned}$$

Hence (4) holds.

Let $x_1 \in X_1$ be an internal point of $\operatorname{dom}_e f$. Then $-x_1 + \operatorname{dom}_e f$ is absorbing, and so there exist $t_1, s_1 > 0$ such that $t_1 y_0, -s_1 y_0 \in -x_1 + \operatorname{dom}_e f$, that is, $f(x_1 + t_1 y_0), f(x_1 - s_1 y_0) \in \mathbb{R}$. Hence,

$$\begin{aligned}
\infty &> \frac{f(x_1 + t_1 y_0) - L_1(x_1)}{t_1} \geq \inf_{x \in X_1,\, t>0} \frac{f(x + ty_0) - L_1(x)}{t} \\
&\geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x - sy_0)}{s} \geq \frac{L_1(x_1) - f(x_1 - s_1 y_0)}{s_1} > -\infty,
\end{aligned}$$

which shows that the numbers in (4) are real and thus we can choose any real number $c \in \mathbb{R}$ with

$$\inf_{x \in X_1,\, t>0} \frac{f(x+ty_0) - L_1(x)}{t} \geq c \geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x-sy_0)}{s}.$$

By the choice of c, we have that

$$f(x+ty_0) \geq L_1(x) + tc = \hat{L}(x+ty_0)$$

for all $x \in X_1$ and all $t \in \mathbb{R}$. Hence, we have extended L_1 as a linear function $\hat{L}$ to Y in such a way that $\hat{L} \leq f$ in Y.

Step 2: Now we use Zorn's lemma. We consider all extensions $\left(\tilde{L}, \tilde{X}\right)$ of (L_1, X_1), where $\tilde{X}$ is a subspace of X containing X_1, $\tilde{L} : \tilde{X} \to \mathbb{R}$ is linear, $\tilde{L} \leq f$ in $\tilde{X}$, and $\tilde{L} = L_1$ in X_1.

We give a partial order. We say that

$$(L', X') \preccurlyeq (L'', X'')$$

if $X' \subset X''$ and $L'' = L'$ in X'. Given a chain $\{(L_\alpha, X_\alpha)\}$, define

$$X_\infty := \bigcup_\alpha X_\alpha$$

and $L_\infty(x) := L_\alpha(x)$ if $x \in X_\alpha$. Then X_∞ is a subspace of X (by Proposition 6), $L_\infty \leq f$ in X_∞, $L_\infty = L_1$ in X_1, and $(L_\alpha, X_\alpha) \preccurlyeq (L_\infty, X_\infty)$. Hence, every chain has an upper bound. It follows by Zorn's lemma that there exists a maximal element (L, Y). If $Y \neq X$, then we can proceed as in Step 1 to extend L. This would contradict the maximality of (L, Y). Hence $Y = X$ and the proof is complete.

■

Monday, January 18, 2010

Martin Luther King, Jr. No classes.

Wednesday, January 20, 2010

2 Seminorms

Definition 20 *Let X be a vector space. A map $p : X \to \mathbb{R}$ is called*

(i) positively homogeneous of degree $\alpha \geq 0$ *if*

$$p(tx) = t^{\alpha} p(x)$$

for all $x \in X$ and $t > 0$,

(ii) subadditive *if*

$$p(x + y) \leq p(x) + p(y)$$

for all $x, y \in X$,

(iii) sublinear *if it is positively homogeneous of degree one and subadditive*

(iv) a seminorm *if it is subadditive and*

$$p(tx) = |t|\, p(x)$$

for all $x \in X$ and $t \in \mathbb{R}$.

Proposition 21 *Let X be a vector space and let $p : X \to \mathbb{R}$ be sublinear. Then*

(i) $p(0) = 0$,

(ii) p is convex,

(iii) $-p(x) \leq p(-x)$ for all $x \in X$, so that p is linear if and only if $-p(x) = p(-x)$ for all $x \in X$,

(iv) the function $g(x) := \max\{p(x), p(-x)\}$, $x \in X$, is a seminorm.

(v) p is a seminorm if and only if $p(x) = p(-x)$ for all $x \in X$, in which case p is nonnegative.

Proof. (i) For $t > 0$,

$$p(0) = p(t0) = tp(0) \to 0$$

as $t \to 0^+$.
(ii) To see that p is convex, let $x, y \in X$ and $0 < \theta < 1$. Using first subadditivity and then homogeneity, we get

$$p(\theta x + (1 - \theta) y) \leq p(\theta x) + p((1 - \theta) y) = \theta p(x) + (1 - \theta) p(y).$$

(iii) Given $x \in X$, by part (i),

$$0 = p(0) = p(x + (-x)) \leq p(x) + p(-x),$$

and so $-p(x) \leq p(-x)$. To prove the second part, assume that $-p(x) = p(-x)$ for all $x \in X$. Then for $x, y \in X$,

$$-p(x+y) = p(-(x+y)) = p(-x-y) \leq p(-x) + p(-y) = -p(x) - p(y),$$

and so

$$p(x+y) \geq p(x) + p(y).$$

Together with subadditivity, this shows that $p(x+y) = p(x) + p(y)$.

Next, for $t < 0$ and $x \in X$,

$$p(tx) = p((-t)(-x)) = -tp(-x) = tp(x),$$

where in the last equality we used the fact that $-p(x) = p(-x)$.

(iv) Given $x, y \in X$, by subadditivity

$$\begin{aligned} p(x+y) &\leq p(x) + p(y) \leq g(x) + g(y), \\ p(-x-y) &\leq p(-x) + p(-y) \leq g(x) + g(y), \end{aligned}$$

and so $g(x+y) = \max\{p(x+y), p(-x-y)\} \leq g(x) + g(y)$.

If $t \neq 0$ and $x \in X$, by positive homogeneity

$$\begin{aligned} p(tx) &= p(\operatorname{sgn} t\,|t|\,x) = |t|\,p((\operatorname{sgn} t)\,x) \\ p(-tx) &= p(-\operatorname{sgn} t\,|t|\,x) = |t|\,p(-(\operatorname{sgn} t)\,x), \end{aligned}$$

and so

$$\begin{aligned} g(tx) &= \max\{p(tx), p(-tx)\} = \max\{p(\operatorname{sgn} t\,|t|\,x), p(-\operatorname{sgn} t\,|t|\,x)\} \\ &= \max\{|t|\,p((\operatorname{sgn} t)\,x), |t|\,p(-(\operatorname{sgn} t)\,x)\} = |t| \max\{p((\operatorname{sgn} t)\,x), p(-(\operatorname{sgn} t)\,x)\} \\ &= |t| \max\{p(x), p(-x)\} = |t|\,g(x). \end{aligned}$$

(v) If $p(x) = p(-x)$ for all $x \in X$, the $g(x) = p(x)$, and so p is a seminorm by part (iv). On the other hand, if p is a seminorm, then

$$p(-x) = p(-1x) = |-1|\,p(x) = p(x).$$

To prove that $p \geq 0$, note that by subadditivity,

$$0 = p(0) = p(x + (-x)) \leq p(x) + p(-x) = 2p(x).$$

■

In view of part (v), whenever we talk of seminorm, we will consider $p : X \to [0, \infty)$.

Remark 22 *Let X be a vector space and let $p : X \to [0, \infty)$ be a seminorm. Then*

$$\rho(x, y) := p(x - y)$$

is a pseudometric. To be a metric, we need $p(x) > 0$ for all $x \neq 0$. In this case we say that p is a norm.

Example 23 *Some important examples of seminorms are listed below.*

(i) *If X is a vector space and $L : X \to \mathbb{R}$ is linear, then*

$$p(x) = |L(x)|, \quad x \in X,$$

is a seminorm.

(ii) *If $\Omega \subset \mathbb{R}^N$ is an open set and $K \subset \Omega$ is a compact set, then in the space of continuous functions $C(\Omega)$ we can consider the seminorm*

$$p_K(f) := \max_{x \in K} |f(x)|.$$

There exist continuous functions that are zero in K but not in Ω.

(iii) *If $E \subset \mathbb{R}^N$ is a Lebesgue measurable set and $1 \leq q < \infty$, then in the space*

$$\mathfrak{L}^q(E) := \left\{ f : E \to \mathbb{R} \text{ measurable such that } \int_E |f(x)|^q \, dx < \infty \right\}$$

we can consider the seminorm

$$p(f) := \left(\int_E |f(x)|^q \, dx \right)^{\frac{1}{q}}.$$

If $p(f) = 0$, then f is zero except on a set of Lebesgue measure zero.

Friday, January 22, 2010

Doherty Hall evacuated for a security threat. Let's just hope it was not my students :)

Monday, January 25, 2010

Given a nonempty set X, a set $\mathcal{R} \subset X \times X$ is called an *equivalence relation* if

(i) (Reflexivity) $(x, x) \in \mathcal{R}$ for every $x \in X$.

(ii) (Symmetry) For all $x, y \in X$, if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.

(iii) (Transitivity) For all $x, y, z \in X$, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

We write $x \sim y$ if $(x, y) \in \mathcal{R}$. Given $x \in X$, the *equivalence class* determined by x is given by

$$[x] := \{z \in X : x \sim z\}.$$

We define the *quotient space*

$$Y = X \diagup \sim := \{[x] : x \in X\}.$$

If X is a vector space and $p : X \to [0, \infty)$ is a seminorm, we can define an equivalence relation $\sim$ on X. Given $x, y \in X$, we say that $x \sim y$ if $p(x - y) = 0$. Consider the quotient space $Y = X \diagup \sim$. The space Y is a vector space with the operations

$$[x] + [y] := [x + y],$$
$$t[x] := [tx]$$

for $x, y \in X$, $t \in \mathbb{R}$. In Y we can define

$$\|[x]\| := p(x), \quad [x] \in Y. \tag{5}$$

Then $\|\cdot\|$ is well-defined and is a norm in Y.

Example 24 *If $E \subset \mathbb{R}^N$ is a Lebesgue measurable set and $1 \leq q < \infty$, then in the space*

$$\mathfrak{L}^q(E) := \left\{ f : E \to \mathbb{R} \text{ measurable such that } \int_E |f(x)|^q \, dx < \infty \right\}$$

we have the seminorm

$$p(f) := \left(\int_E |f(x)|^q \, dx \right)^{\frac{1}{q}}.$$

In this case, given $f, g \in \mathfrak{L}^q(E)$, we have that $f \sim g$ if $f(x) = g(x)$ for all $x \in E$ except on a set of Lebesgue measure zero. Then the space

$$L^q(E) := \mathfrak{L}^p(E) \diagup \sim$$

is a normed space.

Let X be a vector space and let $E \subset X$. The function $p_E : X \to [0, \infty]$, defined by

$$p_E(x) := \inf \{s > 0 : x \in sE\}, \quad x \in X,$$

is called the *gauge*, or *Minkowski functional,* of E. Note that if $x \in E$, then $p_E(x) \leq 1$. Hence,

$$E \subset \{x \in X : p_E(x) \leq 1\}. \tag{6}$$

Definition 25 *Let X be a vector space over $\mathbb{R}$ and let $E \subset X$. The set E is said to be* symmetric *if whenever $x \in E$, $-x \in E$.*

Proposition 26 *Let X be a vector space and let $E \subset X$ be a nonempty set.*

(i) If E is absorbing, then p_E is real-valued and positively homogeneous of degree one.

(ii) If E is convex and absorbing, then p_E is sublinear.

(iii) If E is convex, absorbing, and symmetric, then p_E is a seminorm and

$$\{x \in X : p_E(x) < 1\} \subset E \subset \{x \in X : p_E(x) \leq 1\}. \tag{7}$$

Proof. (i) Assume that E is absorbing. Using the fact that E is absorbing, for every $x \in X$ there exists $t_x > 0$ such that $x \in tE$ for all $0 \leq t \leq t_x$. Hence, $p_E(x) \leq \frac{1}{t_x} < \infty$. Thus, p_E does not take the value ∞.

Fix $x \in X$ and $r > 0$. Let $s > 0$ be such that $rx \in sE$ (note that s exists by what we just proved). Then $x \in \frac{s}{r}E$, and so

$$p_E(x) \leq \frac{s}{r}.$$

Taking the infimum over all $s > 0$ such that $rx \in sE$ gives

$$p_E(x) \leq \frac{1}{r} p_E(rx).$$

The other inequality can be obtained in a similar way.

(ii) Assume that E is convex and absorbing and let $x, y \in X$. We claim that

$$p_E(x + y) \leq p_E(x) + p_E(y).$$

By part (i) $p_E(x)$ and $p_E(y)$ are finite. Let $s_1, s_2 > 0$ be such that $x \in s_1 E$ and $y \in s_2 E$. Then $x = s_1 y_1$ and $y = s_2 y_2$ for some $y_1, y_2 \in E$. Then

$$\begin{aligned} x + y = s_1 y_1 + s_2 y_2 &= (s_1 + s_2) \left(\frac{s_1}{s_1 + s_2} y_1 + \frac{s_2}{s_1 + s_2} y_2 \right) \\ &\in (s_1 + s_2) E \end{aligned}$$

by the convexity of E, and so

$$p_E(x + y) \leq s_1 + s_2.$$

Taking first the infimum over all $s_1 > 0$ such that $x \in s_1 E$ and then over all $s_2 > 0$ such that $y \in s_2 E$ gives

$$p_E(x+y) \le p_E(x) + p_E(y).$$

(iii) Assume that E is convex, absorbing, and symmetric. To prove (**??**), in view of (i) and (ii), it remains to show that

$$p_E(-x) = p_E(x)$$

for every $x \in X$. Since $-E = E$,

$$\begin{aligned} p_E(-x) &= \inf\{s > 0 : -x \in sE\} = \inf\{s > 0 : x \in s(-E)\} \\ &= \inf\{s > 0 : x \in sE\} = p_E(x). \end{aligned}$$

Finally, we show (7). If $x \in X$ is such that $p_E(x) < 1$, then there exists $t \in (0,1)$ such that $x \in tE$. Write $x = ty$ for some $y \in E$. Since E is absorbing, we have that $0 \in E$, and so by the convexity of E, $x = ty + (1-t)\,0 \in E$. This shows that

$$\{x \in X : p_E(x) < 1\} \subset E.$$

■

The converse of (ii) and (iii) is true.

Definition 27 *Let X be a vector space over $\mathbb{R}$ and let $E \subset X$. The set E is said to be* balanced*, or* circled*, if $tx \in E$ for all $x \in E$ and $t \in [-1,1]$.*

Remark 28 *Note that if E is convex, then it is balanced if and only if it is symmetric.*

Corollary 29 *Let X be a vector space ane let $p : X \to \mathbb{R}$. Then*

(i) p is sublinear if and only if it is the Minkowski functional of a convex and absorbing set E.

(ii) p is a seminorm if and only if it is the Minkowski functional of a convex, absorbing, and balanced set E.

Proof. (i) It remains to prove one implication. Let p be sublinear and define

$$E := \{x \in X : p(x) < 1\}.$$

We claim that E is convex. Take $x, y \in E$ and $\theta \in (0,1)$. Since p is convex (see Proposition 21),

$$p(\theta x + (1-\theta)y) \le \theta p(x) + (1-\theta)p(y) < \theta 1 + (1-\theta)1 = 1,$$

and so $\theta x + (1-\theta)y$ belongs to E.

To see that E is absorbing, fix $x \in X$ and let $p(x) = s \in [0, \infty)$. If $s < 1$, then $x \in E$ and there is nothing to prove. Thus assume that $s \geq 1$. Then, by positive homogeneity, for all $0 < t < \frac{1}{s}$,

$$p(tx) = tp(x) = ts < 1,$$

and so $tx \in E$.

(ii) Let p be a seminorm. Since E is convex, to prove that E is balanced, it is enough to show that E is symmetric. This follows from the fact that $p(-x) = p(x)$ for all $x \in X$ (see Proposition 21).

It remains to prove that $p = p_E$. Note that, by positive homogeneity, for $t > 0$ and $x \in X$, we have that $x \in tE$ if and only if $p(x) < t$, and so

$$p_E(x) = \inf\{t > 0 : x \in tE\} = \inf\{t > 0 : p(x) < t\} = p(x).$$

■

Wednesday, January 27, 2010

As a corollary of the Hahn–Banach theorem we have the following separation theorem.

Theorem 30 (Hahn–Banach separation theorem) *Let X be a vector space and let $C_1, C_2 \subset X$ be nonempty disjoint convex sets. Assume that $C_1 - C_2$ has an internal point. Then there exist a nonzero linear functional $L : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that*

$$L(x) \leq \alpha \quad \text{for all } x \in C_1 \text{ and } L(x) \geq \alpha \quad \text{for all } x \in C_2,$$

and $C_1 \cup C_2$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$.

Proof. Let $z \in C_1 - C_2$ be an internal point. Then the set $C := -z + C_1 - C_2$ is convex and absorbing. In turn, p_C is convex and real-valued. Note that since C_1, C_2 are disjoint, $0 \notin C_1 - C_2$, and so $-z \notin C$.

We claim that $p_C(-z) \geq 1$.[1] Indeed, if $p_C(-z) < 1$, then we may find $t \in (0,1)$ such that $-z \in tC$, that is $-z = tx$ for some $x \in C$. Since C is absorbing, we have that $0 \in C$, and so, by convexity, $-z = tx + (1-t)\,0 \in C$, which is a contradiction.

Consider the subspace $X_1 := \{-tz : t \in \mathbb{R}\}$ and define $L_1(-tz) := t$. Since p_C is positively homogeneous of degree one, if $t > 0$,

$$p_C(-tz) = t p_C(-z) \geq t = L_1(-tz),$$

while if $t \leq 0$,

$$p_C(-tz) \geq 0 \geq t = L_1(-tz).$$

Thus we are in a position to apply the Hahn-Banach theorem to find a linear extension $L : X \to \mathbb{R}$ of L_1 such that

$$p_C(x) \geq L(x) \quad \text{for all } x \in X.$$

Note that $L(z) = L_1(z) = -1$, and so $L \neq 0$.

Now let $x_1 \in C_1$ and $x_2 \in C_2$. Then

$$\begin{aligned} L(x_1) &= L(x_1 - x_2 - z) + L(x_2) + L(z) \\ &\leq p_C(x_1 - x_2 - z) + L(x_2) - 1 \\ &\leq 1 + L(x_2) - 1, \end{aligned}$$

where we have used the fact that $p_C(x_1 - x_2 - z) \leq 1$, since $1(x_1 - x_2 - z) = x_1 - x_2 - z \in C$. Hence

$$\sup_{x_1 \in C_1} L(x_1) \leq \inf_{x_2 \in C_2} L(x_2).$$

It remains to show that $C_1 \cup C_2$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$. Assume the contrary. Since $z \in C_1 - C_2$, we may write $z \in y_1 - y_2$, where $y_1 \in C_1$ and $y_2 \in C_2$. Then

$$-1 = L(z) = L(y_1) - L(y_2) = \alpha - \alpha = 0,$$

which is a contradiction. ■

[1] $p_C(-z) > 0$ would be enough

Wednesday, January 27, 2010

(Make up class)

3 Topological Vector Spaces

Definition 31 *Given a vector space X over $\mathbb{R}$ endowed with a topology τ, the pair (X,τ) is called a* topological vector space *if the functions*

$$\begin{array}{c} X\times X\to X, \\ (x,y)\mapsto x+y, \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{R}\times X\to X, \\ (t,x)\mapsto tx, \end{array}$$

are continuous.

We begin by showing that in a topological vector space, the topology τ is translation-invariant.

Proposition 32 *Let (X,τ) be a topological vector space. Then $U\subset X$ is open if and only if x_0+U is open for all $x_0\in X$.*

Proof. Fix $x_0\in X$. Since addition is continuous, the function

$$\begin{aligned} X &\to X \\ x &\mapsto x+x_0 \end{aligned}$$

is a homeomorphism. ■

Remark 33 *In view of the previous proposition, to give a base it is enough to give a local base at the origin.*

Corollary 34 *Let (X,τ_X) and (Y,τ_Y) be topological vector spaces and let $T:X\to Y$ be linear. Then T is continuous if and only if it is continuous at* 0

Proof. If T is continuous, then it is continuous at 0. Assume that T is continuous at zero. Then for every neighborhood $V\subset Y$ of 0_Y, there exists a neighborhood $U\subset X$ of 0_X such that $T(x)\in V$ for all $x\in U$. Fix $x_0\in X$, then by the previous proposition any neighborhood of $T(x_0)$ is of the form $T(x_0)+V$ for some neighborhood $V\subset Y$ of 0_Y. Let U be as above. Then x_0+U is a neighborhood of x_0 and by the linearity of T, $T(x_0+x)=T(x_0)+T(x)\in T(x_0)+V$ for all $x\in U$, which proves continuity at x_0. ■

Proposition 35 *Let (X,τ) be a topological vector space, let U be a neighborhood of zero, and let*

$$0<t_1<t_2<\cdots<t_n\to\infty.$$

Then

$$X=\bigcup_{n=1}^{\infty}t_nU.$$

In particular, U is absorbing.

Proof. Fix $x \in X \setminus \{0\}$. Since multiplication by scalar is continuous is continuous, the function

$$\begin{aligned} \mathbb{R} &\to X \\ t &\mapsto tx \end{aligned}$$

is a homeomorphism. Hence, the set $\{t \in \mathbb{R} : tx \in U\}$ is open and contains 0. Thus, it contains $\frac{1}{t_n}$ for all $n \in \mathbb{N}$ sufficiently large. This shows that $\frac{1}{t_n} x \in U$ for all $n \in \mathbb{N}$ sufficiently large, or, equivalently, that $x \in t_n U$ for all $n \in \mathbb{N}$ sufficiently large. ■

Proposition 36 *Let (X, τ) be a topological vector space and let U be a neighborhood of zero. Then U contains a symmetric neighborhood of zero V such that $V + V \subset U$ and $\overline{V} \subset U$, as well as a balanced neighborhood W of zero.*

Proof. Since addition is continuous at $(0,0)$, there exist two neighborhoods V_1 and V_2 of zero such that $V_1 + V_2 \subset U$. Define

$$V := (V_1 \cap V_2) \cap ((-V_1) \cap (-V_2)).$$

Indeed, by construction, it is symmetric, while $V + V \subset V_1 + V_2 \subset U$. Moreover, if $x \in \overline{V}$, then $x - V$ is a neighborhood of x. Since $x \in \overline{V}$, we have that $(x - V) \cap V \neq \emptyset$. Let $y \in (x - V) \cap V$, then $y = x - z$, where $z \in V$. Hence, $x = y + z \in V + V$, which shows that $\overline{V} \subset U$.

To prove the second part of the statement, since multiplication by scalar is continuous at $(0,0) \in \mathbb{R} \times X$, there exists a neighborhood V_3 of zero and $\delta > 0$ such that $tV_3 \subset U$ for all $t \in \mathbb{R}$ with $|t| \leq \delta$. Define

$$W := \bigcup_{|t| \leq \delta} tV_3.$$

Since multiplication by scalar is continuous, for every $t \neq 0$ the function

$$\begin{aligned} X &\to X \\ x &\mapsto tx \end{aligned}$$

is a homeomorphism. Thus tV_3 is open. This shows that $W \subset U$. Moreover, if $s \in [-1, 1]$ and $x \in W$, then there exists $t \in \mathbb{R}$ with $|t| \leq \delta$ such that $x \in tV_3$. But then $sx \in stV_3$, and since $|st| \leq \delta$, $stV_3 \subset W$, which shows that W is balanced. ■

Friday, January 29, 2010

Corollary 37 *Let (X, τ) be a topological vector space, let $C \subset X$ be closed, and let $K \subset X$ be compact. Then $C + K$ is closed.*

Proof. Let $x \in X \setminus (C + K)$. For every $y \in K$, the set $y + C$ is closed (see Proposition 32). Since $x \notin y + C$ and the complement of $y + C$ is open, there exists a symmetric neighborhood U_y of zero such that

$$(x + U_y) \cap (y + C) = \emptyset.$$

Since U_y is symmetric, $x \notin y + U_y + C$ (indeed, if $x = y + z + w$, where $z \in U_y$ and $w \in C$, then $x - z = y + w$, but $-z \in U_y$, which is a contradiction). By the previous proposition, there exists a symmetric neighborhood of zero V_y such that $V_y + V_y \subset U_y$. The family $\{y + V_y\}_{y \in K}$ covers K, and so there exist $y_1, \ldots, y_m \in K$ such that $\{y_i + V_{y_i}\}_{i=1}^m$ still covers K. Set

$$V := \bigcap_{i=1}^m V_{y_i}.$$

Then

$$K + V \subset \bigcup_{i=1}^m (y_i + V_{y_i} + V_{y_i}) \subset \bigcup_{y \in K} (y + U_y).$$

Since $x \notin y + U_y + C$ for any $y \in K$, we have that $x \notin K + V + C$. Thus $x + V$ does not intersect $K + C$ (indeed, if $x + z = z_1 + z_2$, where $z \in V$, $z_1 \in K$ and $z_2 \in C$, then $x = z_1 - z + z_2 \in K + V + C$, since $-z \in V$ by the symmetry of each V_{y_i}). In turn, $x \notin \overline{K + C}$, and so there is a neighborhood of x contained in the complement of $K + C$. This shows that $X \setminus (C + K)$ is open. ■

Exercise 38 *Prove that the sum of two closed sets in a topological vector space is not closed in general.*

Theorem 39 *Let (X, τ) be a topological vector space and let $C \subset X$ be a convex set. Then $\overline{C}$ and C° are convex.*

Lemma 40 *Let X be a topological vector space.*

(i) If $E \subset X$, then

$$\overline{E} = \bigcap_{U \text{ neighborhood of } 0} (E + U).$$

(ii) If $E, F \subset X$, then $\overline{E} + \overline{F} \subset \overline{E + F}$.

(iii) If $E, F \subset X$ are open then $E + F$ is open.

Proof. (i) By definition of $\overline{E}$, a point $x \in \overline{E}$ if and only if $(x+U) \cap E \neq \emptyset$ for every neighborhood U of 0, and this happens if and only if $x \in E - U$. On the other hand, U is a neighborhood of 0 if and only if $-U$ is. Hence we have proved that $x \in \overline{E}$ if and only if $x \in E + U$ for every neighborhood U of 0.

(ii) Take $x \in \overline{E}$ and $y \in \overline{F}$. By part (a), to show that $x + y \in \overline{E+F}$, it is enough to prove that

$$x + y \in E + F + U$$

for every neighborhood U of 0. Find a neighborhood U_1 of zero such that $U_1 + U_1 \subset U$. Since $x \in \overline{E}$ and $y \in \overline{F}$, by (i) $x \in E + U_1$, while $y \in F + U_1$. Hence, $x + y \in E + F + U_1 + U_1 \subset E + F + U$.

(iii) Write

$$E + F = \bigcup_{x \in E} (x + F).$$

Since F is open, by Proposition 32, $x + F$ is open, and so $E + F$ is the union of open sets, and hence it is open. ■

Proof of Theorem 39. Step 1: Since $C^\circ \subset C$ and C is convex, we have

$$tC^\circ + (1-t)\, C^\circ \subset C$$

for all $t \in (0,1)$. Since the sets tC° and $(1-t)\, C^\circ$ are open, so is their sum $tC^\circ + (1-t)\, C^\circ$. This implies that $tC^\circ + (1-t)\, C^\circ \subset C^\circ$. Hence C° is convex.
Step 2: For any $t \in \mathbb{R}$, $t \neq 0$, the multiplication operator $M_t(x) := tx$ is a homeomorphism, and so $\overline{tC} = t\overline{C}$. Since C is convex,

$$tC + (1-t)\, C \subset C$$

for all $t \in (0,1)$, and so

$$\overline{tC + (1-t)\, C} \subset \overline{C}$$

for all $t \in (0,1)$. On the other hand, by part (ii) of the previous lemma,

$$\begin{aligned} t\overline{C} + (1-t)\, \overline{C} &= \overline{tC} + \overline{(1-t)\, C} \\ &\subset \overline{tC + (1-t)\, C} \subset \overline{C}. \end{aligned}$$

Hence, $t\overline{C} + (1-t)\, \overline{C} \subset \overline{C}$, which shows that $\overline{C}$ is convex. ■

3.1 Separating Theorems

In this subsection we study the regularity of sublinear functions and then apply these results to separate sets with closed hyperplanes.

Exercise 41 *We construct an example of a linear function that is not continuous. Let X be an infinite-dimensional normed space and let $\{e_n\} \subset X$ be a sequence of linearly independent vectors of norm one. Let $L : X \to \mathbb{R}$ be the linear function satisfying $L(e_n) := n$ for all $n \in \mathbb{N}$ and $L :\equiv 0$ outside the span of $\{e_n\}$. Prove that L is linear but not continuous.*

Monday, February 1, 2010

Exercise 42 *Let* $X = \{f \in C([0,1]) : f \text{ is differentiable in } 0\}$ *and let*

$$\|f\|_\infty := \max_{x\in[0,1]} |f(x)|.$$

Let $L : X \to \mathbb{R}$ *be the linear function defined by*

$$L(f) := f'(0).$$

Prove that L *is linear but not continuous.*

Definition 43 *Let* (X, τ) *be a topological vector space and let* $f : E \to \mathbb{R}$*, where* $E \subset X$*. Then* f *is called* uniformly continuous *if for every* $\varepsilon > 0$ *there exists a neighborhood* U *of zero such that*

$$|f(x) - f(y)| \le \varepsilon$$

for all $x, y \in X$ *such that* $x - y \in U$*.*

Theorem 44 *Let* (X, τ) *be a topological vector space. A sublinear function* $p : X \to \mathbb{R}$ *is (uniformly) continuous if and only if it is bounded in a neighborhood of zero.*

Proof. Assume that there exists $M > 0$ such that

$$|p(x)| \le M$$

for all x in some neighborhood U of zero. By taking U smaller, we can assume that U is balanced (see Proposition 36).

Then for every $x, y \in X$,

$$p(x) = p(x - y + y) \le p(x - y) + p(y),$$

and so

$$p(x) - p(y) \le p(x - y).$$

By interchanging x and y, we obtain that

$$|p(x) - p(y)| \le \max\{p(x - y), p(y - x)\}$$

for all $x, y \in X$. In particular, if $x - y \in \frac{\varepsilon}{M}U$, then $\frac{M}{\varepsilon}(x - y) \in U$ and so by positive homogeneity,

$$p(x - y) = \frac{\varepsilon}{M} p\left(\frac{M}{\varepsilon}(x - y)\right) \le \frac{\varepsilon}{M} M = \varepsilon.$$

Similarly, $p(y - x) \le \varepsilon$. Thus, if $x, y \in X$ are such that $x - y \in \frac{\varepsilon}{M}U$, then $|p(x) - p(y)| \le \varepsilon$, which shows uniform continuity.

Conversely, assume that p is continuous. Then the set $U := p^{-1}((-1, 1))$ is a neighborhood of zero and

$$|p(x)| < 1$$

for all $x \in U$. ■

Corollary 45 *Let (X, τ) be a topological vector space. A nonnegative sublinear function $p : X \to [0, \infty)$ is continuous if and only if it is the Minkowski functional of a convex neighborhood of zero.*

Proof. Let $p : E \to [0, \infty)$ be continuous and sublinear. By Corollary 29, p is the Minkowski functional of the convex, absorbing set

$$U := \{x \in X : p(x) < 1\} = p^{-1}((-\infty, 1)).$$

Since p is continuous, $p^{-1}((-\infty, 1))$ is open and it contains zero.

Conversely, assume that $U \subset X$ is a convex neighborhood of zero. Then U is absorbing. Again by Corollary 29, p_U is sublinear. Moreover by (6),

$$U \subset \{x \in X : p_U(x) \leq 1\}.$$

Thus $0 \leq p_U(x) \leq 1$ for all $x \in U$. By the previous theorem p_U is continuous. ■

Exercise 46 *Let (X, τ) be a topological vector space and let $p : E \to [0, \infty)$ be sublinear. What is the analogous statement of the previous corollary if one replaces continuity with lower semicontinuity?*

Corollary 47 *Let (X, τ) be a topological vector space. A linear function $L : X \to \mathbb{R}$ is (uniformly) continuous if and only if it is bounded from above (or from below) in a neighborhood of some point.*

Proof. If L is continuous, then the result follows from Theorem 44. Conversely, assume that there exists $M > 0$ such that

$$L(x) \leq M$$

for all x in some neighborhood V of x_0. Then $V = x_0 + U$ for some neighborhood U of zero and by linearity for all $x \in U$,

$$L(x) = L(x \pm x_0) = L(x_0 + x) - L(x_0) \leq M - L(x_0) := M_1$$

Thus L is bounded from above in a neighborhood U of zero. By taking U smaller, we can assume that U is balanced (see Proposition 36). Since $U = -U$, for all $x \in U$,

$$-L(x) = L(-x) \leq M_1,$$

and so $|L(x)| \leq M_1$ for all $x \in U$. We can now apply Theorem 44. ■

Corollary 48 *Let (X, τ) be a topological vector space, let $L : X \to \mathbb{R}$ be a linear function, $L \neq 0$, and let $\alpha \in \mathbb{R}$. Then the hyperplane $H := \{x \in X : L(x) = \alpha\}$ is closed if and only if L is continuous, while it is dense in X if and only if L is (everywhere) discontinuous.*

Proof. If L is continuous, then the set $L^{-1}(\{\alpha\})$ is closed. Conversely, assume that $L^{-1}(\{\alpha\})$ is closed. Then $X \setminus L^{-1}(\{\alpha\})$ is open. Fix $x_0 \in X \setminus L^{-1}(\{\alpha\})$, with, say $L(x_0) < \alpha$ and find a neighborhood V of 0 such that $x_0 + V \subset X \setminus L^{-1}(\{\alpha\})$. Without loss of generality, we may assume that V is balanced. We claim that $L(x) < \alpha$ for all $x \in x_0 + V$. Indeed, if $L(x_0 + y_0) > \alpha$ for some $y_0 \in V$, consider the function

$$g(t) := L(x_0 + ty), \quad t \in \mathbb{R}.$$

The function g is continuous (why?), $g(0) < \alpha$ and $g(1) > \alpha$. Hence, by the mean value theorem there exists $t_0 \in (0,1)$ such that $g(t_0) = \alpha$. But since V is balanced, $t_0 y \in V$ and so $x_0 + t_0 y \in x_0 + V$, which is a contradiction.

If a dense set is closed, then it is the entire space. Thus if $\{x \in X : L(x) = \alpha\}$ is dense, since it cannot be X, it follows that it cannot be closed, and so L cannot be continuous, by the previous part.

Conversely, assume that L is discontinuous. To prove that H is dense, let $x \in X \setminus H$ and let V be a neighborhood of 0. We claim that H intersects V. Indeed, if not, then $x_0 + V \subset X \setminus L^{-1}(\{\alpha\})$ and so, as in the previous part, we can conclude that L is continuous. ■

Theorem 49 (Hahn–Banach, first geometric form) *Let (X, τ) be a topological vector space and let $C_1, C_2 \subset X$ be nonempty disjoint convex sets. Assume that C_1 has an interior point. Then there exist a continuous linear functional $L : X \to \mathbb{R}$, $L \neq 0$, and a number $\alpha \in \mathbb{R}$ such that*

$$L(x) \leq \alpha \quad \textit{for all } x \in C_1 \textit{ and } L(x) \geq \alpha \quad \textit{for all } x \in C_2,$$

and $C_1 \cup C_2$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$.

Proof. We only need to check that the hypotheses of the previous theorem are satisfied, namely that $C_1 - C_2$ has an internal point. Since C_1 has an interior point x_1, there exists a neighborhood U of 0 such that $x_1 + U \subset C_1$. Let $x_2 \in C_2$. Then $x_1 - x_2 + U \subset C_1 - C_2$. Thus, $x_1 - x_2$ is an interior point of $C_1 - C_2$. Since every neighborhood of 0 is absorbing, every interior point is an internal point. Thus we are in a position to apply the Hahn–Banch separation theorem to conclude that there exist a nonzero linear functional $L : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$L(x) \leq \alpha \quad \text{for all } x \in C_1 \text{ and } L(x) \geq \alpha \quad \text{for all } x \in C_2,$$

and $C_1 \cup C_2$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$. Since L is bounded from above in a neighborhood of x_1, by the previous corollary, L is continuous. ■

Exercise 50 *Let (X, τ) be a topological vector space, let $E \subset X$ be a nonempty set, let $L : X \to \mathbb{R}$ be a linear function, $L \neq 0$, and let $\alpha \in \mathbb{R}$. Assume that $L(x) \geq \alpha$ for all $x \in E$. Prove that $L(x) > \alpha$ for all $x \in E^\circ$.*

Exercise 51 *Let* (X,τ) *be a topological vector space and let* $C_1, C_2 \subset X$ *be nonempty disjoint convex sets. Assume that* C_1 *has an interior point and let* L *be the functional given in the previous theorem. Prove that*

$$L(x) \leq \alpha \quad \text{for all } x \in \overline{C_1} \text{ and } L(x) \geq \alpha \quad \text{for all } x \in \overline{C_2}.$$

The next exercise shows that the previous theorem fails if C_1 and C_2 have no interior points.

Exercise 52 *Consider the space* ℓ^2 *and define the two sets*

$$C_1 := \left\{ x = \{x_n\}_n \in \ell^2 : x_1 \geq n \left| x_n - \frac{1}{n^{2/3}} \right| \text{ for all } n \geq 2 \right\},$$
$$C_2 := \left\{ x = \{x_n\}_n \in \ell^2 : x_n = 0 \text{ for all } n \geq 2 \right\}.$$

(i) Prove that C_1 *and* C_2 *are convex and nonempty.*

(ii) Prove that $C_1 - C_2$ *is dense in* ℓ^2*.*

(iii) Prove that C_1 *and* C_2 *cannot be separated by a closed hyperplane.*

Exercise 53 *In a topological vector space* (X,τ) *every neighborhood of* 0 *is absorbing, and so every interior point is an internal point.*

(i) Prove that if X *is finite-dimensional, then every internal point of a convex set is also an interior point.*

(ii) Consider $X = C([0,1])$ *with the metric* $d_1(f,g) := \int_0^1 |f(x) - g(x)|\, dx$ *and let*

$$E := \left\{ f \in C([0,1]) : \max_{x \in [0,1]} |f(x)|\, dx < 1 \right\}.$$

Prove that 0 *is an internal point of* E *but not an interior point of* E*.*

To obtain stronger types of separation, we need to assume additional properties on the space X.

We recall that given a topological space (X,τ), a family β of open sets of X is a *base* for the topology τ if every open set $U \in \tau$ may be written as the union of elements of β. Given a point $x \in X$, a family β_x of neighborhoods of x is a *local base at* x if every neighborhood of x contains an element of β_x.

Definition 54 *A topological vector space* (X,τ) *is* locally convex *if it has a local base at* 0 *consisting of convex sets.*

Proposition 55 *A locally convex topological vector space admits a local base at the origin consisting of balanced convex neighborhoods of zero.*

Proof. Let U be a convex neighborhood of 0 and define

$$V := U \cap (-U) .$$

Then V is convex, since intersection of convex sets. Moreover, if $x \in V$, then $-x \in -V$, and so by convexity, the segment joining x and $-x$ is contained in V. This shows that V is balanced. ■

If X and Y are topological vector spaces, then the vector space of all continuous linear operators from X to Y is denoted by $\mathcal{L}(X;Y)$. In the special case $Y = \mathbb{R}$, the space $\mathcal{L}(X;\mathbb{R})$ is called the *dual space* of X and it is denoted by X'. The elements of X' are also called continuous linear *functionals.*

The bilinear (i.e., linear in each variable) mapping

$$\begin{aligned} \langle \cdot, \cdot \rangle_{X',X} : X' \times X &\to \mathbb{R} \\ (L, x) &\mapsto L(x) \end{aligned} \tag{8}$$

is called the *duality pairing.*

Friday, February 5, 2010

Example 56 *Some important spaces and their duals.*

(i) *The dual of* $\mathbb{R}^N$ *may be identified with* $\mathbb{R}^N$*.*

(ii) *Given* $1 \leq p < \infty$ *and a Lebesgue measurable set* $E \subset \mathbb{R}^N$*, and consider the space*

$$L^p(E) = \left\{[f] : f : E \to \mathbb{R} : f \text{ measurable and } \int_E |f(x)|^p \, dx < \infty < \infty\right\}$$

with the norm

$$\|[f]\|_{L^p(E)} := \left(\int_E |f(x)|^p \, dx\right)^{1/p}.$$

In what follows, we identify $[f]$ *with* f*.*

Assume that $1 < p < \infty$*. Fix* $g \in L^{p'}(E)$*, where* $p' := \frac{p}{p-1}$*, and consider the linear function* $L_g : L^p(E) \to \mathbb{R}$ *defined by*

$$L_g(f) := \int_E f(x)\, g(x) \, dx, \quad f \in L^p(E).$$

Note that by Hölder's inequality,

$$|L_g(f)| \leq \int_E |f(x)| \, |g(x)| \, dx \leq \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}.$$

Thus L *is well-defined. Moreover, if* $f \in B_{L^p(E)}(0, r)$*, then*

$$|L_g(f)| \leq r \|g\|_{L^{p'}(E)}$$

for all $f \in B_{L^p(E)}(0, r)$*, which shows that* L_g *is bounded in a neighborhood of* 0*. Hence, by Corollary 47,* L_g *is continuous. We will show later on that the dual of* $L^p(E)$ *may be identified with* $L^{p'}(E)$*.*

(iii) *Given a Lebesgue measurable set* $E \subset \mathbb{R}^N$*, we define*

$$L^\infty(E) := \{[f] : f : E \to \mathbb{R} : f \text{ measurable and } \operatorname{esssup} |f| < \infty\},$$

with the norm

$$\|[f]\|_{L^\infty(E)} := \operatorname{esssup} |f|.$$

Here, esssup $|f|$ *is the* essential supremum *of the function* $|f|$*, that is,*

$$\operatorname{esssup} |f| := \inf \{t \in \mathbb{R} : |f(x)| < t \text{ for all } x \in E \text{ except for a set of measure zero}\}.$$

In what follows, we identify $[f]$ *with* f*.*

Fix $g \in L^{\infty}(E)$, and consider the linear function $L_g : L^1(E) \to \mathbb{R}$ defined by

$$L_g(f) := \int_E f(x)\, g(x)\, dx, \quad f \in L^1(E).$$

Note that by Hölder's inequality,

$$|L_g(f)| \le \int_E |f(x)|\, |g(x)|\, dx \le \|f\|_{L^1(E)} \operatorname{esssup} |g|.$$

Thus L is well-defined. Moreover, if $f \in B_{L^1(E)}(0, r)$, then

$$|L_g(f)| \le r \operatorname{esssup} |g|$$

for all $f \in B_{L^1(E)}(0, r)$, which shows that L_g is bounded in a neighborhood of 0. Hence, by Corollary 47, L_g is continuous. We will show later on that the dual of $L^1(E)$ may be identified with $L^{\infty}(E)$.

(iii) If $K \subset \mathbb{R}^N$ is compact, the dual space of $C(K)$ may be identified with the space $\mathcal{M}_{\mathrm{b}}(K)$ of all bounded signed measures $\lambda : \mathcal{B}(K) \to \mathbb{R}$, where $\mathcal{B}(K)$ is the Borel σ-algebra. The norm in $\mathcal{M}_{\mathrm{b}}(K)$ is given by

$$\|\lambda\|_{\mathcal{M}_{\mathrm{b}}(K)} = \sup \left\{ \sum_{n=1}^{\infty} |\lambda(E_n)| : \{E_n\} \subset \mathcal{B}(K) \text{ partition of } K \right\}.$$

We will show that for every $L \in (C(K))'$, we can find a unique measure $\lambda \in \mathcal{M}_{\mathrm{b}}(K)$ such that

$$L(f) = \int_K f \, d\lambda, \quad f \in C(K).$$

We recall that

Definition 57 *Let X be a nonempty set. A collection $\mathfrak{M} \subset \mathcal{P}(X)$ is a σ-*algebra *if*

(i) $\emptyset \in \mathfrak{M}$;

(ii) if $E \in \mathfrak{M}$ then $X \setminus E \in \mathfrak{M}$;

(iii) if $\{E_n\} \subset \mathfrak{M}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$.

If X is a topological vector space, the smallest σ-algebra that contains all open sets is called the *Borel σ-algebra* and is denoted $\mathcal{B}(X)$.

Definition 58 *Let X be a nonempty set and let $\mathfrak{M} \subset \mathcal{P}(X)$ be σ-algebra. A function $\lambda : \mathfrak{M} \to \mathbb{R}$ is called a* signed measure *if*

$$\lambda(\emptyset) = 0, \quad \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n)$$

for every countable collection $\{E_n\} \subset \mathfrak{M}$ *of pairwise disjoint sets.*

A signed measure $\lambda : \mathfrak{M} \to \mathbb{R}$ *is* bounded *if*

$$\sup\left\{\sum_{n=1}^{\infty} |\lambda(E_n)| : \{E_n\} \subset \mathfrak{M} \text{ partition of } X\right\} < \infty.$$

The next result is a consequence of the Hahn–Banach theorem (in analytical form).

Corollary 59 (Dual of a subspace) *Let* (X, τ) *be a locally convex topological vector space and let* Y *be a subspace of* X*. If* $L : Y \to \mathbb{R}$ *is linear and continuous, then* L *can be extended to a continuous linear functional on* X*. In particular, the topological dual* Y' *of* Y *(with the induced topology) is given by the restriction to* Y *of all elements of* X'*.*

Proof. We recall that the induced topology is given by

$$\tau_Y := \{U \cap Y : U \in \tau\}.$$

Since $L : Y \to \mathbb{R}$ is linear and continuous with respect to τ_Y, by Corollary 47 there exist $V \in \tau_Y$ and $\ell > 0$ such that $|L(y)| \le \ell$ for all $y \in V$. Find a neighborhood $U \in \tau$ of zero such that $V = U \cap Y$. By the previous proposition, we may assume that U is convex and balanced. Moreover, by the linearity of L,

$$\left|L\left(\frac{y}{\ell}\right)\right| \le 1 \quad \text{for all } y \in U \cap Y,$$

or, equivalently,

$$|L(x)| \le 1 \quad \text{for all } x \in \left(\frac{1}{\ell}U\right) \cap Y.$$

Set $W := \frac{1}{\ell}U$. Then W is convex and balanced. We claim that

$$|L(x)| \le p_W(x) \quad \text{for all } x \in Y.$$

■

Monday, February 8, 2010

Snow storm (possibly provoked by students). No classes.

Wednesday, February 10, 2010

Snow storm (definitely provoked by students). No classes.

Friday, February 12, 2010

Proof, continued. To see this, let $x \in Y$ and let $t > 0$ be such that $x \in tW$. Then $\frac{x}{t} \in W$, and so $\left|L\left(\frac{x}{t}\right)\right| \leq 1$, that is,

$$|L(x)| \leq t.$$

Taking the infimum over all such t we get

$$|L(x)| \leq \inf\{t > 0 : x \in tW\} = p_W(x),$$

which proves the claim. Since W is convex, balanced, and absorbing, p_W is a seminorm by Corollary 29. Hence, we are in a position to apply the Hahn–Banach theorem to find a linear functional $L_1 : X \to \mathbb{R}$ such that $L_1(x) = L(x)$ for all $x \in Y$ and

$$|L_1(x)| \leq p_W(x) \quad \text{for all } x \in X.$$

Since $p_W(x) \leq 1$ for all $x \in W$ (see (6)), it follows from Corollary 47 that L_1 is continuous. ■

The next theorem is very important for applications.

Theorem 60 *If $\mathcal{F}$ is a balanced, convex local base of 0 for a locally convex topological vector space (X, τ), then the family $\{p_U : U \in \mathcal{F}\}$ is a family of continuous seminorms. Conversely, given a family $\mathcal{P}$ of seminorms on a vector space X, let $\mathcal{B}$ be the collection of all finite intersections of sets of the form*

$$B_p(0, r) := \{x \in X : p(x) < r\}, \quad p \in \mathcal{P},\ r > 0.$$

Then $\mathcal{B}$ is a balanced, convex local base of 0 for a topology τ that turns X into a locally convex topological vector space such that each p is continuous with respect to τ.

Proof. The first part follows from Corollaries 29 and 45. To prove the second part, let $\mathcal{P}$ be a family of seminorms on a vector space X. For every $p \in \mathcal{P}$, $x_0 \in X$, and $r > 0$, define

$$B_p(x_0, r) := \{x \in X : p(x - x_0) < r\}.$$

Note that

$$B_p(x_0, r) = x_0 + B_p(0, r). \tag{9}$$

Let τ be the topology generated by the family

$$\mathcal{F} := \{B_p(x_0, r) : p \in \mathcal{P},\, x_0 \in X,\, r > 0\}.$$

By Proposition 119 in the topology notes, τ consists of the empty set and of arbitrary unions of finite intersections of elements of $\mathcal{F}$. Moreover, by Example 23 in the topology notes, a base $\mathcal{B}'$ for τ is given by all finite intersections of elements of $\mathcal{F}$. We claim that the family

$$\mathcal{B}'' := \left\{\bigcap_{i=1}^{m} B_{p_i}(x_0, r) : p_1, \ldots, p_m \in \mathcal{P},\, m \in \mathbb{N},\, x_0 \in X,\, r > 0\right\}$$

is also a base for τ. To see this, given $U \in \tau$ and $x \in U$, it is enough to show that there is an element of $\mathcal{B}''$ contained in U and containing x. Since $\mathcal{B}'$ is a base, there exist $p_1, \ldots, p_m \in \mathcal{P}$, $x_1, \ldots, x_m \in X$, and $r_1, \ldots, r_m > 0$ such that

$$x \in \bigcap_{i=1}^{m} B_{p_i}(x_i, r_i) \subset U.$$

Since $x \in B_{p_i}(x_i, r_i)$, we have that $p_i(x - x_i) < r_i$. Hence, if we take

$$r := \min\{r_i - p_i(x - x_i) : i = 1, \ldots, m\} > 0,$$

we have that $B_{p_i}(x, r) \subset B_{p_i}(x_i, r_i)$, and so

$$x \in \bigcap_{i=1}^{m} B_{p_i}(x, r) \subset \bigcap_{i=1}^{m} B_{p_i}(x_i, r_i) \subset U.$$

Note that $\bigcap_{i=1}^{m} B_{p_i}(x, r)$ belongs to $\mathcal{B}''$.

This shows that $\mathcal{B}''$ is a base for τ. In particular, for every $x \in X$, the family

$$\mathcal{B}''_x := \left\{ \bigcap_{i=1}^{m} B_{p_i}(x, r) : p_1, \ldots, p_m \in \mathcal{P},\ m \in \mathbb{N}, r > 0 \right\}$$

is a local base at x. Take $\mathcal{B} := \mathcal{B}''_0$. Then $\mathcal{B}$ is a local base at 0. Moreover, in view of (9), every element of $\mathcal{B}''_x$ is simply given by translating by x an element of $\mathcal{B}$. It follows that τ is traslation invariant and that it consists of all unions of translates of elements of $\mathcal{B}$.

Next we prove that addition is continuous. Let U be a neighborhood of 0. Then there exist $p_1, \ldots, p_m \in \mathcal{P}$ and $r_1, \ldots, r_m > 0$ such that

$$B_{p_1}(0, r_1) \cap \cdots \cap B_{p_m}(0, r_m) \subset U.$$

Let $V := B_{p_1}\left(0, \frac{r_1}{2}\right) \cap \cdots \cap B_{p_m}\left(0, \frac{r_m}{2}\right)$. Since each p_i is subadditive, we have that

$$V + V \subset B_{p_1}(0, r_1) \cap \cdots \cap B_{p_m}(0, r_m) \subset U.$$

This proves that addition is continuous at $(0, 0)$. Since τ is a translation-invariant topology, we obtain that addition is continuous. ■

Monday, February 15, 2010

Proof. To prove that multiplication by scalar is continuous, let $x_0 \in X$ and $t_0 \in \mathbb{R}$ and let U and V be as above. By taking $s > 0$ so large that $p_i(x_0) < s\frac{r_i}{2}$ for every $i = 1, \dots, m$, we have that $x_0 \in sV$. Let

$$r := \frac{s}{1 + |t_0|\, s}.$$

If $x \in x_0 + rV$ and $|t - t_0| < \frac{1}{s}$, then we have

$$\begin{aligned} tx - t_0 x_0 &= t(x - x_0) + (t - t_0)x_0 \in |t|\, rV + |t - t_0|\, sV \\ &\subset V + V \subset U, \end{aligned}$$

where to prove the last inclusion we have used the facts that

$$|t|\, r < \left(|t_0| + \frac{1}{s}\right)\frac{s}{1 + |t_0|\, s} = 1,$$

that $|t - t_0|\, s < \frac{1}{s}s = 1$, and that V is balanced (since intersection of balanced sets). This proves that multiplication by scalar is continuous at (x_0, t_0).

Hence, (X, τ) is a topological vector space and by construction, $\mathcal{B}$ is a balanced, convex local base of 0. Since each $p \in \mathcal{P}$ is bounded in $B_p(0, r)$ for every $r > 0$, it follows from Corollary 45 that p is continuous with respect to τ. ■

Remark 61 *If X is a vector space and $p : X \to [0, \infty)$ a seminorm, consider*

$$B_p(0, r) := \{x \in X : p(x) < r\}, \quad p \in \mathcal{P},\ r > 0.$$

Then for $t > 0$,

$$tB_p(0, r) = B_p(0, tr).$$

Indeed, if $x \in tB_p(0, r)$, then $x = ty$, where $y \in B_p(0, r)$, so that $p(y) < r$. By positive homogeneity,

$$p(x) = p(ty) = tp(y) < rt,$$

so $x \in B_p(0, tr)$.

Conversely, if $x \in B_p(0, tr)$, then $p(x) < rt$. Take $y = \frac{x}{t}$. Then $x = ty$ and by positive homogeneity,

$$p(y) = p\left(\frac{x}{t}\right) = \frac{1}{t}p(x) < \frac{1}{t}rt = r,$$

so $y \in B_p(0, r)$, and in turn $x \in tB_p(0, r)$.

Corollary 62 *A normed space is a locally convex topological vector space.*

We now give some necessary and sufficient conditions for the topology τ given in the previous theorem to be Hausdorff and for a set to be topologically bounded.

Definition 63 *Let (X, τ) be a topological vector space. A set $E \subset X$ is said to be* topologically bounded *if for each neighborhood U of 0 there exists $t > 0$ such that $E \subset tU$.*

Note that when the topology τ is generated by a metric d, sets bounded in the topological sense and in the metric sense may be different. To see this, it suffices to observe that the metric $d_1 := \frac{d}{d+1}$ generates the same topology as d, but since $d_1 \leq 1$, every set in X is bounded with respect to d_1.

Topologically bounded sets play an important role in the normability of locally convex topological vector spaces (see Theorem 81 below).

Corollary 64 *Let $\mathcal{P}$ be a family of seminorms on a vector space X and let τ be the locally convex topology generated by $\mathcal{P}$. Then*

(i) τ is Hausdorff if and only if for every $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$;

(ii) a set $E \subset X$ is topologically bounded if and only if the set $p(E)$ is bounded in $\mathbb{R}$ for all $p \in \mathcal{P}$.

Proof. (i) If τ is Hausdorff, then for every $x \neq 0$ there exists a neighborhood U of 0 that does not contain x. Then there exist $p_1, \ldots, p_m \in \mathcal{P}$ and $r_1, \ldots, r_m > 0$ such that

$$B_{p_1}(0, r_1) \cap \cdots \cap B_{p_m}(0, r_m) \subset U.$$

Then there is $i = 1, \ldots, m$ such that x does not belong to $B_{p_i}(0, r_i)$, and so $p_i(x) \geq r_i > 0$. Conversely, assume that for every $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$. Consider $y, z \in X$ with $y \neq z$. Let $x := y - z$ and find $p \in \mathcal{P}$ such that $p(x) = r > 0$. Consider $B_p\left(y, \frac{r}{2}\right)$ and $B_p\left(z, \frac{r}{2}\right)$. We claim that these two neighborhoods are disjoint. Indeed, if $w \in B_p\left(y, \frac{r}{2}\right) \cap B_p\left(z, \frac{r}{2}\right)$, then

$$w = y + x_1 = z + x_2$$

for some $x_1, x_2 \in B_p\left(0, \frac{r}{2}\right)$. In turn, $y - z = x_2 - x_1$, and so by the properties of a seminorm

$$r = p(y - z) = p(x_2 - x_1) \leq p(x_2) + p(-x_1) = p(x_2) + p(x_1) < \frac{r}{2} + \frac{r}{2} = r.$$

(ii) Assume that $E \subset X$ is topologically bounded. The for every $p \in \mathcal{P}$, the set $B_p(0, 1)$ is a neighborhood U of 0, and so there exists $t > 0$ such that $E \subset tB_p(0, 1)$. It follows that $0 \leq p(x) < t$ for all $x \in E$.

Conversely, assume that the set $p(E)$ is bounded in $\mathbb{R}$ for all $p \in \mathcal{P}$. Consider a neighborhood U of 0. Then there exist $p_1, \ldots, p_m \in \mathcal{P}$ and $r_1, \ldots, r_m > 0$ such that

$$B_{p_1}(0, r_1) \cap \cdots \cap B_{p_m}(0, r_m) \subset U.$$

Find $M_1, \ldots, M_m$ such that $0 \leq p_i(x) < M_i$ for all $x \in E$ and $i = 1, \ldots, m$. If $t > \frac{M_i}{r_i}$ for every $1, \ldots, m$, then

$$E \subset t(B_{p_1}(0, r_1) \cap \cdots \cap B_{p_m}(0, r_m)) \subset tU.$$

■

In the case that the family of seminorms is countable and has property (i), then (X,τ) is actually metrizable.

Definition 65 *If X is a vector space, a metric $d : X \times X \to [0,\infty)$ is* translation-invariant *if $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in X$.*

Corollary 66 *Let $\mathcal{P}$ be a countable family of seminorms on a vector space X with the property that for every $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$ and let τ be the locally convex topology generated by $\mathcal{P}$. Then there exists a translation-invariant metric d that generates τ.*

Thus a locally convex topological vector space (X,τ) is metrizable if and only if τ is determined by a countable family $\mathcal{P}$ of seminorms with the property that for every $x \neq 0$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$.

Example 67 *Some important examples and counterexamples of locally convex spaces are presented next.*

(i) Let $\Omega \subset \mathbb{R}^N$ be an open set and consider the space $C(\Omega)$. Construct an increasing sequence of compact sets $\{K_n\} \subset \Omega$ such that

$$\bigcup_{n=1}^{\infty} K_n = \Omega \tag{10}$$

and define the seminorms

$$p_n(f) := \max_{x \in K_n} |f(x)|, \quad f \in C(\Omega).$$

Note that if $f \neq 0$, then there exists $x \in \Omega$ such that $f(x) \neq 0$, and so if n is so large that $x \in K_n$, then $p_n(f) > 0$. Hence, by Theorem 60 and Corollary 64, the countable family of seminorms $\{p_n\}_n$ turns $C(\Omega)$ into a Hausdorff locally convex topological vector space. Moreover, $C(\Omega)$ is metrizable by Corollary 66. Note that convergence with respect to τ is given by uniform convergence on compact sets. Indeed, assume that $\{f_i\} \subset C(\Omega)$ converges to some function f with respect to τ. We claim that $\{f_i\}$ converges uniformly to f on every compact set $K \subset \Omega$. To see this, fix a compact $K \subset \Omega$ and $\varepsilon > 0$. By (10), there exists an integer $n_0 \in \mathbb{N}$ such that $K \subset K_{n_0}$. Since $f + B_{p_{n_0}}(0, \varepsilon)$ is a neighborhood of f with respect to τ, there exists $i_0 \in \mathbb{N}$ such that

$$f_i \in f + B_{p_{n_0}}(0, \varepsilon)$$

for all $i \geq i_0$, that is, $f_i - f \in B_{p_{n_0}}(0, \varepsilon)$ for all $i \geq i_0$, which implies that

$$\max_{x \in K} |f_i(x) - f(x)| \leq \max_{x \in K_{n_0}} |f_i(x) - f(x)| = p_{n_0}(f_i - f) < \varepsilon$$

for all $i \geq i_0$. This shows that $\{f_i\}$ converges uniformly to f on K.

Conversely, assume that $\{f_i\}$ converges uniformly to f on every compact set $K \subset \Omega$. We claim that $\{f_i\}$ converges to f with respect to τ. To see this, let U be a neighborhood of f with respect to τ. Then there exist $K_{n_1}, \ldots, K_{n_m}$ and $r_1, \ldots, r_m$ such that

$$f + B_{p_{n_1}}(0, r_1) \cap \cdots \cap B_{p_{n_m}}(0, r_m) \subset U.$$

Let $r_0 = \min\{r_1, \ldots, r_m\} > 0$ and let $n_0 := \max\{n_1, \ldots, n_m\}$. Then

$$f + B_{p_{n_0}}(0, r_0) \subset f + B_{p_{n_1}}(0, r_1) \cap \cdots \cap B_{p_{n_m}}(0, r_m) \subset U.$$

Since $\{f_i\}$ converges uniformly to f on K_{n_0}, there exists $i_0 \in \mathbb{N}$ such that

$$\max_{x \in K_{n_0}} |f_i(x) - f(x)| < r_0$$

for all $i \geq i_0$*. This implies that*

$$f_i \in f + B_{p_{n_0}}(0, r_0) \subset U$$

for all $i \geq i_0$*, that is,* $\{f_i\}$ *converges to* f *with respect to* τ*.*

(ii) Consider the space $C^\infty([a,b])$*. For every* $n \in \mathbb{N}_0$*, define the seminorm*

$$p_n(f) := \max_{x\in[a,b]} |f(x)| + \cdots + \max_{x\in[a,b]} \left|f^{(n)}(x)\right|, \quad f \in C^\infty([a,b]).$$

Note that if $f \neq 0$*, then* $p_0(f) > 0$*. Hence, by Theorem 60 and Corollary 64, the countable family of seminorms* $\{p_n\}_n$ *turns* $C^\infty([a,b])$ *into a Hausdorff locally convex topological vector space. Moreover,* $C^\infty([a,b])$ *is metrizable by Corollary 66.*

Reasoning as in the previous example, it can be shown that a sequence $\{f_i\} \subset C^\infty([a,b])$ *converges to a function* f *with respect to* τ *if and only if for every* $n \in \mathbb{N}_0$*, the sequence* $\left\{\frac{d^n f_i}{dx^n}\right\}$ *converges uniformly to* $\left\{\frac{d^n f_i}{dx^n}\right\}$ *in* $[a,b]$*. Here,* $\frac{d^0 g}{dx^0} := g$*.*

(iii) Let $\Omega \subset \mathbb{R}^N$ *be an open set and let* $C^\infty(\Omega)$ *be the space of all functions that are continuous together with all their partial derivatives of every order. Given a* multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$*, we set*

$$\frac{\partial^\alpha f}{\partial x^\alpha}(x) := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\left(\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\cdots\left(\frac{\partial^{\alpha_N} f}{\partial x_N^{\alpha_N}}(x)\right)\right)\right).$$

If $\alpha = 0$*, we set* $\dfrac{\partial^0 f}{\partial x^0} := f$*. The* length *of* α *is* $|\alpha| := |\alpha_1| + \cdots + |\alpha_N|$*. Let* $\{K_n\}_{n\in\mathbb{N}} \subset \Omega$ *be an increasing sequence of compact sets such that*

$$\bigcup_{n=1}^{\infty} K_n = \Omega.$$

For every $n \in \mathbb{N}_0$ *define*

$$p_n(f) := \sum_{|\alpha|=0}^{n} \max_{x\in K_n} \left|\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|.$$

Note that if $f \neq 0$*, then* $p_0(f) > 0$*. Hence, by Theorem 60 and Corollary 64, the countable family of seminorms* $\{p_n\}_n$ *turns* $C^\infty(\Omega)$ *into a Hausdorff locally convex topological vector space. Moreover,* $C^\infty(\Omega)$ *is metrizable by Corollary 66.*

Reasoning as in the previous example, it can be shown that a sequence $\{f_i\} \subset C^\infty(\Omega)$ *converges to a function* f *with respect to* τ *if and only if for every multi-index* $\alpha \in \mathbb{N}_0^N$ *and every compact set* $K \subset \Omega$*, the sequence* $\left\{\dfrac{\partial^\alpha f_i}{\partial x^\alpha}\right\}$ *converges uniformly to* $\left\{\dfrac{\partial^\alpha f}{\partial x^\alpha}\right\}$ *in* K*.*

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Example 68 *(iv) Let* $\Omega \subset \mathbb{R}^N$ *be an open set and let* $C_c^\infty(\Omega)$ *be the space of all* C^∞ *functions with compact support contained in* Ω. *Let* $\{K_n\}_{n\in\mathbb{N}_0} \subset \Omega$ *be an increasing sequence of compact sets, with* $K_0 := \emptyset$, *such that*

$$\operatorname{dist}(K_n, \partial K_{n+1}) > 0$$

and

$$\bigcup_{n=1}^{\infty} K_n = \Omega.$$

Given two sequences $\mathbf{m} := \{m_n\}_{n\in\mathbb{N}_0} \subset \mathbb{N}_0$ *and* $\mathbf{a} := \{a_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$, *with* $m_n \to \infty$ *and* $a_n \to 0$, *for every* $\phi \in \mathcal{D}(\Omega)$ *define*

$$p_{\mathbf{m},\mathbf{a}}(\phi) := \sup_{n\in\mathbb{N}_0} \sup_{x\in\Omega\setminus K_n} \frac{1}{a_n} \sum_{|\alpha|\le m_n} \left| \frac{\partial^\alpha \phi}{\partial x^\alpha}(x) \right|.$$

Then $p_{\mathbf{m},\mathbf{a}}$ *is a seminorm. By Theorem 60 and Corollary 64, the family of seminorms* $\{p_{\mathbf{m},\mathbf{a}}\}_{\mathbf{m},\mathbf{a}}$, *where* $\mathbf{m}$ *and* $\mathbf{a}$ *vary among all sequences as above, generates a topology* τ. *The space* $(C_c^\infty(\Omega), \tau)$ *is denoted* $\mathcal{D}(\Omega)$ *and its dual* $\mathcal{D}'(\Omega)$ *is called the* space of distributions. *Note that the family of seminorms* $\{p_{\mathbf{m},\mathbf{a}}\}_{\mathbf{m},\mathbf{a}}$ *is not countable. Actually it can be shown that the topology* τ *is not metrizable.*

(v) Given a vector space X, *let* $X^* := \{L : X \to \mathbb{R} \text{ linear}\}$ *and consider a subspace* $Z \subset X^*$. *For every* $L \in Z$ *define the seminorm*

$$p_L(x) := |L(x)|, \quad x \in X.$$

Then the family of seminorms $\{p_L\}_{L\in Z}$ *turns* X *into a locally convex topological vector space. The topology generated by this family of seminorms is denoted* $\sigma(X, Z)$. *With this topology, each seminorm* p_L *is continuous. In turn, every functional* $L \in Z$ *is continuous, since* $|L| = p_L$.

An important special case is when X *is a topological vector space with topology* τ *and we take* Z *to be the topological dual of* X, *that is,*

$$Z = X' = \{L : X \to \mathbb{R} \text{ linear and } \tau\text{-continuous}\}.$$

In this case $\sigma(X, X')$ *is called the* weak topology *of* X, *as opposed to* τ, *which is referred to as the* strong topology *of* X.

Note that

$$\begin{aligned} B_{p_L}(0,r) &= \{x \in X : p_L(x) < r\} = \{x \in X : |L(x)| < r\} \\ &= L^{-1}((-r,r)) \in \tau, \end{aligned}$$

since L is τ-continuous. Hence, every finite intersection of $B_{p_L}(0,r)$ is contained in τ, and so

$$\sigma(X,X') \subset \tau.$$

An important property of the weak topology is that it does not change the topological dual of X. Indeed, if $L : X \to \mathbb{R}$ is linear and $\sigma(X,X')$-continuous, then for every open set $W \subset \mathbb{R}$, the set $L^{-1}(W)$ belongs to $\sigma(X,X') \subset \tau$, and so L is also τ continuous. Thus, the topological dual of $(X,\sigma(X,X'))$ is still X'.

Given a sequence $\{x_n\} \subset X$, we say that $\{x_n\}$ converges weakly to $x \in X$ and we write $x_n \rightharpoonup x$ if $\{x_n\}$ converges to x with respect to $\sigma(X,X')$. We claim that $x_n \rightharpoonup x$ if and only if

$$\lim_{n\to\infty} L(x_n) = L(x) \tag{11}$$

for every $L \in X'$. Indeed, assume that $\{x_n\} \subset X$ converges to x with respect to $\sigma(X,X')$. We claim that (11) holds. To see this, fix $L \in X'$ and $\varepsilon > 0$. Since $x + B_{p_L}(0,\varepsilon)$ is a neighborhood of x with respect to $\sigma(X,X')$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in x + B_{p_L}(0,\varepsilon)$$

for all $n \geq n_0$, that is, $x_n - x \in B_{p_L}(0,\varepsilon)$ for all $n \geq n_0$, which implies that

$$|L(x_n) - L(x)| = |L(x_n - x)| = p_L(x_n - x) < \varepsilon$$

for all $n \geq n_0$. This shows that (11) holds.

Conversely, assume that (11) holds. We claim that $\{x_n\}$ converges to x with respect to $\sigma(X,X')$. To see this, let U be a neighborhood of x with respect to $\sigma(X,X')$. Then there exist $L_1,\ldots,L_m \in X'$ and $r_1,\ldots,r_m$ such that

$$x + B_{p_{L_1}}(0,r_1) \cap \cdots \cap B_{p_{L_m}}(0,r_m) \subset U.$$

Let $r_0 = \min\{r_1,\ldots,r_m\} > 0$. Since $L_i(x_n) \to L_i(x)$ as $n \to \infty$ for every $i = 1,\ldots,m$, there exists $n_0 \in \mathbb{N}$ such that

$$|L_i(x_n) - L_i(x)| < r_0$$

for all $n \geq n_0$ and for all $i = 1,\ldots,m$. This implies that

$$x_n \in x + B_{p_{L_1}}(0,r_1) \cap \cdots \cap B_{p_{L_m}}(0,r_m) \subset U$$

for all $n \geq n_0$, that is, $\{x_n\}$ converges to x with respect to $\sigma(X,X')$.

Example 69 *Given $1 < p < \infty$ and a Lebesgue measurable set $E \subset \mathbb{R}^N$, by what we proved in part (v) and by Example* **??***, we have that a sequence $\{f_n\} \subset L^p(E)$ converges weakly to f if*

$$\lim_{n\to\infty} \int_E f_n(x)\, g(x)\, dx = \int_E f(x)\, g(x)\, dx$$

for every $g \in L^{p'}(E)$.

Similarly, a sequence $\{f_n\} \subset L^1(E)$ *converges weakly to* f *if*

$$\lim_{n\to\infty} \int_E f_n(x)\, g(x)\, dx = \int_E f(x)\, g(x)\, dx$$

for every $g \in L^\infty(E)$.

Example 70 *If* $K \subset \mathbb{R}^N$ *is compact, by what we proved in part (v) and by Example* **??**, *we have that a sequence* $\{f_n\} \subset C(K)$ *converges weakly to* f *if*

$$\lim_{n\to\infty} \int_K f_n\, d\lambda = \int_K f\, d\lambda$$

for every $\lambda \in \mathcal{M}_{\mathrm{b}}(K)$ *(the space of all bounded signed measures).*

Exercise 71 *Consider the sequence*

$$f_n(x) := \sin(\pi n x), \quad x \in [0,1].$$

Prove that $f_n \rightharpoonup 0$ *in* $L^p([0,1])$, $1 \le p < \infty$, *but the limit*

$$\lim_{n\to\infty} f_n(x)$$

does not exist for every $x \in (0,1]$. *So weak convergence does not imply pointwise convergence.*

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Example 72 *Given a topological vector space* (X,τ)*, for each* $x\in X$ *the function* $p_x : X' \to [0,\infty)$ *defined by*

$$p_x(L) := |L(x)|, \quad L \in X',$$

is a seminorm. In view of Theorem 60, the family of seminorms $\{p_x\}_{x\in X}$ *generates a locally convex topology* $\sigma(X',X)$ *on the space* X'*, called the* weak star topology*, such that each* p_x *is continuous with respect to* $\sigma(X',X)$*. Given a sequence* $\{L_n\}\subset X'$*, we say that* $\{L_n\}$ *converges weakly star to* $L\in X'$ *and we write* $L_n \overset{*}{\rightharpoonup} L$ *if* $\{L_n\}$ *converges to* L *with respect to* $\sigma(X',X)$*. Reasoning as in part (v), it is possible to show that* $L_n \overset{*}{\rightharpoonup} L$ *if and only if*

$$\lim_{n\to\infty} L_n(x) = L(x)$$

for every $x\in X$*.*

Example 73 *Given a Lebesgue measurable set* $E\subset\mathbb{R}^N$*, we have seen in Example* **??** *that the dual of* $L^1(E)$ *may be identified with* $L^\infty(E)$*, through the mapping* $g\in L^\infty(E)\mapsto L_g$*, where* $L_g : L^1(E)\to\mathbb{R}$ *is the linear continuous function defined by*

$$L_g(f) := \int_E f(x)\,g(x)\,dx, \quad f\in L^1(E).$$

Hence, with a slight abuse of notation, we say that a sequence $\{g_n\}\subset L^\infty(E)$ *converges weakly star to* g *and we write* $g_n \overset{*}{\rightharpoonup} g$ *if* $L_{g_n} \overset{*}{\rightharpoonup} L_g$*, that is, if*

$$\lim_{n\to\infty}\int_E f(x)\,g_n(x)\,dx = \int_E f(x)\,g(x)\,dx$$

for every $f\in L^1(E)$*.*

Example 74 *Given a compact set* $K\subset\mathbb{R}^N$*, we have seen in Example* **??** *that the dual of* $C(K)$ *may be identified with* $\mathcal{M}_{\mathrm{b}}(K)$ *(the space of all bounded signed measures), through the mapping* $\lambda\in C(K)\mapsto L_\lambda$*, where* $L_\lambda : C(K)\to\mathbb{R}$ *is the linear continuous function defined by*

$$L_\lambda(f) := \int_K f\,d\mu \quad f\in C(K).$$

Hence, with a slight abuse of notation, we say that a sequence $\{\lambda_n\}\subset\mathcal{M}_{\mathrm{b}}(K)$ *converges weakly star to* λ *and we write* $\lambda_n \overset{*}{\rightharpoonup} \lambda$ *if* $L_{\lambda_n} \overset{*}{\rightharpoonup} L_\lambda$*, that is, if*

$$\lim_{n\to\infty}\int_K f\,d\lambda_n = \int_K f\,d\lambda$$

for every $f\in C(K)$*.*

Proof of Corollary 66. Let $\mathcal{P} = \{p_n\}_n$ and for $x, y \in X$ define

$$d(x,y) := \sup_n \frac{1}{n} \frac{p_n(x-y)}{1+p_n(x-y)}. \tag{12}$$

We leave as an exercise to prove that d is a metric. Note that $d(x,y) = d(x-y,0)$, and so d is translation-invariant. We now prove that the family of balls

$$B(0,r) := \{x \in X : d(x,0) < r\},$$

where $r > 0$, is a convex, balanced, local base for τ. We begin by showing that each $B(0,r)$ is open with respect to τ. If $r \geq 1$, then $B(0,r) = X$. To see this, note that it can never be that $d(x,0) = 1$. Indeed, if $d(x,0) = 1$, then since $\frac{1}{n} < 1$ for all $n \geq 2$, then necessarily,

$$d(x,0) = 1 \frac{p_1(x)}{1+p_1(x)} = 1,$$

which is impossible. Hence, $d(x,0) < 1$ for all $x \in X$, which implies that $B(0,r) = X \in \tau$ for all $r \geq 1$.

Next, fix $0 < r < 1$. Let $n_1 \in \mathbb{N}$ be so large that $\frac{1}{n} \leq r$ for all $n > n_1$ and $\frac{1}{n} > r$ for $n \leq n_1$. If $d(x,0) < r$, then

$$\sup_n \frac{1}{n} \frac{p_n(x)}{1+p_n(x)} < r \quad \text{if and only if} \quad \max_{1 \leq n \leq n_1} \frac{1}{n} \frac{p_n(x)}{1+p_n(x)} < r.$$

For $1 \leq n \leq n_1$, $\frac{1}{n} \frac{p_n(x)}{1+p_n(x)} < r$ if and only if $p_n(x) < \frac{r}{\frac{1}{n} - r}$. Hence,

$$B(0,r) = \bigcap_{n=1}^{n_1} B_{p_n}\left(0, \frac{r}{\frac{1}{n} - r}\right) \in \tau.$$

Since each $B_{p_n}(0,s)$ is convex and balanced, so is $B(0,r)$.

To prove that we have a local base, let U be a neighborhood of 0. Then there exist $p_{n_1}, \ldots, p_{n_m} \in \mathcal{P}$ and $r_1, \ldots, r_m \in (0,1)$ such that

$$B_{p_{n_1}}(0,r_1) \cap \cdots \cap B_{p_{n_m}}(0,r_m) \subset U.$$

Let

$$0 < r < \frac{1}{2} \min\left\{\frac{1}{n_1} r_1, \ldots, \frac{1}{n_m} r_m\right\}.$$

If $x \in B(0,r)$, then for every $i = 1, \ldots, m$, we have

$$\frac{1}{n_i} \frac{p_{n_i}(x)}{1+p_{n_i}(x)} < r \leq \frac{1}{2} \frac{1}{n_i} r_i,$$

that is

$$p_{n_i}(x) \leq (2 - r_i) p_{n_i}(x) < r_i.$$

Hence, $x \in B_{p_{n_i}}(0, r_i)$, and so $B(0, r) \subset U$.

To prove the last part of the statement, let (X, τ) be a locally convex topological vector space and assume that it is metrizable with metric d_1. Consider

$$B_{d_1}\left(0, \frac{1}{n}\right) = \left\{x \in X : d_1(x, 0) < \frac{1}{n}\right\}.$$

Then $\left\{B_{d_1}\left(0, \frac{1}{n}\right)\right\}_n$ is a local base at 0 for the topology τ. Note that we do not know if d_1 is translation-invariant, but since X is a topological vector space, for every $x \in X$, $\left\{x + B_{d_1}\left(0, \frac{1}{n}\right)\right\}_n$ is a local base at x for the topology τ. Thus, we need to concentrate at $x = 0$.

Since X is locally convex, for each $n \in \mathbb{N}$ there exist continuous (with respect to τ, or, equivalently, d_1) seminorms $p_{1,n}, \ldots, p_{m_n,n}$ and $r_{1,n}, \ldots, r_{m_n,n} > 0$ such that

$$B_{p_{1,n}}(0, r_{1,n}) \cap \cdots \cap B_{p_{m_n,n}}(0, r_{m_n,n}) \subset B_{d_1}\left(0, \frac{1}{n}\right).$$

Define

$$p_n := \frac{1}{r_{1,n}} p_{1,n} + \cdots + \frac{1}{r_{m_n,n}} p_{m_n,n}.$$

Note that if $p_n(x) < 1$, then $x \in B_{d_1}\left(0, \frac{1}{n}\right)$. Hence,

$$B_{p_n}(0, 1) \subset B_{d_1}\left(0, \frac{1}{n}\right).$$

Moreover, p_n is a d_1-continuous seminorm, and so $B_{p_n}(0, 1)$ is open with respect to d_1.

If $x \neq 0$, then there exists $n \in \mathbb{N}$ such that $p_n(x) > 0$. Indeed, if not, then $p_n(x) = 0$ for all $n \in \mathbb{N}$. By what we just proved, $x \in B_{p_n}(0, 1) \subset B_{d_1}\left(0, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$, which means that $x = 0$. ■

Remark 75 *Why did we not take p_n to be the Minkowski functionals of $B_{d_1}\left(0, \frac{1}{n}\right)$?*

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Make up class

Exercise 76 *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set and consider the space*

$$\mathfrak{L}^0(E) := \{f : E \to \mathbb{R} \text{ measurable}\}.$$

(i) Prove that the function

$$\rho(f,g) := \int_E \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx, \quad f, g \in \mathfrak{L}^0(E),$$

is a pseudometric.

(ii) Prove that the space $\mathfrak{L}^0(E)$ is not locally convex.

(ii) Prove that the only linear continuous functional $L : \mathfrak{L}^0(E) \to \mathbb{R}$ is the zero functional.

Exercise 77 *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set and for $0 < p < 1$ consider the space*

$$\mathfrak{L}^p(E) := \left\{ f : E \to \mathbb{R} \text{ measurable} : \int_E |f(x)|^p\, dx < \infty \right\}.$$

(i) Prove that the function

$$\rho(f,g) := \int_E |f(x) - g(x)|^p\, dx, \quad f, g \in \mathfrak{L}^p(E),$$

is a pseudometric.

(ii) Prove that the space $\mathfrak{L}^p(E)$ is not locally convex.

(ii) Prove that the only linear continuous functional $L : \mathfrak{L}^p(E) \to \mathbb{R}$ is the zero functional.

We recall the definition of normed spaces.

Definition 78 *A* norm *on a vector space X is a map*

$$\|\cdot\| : X \to [0, \infty)$$

such that

(i) $\|x\| = 0$ implies $x = 0$;

(ii) $\|tx\| = |t|\, \|x\|$ for all $x \in X$ and $t \in \mathbb{R}$;

(iii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$.

A *normed space* $(X, \|\cdot\|)$ is a vector space X endowed with a norm $\|\cdot\|$. For simplicity, we often say that X is a normed space. If for every $x, y \in X$ we define

$$d(x, y) := \|x - y\|,$$

then (X, d) is a metric space.

Exercise 79 *Let $(X, \|\cdot\|)$ be a normed space. Prove that for every $x_0 \in X$ and $r > 0$,*

$$\overline{B(x_0, r)} = \{x \in X : \|x - x_0\| \leq r\}$$

Exercise 80 *Let $(X, \|\cdot\|)$ be a normed space. Prove that a set $E \subset X$ is topologically bounded if and only it is contained in some ball.*

A topological vector space is *normable* if its topology can be determined by a norm. Note that if $(X, \|\cdot\|)$ is a normed space, then $B(x_0, r)$ is convex for every $x_0 \in X$ and $r > 0$. Hence, a normed space is a Hausdorff locally convex topological vector space.

Theorem 81 *A topological vector space (X, τ) is normable if and only if it is Hausdorff, locally convex, and it has a topologically bounded neighborhood of* 0.

Proof. If (X, τ) is normable, let $\|\cdot\|$ be a norm compatible with τ. Then $B(0, 1)$ is open and convex. Moreover, it is topologically bounded. Indeed, every neighborhood U of 0 contains a ball $B(0, r)$ for some $r > 0$. Hence, there exists $B(0, 1) \subset \frac{1}{r}U$.

Conversely, assume that (X, τ) is Hausdorff, locally convex, and that there exists a topologically bounded neighborhood W of 0. Since (X, τ) is locally convex, by Proposition 55 we may find a convex, balanced neighborhood V of 0 with $V \subset W$. Note that V is absorbing. Since W is topologically bounded, it follows that V is topologically bounded.

Define

$$\|x\| := p_V(x), \quad x \in X,$$

where p_V is the Minkowski functional of V. By Proposition 26, $\|\cdot\|$ is a seminorm.

We claim that the sets $\{rV\}_{r>0}$ are a local base at 0. To see this, let U be a neighborhood of 0. Since V is topologically bounded, there exists $t > 0$ such that $V \subset tU$, or, equivalently, $\frac{1}{t}V \subset U$, which proves the claim. Note that

$$rV = \{x \in X : \|x\| < r\}.$$

So the balls generates the topology.

Finally, we prove that $\|\cdot\|$ is a norm. It remains to show that if $x \neq 0$, then $\|x\| > 0$. Since X is a Hausdorff space, there exists a neighborhood U of 0 that does not contain x. By what we just proved, we can find $r > 0$ such that $rV \subset U$, and so $x \notin rV$, that is, $\|x\| \geq r$. ■

Exercise 82 *Let $C(\Omega)$ be as in Example 67 and let d be the metric defined in (12).*

(i) Prove that $(C(\Omega), d)$ is a complete metric space.

(ii) Characterize the convergence with respect to d.

(iii) Prove that a set $E \subset C(\Omega)$ is topologically bounded if and only if for every $n \in \mathbb{N}$ there exists a contant $M_n > 0$ such that $p_n(f) \le M_n$ for all $f \in E$.

(iv) Prove that the set $B_{p_n}(0, r)$, where $n \in \mathbb{N}$ and $r > 0$, is not topologically bounded.

Example 83 *If $\Omega \subset \mathbb{R}^N$ is an open set, then by Exercise 82 the space $C(\Omega)$ is not normable.*

Wednesday, February 24, 2010

Theorem 84 (Hahn–Banach, second geometric form) *Let (X,τ) be a locally convex topological vector space, and let C, $K \subset X$ be nonempty disjoint convex sets, with C closed and K compact. Then there exist a continuous linear functional $L : X \to \mathbb{R}$ and two numbers $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$L(x) \leq \alpha - \varepsilon \quad \text{for all } x \in C \text{ and } L(x) \geq \alpha + \varepsilon \quad \text{for all } x \in K.$$

Proof. The set $K - C$ is nonempty and convex and by Corollary 37 it is closed. Moreover, $0 \notin K - C$, since C and K are disjoint. Since $X \setminus (K - C)$ is open and contains zero, by Proposition 55 we may find a balanced convex neighborhood U of zero disjoint from $K - C$. By the first geometric form of the Hahn–Banach theorem, there exist a nonzero continuous linear functional $L : X \to \mathbb{R}$ such that

$$L(x) \leq L(y - z) = L(y) - L(z) \quad \text{for all } x \in U,\ y \in K, \text{ and } z \in C.$$

Since $L \neq 0$ and U is absorbing, there exists $x_0 \in U$ such that $L(x_0) \neq 0$. Since $-x_0 \in U$, we have that either $L(x_0)$ or $L(-x_0)$ has positive sign, say, $L(x_0)$. Then

$$L(z) + L(x_0) \leq L(y) \quad \text{for all } y \in K \text{ and } z \in C,$$

that is,

$$\sup_{z \in C} L(z) + L(x_0) \leq \inf_{y \in K} L(y).$$

Define $\varepsilon := \frac{1}{2} L(x_0)$ and $\alpha := \varepsilon + \sup_{z \in C} L(z)$. ■

We present some interesting consequences of the previous theorem.

Corollary 85 *Let (X,τ) be a locally convex topological vector space.*

(i) Let $C \subset X$ be a nonempty closed convex set and let $x_0 \notin C$. Then there exist a continuous linear functional $L : X \to \mathbb{R}$ and two numbers $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$L(x) \leq \alpha - \varepsilon \quad \text{for all } x \in C \quad \text{and } L(x_0) \geq \alpha + \varepsilon.$$

(ii) Assume that X is also Hausdorff. Then for all $x, y \in X$ with $x \neq y$ there exists a continuous linear functional $L : X \to \mathbb{R}$ such that

$$L(x) < L(y).$$

Proof. For (i) it suffices to observe that the singleton $\{x\}$ is compact and convex, while for (ii) we note that $\{x\}$ and $\{y\}$ are closed, since they are compact and X is Hausdorff. ■

4 Completeness

Definition 86 *Given a topological vector space, (X, τ), we say that a sequence $\{x_n\} \subset X$ is a* Cauchy sequence, *if for every neighborhood U of 0, there exists $n_U \in \mathbb{N}$ such that $x_n - x_m \in U$ for all $n, m \geq n_U$. The space X is said to be* complete *if every Cauchy sequence converges to an element of X.*

We say that a normed space X is a *Banach space* if it is complete.

Example 87 *Some important Banach spaces are the following:*

(i) $\mathbb{R}^N$ *is a Banach space.*

(ii) If $K \subset \mathbb{R}^N$ is compact, then $C(K)$ with the usual norm

$$\|f\|_\infty := \max_{x \in K} |f(x)|$$

is a Banach space.

(iii) Given $1 \leq p < \infty$ and a Lebesgue measurable set $E \subset \mathbb{R}^N$, the space $L^p(E)$ is a Banach space. To see this we use your homeowerk. Let $\{f_n\} \subset L^p(E)$ be such that

$$\sum_{n=1}^{\infty} \|f_n\|_{L^p} < \infty.$$

Define

$$g(x) := \left(\sum_{n=1}^{\infty} |f_n(x)| \right)^p, \quad x \in E.$$

By Minkowski's inequality, for each $\ell \in \mathbb{N}$,

$$\left(\int_E \left(\sum_{n=1}^{\ell} |f_n(x)| \right)^p dx \right)^{\frac{1}{p}} = \left\| \sum_{n=1}^{\ell} |f_n| \right\|_{L^p} \leq \sum_{n=1}^{\ell} \|f_n\|_{L^p},$$

and so by the Lebesgue monotone convergence theorem,

$$\int_E g(x)\, dx = \lim_{\ell \to \infty} \int_E \left(\sum_{n=1}^{\ell} |f_n(x)| \right)^p dx \leq \left(\sum_{n=1}^{\infty} \|f_n\|_{L^p} \right)^p < \infty.$$

Hence by a property of integrable functions, $g(x) < \infty$ for a.e. $x \in E$. At any such point $x \in E$ we have that the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent. Define the function

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} f_n(x) & \text{if } g(x) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is measurable and $|f(x)|^p \leq g(x)$ for all $x \in E$. Hence $f \in L^p(E)$. Since

$$\lim_{\ell \to \infty} \sum_{n=1}^{\ell} f_n(x) = f(x)$$

for a.e. $x \in E$ *and*

$$\left|\sum_{n=1}^{\ell} f_n(x) - f(x)\right|^p \leq g(x)$$

for a.e. $x \in E$ *and for all* $\ell \in \mathbb{N}$, *it follows by the Lebesgue dominated convergence theorem that*

$$\lim_{\ell \to \infty} \int_E \left|\sum_{n=1}^{\ell} f_n(x) - f(x)\right|^p \, dx = 0.$$

This concludes the proof.

Friday, February 26, 2010

Remark 88 *If a topological vector space (X, τ) is metrizable and the metric d is translation invariant, then since $B(0, r)$, $r > 0$, are a local base for τ, it follows that $\{x_n\} \subset X$ is a Cauchy sequence (in the sense of the previous definition) if and only if for $r > 0$, there exists $n_r \in \mathbb{N}$ such that $x_n - x_m \in B(0, r)$ for all $n, m \geq n_r$. Since d is translation invariant, we have that*

$$d(x_n - x_m, 0) = d(x_n, x_m),$$

and so $\{x_n\} \subset X$ is a Cauchy sequence (in the sense of the previous definition) if and only if $\{x_n\} \subset X$ is a Cauchy sequence in the metric sense. However, if the metric d is not translation invariant, then this is no longer true.

Exercise 89 *Prove that in $\mathbb{R}$ the euclidean metric $d(x, y) := |x - y|$ and the metric*

$$d_1(x, y) := \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right|$$

induce the same topology on $\mathbb{R}$ but $(\mathbb{R}, d)$ is complete, while $(\mathbb{R}, d_1)$ is not.

Exercise 90 *Prove that $L^\infty(E)$ is a Banach space.*

Exercise 91 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Prove that $C^\infty(\Omega)$ is complete with the topology given in Example 67(iii) .*

Exercise 92 *Let*

$$f(x) := \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Prove that $f \in C_c^\infty(\mathbb{R})$. Consider the sequence

$$f_n(x) := f(x - 1) + \frac{1}{2} f(x - 2) + \cdots + \frac{1}{n} f(x - n).$$

Prove that $\{f_n\}$ is a Cauchy sequence in $(C_c^\infty(\mathbb{R}), \tau)$ with the topology inherited from $C^\infty(\mathbb{R})$ (see Example 67(iii)) but that $\lim_n f_n$ does not have compact support. Hence, $(C_c^\infty(\mathbb{R}), \tau)$ is not complete.

Exercise 93 *Let $\Omega \subset \mathbb{R}^N$ be an open set. Prove that $C_c^\infty(\Omega)$ is complete with the topology given in Example 67(iv).*

Proposition 94 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces.*

(i) A linear operator $T : X \to Y$ is continuous if and only if

$$\|T\|_{\mathcal{L}(X;Y)} := \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} < \infty;$$

(ii) the mapping $T \in \mathcal{L}(X; Y) \mapsto \|T\|_{\mathcal{L}(X;Y)}$ is a norm;

(iii) if Y is a Banach space, then so is $\mathcal{L}(X;Y)$; conversely, if $X \neq \{0\}$ and $\mathcal{L}(X;Y)$ is a Banach space, then so is Y.

Proof. (i) By Corollary 34, T is continuous if and only if it is continuous at 0. If T is continuous at 0, then taking $\varepsilon = 1$ we may find $\delta > 0$ such that

$$\|T(x) - 0\|_Y = \|T(x) - T(0)\|_Y \leq 1$$

for all $x \in X$ with $\|x - 0\|_X \leq \delta$. If $x \in X \setminus \{0\}$, then $z := \frac{\delta}{\|x\|_X} x$ has norm δ, and so

$$\frac{\|T(x)\|_Y}{\|x\|_X} = \frac{1}{\delta} \frac{\|T(\delta x)\|_Y}{\|x\|_X} = \frac{1}{\delta} \|T(z)\|_Y \leq \frac{1}{\delta},$$

which implies that $\|T\|_{\mathcal{L}(X;Y)} < \infty$. Conversely, if $\|T\|_{\mathcal{L}(X;Y)} < \infty$, then

$$\|T(x)\|_Y \leq \|T\|_{\mathcal{L}(X;Y)} \|x\|_X$$

for all $x \in X$, which implies continuity at zero.

(ii) Is left as an exercise.

(iii) Assume that Y is a Banach space and let $\{T_n\} \subset \mathcal{L}(X;Y)$ be a Cauchy sequence. Then for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|T_n - T_m\|_{\mathcal{L}(X;Y)} \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. Hence,

$$\|T_n(x) - T_m(x)\|_Y \leq \varepsilon \|x\|_X \tag{13}$$

for all $x \in X$ for all $n, m \geq n_\varepsilon$. Hence, for every $x \in X$, $\{T_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exists $T(x) \in Y$ such that $\lim_{n\to\infty} \|T_n(x) - T(x)\|_Y = 0$. Since each T_n is linear, it follows that T is linear.

Letting $n \to \infty$ in (13) gives

$$\|T(x) - T_m(x)\|_Y \leq \varepsilon \|x\|_X$$

for all $x \in X$ for all $m \geq n_\varepsilon$, which shows that $\|T - T_m\|_{\mathcal{L}(X;Y)} \leq \varepsilon$ for all $m \geq n_\varepsilon$. Since,

$$\begin{aligned}\|T(x)\|_Y = \|T(x) \pm T_{n_\varepsilon}(x)\|_Y &\leq \|T(x) - T_{n_\varepsilon}(x)\|_Y + \|T_{n_\varepsilon}(x)\|_Y \\ &\leq \varepsilon \|x\|_X + \|T_{n_\varepsilon}\|_{\mathcal{L}(X;Y)} \|x\|_X\end{aligned}$$

for all $x \in X$, it follows that $T \in \mathcal{L}(X;Y)$. This shows that $\mathcal{L}(X;Y)$ is a Banach space.

Conversely, assume that $\mathcal{L}(X;Y)$ is a Banach space and that $X \neq \{0\}$. Let $\{y_n\} \subset Y$ be a Cauchy sequence. Since $X \setminus \{0\}$ is nonempty, let $x_0 \in X \setminus \{0\}$. By Corollary 85, there exists a continuous functional $L : X \to \mathbb{R}$ such that $L(x_0) = 1$. Define

$$T_n(x) := L(x)\, y_n, \quad x \in X.$$

Then $T_n : X \to Y$ is linear and

$$\|T_n\|_{\mathcal{L}(X;Y)} = \sup_{x \in X \setminus \{0\}} \frac{\|L(x)\, y_n\|_Y}{\|x\|_X} = \|y_n\|_Y \sup_{x \in X \setminus \{0\}} \frac{|L(x)|_Y}{\|x\|_X} = \|y_n\|_Y \|L\|_{X'}$$

and, similarly, $\|T_n - T_m\|_{\mathcal{L}(X;Y)} = \|y_n - y_m\|_Y \|L\|_{X'}$. This implies that $\{T_n\}$ is a Cauchy sequence in $\mathcal{L}(X;Y)$. By hypothesis, there exists $T \in \mathcal{L}(X;Y)$ such that $T_n \to T$ in $\mathcal{L}(X;Y)$. In particular, $T_n(x) \to T(x)$ for all $x \in X$. Since

$$y_n = L(x_0)\, y_n = T_n(x_0) \to T(x_0),$$

and so Y is complete. ■

Exercise 95 *Given a vector space X, prove that a nonempty family $\mathcal{B}$ of subsets of X is a local base at 0 for a topology τ that makes X a topological vector space if and only if*

(i) for every $B_1, B_2 \in \mathcal{B}$ there exists $B_3 \in \mathcal{B}$ such that $B_3 \subset B_1 \cap B_2$,

(ii) each $B \in \mathcal{B}$ is absorbing and balanced,

(iii) for every $B_1 \in \mathcal{B}$ there exists $B_2 \in \mathcal{B}$ such that $B_2 + B_2 \subset B_1$.

(iv) for every $B_1 \in \mathcal{B}$ and every $x \in B_1$, there exists $B_2 \in \mathcal{B}$ such that $x + B_2 \subset B_1$.

Remark 96 *Note that*

$$\|T\|_{\mathcal{L}(X;Y)} = \sup_{x\in X:\, \|x\|_X=1} \|T(x)\|_Y = \sup_{x\in X:\, 0<\|x\|_X\le 1} \frac{\|T(x)\|_Y}{\|x\|_X}$$
$$= \sup_{x\in X:\, 0<\|x\|_X<1} \frac{\|T(x)\|_Y}{\|x\|_X}.$$

Moreover, if $Y=\mathbb{R}$, then

$$\|T\|_{X'} = \sup_{x\in X\setminus\{0\}} \frac{T(x)}{\|x\|}.$$

Indeed, it suffices to observe that if and $T(x)<0$, then $|T(x)| = -T(x) = T(-x)$.

Example 97 *Given $1<p<\infty$ and a Lebesgue measurable set $E\subset\mathbb{R}^N$, consider the space $L^p(E)$. Given $g\in L^{p'}(E)$, where $p' := \frac{p}{p-1}$ consider the linear function $L_g : L^p(E)\to\mathbb{R}$ defined by*

$$L_g(f) := \int_E f(x)\,g(x)\,dx, \quad f\in L^p(E).$$

We have seen that by Hölder's inequality,

$$|L_g(f)| \le \int_E |f(x)|\,|g(x)|\,dx \le \|f\|_{L^p(E)}\,\|g\|_{L^{p'}(E)}.$$

Hence,

$$\|L\|_{(L^p(E))'} = \sup_{f\in L^p(E)\setminus\{0\}} \frac{|L_g(f)|}{\|f\|_{L^p(E)}} \le \|g\|_{L^{p'}(E)}.$$

To prove the converse inequality, if $1<p<\infty$, take $f := g\,|g|^{p'-2}$. Then

$$L_g(f) = \int_E |g(x)|^{p'}\,dx,$$

while

$$\|f\|_{L^p(E)} = \left(\int_E |g(x)|^{(p'-1)p}\,dx\right)^{1/p} = \left(\int_E |g(x)|^{p'}\,dx\right)^{1/p}$$

where we have used the fact that $\frac{1}{p}+\frac{1}{p'}=1$. Hence,

$$\frac{|L_g(f)|}{\|f\|_{L^p(E)}} = \frac{\int_E |g(x)|^{p'}\,dx}{\left(\int_E |g(x)|^{p'}\,dx\right)^{1/p}} = \left(\int_E |g(x)|^{p'}\,dx\right)^{1/p'}.$$

Thus,

$$L^{p'}(E) \to (L^p(E))'$$
$$g \in L^{p'}(E) \mapsto L_g$$

preserves the norm.

Exercise 98 *Given a Lebesgue measurable set $E \subset \mathbb{R}^N$, consider the space $L^1(E)$. Given $g \in L^\infty(E)$, consider the linear function $L_g : L^\infty(E) \to \mathbb{R}$ defined by*

$$L_g(f) := \int_E f(x) g(x)\, dx, \quad f \in L^1(E).$$

Prove that

$$\|L\|_{(L^1(E))'} = \sup_{f \in L^1(E) \setminus \{0\}} \frac{|L_g(f)|}{\|f\|_{L^1(E)}} = \|g\|_{L^\infty(E)}.$$

Exercise 99 *Given a compact set $K \subset \mathbb{R}^N$, consider the space $C(K)$. Given $\lambda \in \mathcal{M}_{\mathrm{b}}(K)$, consider the linear function $L_\lambda : C(K) \to \mathbb{R}$ defined by*

$$L_\lambda(f) = \int_K f\, d\lambda, \quad f \in C(K).$$

Prove that

$$\|L\|_{(C(K))'} = \sup_{f \in C(K) \setminus \{0\}} \frac{|L_\lambda(f)|}{\|f\|_{C(K)}} = \|\lambda\|_{\mathcal{M}_{\mathrm{b}}(K)}.$$

As a corollary of the Hahn–Banach theorem one has the following results.

Corollary 100 (Dual of a subspace) *Let $(X, \|\cdot\|)$ be a normed space and let Y be a subspace of X. If $L \in Y'$, then L can be extended to functional $L_1 \in X'$ with*

$$\|L_1\|_{X'} = \|L\|_{Y'}.$$

Proof. See Corollary 59. ■

Corollary 101 *Let $(X, \|\cdot\|)$ be a normed space. Then for every $x \in X \setminus \{0\}$ there exists $L \in X'$ such that*

$$L(x) = \|x\| \quad \textit{and} \quad \|L\|_{X'} = 1.$$

Proof. Let $Y = \operatorname{span}\{x\}$ and define

$$L'(tx) = t\|x\|, \quad t \in \mathbb{R}.$$

Then $L' \in Y'$, $L'(x) = \|x\|$, and $\|L'\|_{Y'} = 1$. By the previous corollary, we can extend L' to a functional $L \in X'$ with $\|L\|_{X'} = 1$. ■

Corollary 102 *Let* $(X, \|\cdot\|)$ *be a normed space. Then for all* $x_0 \in X \setminus \{0\}$*,*

$$\|x_0\| = \max_{L \in X',\, \|L\|_{X'} \le 1} \frac{|L(x_0)|}{\|x_0\|}.$$

Proof. If $L \in X'$ is such that $\|L\|_{X'} \le 1$, then $|L(x_0)| \le \|L\|_{X'} \|x_0\| \le \|x_0\|$, and so

$$\sup_{L \in X',\, \|L\|_{X'} \le 1} \frac{|L(x_0)|}{\|x_0\|} \le \|x_0\|.$$

By the previous corollary the supremum is attained, so it is a maximum. ■

Remark 103 *The previous corollary is especially useful for* L^p *spaces.*

Theorem 104 (Banach–Steinhaus) *Let* $(X, \|\cdot\|_X)$ *be a Banach space,* $(y, \|\cdot\|_Y)$ *be a normed space, and let* $\{T_\alpha\}_{\alpha\in\Lambda}$ *be a family of linear continuous operators* $T_\alpha : X \to Y$ *such that*

$$\sup_{\alpha\in\Lambda} \|T_\alpha(x)\|_Y < \infty$$

for every $x \in X$*. Then*

$$\sup_{\alpha\in\Lambda} \|T_\alpha\|_{\mathcal{L}(X;Y)} < \infty.$$

Proof. For every $n \in \mathbb{N}$ consider the closed set

$$\begin{aligned} C_n : &= \{x \in X : \|T_\alpha(x)\|_Y \le n \text{ for every } \alpha \in \Lambda\} \\ &= \bigcap_{\alpha\in\Lambda} T_\alpha^{-1}\left(\overline{B_Y(0,n)}\right). \end{aligned}$$

We claim that

$$\bigcup_{n=1}^{\infty} C_n = X.$$

Indeed, if $x \in X$, then by hypothesis,

$$m := \sup_{\alpha\in\Lambda} \|T_\alpha(x)\|_Y < \infty.$$

Let $n \in \mathbb{N}$ be such that $n \ge m$. Then x belongs to C_n. This proves the claim.

Since X is complete, by Baire's theorem, there exists $n_0 \in \mathbb{N}$ such that C_{n_0} has nonempty interior. Thus, there exists $B_X(x_0, r) \subset C_{n_0}$. This implies that for all $z \in B_X(0,1)$ and for every $\alpha \in \Lambda$,

$$\|T_\alpha(x_0) + rT_\alpha(z)\|_Y = \|T_\alpha(x_0 + rz)\|_Y \le n_0.$$

Thus,

$$r\|T_\alpha(z)\|_Y = \|T_\alpha(rz) \pm T_\alpha(x_0)\|_Y \le \|T_\alpha(x_0)\|_Y + n_0,$$

and so

$$\|T_\alpha\|_{\mathcal{L}(X;Y)} = \sup_{z \in B_X(0,1)} \|T_\alpha(z)\|_Y \le \frac{1}{r}\left(\|T_\alpha(x_0)\|_Y + n_0\right)$$

for all $\alpha \in \Lambda$. Hence,

$$\sup_{\alpha \in \Lambda} \|T_\alpha\|_{\mathcal{L}(X;Y)} \leq \frac{1}{r} \left(\sup_{\alpha \in \Lambda} \|T_\alpha (x_0)\|_Y + n_0 \right) < \infty$$

and the proof is complete. ■

Exercise 105 *Let $(X, \|\cdot\|)$ be a Banach space and let $E \subset X$. Prove that if for every $L \in X'$ the set $L(E) \subset \mathbb{R}$ is bounded then E is (norm) bounded. Hint: use the Banach-Steinhaus Theorem.*

Theorem 106 (Open mapping theorem) *Let $(X, \|\cdot\|_X)$ and $(y, \|\cdot\|_Y)$ be two Banach spaces and let $T : X \to Y$ be linear, continuous, and onto. Then T is open; that is $T(U)$ is open for every open set $U \subset X$.*

Proof. Step 1: We claim that there exists $c > 0$ such that

$$\overline{T\left(B_X\left(0, \frac{1}{4}\right)\right)} \supset B_Y(0, c).$$

For every $n \in \mathbb{N}$ consider the closed set

$$C_n := \overline{T(B_X(0, n))}.$$

We claim that

$$\bigcup_{n=1}^{\infty} C_n = Y.$$

Indeed, if $y \in Y$, then since T is onto, there exists $x \in X$ such that $T(x) = y$. Let $m := \|x\|_X + 1$ and let $n \in \mathbb{N}$ be such that $n \geq m$. Then $x \in B_X(0, n)$, and so $T(x) = y$ belongs to C_n. This proves the claim.

Since Y is complete, by Baire's theorem, there exists $n_0 \in \mathbb{N}$ such that C_{n_0} has nonempty interior. Thus, there exists $B_Y(y_0, r) \subset C_{n_0} = \overline{T(B_X(0, n_0))}$. This implies that

$$B_Y\left(\frac{y_0}{n_0}, \frac{r}{n_0}\right) \subset \overline{T(B_X(0, 1))}.$$

Let $z_0 := \frac{y_0}{n_0}$ and $8c := \frac{r}{n_0}$. Then $z_0 \in \overline{T(B_X(0, 1))}$. Since $-B_X(0, 1) = B_X(0, 1)$, we have that

$$-z_0 \in -\overline{T(B_X(0, 1))} = \overline{T(-B_X(0, 1))} = \overline{T(B_X(0, 1))},$$

and so

$$B_Y(0, 8c) = -z_0 + B_Y(z_0, 8c) \subset \overline{T(B_X(0, 1))} + \overline{T(B_X(0, 1))} = 2\overline{T(B_X(0, 1))},$$

or, equivalently,

$$B_Y(0, c) \subset \overline{T\left(B_X\left(0, \frac{1}{4}\right)\right)},$$

which prove the claim. Note that by rescaling, this implies that

$$B_Y\left(0, \frac{c}{2^{n-1}}\right) \subset \overline{T\left(B_X\left(0, \frac{1}{2^{n+1}}\right)\right)}$$

for all $n \in \mathbb{N}$.

Step 2: We claim that

$$T(B_X(0, 1)) \supset B_Y(0, c).$$

Fix $y \in Y$ with $\|y\|_Y < c$. By the previous step for every $\varepsilon > 0$ there exists $x \in X$ with $\|x\|_X < \frac{1}{4}$ such that

$$\|y - T(x)\|_Y < \varepsilon.$$

Take $\varepsilon := \frac{c}{4}$ and find $x_1 \in X$ with $\|x_1\|_X < \frac{1}{4}$ such that

$$\|y - T(x_1)\|_Y < \frac{c}{4}.$$

Repeat the same procedure with $y - T(x_1)$ in place of y and with $\varepsilon := \frac{c}{8}$ to find $x_2 \in X$ with $\|x_2\|_X < \frac{1}{8}$ such that

$$\|(y - T(x_1)) - T(x_2)\|_Y < \frac{c}{8}.$$

By induction, construct $\{x_n\} \subset X$ such that

$$\|x_n\|_X < \frac{1}{2^{n+1}}, \quad \|y - T(x_1 + \cdots + x_n)\|_Y < \frac{c}{2^{n+1}}$$

for all $n \in \mathbb{N}$. Define $z_n := x_1 + \cdots + x_n$. The sequence $\{z_n\}$ is a Cauchy sequence, since

$$\|z_{n+k} - z_n\|_X = \|x_{n+1} + \cdots + x_{n+k}\|_X \leq \sum_{i=1}^{k} \frac{1}{2^{n+i}} \leq \frac{1}{2^{n+1}} \to 0$$

as $n \to \infty$. Since X is complete, there exists $z \in X$ such that $z_n \to z$. By the continuity of the norm and of T,

$$\|y - T(z)\|_Y \leftarrow \|y - T(z_n)\|_Y < \frac{c}{2^{n+1}} \to 0,$$

and so $y = T(z)$. Moreoever,

$$\|z\|_X = \left\| \sum_{n=1}^{\infty} x_n \right\|_X \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} < 1,$$

which proves the claim.

Step 3: We claim that T is open. Let $U \subset X$ be an open set and let $y_0 \in T(U)$. Then there exist $x_0 \in U$ such that $T(x_0) = y_0$. Since U is open, we may find $B_X(x_0, r) \subset U$. Hence, $x_0 + B_X(0, r) \subset U$, and so

$$y_0 + T(B_X(0, r)) \subset T(U).$$

On the other hand, by Step 2, $T(B_X(0, r)) \supset B_Y(0, rc)$, and so

$$y_0 + B_Y(0, rc) \subset y_0 + T(B_X(0, r)) \subset T(U),$$

which proves that y_0 is an interior point of $T(U)$, and in turn, that $T(U)$ is open. ■

Remark 107 *If* $X = \mathbb{R}^N$*, then the proof of Step 2 is much simpler. Indeed, one can show that*

$$\overline{T\left(B_X\left(0,\frac{1}{4}\right)\right)} \subset T\left(B_X\left(0,1\right)\right).$$

Take $y \in \overline{T\left(B_X\left(0,\frac{1}{4}\right)\right)}$ *and find a sequence* $\{y_n\} \subset T\left(B_X\left(0,\frac{1}{4}\right)\right)$ *such that* $y_n \to y$*. Then there eare* $x_n \in B_X\left(0,\frac{1}{4}\right)$ *such that* $T\left(x_n\right) = y_n$*. Since in* $\mathbb{R}^N$*,* $\overline{B_X\left(0,\frac{1}{4}\right)}$ *is compact, there exist a subsequence* $\{x_{n_k}\}$ *and* $x \in \overline{B_X\left(0,\frac{1}{4}\right)}$ *such that* $x_{n_k} \to x$*. By the continuity of* T*,* $T\left(x\right) = y$*. Unfortunately, if instead* X *is infinite dimensional,* $\overline{B_X\left(0,\frac{1}{4}\right)}$ *is never compact.*

Corollary 108 *Let* $\left(X, \|\cdot\|_X\right)$ *and* $\left(y, \|\cdot\|_Y\right)$ *be two Banach spaces and let* $T : X \to Y$ *be linear, continuous, onto, and one-to-one. Then* T^{-1} *is continuous; that is* T *is a homeomorphism.*

Proof. Let $S = T^{-1}$. To prove that S is continuous, it is enough to show that $S^{-1}\left(U\right)$ is open for every $U \subset X$ open. But $S^{-1}\left(U\right) = T\left(U\right)$, which is open by the previous theorem. ■

Definition 109 *Given a vector space* X*, two norms* $\|\cdot\|_1 : X \to [0,\infty)$ *and* $\|\cdot\|_2 : X \to [0,\infty)$ *are said to be* equivalent *if there exists a constant* $c > 0$ *such that*

$$\frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1$$

for all $x \in X$*.*

Note that equivalent norms generate the same topology.

Corollary 110 *Given a vector space X, and two norms $\|\cdot\|_1 : X \to [0,\infty)$ and $\|\cdot\|_2 : X \to [0,\infty)$, assume that there exists a constant $c > 0$ such that*

$$\|x\|_2 \le c \|x\|_1$$

for all $x \in X$. If $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces, then the two norms are equivalent.

Proof. The operator

$$\begin{aligned} T : (X, \|\cdot\|_1) &\to (X, \|\cdot\|_2) \\ x &\mapsto x \end{aligned}$$

is linear, onto, and one-to-one. Moreover,

$$\|T(x)\|_2 = \|x\|_2 \le c \|x\|_1$$

for all $x \in X$, and so T is continuous. By Corollary 108, T^{-1} is continuous. Hence, there exists $C > 0$ such that

$$\left\|T^{-1}(x)\right\|_1 = \|x\|_1 \le C \|x\|_2$$

for all $x \in X$, which shows the desired result. ■

Example 111 *In the previous theorem, completeness plays an important role. In $X = C([0,1])$ consider the two norms*

$$\|f\|_1 := \max_{x\in[0,1]} |f(x)|, \qquad \|f\|_2 := \int_0^1 |f(x)|\, dx.$$

Then $\|f\|_2 \le \|f\|_1$, but the two norms are not equivalent. The problem here is that $(C([0,1]), \|\cdot\|_2)$ is not a Banach space.

Corollary 112 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in \mathcal{L}(X;Y)$ be such that $T(X)$ is a closed set. Then there exists a constant $C > 0$ such that for every $y \in T(X)$ there exists an $x \in X$ with $T(x) = y$ and*

$$\|x\|_X \le C \|y\|_Y .$$

Proof. Since $T(X)$ is a closed set of a Banach space, it is a Banach space. Thus we can apply the open mapping theorem to $T : X \to T(X)$. Hence, $T(B_X(0,1))$ is relatively open in Y, and so there exists $r > 0$ such that $B_Y(0,r) \cap T(X) \subset T(B_X(0,1))$. Now if $y \in T(X)$ and $y \ne 0$, then $\frac{r}{2}\frac{y}{\|y\|_Y} \in B_Y(0,r) \cap T(X)$ and so there exists $z \in B_X(0,1)$ such that $T(z) = \frac{r}{2}\frac{y}{\|y\|_Y}$. Define $x := \frac{2}{r}\|y\|_Y z \in X$. Then $T(x) = y$ and

$$\|x\|_X = \left\|\frac{2}{r}\|y\|_Y z\right\|_X = \frac{2}{r}\|z\|_X \|y\|_Y \le \frac{2}{r}\|y\|_Y .$$

■

Theorem 113 (Closed graph theorem) *Let* $(X, \|\cdot\|_X)$ *and* $(y, \|\cdot\|_Y)$ *be two Banach spaces and let* $T : X \to Y$ *be a linear operator such that*

$$\operatorname{graph} T = \{(x, T(x)) : x \in X\}$$

is closed. Then T *is continuous,*

Proof. Consider the vector space

$$Z := \{(x, T(x)) : x \in X\}$$

and in Z define

$$\|(x, T(x))\|_Z := \|x\|_X + \|T(x)\|_Y .$$

Since T is linear, we have that $\|\cdot\|_Z$ is a norm. We claim that $(Z, \|\cdot\|_Z)$ is a Banach space. To see this, let $\{(x_n, T(x_n))\}$ be a Cauchy sequence. It follows that the two sequences $\{x_n\} \subset X$ and $\{T(x_n)\} \subset Y$ are Cauchy sequences. Since X and Y are complete, there exist $x \in X$ and $y \in Y$ such that $x_n \to x$ and $T(x_n) \to y$. Since in a metric space, closed sets and sequentially closed sets are the same, and since $(x_n, T(x_n)) \in \operatorname{graph} T$ and $(x_n, T(x_n)) \to (x, y)$, it follows that $(x, y) \in \operatorname{graph} T$, that is, $T(x) = y$. This shows that $(x_n, T(x_n)) \to (x, T(x))$ in Z. Hence, Z is a Banach space.

The projection operator

$$\begin{aligned} P_1 : Z &\to X \\ (x, T(x)) &\mapsto x \end{aligned}$$

is linear, onto, and one-to-one. Moreover,

$$\|P_1((x, T(x)))\|_X = \|x\|_X \le \|x\|_X + \|T(x)\|_Y = \|(x, T(x))\|_Z$$

for all $x \in X$, and so P_1 is continuous. By Corollary 108, $P_1^{-1} : X \to Z$ is continuous. Hence, there exists $C > 0$ such that

$$\|x\|_X + \|T(x)\|_Y = \left\|P_1^{-1}((x, T(x)))\right\|_Z \le C \|x\|_X$$

for all $x \in X$. In particular,

$$\|T(x)\|_Y \le C \|x\|_X$$

for all $x \in X$, which shows the desired resuly. ■

Remark 114 *The open mapping theorem and the closed graph theorem are actually equivalent, in the sense that one could alternatively prove directly the closed graph and then use it to prove the open mapping theorem.*

5 Weak Topologies

5.1 Loss of Compactness

In $\mathbb{R}^N$, given a sequence $\{x_n\}$ such that

$$\sup_n \|x_n\| < \infty,$$

by the Bolzano–Weierstrass theorem there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in \mathbb{R}^N$ such that $x_{n_k} \to x$ as $k \to \infty$. Unfortunately, the same result fails in infinite-dimensional normed spaces.

Indeed, in $L^1([0,1])$ consider the sequence

$$f_n(x) := \begin{cases} n & \text{if } 0 \le x \le \frac{1}{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|f_n\|_{L^1([0,1])} = \int_0^1 |f_n(x)|\, dx = 1.$$

If there existed a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in L^1([0,1])$ such that $f_{n_k} \to f$ in $L^1([0,1])$ as $k \to \infty$, then by a theorem of measure theory, up to a further subsequence, we could assume that $f_{n_k}(x) \to f(x)$ for all $x \in [0,1]$ except on a set of measure zero. But for $x > 0$, $f_n(x) = 0$ for all $n > \frac{1}{x}$, and so $f(x) = 0$. However,

$$\|f_{n_k} - f\|_{L^1([0,1])} = \int_0^1 |f_{n_k}(x) - 0|\, dx = 1 \nrightarrow 0,$$

which is a contradiction.

Similarly, in $C([0,1])$ consider the sequence

$$f_n(x) := \sin(2\pi n x).$$

Then

$$\|f_n\|_{C([0,1])} = 1$$

but for every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and for every $x \in [0,1]$ irrational, the limit

$$\lim_{k\to\infty} \sin(2\pi n_k x)$$

does not exist. Since uniform convergence implies pointwise convergence, no subsequence can converge uniformly to a function.

Proposition 115 *Let (X, τ) be a topological vector space and let $K \subset X$ be compact. Then K is topologically bounded.*

Proof. Let V be a neighborhood of 0. Find a balanced neighborhood of 0, $W \subset V$. We proved that

$$X = \bigcup_{n=1}^{\infty} nW.$$

Since $K \subset \bigcup_{n=1}^{\infty} nW$, by compactness there exists $\overline{n} \in \mathbb{N}$ such that

$$K \subset \bigcup_{n=1}^{\overline{n}} nW = \overline{n}W,$$

where in the last equality we have used the fact that W is balanced. Thus, $K \subset \overline{n}V$. ■

Remark 116 *Thus in a Hausdorff topological vector space, every compact set is closed and topologically bounded.*

We recall the following definition.

Proposition 117 *Let (X, τ) be a topological vector space.*

(i) If $E \subset X$ is topologically bounded, then for every neighborhood V of 0 there exists $t_0 > 0$ such that $E \subset tV$ for all $t > t_0$.

(ii) If $V \subset X$ is a topologically bounded neighborhood of 0, then for every strictly decreasing sequence $\{\delta_n\} \subset (0, \infty)$ such that $\delta_n \to 0$, the family $\{\delta_n V\}_n$ is a local base of 0.

Proof. (i) Let $W \subset V$ be a balanced neighborhood of 0. Since E is topologically bounded, there exists $t_0 > 0$ such that $E \subset t_0 W$. But since W is balanced, if $t > t_0$, then $t_0 W \subset tW$, and so $E \subset tW \subset tV$.

(ii) Let U be a neighborhood of 0. By part (i) there exists $t_0 > 0$ such that $V \subset tU$ for all $t > t_0$. Let $n \in \mathbb{N}$ be so large that $\delta_n t_0 < 1$. Then $V \subset \frac{1}{\delta_n} U$, or, equivalently, $\delta_n V \subset U$. ■

Theorem 118 *Let (X, τ) be a Hausdorff topological vector space and let $Y \subset X$ be a finite-dimensional subspace. Then Y is closed.*

Proof. Step 1: We prove that if $T : \mathbb{R}^N \to X$ is linear, then T is continuous. Consider the canonical basis $\{e_1, \ldots, e_N\}$ in $\mathbb{R}^N$ and define $w_i := T(e_i)$. Then by linearity,

$$T(x) = T\left(\sum_{i=1}^{N} x_i e_i\right) = \sum_{i=1}^{N} x_i w_i.$$

Since in X multiplication by scalar is continuous and addition is continuous, we have that T is continuous. ■

Proof. Step 2: Let $Y \subset X$ be an N-dimensional subspace and let $T : \mathbb{R}^N \to Y$ be linear and invertible. We claim that T is a homeomorphism. By the previous step, $T : \mathbb{R}^N \to X$ is continuous. Since the unit sphere $\partial B(0,1) \subset \mathbb{R}^N$ is compact, it follows that $K := T(\partial B(0,1)) \subset X$ is compact. Moroever, $0 \notin K$ (since $T(0) = 0$ and T is invertible). Thus there exists a balanced neighborhood V of 0 that does not intersect K. Let $W := T^{-1}(V \cap Y)$. Then W is a open neighborhood of 0. Moreover, W is balanced. Indeed, if $x \in W$ and $t \in [-1,1]$, then $T(tx) = tT(x) \in tV \subset V$. Since V does not intersect K and T is invertible, we have that W does not intersect $\partial B(0,1)$. This implies that $W \subset B(0,1)$ (otherwise if there existed $x \in W$ such that $\|x\| > 1$, then $\frac{x}{\|x\|} \in W$, since W is balanced and we have a contradiction).

Consider $T^{-1} : Y \to \mathbb{R}^N$. Since T^{-1} is linear and maps the relatively open set $V \cap Y$ into $B(0,1)$, we have that each component is bounded in a neighborhood of 0, and so by Corollary 47, T^{-1} is continuous.

Step 3: Let $Y \subset X$ be an N-dimensional subspace and let $\{w_1, \ldots, w_N\}$ be a basis in Y. Define

$$T(x) := \sum_{i=1}^{N} x_i w_i.$$

By the previous two steps, T is a homeomorphism. Let $y \in \overline{Y}$ and let V be as in Step 2. Since V is balanced, $y \in tV$ for some $t > 0$. Hence,

$$y \in \overline{Y \cap tV} \subset \overline{T(B(0,t))} \subset \overline{T\left(\overline{B(0,t)}\right)} = T\left(\overline{B(0,t)}\right) \subset Y,$$

where we have used the fact that $T\left(\overline{B(0,t)}\right)$ is closed, since it is compact ($\overline{B(0,t)}$ is compact and T continuous). Hence, Y contains its closure, and so it is closed. ■

Exercise 119 *Let (X, τ) be topological vector space, let $E \subset X$ and $\mathcal{B}$ be a local base of 0. Prove that*

$$\overline{E} = \bigcap_{B \in \mathcal{B}} (E + B).$$

Definition 120 *A topological space (X, τ) is* locally compact *if every point has a neighborhood whose closure is compact.*

Theorem 121 *Every locally compact, Hausdorff topological vector space (X, τ) is finite-dimensional.*

Proof. Let $V \subset X$ be a neighborhood of 0 with $\overline{V}$ compact. By Proposition 115, $\overline{V}$ is topologically bounded. In turn, by Proposition 117 the family $\left\{\frac{1}{2^n}V\right\}_{n\in\mathbb{N}}$ is a local base at 0.

Consider the family of open sets $\left\{x + \frac{1}{2}V\right\}_{x\in X}$. Since it covers the compact set $\overline{V}$, there exist $x_1, \ldots, x_m \in X$ such that

$$\overline{V} \subset \bigcup_{i=1}^{m} \left(x_i + \frac{1}{2}V\right). \tag{14}$$

Let $Y = \operatorname{span}\{x_1, \dots, x_m\}$. Then Y has finite dimension $N \le m$, and so Y is closed, by the previous theorem. By (14), $V \subset Y + \frac{1}{2}V$, and since Y is a subspace, we have that

$$\frac{1}{2}V \subset \frac{1}{2}Y + \frac{1}{4}V = Y + \frac{1}{4}V,$$

and so $V \subset Y + \frac{1}{2}V \subset Y + Y + \frac{1}{4}V = Y + \frac{1}{4}V$. An induction argument shows that $V \subset Y + \frac{1}{2^n}V$ for every $n \in \mathbb{N}$. Hence,

$$V \subset \bigcap_{n=1}^{\infty} \left(Y + \frac{1}{2^n}V \right) = \overline{Y} = Y,$$

where we have used the previous exercise. In turn $kV \subset kY = Y$ for every $k \in \mathbb{N}$. By Proposition 35

$$X = \bigcup_{k=1}^{\infty} kV \subset Y,$$

which shows that X is finite-dimensional. ■

Corollary 122 *In an infinite-dimensional normed space $\overline{B(0,r)}$ is never compact.*

Monday, March 22, 2010

5.2 Weak Star Topology

Given a topological vector space (X, τ), for each $x \in X$ the function $p_x : X' \to [0, \infty)$ defined by

$$p_x(L) := |L(x)|, \quad L \in X', \tag{15}$$

is a seminorm. In view of Theorem 60, the family of seminorms $\{p_x\}_{x \in X}$ generates a locally convex topology $\sigma(X', X)$ on the space X', called the *weak star topology*, such that each p_x is continuous with respect to $\sigma(X', X)$.

Exercise 123 *Let (X, τ) be a locally convex topological vector space. Prove that $(X', \sigma(X', X))$ is a Hausdorff space.*

Theorem 124 (Banach–Alaoglu) *If V is a neighborhood of 0 in a locally convex topological vector space (X, τ), then*

$$K := \{L \in X' : |L(x)| \leq 1 \text{ for every } x \in V\}$$

is weak star compact.

Proof. Since V is absorbing, for every $x \in X$ there exists $t_x > 0$ such that $x \in t_x V$, or, equivalently $\frac{1}{t_x} x \in V$. Hence, if $L \in K$, then $\left|L\left(\frac{1}{t_x} x\right)\right| \leq 1$, that is, $|L(x)| \leq t_x$, so that $L(x) \in [-t_x, t_x]$, which is a compact set of $\mathbb{R}$. Define

$$Y := \prod_{x \in X} [-t_x, t_x] = \{f : X \to \mathbb{R} : f(x) \in [-t_x, t_x] \text{ for every } x \in X\}.$$

By Tychonoff's theorem, Y is compact with the product topology τ_Y. Since $K \subset X' \cap Y$, in K we have two topologies, $\sigma(X', X)$ and the product topology τ_Y. We claim that in K these two topologies coincide. To see this, fix $L_0 \in K$. A local base neighborhood of L_0 in $\sigma(X', X)$ is a set of the form

$$W_1 = \{L \in X' : |L(x_i) - L_0(x_i)| < r, \, i = 1, \ldots, m\}$$

for some $\{x_1, \ldots, x_m\} \subset X$, while a local base neighborhood of L_0 in τ_Y is a set of the form

$$W_2 = \{f \in Y : |f(x_i) - L_0(x_i)| < r, \, i = 1, \ldots, m\}.$$

Since $K \subset X' \cap Y$, we have that $W_1 \cap K = W_2 \cap K$. Hence, the relatives topologies coincide.

Next we claim that K is a closed subset of Y with respect to τ_Y. To see this, let $f_0 \in \overline{K}^{\tau_Y}$. Let $x, y \in X$, $s, t \in \mathbb{R}$, and $\varepsilon > 0$. Consider the set U of all $f \in Y$ such that

$$|f(x) - f_0(x)| < \varepsilon, \quad |f(y) - f_0(y)| < \varepsilon, \quad |f(sx + ty) - f_0(sx + ty)| < \varepsilon.$$

The set U is a neighborhood of f_0 with respect to τ_Y. Since $f_0 \in \overline{K}^{\tau_Y}$, it follows that $K \cap U \neq \emptyset$. Let $L \in K \cap U$. Since L is linear, we have

$$\begin{aligned}&|f_0(sx+ty) - sf_0(x) - tf_0(y)| \\ &= |f_0(sx+ty) - L(sx+ty) - sf_0(x) + sL(x) - tf_0(y) + tL(y)| \\ &\leq \varepsilon(1+|s|+|t|).\end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain that f_0 is linear. Next we prove that f_0 is continuous. Fix $x \in V$ and $\varepsilon > 0$ and define

$$W := \{f \in Y : |f(x) - f_0(x)| < \varepsilon\}.$$

The set W is a neighborhood of f_0 with respect to τ_Y. Since $f_0 \in \overline{K}^{\tau_Y}$, it follows that $K \cap W \neq \emptyset$. Let $L \in K \cap W$. Then $|L(x)| \leq 1$, and so

$$|f_0(x)| = |f_0(x) \pm L(x)| \leq 1 + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we obtain that $|f_0(x)| \leq 1$ for all $x \in V$. Hence, by Corollary 47, f_0 is continuos. In turn, $f_0 \in K$. This shows that $\overline{K}^{\tau_Y} = K$.

Since K is closed in τ_Y and $K \subset Y$, which is compact with respect to τ_Y, we have that K is compact with respect to τ_Y. But in K the toplogy τ_Y coincides with $\sigma(X', X)$, and so K is compact with respect to $\sigma(X', X)$. ■

Corollary 125 *If $(X, \|\cdot\|)$ is a normed space then the closed unit ball of X',*

$$\{L \in X' : \|L\|_{X'} \leq 1\},$$

is weak star compact.

Proof. We apply the Banach–Alaoglu theorem taking $V = B(0,1)$. Then the set

$$\begin{aligned}K &= \{L \in X' : |L(x)| \leq 1 \text{ for every } x \in B(0,1)\} \\ &= \left\{L \in X' : \sup_{x \in B(0,1)} |L(x)| \leq 1\right\} = \left\{L \in X' : \sup_{x \in X \setminus \{0\}} \frac{|L(x)|}{\|x\|} \leq 1\right\}\end{aligned}$$

is weak star compact. ■

Example 126 *Two important applications are the following.*

(i) Since $L^\infty([a,b])$ is the dual of $L^1([a,b])$, given $f_0 \in L^\infty([a,b])$ and $r > 0$, the closed unit ball

$$B(f_0, r) = \left\{f \in L^\infty([a,b]) : \|f - f_0\|_{L^\infty([a,b])} \leq r\right\}$$

is weak star compact.

(ii) Since the space of Radon measures $\mathcal{M}_b([a,b])$ is the dual of $C([a,b])$, given $\lambda_0 \in \mathcal{M}_b([a,b])$ and $r > 0$, the closed unit ball

$$B(\lambda_0, r) = \left\{\lambda \in \mathcal{M}_b([a,b]) : \|\lambda - \lambda_0\|_{\mathcal{M}_b([a,b])} \leq r\right\}$$

is weak star compact.

Monday, March 22, 2010

Makeup class

If X is separable, it actually turns out that weak star compact sets are metrizable, and thus one can work with the friendlier notion of sequential compactness.

Theorem 127 *Let (X, τ) be a separable, locally convex topological vector space and let $K \subset X'$ be weak star compact. Then $(K, \sigma(X', X))$ is metrizable.*

Proof. Since X is separable, there exists $\{x_n\} \subset X$ such that $\{x_n\}$ is dense in X. For every n define the linear functional

$$\begin{aligned} T_n : X' &\to \mathbb{R} \\ L &\mapsto L(x_n) \end{aligned}$$

We claim that T_n is $\sigma(X', X)$-continuous. To see this, let $\varepsilon > 0$. Then

$$\begin{aligned} T_n^{-1}((-\varepsilon, \varepsilon)) &= \{L \in X' : T_n(L) \in (-\varepsilon, \varepsilon)\} = \{L \in X' : |L(x_n)| < \varepsilon\} \\ &= B_{p_{x_n}}(0, \varepsilon) \in \sigma(X', X), \end{aligned}$$

which proves the claim.

Since K is $\sigma(X', X)$-compact, we have that T_n is bounded on K. Thus, there exists $M_n > 0$ such that

$$|T_n(L)| \le M_n$$

for all $L \in K$. By replacing T_n with $\frac{1}{M_n} T_n$, without loss of generality we may assume that $|T_n(L)| \le 1$ for all $L \in K$. Hence, we may define the function $d : K \times K \to [0, \infty)$ as

$$d(L, L_1) := \sum_n \frac{1}{2^n} |T_n(L) - T_n(L_1)|.$$

We claim that d is a metric. Indeed, if $d(L, L_1) = 0$, then $|T_n(L) - T_n(L_1)| = 0$ for all n, that is, $L(x_n) = L_1(x_n)$ for all n. Since $\{x_n\}$ is dense in X, it follows (exercise) that $L = L_1$. The properties $d(L, L_1) = d(L_1, L)$ and the triangle inequality are easy to verify.

Next we claim that d is $\sigma(X', X)$-continuous. Let $L, L_1 \in K$. Fix $\varepsilon > 0$ and find $m \in \mathbb{N}$ so large that

$$\sum_{n=m+1}^{\infty} \frac{1}{2^n} \le \varepsilon.$$

Since the functionals $T_1, \ldots, T_m$ are $\sigma(X', X)$-continuous at L and L_1, there exist $V_1, \ldots, V_m$ neighborhoods of L and $W_1, \ldots, W_m$ neighborhoods of L_1 such that if $M \in V_n$, then $|T_n(M) - T_n(L)| \le \varepsilon$, while if $P \in W_n$, then $|T_n(P) - T_n(L_1)| \le \varepsilon$. Set

$$V := \bigcap_{n=1}^{m} V_n, \quad W := \bigcap_{n=1}^{m} W_n.$$

If $M \in V \cap K$ and $P \in W \cap K$, then

$$\begin{aligned}|d(M,P) - d(L,L_1)| &= \left| \sum_n \frac{1}{2^n} |T_n(M) - T_n(P)| - \sum_n \frac{1}{2^n} |T_n(L) - T_n(L_1)| \right| \\ &\le 4 \sum_{n=m+1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{m} \frac{1}{2^n} |T_n(M) - T_n(L)| + \sum_{n=1}^{m} \frac{1}{2^n} |T_n(P) - T_n(L_1)| \\ &\le 4\varepsilon + \sum_{n=1}^{m} \frac{1}{2^n} 2\varepsilon = 6\varepsilon,\end{aligned}$$

where we have used the fact that $|T_n| \le 1$ in K and the inequality $||a| - |b|| \le |a - b|$. This shows that d is $\sigma(X', X)$-continuous. In particular, the set

$$B_d(L_0, r) = \{L \in K : d(L, L_0) < r\}$$

belongs to $\sigma(X', X)$. This proves that the topology τ_d determined by d is contained in the relative topology τ_w induced by $\sigma(X', X)$ on K. We claim that τ_d coincides with τ_w in K. Let $C \subset K$ be τ_w closed. Since K is τ_w compact, the set C is τ_w compact.

Since $\tau_d \subset \tau_w$, every open cover of K with respect to τ_d is an open cover of K with respect to τ_w. Thus C is compact with respect to τ_d. It follows that C is closed with respect to τ_d. Thus, τ_d and τ_w have the same closed sets, and, in turn, the same open sets. Thus, they coincide. ■

The reason the following theorem is important, is that in a metric space compactness and sequentiall compactness are equivalent. Hence, we are able to work with sequences. We recall that if (X, τ) is a locally convex topological vector space, then a sequence $\{L_n\} \subset X'$ is weakly star convergent to L in X' if and only if

$$\lim_{n\to\infty} L_n(x) = L(x)$$

for every $x \in X$. In particular, if $(X, \|\cdot\|)$ is a normed space, and if $\{L_n\} \subset X'$ converges to L strongly, then it weakly star converges to L.

As a corollary of Theorem 127 we have the following result.

Corollary 128 (Bolzano-Weierstrass) *Let V be a neighborhood of 0 in a separable locally convex topological vector space (X, τ) and let $\{L_n\} \subset X'$ be such that*

$$|L_n(x)| \le 1 \text{ for every } x \in V \text{ and for all } n \in \mathbb{N}.$$

Then there exists a subsequence $\{L_{n_k}\}$ that is weakly star convergent. In particular, if X is a separable normed space and $\{L_n\} \subset X'$ is any bounded sequence in X', then there exists a subsequence that is weakly star convergent.

Proof. By the Banach–Alaoglu theorem, the set

$$K := \{L \in X' : |L(x)| \le 1 \text{ for every } x \in V\}$$

is weak star compact. By Theorem 127 it is weak star sequentially compact. Hence, every sequence in K admits a weakly star convergent subsequence. The second part of the statement follows from Corollary 125. ■

Example 129 *Two important applications are the following.*

(i) Since $L^1([a,b])$ is separable, given a sequence $\{f_n\} \subset L^\infty([a,b])$ such that

$$\sup_n \|f_n\|_{L^\infty([a,b])} < \infty,$$

there exists a subsequence $\{f_{n_k}\}$ that is weakly star convergent to some function $f \in L^\infty([a,b])$, that is

$$\lim_{k\to\infty} \int_a^b g(x) f_{n_k}(x)\, dx = \int_a^b g(x) f(x)\, dx$$

for every $g \in L^1([a,b])$.

(ii) Since $C([a,b])$ is separable, given a sequence $\{\lambda_n\} \subset \mathcal{M}_b([a,b])$ such that

$$\sup_n \|\lambda_n\|_{\mathcal{M}_b([a,b])} < \infty,$$

there exists a subsequence $\{\lambda_{n_k}\}$ that is weakly star convergent to some $\lambda \in \mathcal{M}_b([a,b])$, that is

$$\lim_{k\to\infty} \int_{[a,b]} g\, d\lambda_{n_k} = \int_{[a,b]} g\, d\lambda$$

for every $g \in C([a,b])$. An important application is the following. Consider a sequence $\{f_n\} \subset L^1([a,b])$ such that

$$\sup_n \|f_n\|_{L^1([a,b])} < \infty.$$

For every $n \in \mathbb{N}$ define the Radon measure

$$\lambda_n(E) := \int_E f_n(x)\, dx, \quad E \in \mathcal{B}([a,b]).$$

Then (exercise) $\|f_n\|_{L^1([a,b])} = \|\lambda_n\|_{\mathcal{M}_b([a,b])}$ and so there exists a subsequence $\{\lambda_{n_k}\}$ that is weakly star convergent to some $\lambda \in \mathcal{M}_b([a,b])$, that is

$$\lim_{k\to\infty} \int_a^b g(x) f_{n_k}(x)\, dx = \int_{[a,b]} g\, d\lambda$$

for every $g \in C([a,b])$.

Wednesday, March 24, 2010

For Banach spaces the converse of Theorem 127 holds:

Theorem 130 *Let $(X, \|\cdot\|)$ be a normed space. Then $(B_{X'}(0;1), \sigma(X'.X))$ is metrizable if and only if X is separable.*

Proof. In view of Theorem 127, it remains to show that if $(B_{X'}(0;1), \sigma(X'.X))$ is metrizable with metric d, then X is separable. Since $B_d\left(0, \frac{1}{n}\right)$ is open in $\sigma(X'.X)$, there exists $V_n \subset B_d\left(0, \frac{1}{n}\right)$ of the form

$$V_n = \{L \in B_{X'}(0;1) : |L(x_i)| < \varepsilon_n, \, i = 1, \ldots, m_n\}$$

for some $\{x_1, \ldots, x_{m_n}\} \subset X$ and some $\varepsilon_n > 0$. Consider the countable set

$$E := \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{m_n} \{x_i\}.$$

Let $Y = \overline{\operatorname{span} E}$. Note that the set F of all finite linear combinations of elements of E with rational coefficients is still countable and is dense in Y. Thus, Y is separable. We claim that $Y = X$. Indeed, if not then let $x_0 \in X \setminus Y$. By Corollary 85 there exist a continuous linear functional $L : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$L(x) \leq \alpha \quad \text{for all } x \in Y \quad \text{and } L(x_0) > \alpha.$$

Since $\operatorname{span} E$ is a subspace, we have that $L(x) = 0$ for all $x \in \operatorname{span} E$ (why?). In particular, $L(x) = 0$ for all $x \in E$. We claim that $L = 0$. Indeed, if not, then $\frac{L}{2\|L\|_{X'}} \in B_{X'}(0;1)$ and so $\frac{L}{2\|L\|_{X'}} \in V_n$ for all $n \in \mathbb{N}$, but since

$$\{0\} \subset \bigcap_{n=1}^{\infty} V_n \subset \bigcap_{n=1}^{\infty} B_d\left(0, \frac{1}{n}\right) = \{0\},$$

it follows that $L = 0$. ■

Proposition 131 *Let $(X, \|\cdot\|)$ be a normed space. If X' is separable, then so is X.*

Proof. Since X' is separable, there exists $\{L_n\} \subset X'$ such that $\{L_n\}$ is dense in X'. Since

$$\|L_n\|_{X'} = \sup_{x \in \overline{B(0,1)}} L_n(x)$$

there exists $x_n \in \overline{B(0,1)}$ such that

$$L_n(x_n) \geq \frac{1}{2} \|L_n\|_{X'}.$$

Let $E = \{x_n\}$ and let $Y = \overline{\operatorname{span} E}$. Note that the set F of all finite linear combinations of elements of E with rational coefficients is still countable and is dense in Y. Thus, Y is separable. We claim that $Y = X$. Indeed, if not then let $x_0 \in X \setminus Y$. By Corollary 85 there exist a continuous linear functional $L : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$L(x) \leq \alpha \quad \text{for all } x \in Y \quad \text{and } L(x_0) > \alpha.$$

Since $\operatorname{span} E$ is a subspace, we have that $L(x) = 0$ for all $x \in \operatorname{span} E$. We claim that $L = 0$. Indeed, by the density of $\{L_n\}$, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|L_n - L\|_{X'} < \varepsilon$. It follows that

$$\frac{1}{2}\|L_n\|_{X'} \leq L_n(x_n) = L_n(x_n) - L(x_n) \leq \|L_n - L\|_{X'}\|x_n\| \leq \|L_n - L\|_{X'} 1 \leq \varepsilon,$$

and

$$\|L\|_{X'} = \|L \pm L_n\|_{X'} \leq \|L_n - L\|_{X'} + \|L_n\|_{X'} \leq 3\varepsilon.$$

Letting $\varepsilon \to 0$, it follows that $L = 0$, which is a contradiction. ■

The converse is false in general (take, for example, the separable space $L^1([a, b])$ and its dual $L^\infty([a, b])$, which is not separable).

Proposition 132 *Let X be a Banach space. If a sequence $\{L_n\} \subset X'$ weakly star converges to $L \in X'$, then it is bounded and*

$$\|L\|_{X'} \leq \liminf_{n \to \infty} \|L_n\|_{X'} . \tag{16}$$

Moreover, if $\{x_n\} \subset X$ converges to $x \in X$, then $L_n(x_n) \to L(x)$.

Proof. By Exercise **??**,

$$\lim_{n \to \infty} L_n(x) = L(x)$$

for every $x \in X$. Hence, by the Banach–Steinhaus theorem,

$$M := \sup_n \|L_n\|_{X'} < \infty.$$

Moreover, for every $x \in X$, $x \neq 0$,

$$|L_n(x)| \leq \|L_n\|_{X'} \|x\| .$$

Taking the limit inferior on both sides gives

$$|L(x)| \leq \liminf_{n \to \infty} \|L_n\|_{X'} \|x\| .$$

Dividing by $x \neq 0$ and taking the supremum gives (16).

To prove the last statement, note that

$$\begin{aligned} |L_n(x_n) - L(x)| &= |L_n(x_n) \pm L_n(x) - L(x)| \\ &\leq |L_n(x_n - x)| + |L_n(x) - L(x)| \\ &\leq \|L_n\|_{X'} \|x_n - x\| + |L_n(x) - L(x)| \\ &\leq M \|x_n - x\| + |L_n(x) - L(x)| \to 0. \end{aligned}$$

■

Friday, March 26, 2010

5.3 Weak Topology

Given a locally convex topological vector space X, for each $L \in X'$ the function $p_L : X \to [0, \infty)$ defined by

$$p_L(x) := |L(x)|, \quad x \in X, \tag{17}$$

is a seminorm. In view of Theorem 60, the family of seminorms $\{p_L\}_{L \in X'}$ generates a locally convex topology $\sigma(X, X')$ on the space X, called the *weak topology*, such that each p_L is continuous with respect to $\sigma(X, X')$. In turn, this implies that every $L \in X'$ is $\sigma(X, X')$ continuous.

Exercise 133 *Let (X, τ) be a Hausdorff, locally convex topological vector space. Prove that $(X, \sigma(X, X'))$ is a Hausdorff space.*

Exercise 134 *Let $(X, \|\cdot\|)$ be a normed space and let $C \subset X$ be closed and convex. Prove that E is weakly closed. Hint: use Hahn-Banach Theorem.*

Next we study the relation between weak compactness and weak sequential compactness.

Given a topological vector space (X, τ), we recall that a sequence $\{x_n\} \subset X$ converges weakly to $x \in X$ if and only if

$$\lim_{n \to \infty} L(x_n) = L(x)$$

for every $L \in X'$.

An infinite-dimensional Banach space when endowed with the weak topology is never metrizable. However, we have the following result.

Proposition 135 *Let $(X, \|\cdot\|)$ be a Banach space. Assume that there exists $\{L_n\} \subset X' \setminus \{0\}$ with the property that*

$$\bigcap_{n=1}^{\infty} \ker L_n = \{0\}. \tag{18}$$

Then every weak compact set $K \subset X$ is metrizable.

Proof. By replacing L_n with $\frac{L_n}{\|L_n\|_{X'}}$, we can assume that $\|L_n\|_{X'} = 1$. Define

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} |L_n(x - y)|, \quad x, y \in X.$$

We leave as an exercise to check that d is a distance. Note that (18) guarantees that $d(x, y) = 0$ if and only if $x = y$.

For every $L \in X'$, since L is weakly continuous and K is weakly compact, we have that $L(K)$ is compact. By Exercise 105, there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in K$.

Let $\tau_K := \{U \cap K : U \in \sigma(X, X')\}$ be the induced topology from $\sigma(X, X')$ and let d_K be the restriction of d to $K \times K$. Fix $x_0 \in K$ and consider

$$B_{d,K}(x_0, r) = \left\{ x \in K : \sum_{n=1}^{\infty} \frac{1}{2^n} |L_n(x - x_0)| < r \right\}.$$

We claim that there exists a neighborhhod $V \in \tau_K$ of x_0 such that $V \subset B_{d,K}(x_0, r)$. Let $m \in \mathbb{N}$ be so large that $\sum_{n=m+1}^{\infty} \frac{1}{2^n} < \frac{r}{4M}$. Since $\|L_n\|_{X'} = 1$,

$$\sum_{n=m+1}^{\infty} \frac{1}{2^n} |L_n(x - x_0)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|L_n\|_{X'} \|x - x_0\| \leq 2M \sum_{n=1}^{\infty} \frac{1}{2^n} < \frac{r}{2}.$$

Consider the (relatively) weak neighborhood of x_0,

$$V := \left\{ x \in K : |L_n(x - x_0)| < \frac{r}{2m}, \, n = 1, \ldots, m \right\}.$$

If $x \in V$, then

$$d_K(x, x_0) = \sum_{n=1}^{m} \frac{1}{2^n} |L_n(x - x_0)| + \sum_{n=m+1}^{\infty} \frac{1}{2^n} |L_n(x - x_0)| < m \frac{r}{2m} + \frac{r}{2} = r.$$

Hence, $V \subset B_{d,K}(x_0, r)$. This proves that τ_K is a base for d_K. Hence, $\tau_{d_K} \subset \tau_K$, where τ_{d_K} is the topology induced on K by d_K. Note that this also shows that the identity

$$\mathrm{Id} : (K, \sigma(X, X')) \to (K, d_k)$$

is continuous.

Let $V \in \tau_K$ and consider $C := K \setminus V$. Then C is weakly compact. Since Id is continuous, $\mathrm{Id}(C) = C$ is compact with respect to d_K, and so C is closed with respect to d_K. Thus, V is open with respect to d_K. This shows that $\tau_{d_K} = \tau_K$. ■

Corollary 136 *Let $(X, \|\cdot\|)$ be a Banach space such that X or X' is separable. Then every weak compact set $K \subset X$ is metrizable.*

Proof. In view of Proposition 131, it is enought to consider the case in which X is separable. Then $\partial B(0,1)$ is separable, and so we may find $\{x_n\} \subset \partial B(0,1)$ such that $\{x_n\}$ is dense in $\partial B(0,1)$. By Corollary 101 for every n there exists $L_n \in X'$ such that

$$L_n(x_n) = \|x_n\| = 1 \quad \text{and} \quad \|L_n\|_{X'} = 1.$$

We claim that (18) holds. To see this, it is enough to show that for every $y \in X \setminus \{0\}$,

$$\|y\| = \sup_n |L_n(y)|.$$

Indeed, since $x := \frac{y}{\|y\|} \in \partial B(0,1)$ by density, for every $\varepsilon > 0$ we can find x_{n_ε} such that $\|x - x_{n_\varepsilon}\| < \varepsilon$. Hence,

$$\begin{aligned}1 = \|x\| = \|x \pm x_{n_\varepsilon}\| &\leq \|x_{n_\varepsilon}\| + \|x - x_{n_\varepsilon}\| \leq L_{n_\varepsilon}(x_{n_\varepsilon}) + \varepsilon \\ &= L_{n_\varepsilon}(x_{n_\varepsilon} \pm x) + \varepsilon \leq \|L_{n_\varepsilon}\|_{X'} \|x - x_{n_\varepsilon}\| + L_{n_\varepsilon}(x) + \varepsilon \\ &\leq L_{n_\varepsilon}(x) + 2\varepsilon \leq \sup_n |L_n(y)| + 2\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude that $1 \leq \sup_n |L_n(x)|$, and by linearity, it follows that

$$\|y\| \leq \sup_n |L_n(y)| \leq \sup_n \|L_n\|_{X'} \|y\| \leq 1 \|y\|.$$

■

Corollary 137 *Let $(X, \|\cdot\|)$ be a Banach space. If $K \subset X$ is weakly compact, then it is weakly sequentially compact.*

Proof. Let $\{x_n\} \subset K$ and let $Y = \overline{\operatorname{span}\{x_n\}}$. Then Y is closed, convex, and separable. By Exercise 134, Y is weakly closed.

We claim that $\sigma(Y, Y')$ is the induced topology on Y from $\sigma(X, X')$. Indeed, a local base neighborhood at 0 in the induced topology from $\sigma(X, X')$ is given by

$$\begin{aligned} V &:= \{x \in X : |L_n(x)| < r_n,\, n = 1, \ldots, m\} \cap Y \\ &= \{x \in Y : |L_n(x)| < r_n,\, n = 1, \ldots, m\} \end{aligned}$$

for some $L_1, \ldots, L_m \in X'$. But the restriction of $L_i|_Y \in Y'$, and thus $V \in \sigma(Y, Y')$. On the other hand, a local base neighborhood at 0 in the topology $\sigma(Y, Y')$ is given by

$$W := \{x \in Y : |T_n(x)| < r_n,\, n = 1, \ldots, m\}$$

for some $T_1, \ldots, T_m \in Y'$. In view of Corollary 100, each T_n can be extended to some $\widehat{T}_n \in X'$ and thus we may write

$$W = \left\{x \in Y : \left|\widehat{T}_n(x)\right| < r_n,\, n = 1, \ldots, m\right\} \cap Y,$$

which shows that W belongs to the induced topology from $\sigma(X, X')$. Thus the claim holds.

In view of the claim, we have that $K \cap Y$ is compact with respect to $\sigma(Y, Y')$. Thus, by the previous corollary, $K \cap Y$ with the induced topology τ_K obtained from $\sigma(Y, Y')$ is metrizable. On the other hand, in a metric space, compactness and sequential compactness coincide. Hence, since $\{x_n\} \subset K \cap Y$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to some $x \in K$ in with respect to τ_K. We claim that $\{x_{n_k}\}$ converges to x with respect to $\sigma(X, X')$. Indeed, $L \in X'$, then $L|_Y \in Y'$, and so $\lim_{k \to \infty} L(x_{n_k}) = L(x)$. Thus, $\{x_{n_k}\}$ converges to x with respect to $\sigma(X, X')$. ■

Using Banach–Alaoglu's theorem one can prove the following theorem:

Theorem 138 (Eberlein–Šmulian) *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be weakly sequentially compact. Then its weak closure is weakly compact.*

The proof relies on a few auxiliary results, which are of interest in themselves. The dual space $\mathcal{L}(X'; \mathbb{R})$ of X' is called *bidual space* of X and it is denoted by X''. As an immediate application of the Hahn–Banach theorem we have the following:

Lemma 139 *Let $(X, \|\cdot\|)$ be a normed space and consider the linear operator mapping*

$$J : (X, \|\cdot\|) \to (X'', \|\cdot\|_{X''})$$

defined by

$$J(x)(L) := L(x), \quad L \in X'.$$

Then

$$\|J(x)\|_{X''} = \|x\| \quad \textit{for all } x \in X. \tag{19}$$

In particular, J is injective and continuous. Moreover,

$$J : (X, \sigma(X, X')) \to (J(X), \sigma(X'', X'))$$

is a homeomorphism.

Proof. For $x \in X$,

$$\|J(x)\|_{X''} = \sup_{L \in X' \setminus \{0\}} \frac{|J(x)(L)|}{\|L\|_{X'}} = \sup_{L \in X' \setminus \{0\}} \frac{|L(x)|}{\|L\|_{X'}} = \|x\|$$

by Corollary 102. Since J is injective, $J : X \to J(X)$ is a bijection.

To prove the last part of the statement, observe that we are putting the weak star topology on X''. Thus, if $T_0 \in X''$, a local base neighborhood of T_0 in the $\sigma(X'', X')$ topology is a set $V \subset X''$ of the form

$$V = \{T \in X'' : |T(L_i) - T_0(L_i)| < \varepsilon_i, \, i = 1, \dots, m\}$$

for some $L_i \in X'$ and $\varepsilon_i > 0$, $i = 1, \dots, m$.

Let $x_0 \in X$ and let V be a neighborhood of $J(x_0)$ with respect to $\sigma(X'', X')$. Without loss of generality, we may assume that

$$V = \{T \in X'' : |T(L_i) - J(x_0)(L_i)| < \varepsilon_i, \, i = 1, \dots, m\}$$

for some $L_i \in X'$ and $\varepsilon_i > 0$, $i = 1, \dots, m$. Define

$$W := \{x \in X : |L_i(x) - L_i(x_0)| < \varepsilon_i, \, i = 1, \dots, m\}.$$

Since $J(x)(L_i) = L_i(x)$ for every $x \in X$ and $i = 1, \dots, m$, we have that $J^{-1}(V) \subset W$. Thus J is continuous.

Let $\Psi := J^{-1}$. To prove continuity at a point $T_0 \in J(X)$, we need to show that for every neighborhood U of $\Psi(T_0)$, there exists a neighborhood W of T_0 such that $\Psi^{-1}(U) \subset W$.

For every $T \in J(X)$ there is a unique $x \in X$ such that $J(x) = T$. Let $x_0 \in X$ be such that $J(x_0) = T_0$. Then $\Psi(T_0) = x_0$. Consider a neighborhood U of x_0 in the topology $\sigma(X, X')$. Without loss of generality, we may assume that

$$U = \{x \in X : |L_i(x) - L_i(x_0)| < \varepsilon_i, \, i = 1, \dots, m\}$$

for some $L_i \in X'$ and $\varepsilon_i > 0$, $i = 1, \dots, m$. Then

$$\begin{aligned}
\Psi^{-1}(U) &= \{T \in J(X) : \Psi(T) \in U\} \\
&= \left\{T \in J(X) : \left|L_i\left(J^{-1}(T)\right) - L_i\left(J^{-1}(T_0)\right)\right| < \varepsilon_i, \, i = 1, \dots, m\right\} \\
&= \left\{T \in J(X) : \left|J\left(J^{-1}(T)\right)(L_i) - J\left(J^{-1}(T_0)\right)(L_i)\right| < \varepsilon_i, \, i = 1, \dots, m\right\} \\
&= \{T \in J(X) : |T(L_i) - T_0(L_i)| < \varepsilon_i, \, i = 1, \dots, m\} = V \cap J(X),
\end{aligned}$$

where we have used the fact that $J(x)(L) = L(x)$ and where V is as above. Note that the set $W := V \cap J(X)$ is a neighborhood of T_0 in the induced $\sigma(X'', X')$ topology. ■

Wednesday, March 31, 2010

It remains to understand which sets are weakly compact. In particular, we would like to understand if closed balls are weakly compact.

Definition 140 *A normed space* $(X, \|\cdot\|)$ *is* reflexive *if* $J(X) = X''$.

In this case it is possible to identify X with its bidual X''.

Example 141 *The most important examples of reflexive spaces is given by* $L^p([a,b])$ *for* $1 < p < \infty$. *The space* $L^1([a,b])$ *is not reflexive, since the dual of* $L^1([a,b])$ *is* $L^\infty([a,b])$, *while the dual of* $L^\infty([a,b])$ *is a space of measures larger than* $L^1([a,b])$.

Theorem 142 (Kakutani) *A Banach space* $(X, \|\cdot\|)$ *is reflexive if and only if the closed unit ball* $\{x \in X : \|x\| \leq 1\}$ *is weakly compact.*

Proposition 143 (Goldstine) *Let* $(X, \|\cdot\|)$ *be a Banach space. Then* $J\left(\overline{B_X(0,1)}\right)$ *is dense in* $\overline{B_{X''}(0,1)}$ *with respect to the topology* $\sigma(X'', X')$.

Proof. Let $T_0 \in X''$ be such that $\|T_0\|_{X''} \leq 1$ and let V be a neighborhood of T_0 in the $\sigma(X'', X')$ topology. Without loss of generality, we may assume that

$$V = \{T \in X'' : |T(L_i) - T_0(L_i)| < \varepsilon_i,\ i = 1, \ldots, m\}$$

for some $L_i \in X'$ and $\varepsilon_i > 0$, $i = 1, \ldots, m$. We need to prove that $J\left(\overline{B_X(0,1)}\right) \cap V$ is nonempty, that is, that there exists $x_0 \in \overline{B_X(0,1)}$ such that

$$|L_i(x_0) - T_0(L_i)| = |J(x_0)(L_i) - T_0(L_i)| < \varepsilon_i,\ i = 1, \ldots, m. \tag{20}$$

Set $a = (a_1, \ldots, a_m)$, where $a_i := T_0(L_i)$, $i = 1, \ldots, m$ and consider the function $g : X \to \mathbb{R}^m$ defined by

$$g(x) = (L_1(x), \ldots, L_m(x)), \quad x \in X.$$

Then (20) is equivalent to proving that $a \in \overline{g\left(\overline{B_X(0,1)}\right)}$. If $a \notin \overline{g\left(\overline{B_X(0,1)}\right)}$, then since $\overline{g\left(\overline{B_X(0,1)}\right)}$ is a closed convex set of $\mathbb{R}^m$ by Corollary 85 there exist $b \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that

$$b \cdot g(x) < \alpha < b \cdot a \quad \text{for all } x \in \overline{B_X(0,1)}.$$

Hence,

$$\sum_{i=1}^m b_i L_i(x) < \alpha < \sum_{i=1}^m b_i a_i \quad \text{for all } x \in \overline{B_X(0,1)}.$$

Taking $x = 0$, we get $\alpha > 0$. Hence, also by Remark 96,

$$\left\| \sum_{i=1}^m b_i L_i \right\|_{X'} = \sup_{x \in \overline{B_X(0,1)}} \sum_{i=1}^m b_i L_i(x) \leq \alpha < \sum_{i=1}^m b_i a_i.$$

On the other hand,

$$\sum_{i=1}^{m} b_i a_i = \sum_{i=1}^{m} b_i T_0 (L_i) = T_0 \left(\sum_{i=1}^{m} b_i L_i \right)$$
$$\leq \|T_0\|_{X''} \left\| \sum_{i=1}^{m} b_i L_i \right\|_{X'} \leq 1 \left\| \sum_{i=1}^{m} b_i L_i \right\|_{X'},$$

which gives a contradiction. ■

We now turn to the proof of Kakutani's theorem.

Proof of Theorem 142. Assume that X is reflexive. Then by (19), $J\left(\overline{B_X(0,1)}\right) = \overline{B_{X''}(0,1)}$. By the Banach–Alaoglu theorem we have that $\overline{B_{X''}(0,1)}$ is compact with respect to $\sigma(X'', X')$. Since

$$J^{-1} : (X'', \sigma(X'', X')) \to (X, \sigma(X, X'))$$

is continuous by Lemma 139, J^{-1} sends compact sets into compact sets. It follows that $J^{-1}\left(\overline{B_{X''}(0,1)}\right) = \overline{B_X(0,1)}$ is compact with respect to $\sigma(X, X')$.

Conversely, assume that $\overline{B_X(0,1)}$ is compact with respect to $\sigma(X, X')$. Again by by Lemma 139, we have that

$$J : (X, \sigma(X, X')) \to (X'', \sigma(X'', X'))$$

is continuous, and so $J\left(\overline{B_X(0,1)}\right)$ is compact with respect to $\sigma(X'', X')$. Since $(X'', \sigma(X'', X'))$ is Hausdorff, $J\left(\overline{B_X(0,1)}\right)$ is closed with respect to $\sigma(X'', X')$. Hence, by Goldstine's proposition, $J\left(\overline{B_X(0,1)}\right) = \overline{B_{X''}(0,1)}$. By linearity, it follows that

$$J(X) = X''.$$

■

In view of the Kakutani theorem, we have the following corollary:

Corollary 144 *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $\{x_n\} \subset X$ be a bounded sequence. Then there exists a subsequence that is weakly convergent.*

Proof. By Kakutani's theorem, closed balls are weakly compact, and so by Corollary 137 they are weakly sequentially compact. ■

Example 145 *Let $1 < p < \infty$. Since $L^p([a,b])$ is reflexive, given a sequence $\{f_n\} \subset L^p([a,b])$ such that*

$$\sup_n \|f_n\|_{L^p([a,b])} < \infty,$$

there exists a subsequence $\{f_{n_k}\}$ that is weakly star convergent to some function $f \in L^p([a,b])$, that is

$$\lim_{k\to\infty} \int_a^b g(x) f_{n_k}(x)\, dx = \int_a^b g(x) f(x)\, dx$$

for every $g \in L^{p'}([a,b])$. *For* $p = 1$, *this does not work, since* $L^1([a,b])$ *is not reflexive.*

Exercise 146 *Prove that a Banach space* $(X, \|\cdot\|)$ *is reflexive if and only if* $(X', \|\cdot\|_{X'})$ *is reflexive.*

Exercise 147 *Prove that a Banach space* $(X, \|\cdot\|)$ *is reflexive and separable if and only if* $(X', \|\cdot\|_{X'})$ *is reflexive and separable.*

Remark 148 *The previous exercise is often used to prove that a Banach space is not reflexive. Indeed, if* X *is separable (typical examples:* $X = C([a,b])$ *or* $X = L^1([a,b])$*), and one can prove that* X' *is not separable, then* X *cannot be reflexive.*

The following proposition is often used. The proof is similar to the one of Proposition 132 and we omit it.

Proposition 149 *Let* $(X, \|\cdot\|)$ *be a Banach space. If a sequence* $\{x_n\} \subset X$ *converges weakly to* $x \in X$, *then it is bounded and*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\| .$$

Moreover, if $\{L_n\} \subset X'$ *converges in* X' *to some* L, *then*

$$\lim_{n\to\infty} L_n(x_n) = L(x) .$$

Friday, April 02, 2010

6 Uniformly Convex Spaces

An important family of reflexive Banach spaces that includes $L^p(\mathbb{R}^N)$, $1 < p < \infty$, is given by uniformly convex Banach spaces.

Definition 150 *A normed space* $(X, \|\cdot\|)$ *is* uniformly convex *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* $x, y \in X$, *with* $\|x\| \leq 1$, $\|y\| \leq 1$, *and* $\|x - y\| > \varepsilon$,

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Theorem 151 (Milman-Pettis) *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space. Then* X *is reflexive.*

Proof. Note that by the linearity of J and (19), to prove that X is reflexive it is enough to show that $\partial B_{X''}(0,1) = J(\partial B_X(0,1))$. Fix $T_0 \in \partial B_{X''}(0,1)$. We claim that for every $\varepsilon > 0$ there exists $x \in \overline{B_X(0,1)}$ such that

$$\|T_0 - J(x)\|_{X''} \leq \varepsilon. \tag{21}$$

Note that this implies that $T_0 \in \overline{J\left(\overline{B_X(0,1)}\right)}$, but since $J\left(\overline{B_X(0,1)}\right)$ is closed by by Lemma 139, it follows that $T_0 \in J\left(\overline{B_X(0,1)}\right)$. In turn by (19), we have that $T_0 \in J(\partial B_X(0,1))$. Thus it remains to prove the claim.

Fix $\varepsilon > 0$. Since

$$1 = \|T_0\|_{X''} = \sup_{L \in \partial B_{X'}(0,1)} |T_0(L)|,$$

we may find $L_0 \in \partial B_{X'}(0,1)$ such that $|T_0(L_0)| > 1 - \frac{1}{2}\delta$, where $\delta = \delta(\varepsilon) > 0$ is the number given in the definition of uniform convexity. Define

$$V := \left\{ T \in X'' : |T(L_0) - T_0(L_0)| < \frac{1}{2}\delta \right\} \in \sigma(X'', X').$$

Then V is a neighborhood of T_0 with respect to $\sigma(X'', X')$. By the Goldstine theorem, $J\left(\overline{B_X(0,1)}\right)$ is dense in $\overline{B_{X''}(0,1)}$ with respect to the topology $\sigma(X'', X')$. Hence, V intersects $J\left(\overline{B_X(0,1)}\right)$. Let $x \in \overline{B_X(0,1)}$ be such that $J(x), J(x_1) \in V$. We claim that

$$J\left(\overline{B_X(0,1)}\right) \cap V \subset \overline{B_{X''}(J(x), \varepsilon)}. \tag{22}$$

Indeed, if $x_1 \in \overline{B_X(0,1)}$ is such that $J(x_1) \in V$, then

$$\begin{aligned} |J(x)(L_0) + J(x_1)(L_0)| &\geq |2T_0(L_0)| - |J(x)(L_0) - T_0(L_0) + J(x_1)(L_0) - T_0(L_0)| \\ &> 2\left(1 - \frac{1}{2}\delta\right) - \frac{2}{2}\delta = 2 - 2\delta, \end{aligned}$$

and so, recalling that $L_0 \in \partial B_{X'}(0,1)$, $\left\|\frac{J(x)+J(x_1)}{2}\right\|_{X''} > 1-\delta$. By (19) and the linearity of J, it follows that $\left\|\frac{x+x_1}{2}\right\| > 1-\delta$, which, by uniform convexity, implies that $\|x-x_1\| \le \varepsilon$. In turn, again by (19), $\|J(x)-J(x_1)\|_{X''} \le \varepsilon$.

Next we claim that T_0 belongs to the closure of $J\left(\overline{B_X(0,1)}\right) \cap V$ with respect to $\sigma(X'',X')$. Indeed, if $W \in \sigma(X'',X')$ is a neighborhood of T_0, then $W \cap V$ is still a neighborhood of T_0 with respect to $\sigma(X'',X')$, and thus

$$\left(J\left(\overline{B_X(0,1)}\right) \cap V\right) \cap W = J\left(\overline{B_X(0,1)}\right) \cap (V \cap W) \neq \emptyset,$$

where we have used the fact that

$$T_0 \in \partial B_{X''}(0,1) \subset \overline{B_{X''}(0,1)} = \overline{J\left(\overline{B_X(0,1)}\right)}^{\sigma(X'',X')}.$$

Since $\overline{B_{X''}(T,\varepsilon)}$ is closed with respect to $\sigma(X'',X')$, it follows from (22) and what we just proved that

$$T_0 \in \overline{J\left(\overline{B_X(0,1)}\right)}^{\sigma(X'',X')} \subset \overline{B_{X''}(T,\varepsilon)},$$

that is $T_0 \in \overline{B_{X''}(T,\varepsilon)}$. This proves (21). ■

Remark 152 *There exist separable, reflexive, Banach spaces X for which no equivalent norm makes X uniformly convex.*

Proposition 153 *Let $(X, \|\cdot\|)$ be a uniformly convex space. If $\{x_n\} \subset X$ converges weakly to $x \in X$ and*

$$\limsup_{n\to\infty} \|x_n\| \le \|x\|,$$

then $\{x_n\}$ converges strongly to x.

Proof. If $x = 0$, then there is nothing to prove, thus assume that $x \neq 0$. Since

$$0 < \|x\| \le \liminf_{n\to\infty} \|x_n\|$$

by Proposition 149, it follows that $x_n \neq 0$ for all n sufficiently large. Hence, we may define $y_n := \frac{x_n}{\|x_n\|}$ and $y := \frac{x}{\|x\|}$. We claim that $y_n \rightharpoonup y$. To see this, let $L \in X'$. Then by the linearity of L, and the facts that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have

$$L(y_n) = \frac{1}{\|x_n\|} L(x_n) \to \frac{1}{\|x\|} L(x) = L(y),$$

which proves the claim.

In turn, $\frac{y_n+y}{2} \rightharpoonup y$, and so, again by Proposition 149,

$$1 = \|y\| \le \liminf_{n\to\infty} \left\|\frac{y_n+y}{2}\right\| \le \limsup_{n\to\infty} \left\|\frac{y_n+y}{2}\right\| \le 1.$$

Hence, $\lim_{n\to\infty} \left\| \frac{y_n+y}{2} \right\| = 1$. We claim that

$$\lim_{n\to\infty} \|y_n - y\| = 0.$$

Indeed, assume by contradiction that there exists $\varepsilon > 0$ such that $\|y_n - y\| \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Then there exists $\delta > 0$ such that

$$\left\| \frac{y_n + y}{2} \right\| < 1 - \delta$$

for infinitely many $n \in \mathbb{N}$, which is a contradiction. Hence $y_n \to y$. In turn,

$$\begin{aligned}
\|x_n - x\| &= \left\| \frac{x_n}{\|x_n\|} \|x_n\| \pm \|x_n\| \frac{x}{\|x\|} - \frac{x}{\|x_n\|} \|x_n\| \right\| \\
&\leq \|x_n\| \|y_n - y\| + \|x_n\| \left| \frac{1}{\|x\|} - \frac{1}{\|x_n\|} \right| \|x\| \\
&\to \|x\| \, 0 + \|x\| \, 0 = 0.
\end{aligned}$$

This concludes the proof. ■

Monday, April 05, 2010

7 Some Important Dual Spaces

Definition 154 *Let X be a nonempty set. A collection $\mathfrak{M}\subset\mathcal{P}(X)$ is an* algebra *if*

(i) $\emptyset\in\mathfrak{M}$*;*

(ii) if $E\in\mathfrak{M}$ *then* $X\setminus E\in\mathfrak{M}$*;*

(iii) if $E_1,\ E_2\in\mathfrak{M}$ *then* $E_1\cup E_2\in\mathfrak{M}$*.*

$\mathfrak{M}$ *is said to be a* σ-algebra *if it satisfies (i)–(ii) and*

(iii)′ if $\{E_n\}\subset\mathfrak{M}$ *then* $\bigcup_{n=1}^{\infty}E_n\in\mathfrak{M}$*.*

To highlight the dependence of the σ-algebra $\mathfrak{M}$ on X we will sometimes use the notation $\mathfrak{M}(X)$. If $\mathfrak{M}$ is a σ-algebra then the pair $(X,\mathfrak{M})$ is called a *measurable space*. For simplicity we will often apply the term measurable space only to X.

Definition 155 *Let X be a nonempty set and let $\mathfrak{M}\subset\mathcal{P}(X)$ be an algebra. A map $\mu:\mathfrak{M}\to[0,\infty]$ is called a* (positive) finitely additive measure *if*

$$\mu(\emptyset)=0,\quad \mu(E_1\cup E_2)=\mu(E_1)+\mu(E_2)$$

for all $E_1,\ E_2\in\mathfrak{M}$ *with* $E_1\cap E_2=\emptyset$*.*

Definition 156 *Let X be a nonempty set, let $\mathfrak{M}\subset\mathcal{P}(X)$ be a σ-algebra. A map $\mu:\mathfrak{M}\to[0,\infty]$ is called a* (positive) measure *if*

$$\mu(\emptyset)=0,\quad \mu\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\mu(E_n)$$

for every countable collection $\{E_n\}\subset\mathfrak{M}$ of pairwise disjoint sets. The triple $(X,\mathfrak{M},\mu)$ is said to be a measure space.

In the proof of the next theorem we will use Young's inequality

$$\frac{1}{p}a^p+\frac{1}{p'}b^{p'}\geq ab,$$

which holds for every $1<p<\infty$ and for all $a,b\geq 0$. This inequality is strict unless $a^p=b^{p'}$, in which case we have equality.

Theorem 157 *Let $(X,\mathfrak{M},\mu)$ be a measure space. Then $L^p(X)$ is uniformly convex for every $1<p<\infty$. In particular, $L^p(X)$ is reflexive for $1<p<\infty$.*

Proof. Step 1: We claim that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every probability space $(X, \mathfrak{M}, \nu)$ (i.e., $\nu(X) = 1$) and for every measurable function $f : X \to \mathbb{R}$ such that

$$\int_X |f|^p \, d\nu \leq 1 \quad \text{and} \quad \int_X f \, d\nu > 1 - \delta, \tag{23}$$

we have that

$$\int_X |f - 1|^p \, d\nu < \varepsilon^p.$$

To see this, consider the function

$$\varphi(t) := |t|^p - 1 + p(1 - t), \quad t \in \mathbb{R}.$$

Note that by Young's inequality (with $a = |t|$ and $b = 1$), we have that $\varphi(t) > 0$ for all $t \neq 1$. Moreover,

$$\lim_{|t| \to \infty} \frac{\varphi(t)}{|t - 1|^p} = 1.$$

Hence, there exists $C_\varepsilon > 1$ such that

$$|t - 1|^p \leq C_\varepsilon \varphi(t)$$

for all $t \in \mathbb{R}$ with $|t - 1| \geq \varepsilon$. Take $\delta := \frac{\varepsilon^p}{pC_\varepsilon}$ and let f satisfy (23). Define $E := \{x \in X : |f(x) - 1| < \varepsilon\}$. Then

$$\begin{aligned}
\int_X |f - 1|^p \, d\nu &= \int_E |f - 1|^p \, d\nu + \int_{X \setminus E} |f - 1|^p \, d\nu \\
&\leq \varepsilon^p \nu(E) + C_\varepsilon \int_X \varphi(f(x)) \, d\nu \\
&= \varepsilon \nu(E) + C_\varepsilon \left(\int_X |f|^p \, d\nu - 1 + p \int_X (1 - f(x)) \, d\nu \right) \\
&\leq \varepsilon^p + C_\varepsilon p \delta \leq 2\varepsilon^p.
\end{aligned}$$

Step 2: We claim that for every $\varepsilon > 0$ and $1 < p < \infty$ there exists $\delta = \delta(p, \varepsilon) > 0$ such that for every measure space $(X, \mathfrak{M}, \mu)$ and for every $f \in L^p(X)$, $\psi \in L^{p'}(X)$ such that

$$\int_X |f|^p \, d\mu \leq 1, \quad \int_X |\psi|^{p'} \, d\mu = 1, \quad \text{and} \quad \int_X f\psi \, d\nu > 1 - \delta, \tag{24}$$

we have that

$$\int_X |f - h|^p \, d\mu < \varepsilon^p,$$

where $h := \psi |\psi|^{p'-2}$. Given $\varepsilon > 0$ and $1 < p < \infty$, let δ be as in Step 1. Let f, ψ satisfy (24). Assume first that $\psi \geq 0$ and define the function

$$z(x) := \begin{cases} \frac{f(x)}{h(x)} & \text{if } h(x) > 0, \\ 0 & \text{if } h(x) = 0. \end{cases}$$

Then $zh = f$ in the set in which h is strictly positive, while $zh = 0$ in the set in which h is zero. Since in $(f - zh) zh = 0$, we have that

$$\begin{aligned}\int_X |f|^p \, d\mu &= \int_X |f - zh + zh|^p \, d\mu = \int_{f - zh \neq 0} |f - zh|^p \, d\mu + \int_{zh \neq 0} |zh|^p \, d\mu \\ &= \int_X |f - zh|^p \, d\mu + \int_X |zh|^p \, d\mu.\end{aligned}$$

Moreover, by Hölder's inequality

$$\|zh\|_{L^p} = \|zh\|_{L^p} \|\psi\|_{L^{p'}} \geq \int_X zh\psi \, d\mu = \int_X f\psi \, d\mu > 1 - \delta.$$

Hence,

$$\int_X |f - zh|^p \, d\mu \leq \int_X |f|^p \, d\mu - \int_X |zh|^p \, d\mu \leq 1 - (1 - \delta)^p . \tag{25}$$

Consider the probability measure,

$$\nu(E) := \int_E h\psi \, d\mu = \int_E h\psi \, d\mu = \int_E |\psi|^{p'} \, d\mu, \quad E \in \mathfrak{M}.$$

Then

$$\begin{aligned}\int_X |z|^p \, d\nu = \int_X |z|^p \, h\psi \, d\mu = \int_{\{h>0\}} |f|^p \, d\mu \leq 1, \\ \int_X z \, d\nu = \int_X zh\psi \, d\mu = \int_X f\psi \, d\mu > 1 - \delta,\end{aligned}$$

where we have used the fact that $\psi = h^{p-1}$. Hence, by Step 1 we obtain

$$\varepsilon^p > \int_X |z - 1|^p \, d\nu = \int_X |z - 1|^p \, h^p \, d\mu = \int_{\{h>0\}} |f - h|^p \, d\mu.$$

On the other hand, by (25),

$$\int_{\{h=0\}} |f - h|^p \, d\mu = \int_{\{h=0\}} |f - zh|^p \, d \leq 1 - (1 - \delta)^p ,$$

and so

$$\int_X |f - h|^p \, d\mu \leq \varepsilon^p + 1 - (1 - \delta)^p \leq 2\varepsilon^p,$$

provided δ is taken so small that $1 - (1 - \delta)^p \leq \varepsilon^p$. This proves the claim in the case $\psi \geq 0$. ■

Proof. In the general case, given $f \in L^p(X)$, $\psi \in L^{p'}(X)$ satisfying (24), consider the function

$$\widehat{\psi}(x) := \begin{cases} \operatorname{sgn}\psi(x) & \text{if } \psi(x) \neq 0, \\ 1 & \text{if } \psi(x) = 0. \end{cases}$$

Note that $\left|\widehat{\psi}\right| = 1$. The functions $\overline{f} := \widehat{\psi} f$ and $\overline{\psi} := \widehat{\psi}\psi \geq 0$ satisfy (24) and so

$$\int_X |f - h|^p \, d\mu = \int_X |f - h|^p \left|\widehat{\psi}\right|^p \, d\mu = \int_X \left|\overline{f} - \overline{\psi}\right|^p \, d\mu \leq \varepsilon^p.$$

Step 3: Given $\varepsilon > 0$ and $1 < p < \infty$ let δ be as in the previous step. Consider $f, g \in L^p(X)$ with $\|f\|_{L^p} \leq 1$, $\|g\|_{L^p} \leq 1$, and $\|f + g\|_{L^p} > 2 - \delta$. Let $h := (f + g) \diagup \|f + g\|_{L^p}$ and define $\psi := h|h|^{p-2}$. Then $h\psi = |h|^p = |\psi|^{p'}$. In particular, $\|\psi\|_{L^{p'}} = \|h\|_{L^p} = 1$. Moreover,

$$\int_X (f + g)\psi \, d\mu = \|f + g\|_{L^p} \int_X h\psi \, d\mu = \|f + g\|_{L^p} > 2 - \delta.$$

Since by Hölder's inequality $\int_X g\psi \, d\mu \leq \|g\|_{L^p} \|\psi\|_{L^{p'}} \leq 1$, it follows that

$$\int_X f\psi \, d\mu = \int_X (f + g)\psi \, d\mu - \int_X g\psi \, d\mu > 2 - \delta - 1 = 1 - \delta.$$

Hence, by Step 2, $\|f - h\|_{L^p} < \varepsilon$. Similarly, $\|g - h\|_{L^p} < \varepsilon$, and so $\|f - g\|_{L^p} < 2\varepsilon$. ■

As a corollary we obtain the following result.

Theorem 158 (Riesz representation theorem in L^p) *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $1 < p < \infty$. Then every bounded linear functional $L : L^p(X) \to \mathbb{R}$ is represented by a unique $g \in L^{p'}(X)$ in the sense that*

$$L(f) = \int_X fg \, d\mu \quad \text{for every } f \in L^p(X). \tag{26}$$

Moreover, the norm of L coincides with $\|g\|_{L^{p'}}$. Conversely, every functional of the form (26), where $g \in L^{p'}(X)$, is a bounded linear functional on $L^p(X)$.

Proof. Step 1: We first prove that for every $f \in L^p(X)$,

$$\|f\|_{L^p} = \max_{g \in L^{p'}(X)\setminus\{0\}} \frac{\int_X fg \, d\mu}{\|g\|_{L^{p'}}}.$$

It suffices to assume that $f \neq 0$. By Hölder's inequality, we get that

$$\sup_{g \in L^{p'}(X)\setminus\{0\}} \frac{\int_X fg \, d\mu}{\|g\|_{L^{p'}}} \leq \|f\|_{L^p}.$$

To prove that the supremum is reached, define $g := f\,|f|^{p-2}$.
Step 2: We prove that $L^{p'}(X)$ can be embedded into $(L^p(X))'$. Indeed, consider the linear mapping

$$\Psi : L^{p'}(X) \to (L^p(X))'$$

defined by

$$\Psi(g)(f) := \int_X fg\,d\mu, \quad f \in L^p(X).$$

Then exactly as in Step 1 (with the roles of p and p' interchanged)

$$\|\Psi(g)\|_{(L^p(X))'} = \sup_{f\in L^p(X)\setminus\{0\}} \frac{\int_X fg\,d\mu}{\|f\|_{L^p}} = \|g\|_{L^{p'}}.$$

Assume by contradiction, that there exists $L_0 \in (L^p(X))' \setminus \Psi\left(L^{p'}(X)\right)$. By Corollary 85 there exists $T_0 \in (L^p(X))''$ and $\alpha \in \mathbb{R}$ such that

$$T_0(L) \le \alpha \text{ for all } L \in \Psi\left(L^{p'}(X)\right) \text{ and } T_0(L_0) > \alpha.$$

Since $\Psi\left(L^{p'}(X)\right)$ is a subspace, it follows that $T_0(L) = 0$ for all $L \in \Psi\left(L^{p'}(X)\right)$. On the other hand, since $L^p(X)$ is reflexive, there exists $f_0 \in L^p(X)$ such that $T_0 = J(f_0)$. Hence,

$$0 = T_0(L) = J(f_0)(L) = L(f_0)$$

for all $L \in \Psi\left(L^{p'}(X)\right)$, that is $\int_X f_0 g\,d\mu = 0$ for all $g \in L^{p'}(X)$. By Step 1, it follows that $f_0 = 0$. In turn $0 = J(f_0) = T_0$, which is a contradiction. Thus, $(L^p(X))' = \Psi\left(L^{p'}(X)\right)$. ∎

Theorem 159 (Riesz representation theorem in L^1) *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ σ-finite. Then every bounded linear functional $L : L^1(X) \to \mathbb{R}$ is represented by a unique $g \in L^\infty(X)$ in the sense that*

$$L(f) = \int_X fg\,d\mu \quad \textit{for every } f \in L^1(X). \tag{27}$$

Moreover, the norm of L coincides with $\|g\|_{L^\infty}$. Conversely, every functional of the form (27), where $g \in L^\infty(X)$, is a bounded linear functional on $L^1(X)$.

Friday, April 09, 2010

Proof. Step 1: Assume that μ is finite. Then $L^p(X) \subset L^1(X)$ for every $1 \le p \le \infty$. Fix $1 < p < \infty$. Given $L \in \left(L^1(X)\right)'$, consider the restriction of L to $L^p(X)$, $L : L^p(X) \to \mathbb{R}$. Then for every $f \in L^1(X)$,

$$|L(f)| \le \|L\|_{(L^1(X))'} \|f\|_{L^1} .$$

In particular, if $f \in L^p(X)$, by Hölder's inequality

$$|L(f)| \le \|L\|_{(L^1(X))'} \|f\|_{L^p} (\mu(X))^{\frac{1}{p'}} ,$$

and so

$$\|L\|_{(L^p(X))'} = \sup_{f \in L^p(X) \setminus \{0\}} \frac{|L(f)|}{\|f\|_{L^p}} \le \|L\|_{(L^1(X))'} (\mu(X))^{\frac{1}{p'}} .$$

It follows by the Riesz representation theorem in L^p that there exists a unique $g_p \in L^{p'}(X)$ such that

$$L(f) = \int_X f g_p \, d\mu \quad \text{for every } f \in L^p(X)$$

and

$$\|g_p\|_{L^{p'}} = \|L\|_{(L^p(X))'} \le \|L\|_{(L^1(X))'} (\mu(X))^{\frac{1}{p'}} .$$

In particular, since for $1 < p < q < \infty$, $L^q(X) \subset L^p(X)$, it follows that

$$L(f) = \int_X f g_p \, d\mu = \int_X f g_q \, d\mu \quad \text{for every } f \in L^q(X),$$

that is

$$\int_X f(g_p - g_q) \, d\mu = 0 \quad \text{for every } f \in L^q(X),$$

which implies that $g_p(x) = g_q(x)$ for μ-a.e. $x \in X$.

Thus, we have shown that there exists a measurable function g that belongs to $L^{p'}(X)$ for every $1 < p' < \infty$ and such that

$$\|g\|_{L^{p'}} \le \|L\|_{(L^1(X))'} (\mu(X))^{\frac{1}{p'}} .$$

It follows (exercise) that $g \in L^\infty(X)$, with

$$\|g\|_{L^\infty} = \lim_{p' \to \infty} \|g\|_{L^{p'}} \le \lim_{p' \to \infty} \|L\|_{(L^1(X))'} (\mu(X))^{\frac{1}{p'}} = \|L\|_{(L^1(X))'} .$$

Moreover, for every $1 < p < \infty$,

$$L(f) = \int_X f g \, d\mu \quad \text{for every } f \in L^p(X).$$

Since $L^p(X)$ is dense in $L^1(X)$ and both sides of the previous identity are continuous, we obtain that

$$L(f) = \int_X fg\,d\mu \quad \text{for every } f \in L^1(X).$$

Moreover,

$$\begin{aligned} \|L\|_{(L^1(X))'} &= \sup_{f\in L^1(X)\setminus\{0\}} \frac{|L(f)|}{\|f\|_{L^1}} = \sup_{f\in L^1(X)\setminus\{0\}} \frac{\left|\int_X fg\,d\mu\right|}{\|f\|_{L^1}} \\ &\le \|g\|_{L^\infty} \le \|L\|_{(L^1(X))'}. \end{aligned} \tag{28}$$

Step 2: In the general case, we can write

$$X = \bigcup_{n=1}^{\infty} X_n,$$

where the sets $X_n \in \mathfrak{M}$ are pairwise disjoint and $\mu(X_n) < \infty$. We apply Step 1 to each functional

$$L_n(f) := L(f\chi_{X_n}), \quad f \in L^1(X_n)$$

to find a unique function $g^{(n)} \in L^\infty(X_n)$ such that

$$L_n(f) = \int_{X_n} fg^{(n)}\,d\mu \quad \text{for every } f \in L^1(X_n)$$

and

$$\left\|g^{(n)}\right\|_{L^\infty(X_n)} = \|L_n\|_{(L^1(X_n))'} \le \|L\|_{(L^1(X))'}.$$

The function

$$g(x) := g^{(n)}(x) \quad \text{if } x \in X_n$$

belongs to $L^\infty(X)$, $\|g\|_{L^\infty(X)} \le \|L\|_{(L^1(X))'}$, and for every $f \in L^1(X)$, by the continuity of L,

$$\begin{aligned} L(f) &= L\left(\sum_{n=1}^{\infty} f\chi_{X_n}\right) = \sum_{n=1}^{\infty} L(f\chi_{X_n}) \\ &= \sum_{n=1}^{\infty} \int_{X_n} fg^{(n)}\,d\mu = \int_X fg\,d\mu. \end{aligned}$$

As in (28), we have that $\|g\|_{L^\infty(X)} = \|L\|_{(L^1(X))'}$. ■

Definition 160 *Let X be a nonempty set and let $\mathfrak{M} \subset \mathcal{P}(X)$ be an algebra. The space* $\operatorname{ba}(X,\mathfrak{M})$ *of* bounded finitely additive signed measures *is composed of all set functions $\lambda : \mathfrak{M} \to \mathbb{R}$ such that*

(i) $\lambda(\emptyset) = 0$*;*

(ii) λ *is* finitely additive*; that is,*

$$\lambda(E_1 \cup E_2) = \lambda(E_1) + \lambda(E_2)$$

for all $E_1, E_2 \in \mathfrak{M}$ *with* $E_1 \cap E_2 = \emptyset$*;*

(iii) λ *is* bounded*; that is, its* total variation norm

$$\|\lambda\|(X) := \sup\left\{\sum_{n=1}^{l} |\lambda(E_n)| : \{E_n\} \subset \mathfrak{M} \text{ finite partition of } X\right\}$$

is finite.

Exercise 161 *Prove that* $\operatorname{ba}(X, \mathfrak{M})$ *is a Banach space endowed with the total variation norm* $\|\cdot\|(X)$*.*

Given a measure space, $(X, \mathfrak{M}, \mu)$, the dual of $L^\infty(X)$ may be identified with $\operatorname{ba}(X, \mathfrak{M}, \mu)$, which is the space of all *bounded finitely additive signed measures absolutely continuous with respect to* μ, that is, all $\lambda \in \operatorname{ba}(X, \mathfrak{M})$ such that $\lambda(E) = 0$ whenever $E \in \mathfrak{M}$ and $\mu(E) = 0$.

Given a function $u \in L^\infty(X)$, there exists a sequence of of simple functions $\{s_n\} \subset L^\infty(X)$ that converges uniformly to u. We define

$$\int_E u \, d\lambda := \lim_{n \to \infty} \int_E s_n \, d\lambda \tag{29}$$

It may be verified (exercise) that the limit exists and that the integral does not depend on the particular approximating sequence $\{s_n\}$.

Theorem 162 (Riesz representation theorem in L^∞) *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space. Then every continuous linear functional* $L : L^\infty(X) \to \mathbb{R}$ *is represented by a unique* $\lambda \in \operatorname{ba}(X, \mathfrak{M}, \mu)$ *in the sense that*

$$L(f) = \int_X f \, d\lambda \quad \text{for every } f \in L^\infty(X). \tag{30}$$

Moreover, the norm of L *coincides with* $\|\lambda\|$*. Conversely, every functional of the form* (30)*, where* $\lambda \in \operatorname{ba}(X, \mathfrak{M}, \mu)$*, is a bounded linear functional on* $L^\infty(X)$*.*

Proof. Step 1: Let $L \in (L^\infty(X))'$ and for every $E \in \mathfrak{M}$ define

$$\lambda(E) := L(\chi_E).$$

Since L is linear and $L(0) = 0$ it follows that $\lambda : \mathfrak{M} \to \mathbb{R}$ is a finitely additive signed measure. Moreover, if $E \in \mathfrak{M}$ is such that $\mu(E) = 0$, then since χ_E is equivalent to 0, and so $\lambda(E) = L(\chi_E) = L(0) = 0$. Thus, λ is absolutely continuous with respect to μ. To prove that λ is bounded, let $\{E_n\}_{n=1}^{l} \subset \mathfrak{M}$ be a finite partition of X and define

$$s := \sum_{n=1}^{l} c_n \chi_{E_n},$$

where $c_n := \operatorname{sgn}(L(\chi_{E_n}))$. Then $s \in L^\infty(X)$ and

$$\begin{aligned}\sum_{n=1}^{l} |\lambda(E_n)| &= \sum_{n=1}^{l} \operatorname{sgn}(L(\chi_{E_n})) L(\chi_{E_n}) \\ &= \left| \sum_{n=1}^{l} c_n L(\chi_{E_n}) \right| = |L(s)| \\ &\le \|L\|_{(L^\infty(X))'} \|s\|_\infty \le \|L\|_{(L^\infty(X))'},\end{aligned}$$

where we have used the fact that $\|s\|_\infty \le 1$. Hence

$$\|\lambda\|(X) \le \|L\|_{(L^\infty(X))'}, \tag{31}$$

which shows that $\lambda \in \operatorname{ba}(X, \mathfrak{M}, \mu)$. ■

Proof. We now show that (**??**) holds. Fix $f \in L^{\infty}(X)$ and $\varepsilon > 0$, and partition the interval $[-\|f\|_{\infty}, \|f\|_{\infty}]$ into l intervals I_n, $n = 1, \ldots, l$, of length less than or equal to ε. For each $n = 1, \ldots, l$, define

$$E_n := f^{-1}(I_n) \in \mathfrak{M}$$

and let

$$s := \sum_{n=1}^{l} c_n \chi_{E_n},$$

where c_n is any fixed number in I_n. Note that for $x \in E_n$ we have that $f(x) \in I_n$, and so

$$|f(x) - s(x)| \leq \varepsilon.$$

Hence $\|f - s\|_{\infty} \leq \varepsilon$, while by the linearity of L,

$$\begin{aligned} \left| L(f) - \int_X s \, d\lambda \right| &= \left| L(f) - \sum_{n=1}^{l} c_n \lambda(E_n) \right| \\ &= \left| L(f) - \sum_{n=1}^{l} c_n L(\chi_{E_n}) \right| \\ &= |L(f - s)| \leq \|L\|_{(L^{\infty}(X))'} \|f - s\|_{\infty} \\ &\leq \|L\|_{(L^{\infty}(X))'} \varepsilon. \end{aligned} \tag{32}$$

By taking $\varepsilon := \frac{1}{k}$ we can construct a sequence $\{s_k\}$ of simple functions converging uniformly to f such that

$$\lim_{k \to \infty} \int_X s_k \, d\lambda = L(f).$$

By (29) this implies that

$$L(f) = \int_X f \, d\lambda.$$

Moreover, from (32) and the fact that $\|f - s\|_{\infty} \leq \varepsilon$ we have that

$$\begin{aligned} |L(f)| &\leq \left| \int_X s \, d\lambda \right| + \|L\|_{(L^{\infty}(X))'} \varepsilon \\ &\leq \sum_{n=1}^{l} |c_n| \, |\lambda(E_n)| + \|L\|_{(L^{\infty}(X))'} \varepsilon \\ &\leq (\|f\|_{\infty} + \varepsilon) \sum_{n=1}^{l} |\lambda(E_n)| + \|L\|_{(L^{\infty}(X))'} \varepsilon \\ &\leq (\|f\|_{\infty} + \varepsilon) \|\lambda\|(X) + \|L\|_{(L^{\infty}(X))'} \varepsilon, \end{aligned}$$

and so, by the arbitrariness of ε, we get

$$|L(f)| \leq \|f\|_{\infty} \|\lambda\|(X),$$

which, together with (31), implies that $\|\lambda\|(X) = \|L\|_{(L^{\infty}(X))'}$.

Step 2: Conversely, given $\lambda \in \mathrm{ba}(X, \mathfrak{M}, \mu)$, for $f \in L^{\infty}(X)$ define

$$L(f) := \int_X f\, d\lambda.$$

Then $L \in (L^{\infty}(X))'$ and $\|\lambda\|(X) = \|L\|_{(L^{\infty}(X))'}$. We leave the details as an exercise. ■

Note that in general,

$$(L^{\infty}(X))' \supsetneqq L^1(X).$$

In particular, if μ is σ-finite, it follows that $L^1(X)$ is not reflexive, since,

$$\left(L^1(X)\right)'' = (L^{\infty}(X))' \supsetneqq L^1(X).$$

Hence, if

$$\sup_n \|f_n\|_{L^1(X)} < \infty$$

we cannot conclude any compactness of the sequence $\{f_n\}$.

Exercise 163 *Prove that $L^{\infty}([0,1])$ is not separable and conclude that $L^1([0,1])$ is not reflexive.*

8 Linear Operators

8.1 Compact Operators and Spectral Theory

Theorem 1.18 Rudin FA

Definition 164 *Let (X, τ_X) and (Y, τ_Y) be topological spaces. An operator $T : X \to Y$ is* bounded *if it sends topologically bounded sets of X into topologically bounded sets of Y.*

Theorem 165 *Let (X, τ_X) and (Y, τ_Y) be topological vector spaces and let $T : X \to Y$ be linear and continuous. Then T is bounded.*

Proof. Let $E \subset X$ be a topologically bounded set and let $W \subset Y$ be a neighborhood of zero. Since T is continuous and $T(0) = 0$, there exists a neighborhood $U \subset X$ of zero such that $T(U) \subset W$. By the boundedness of E there exists $t > 0$ such that $E \subset tU$. Hence

$$T(E) \subset T(tU) = tT(U) \subset tW.$$

This shows that $T(E)$ is bounded in Y. ■

Sequences of bounded linear operators: weak, strong and uniform convergence

Definition 166 *Let* (X, τ_X) *be a topological vector space and let* (Y, τ_Y) *be a topological space. An operator* $T : X \to Y$ *is said to be* compact *if it maps every topologically bounded subset of* X *onto a relatively compact subset of* Y*.*

Remark 167 *In view of Proposition 115, if both* X *and* Y *are topological vector spaces and if* $T : X \to Y$ *is linear and compact, then* T *is bounded, and so continuous.*

In particular, if Y is a normed space and if T is compact, then from every topologically bounded sequence $\{x_n\} \subset X$, we may extract a subsequence $\{x_{n_k}\}$ such that $\{T(x_{n_k})\}$ converges in Y.

Exercise 168 *Let* $(X, \|\cdot\|_X)$*,* $(Y, \|\cdot\|_Y)$*, and* $(Z, \|\cdot\|_Z)$ *be three normed spaces and let* $T : X \to Y$ *and* $S : Y \to Z$ *be linear operators. Prove that if one is compact and the other continuous, then their composition if compact.*

Exercise 169 *Consider a function* $k \in C([a, b] \times [a, b])$ *(called* kernel*) and define* $T : C([a, b]) \to C([a, b])$ *as follows*

$$T(f)(x) := \int_a^b k(x, y) f(y)\, dy.$$

Prove that T *is compact, linear operator. Hint: use the Ascoli-Arzela theorem.*

Definition 170 *We say that the normed space* $(X, \|\cdot\|_X)$ *is* embedded *in the normed space* $(Y, \|\cdot\|_Y)$ *and we write*

$$X \hookrightarrow Y$$

if X *is a vector subspace of* Y *and the* immersion

$$\begin{aligned} i : X &\to Y \\ x &\mapsto x \end{aligned}$$

is continuous.

Note that since the immersion is linear, in view of Proposition 94 the continuity of i is equivalent to requiring the existence of a constant $M > 0$ such that

$$\|x\|_Y \leq M \|x\|_X \quad \text{for all } x \in X.$$

We say that X is *compactly embedded* in Y if the immersion i is a compact operator.

Wednesday, April 14, 2010

Theorem 171 (Fredholm alternative) *Let $(X, \|\cdot\|)$ be a normed space and let $T : X \to X$ be a compact linear operator. Then either the homogeneous equation*

$$x - T(x) = 0 \tag{33}$$

has a nontrivial solution $x \in X \setminus \{0\}$ or for every $y \in X$ the equation

$$x - T(x) = y \tag{34}$$

has a unique solution $x \in X$ and in this case the operator $(I - T)^{-1} : X \to X$ is bounded.

We begin with an auxiliary result due to Riesz.

Lemma 172 *Let $(X, \|\cdot\|)$ be a normed space and let $Y \subset X$ be a proper closed subspace of X. Then for every $\theta \in (0, 1)$ there exists $x_\theta \in X$ such that $\|x_\theta\| = 1$ and*

$$\operatorname{dist}(x_\theta, Y) \geq \theta.$$

Proof. Let $x \in X \setminus Y$. Since Y is closed, we have

$$d := \operatorname{dist}(x, Y) = \inf \{\|x - y\| : y \in Y\} > 0.$$

Since $\frac{1}{\theta} d > d$, there exists $y_\theta \in Y$ such that $d \leq \|x - y_\theta\| \leq \frac{1}{\theta} d$. Define

$$x_\theta := \frac{x - y_\theta}{\|x - y_\theta\|}.$$

Then $\|x_\theta\| = 1$ and for every $y \in Y$,

$$\begin{aligned} \|x_\theta - y\| &= \frac{1}{\|x - y_\theta\|} \|x - y_\theta - y \|x - y_\theta\|\| \\ &\geq \frac{d}{\|x - y_\theta\|} \geq d \frac{\theta}{d} = \theta, \end{aligned}$$

where we have used the fact that $y_\theta + y \|x - y_\theta\| \in Y$, since Y is a subspace. ■

We now turn to the proof of Theorem 171.

Proof of Theorem 171. Step 1: Let $S := \operatorname{Id} - T$. We claim that there exists $C > 0$ such that

$$\operatorname{dist}(x, \ker S) \leq C \|S(x)\| \quad \text{for all } x \in X.$$

If not, then by rescaling, we can find a sequence $\{x_n\} \subset X$ with $\|S(x_n)\| = 1$ such that

$$d_n := \operatorname{dist}(x_n, \ker S) \to \infty.$$

For every $n \in \mathbb{N}$ let $y_n \in \ker S$ be such that $d_n \leq \|x_n - y_n\| \leq 2 d_n$ and define

$$z_n := \frac{x_n - y_n}{\|x_n - y_n\|}.$$

Then $\|z_n\| = 1$, while

$$\|S(z_n)\| = \frac{\|S(x_n - y_n)\|}{\|x_n - y_n\|} = \frac{\|S(x_n)\|}{\|x_n - y_n\|} = \frac{1}{\|x_n - y_n\|} \le \frac{1}{d_n} \to 0$$

as $n \to \infty$. On the other hand, since T is compact, up to a subsequence, not relabeled, there exists $y_0 \in X$ such that $T(z_n) \to y_0$. Since $\mathrm{Id} = S + T$, we have that

$$z_n = S(z_n) + T(z_n) \to 0 + y_0.$$

Hence, by the continuity of T, $y_0 = T(y_0)$, which means that $y_0 \in \ker S$. On the other hand,

$$\begin{aligned}
\operatorname{dist}(z_n, \ker S) &= \inf\{\|z_n - y\| : y \in \ker S\} \\
&= \inf\left\{\frac{1}{\|x_n - y_n\|}\|x_n - y_n - y\|x_n - y_n\|\| : y \in \ker S\right\} \\
&= \frac{1}{\|x_n - y_n\|}\inf\{\|x_n - y_n - y\|x_n - y_n\|\| : y \in \ker S\} \\
&= \frac{1}{\|x_n - y_n\|}\operatorname{dist}(x_n, \ker S) \ge \frac{d_n}{2d_n} = \frac{1}{2},
\end{aligned}$$

which is a contradiction.

Step 2: We claim that the range $S(X)$ is a closed subspace of X. Let $\{x_n\} \subset X$ be such that $S(x_n) \to y_0$. Let $d_n := \operatorname{dist}(x_n, \ker S)$. By the previous step, the sequence $\{d_n\}$. For every $n \in \mathbb{N}$ let $y_n \in \ker S$ be such that $d_n \le \|x_n - y_n\| \le 2d_n$ and define $w_n := x_n - y_n$. Then $\{w_n\}$ is bounded, while $S(w_n) = S(x_n) \to y_0$. Since T is compact, up to a subsequence, not relabeled, there exists $w_0 \in X$ such that $T(w_n) \to w_0$. Since $\mathrm{Id} = S + T$, we have that

$$w_n = S(w_n) + T(w_n) \to y_0 + w_0.$$

Hence, by the continuity of $S(y_0 + w_0) = y_0$, which shows that $S(X)$ is closed.

Step 3: We claim that if $\ker S = \{0\}$, that is, if (33) does not hold, then $S(X) = X$, that is, (34) holds. Assume by contradiction that $S(X) \neq X$. By the previous step, $X_1 := S(X)$ is closed. Moreover, $T(X_1) \subset X_1$. Indeed, if $y \in X_1$, then there exists $x \in X$ such that $y = x - T(x)$. Then

$$T(y) = T(x - T(x)) = T(x) - T(T(x)) = z - T(z) \in S(X) = X_1,$$

where $z := T(x)$. Hence, $T : X_1 \to X_1$ is well-defined and it is still compact (again because X_1 is closed). By applying Step 2 to $T : X_1 \to X_1$ we obtain that $X_2 := S(X_1)$ is a closed subspace of X_2. Continuing in this way, with $X_n := S^n(X)$, we obtain a decreasing sequence of closed subspaces of X. ■

Friday, April 16, 2010

Carnival, no classes.

Proof. Assume that the sequence is strictly decreasing. Then by the previous lemma with $\theta = \frac{1}{2}$ for every $n \in \mathbb{N}$ we may find $x_n \in X_n$ such that $\|x_n\| = 1$ and

$$\operatorname{dist}(x_n, X_{n+1}) \geq \frac{1}{2}.$$

Then for $n > m$,

$$\begin{aligned} T(x_n) - T(x_m) &= -(x_n - T(x_n)) + (x_m - T(x_m)) + (x_n - x_m) \\ &= (-S(x_n) + S(x_m) + x_n) - x_m. \end{aligned}$$

Since $X_{n+1} \subset X_n \subset \cdots \subset X_m$, we have that $-S(x_n) + S(x_m) + x_n \in X_{m+1}$, and so

$$\|T(x_n) - T(x_m)\| \geq \frac{1}{2}.$$

This contradicts the fact that T is compact.

It remains to consider the case in which the sequence $\{X_n\}$ is not strictly decreasing. Hence, there exists $k \in \mathbb{N}$ such that $X_k = X_{k+1}$. Let $y \in X$, then $S^k(y) \in X_k = X_{k+1}$. This implies that there exists $x \in X$ such that $S^k(y) = S^{k+1}(x)$, that is, $S^k(y - S(x)) = 0$. Since $\ker S = \{0\}$, we have that $\ker S^k = \{0\}$, and so $y = S(x)$, which is again a contradiction and proves the claim.

Step 4: We claim that if $S(X) = X$, that is, if (34) holds, then $\ker S = \{0\}$, that is, (33) does not hold. This time we consider the increasing sequence $\{Y_n\}$ of closed subspaces $Y_n := \ker S^n$. Note that the fact that Y_n is closed follows from the continuity of S. Reasoning as in Step 3, if $\{Y_n\}$ is strictly increasing, by the previous lemma with $\theta = \frac{1}{2}$ for every $n \in \mathbb{N}$ we may find $y_n \in Y_{n+1}$ such that $\|y_n\| = 1$ and

$$\operatorname{dist}(y_n, Y_n) \geq \frac{1}{2}.$$

As before, for $n > m$,

$$T(y_n) - T(y_m) = (-S(y_n) + S(y_m) - y_m) + y_n.$$

Since $Y_n \supset \cdots \supset Y_m$, we have that $-S(y_n) + S(y_m) - y_m \in Y_n$, and so

$$\|T(y_n) - T(y_m)\| \geq \frac{1}{2}.$$

This contradicts the fact that T is compact.

It remains to consider the case in which the sequence $\{Y_n\}$ is not strictly increasing. Hence, there exists $k \in \mathbb{N}$ such that $Y_k = Y_{k+1}$. Then $Y_n = Y_k$ for all $n \geq k$. Indeed, if $y \in Y_{k+2}$, then $0 = S^{k+2}(y) = S^{k+1}(S(y))$, and so $S(y) \in Y_{k+1}$, but $Y_{k+1} = Y_k$, which implies that $S(y) \in Y_k$. Hence, $y \in Y_{k+1} = Y_k$. By induction, $Y_n = Y_k$ for all $n \geq k$.

Let $y \in Y_k$. Since $S(X) = X$, there exists $x \in X$ such that $y = S^k(x)$. Hence, $S^{2k}(x) = S^k(y) = 0$, and so $x \in X_{2k} = X_k$. But then $y = S^k(x) = 0$, which shows that $\ker S^k = \{0\}$. In turn, $\ker S = \{0\}$.

Step 5: To conclude the proof, it remains to show that if $S(X) = X$, then the operator $S^{-1} = (I - T)^{-1} : X \to X$, which well-defined in view of Step 4, is bounded. This follows from Step 1, with $\ker S = \{0\}$. ■

Definition 173 *Given a topological vector space (X, τ) and an operator $T : X \to X$, a number $\lambda \in \mathbb{R}$ is called an* eigenvalue *of T if there exists a nonzero vector x, called* eigenvector*, satisfying $T(x) = \lambda x$. The dimension of the kernel of the operator $S_\lambda := (\lambda I - T)$ is called the* multiplicity *of λ.*

Note that eigenvectors corresponding to different eigenvalues are linearly independent.

The set $\rho(T) = \{\lambda \in \mathbb{R} : \lambda I - T \text{ is invertible}\}$ is called the *resolvent set* of T. If $\lambda \in \rho(T)$, the operator $R_\lambda := (\lambda I - T)^{-1} : X \to X$ is called the *resolvent operator*.

The set $\sigma(T) := \mathbb{R} \setminus \rho(T)$ is called the *spectrum* of T. Note that $\lambda \in \sigma(T)$ if $\lambda I - T$ is not onto or not one-to-one. The set of eigenvalues of T is called the *point spectrum* of T and is denoted $\sigma_p(T)$. Note that $\sigma_p(T) \subset \sigma(T)$, but the inclusion may be strict.

Exercise 174 *Let $T : \ell^2 \to \ell^2$ be the right-shift operator defined by*

$$T(x) := (0, x_1, x_2, \ldots,)$$

for $x = (x_1, x_2, \ldots,) \in \ell^2$. Prove that T is linear, continuous, and that $\sigma_p(T) \neq \sigma(T)$.

Under the hypotheses of Theorem 171, if $\lambda \in \mathbb{R}$, $\lambda \neq 0$, is not an eigenvalue of T, then λ belongs to the resolvent set.

Remark 175 *If we work with vector spaces over $\mathbb{C}$ instead of $\mathbb{R}$, then we can allow complex numbers to be eigenvalues.*

Proposition 176 *Let $(X, \|\cdot\|)$ be a normed space and let $T : X \to X$ be a continuous linear operator. If $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are distinct eigenvalues of T and $x_1, \ldots, x_n$ are corresponding eigenvectors, then $x_1, \ldots, x_n$ are linearly independent.*

Proof. The proof is by induction on n. For $n = 1$, it is true. Assume that the result is true for n and let's prove it for $n+1$. Let $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{R}$ be distinct eigenvalues of T and $x_1, \ldots, x_{n+1}$ corresponding eigenvectors. By the induction hypothesis $x_1, \ldots, x_n$ are linearly independent. Assume that

$$x_{n+1} = \sum_{i=1}^{n} c_i x_i$$

for some $c_i \in \mathbb{R}$. Then

$$\lambda_{n+1} \sum_{i=1}^{n} c_i x_i = \lambda_{n+1} x_{n+1} = T(x_{n+1}) = \sum_{i=1}^{n} c_i T(x_i) = \sum_{i=1}^{n} c_i \lambda_i x_i,$$

that is,

$$\sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) c_i x_i = 0.$$

Since $x_1, \ldots, x_n$ are linearly independent, and $\lambda_{n+1} - \lambda_i \neq 0$, it follows that $c_i = 0$ for all $i = 1, \ldots, n$. ■

Proposition 177 *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be continuous linear operator. Then $\sigma(T)$ is compact and*

$$\sigma(T) \subset \left[- \|T\|_{\mathcal{L}(X;X)}, \|T\|_{\mathcal{L}(X;X)} \right].$$

Proof. Let $\lambda \in \mathbb{R}$, with $|\lambda| > \|T\|_{\mathcal{L}(X;X)}$. We claim that $\lambda I - T$ is invertible, or, equivalently, that given $y \in X$, there exists a unique $x \in X$ such that $\lambda x - T(x) = y$. Consider the function $f : X \to X$ defined by

$$f(x) := \frac{1}{\lambda} (T(x) - y), \quad x \in X.$$

Note that f is a contraction, since for all $x, x_1 \in X$,

$$\|f(x) - f(x_1)\| \leq \frac{1}{|\lambda|} \|T(x) - T(x_1)\| \leq \frac{\|T\|_{\mathcal{L}(X;X)}}{|\lambda|} \|x - x_1\|.$$

Hence, by the Banach contraction principle there exists a unique $x \in X$ such that $x = f(x) = \frac{1}{\lambda} (T(x) - y)$.

Next we prove that $\rho(T)$ is open. Fix $\lambda_0 \in \rho(T)$. As before, given λ sufficiently closed to λ_0 and $y \in X$, we need to find a unique $x \in X$ such that $\lambda x - T(x) = y$. We rewrite this equation as

$$(\lambda_0 I - T)(x) = y + (\lambda_0 - \lambda) x,$$

or, equivalently, $x = (\lambda_0 I - T)^{-1} (y + (\lambda_0 - \lambda) x)$. Consider the function $g : X \to X$ defined by

$$g(x) := (\lambda_0 I - T)^{-1} (y + (\lambda_0 - \lambda) x), \quad x \in X.$$

For all $x, x_1 \in X$, by Corollary 108,

$$\|g(x) - g(x_1)\| \le \left\|(\lambda_0 I - T)^{-1}\right\|_{\mathcal{L}(X;X)} |\lambda_0 - \lambda| \, \|x - x_1\|,$$

and thus, g is a contradiction for all λ so close to λ_0 that $\left\|(\lambda_0 I - T)^{-1}\right\|_{\mathcal{L}(X;X)} |\lambda_0 - \lambda| < 1$. Hence, by the Banach contraction principle there exists a unique $x \in X$ such that $x = g(x)$. ■

Theorem 178 *Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space and let $T : X \to X$ be a compact linear operator. Then*

(i) $0 \in \sigma(T)$.

(ii) $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.

(iii) $\sigma(T)$ *is either* $\{0\}$ *or a finite set or sequence converging to zero.*

Moreover, each nonzero eigenvalue has finite multiplicity.

Proof. (i) If $0 \notin \sigma(T)$, then T is invertible and by Exercise 168, $I = T \circ T^{-1}$ is compact. In particular the closed unit ball is compact, which implies that X has finite dimension by Theorem....

(ii) This follows from the Fredholm alternative.

(iii) Assume that there exists a sequence $\{\lambda_n\}$ of nonzero eigenvalues, not necessarily distinct, such that $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$ (the case $\lambda_n \equiv \lambda$ is allowed). Let $\{x_n\}$ be a sequence of corresponding linearly independent eigenvectors. For every $n \in \mathbb{N}$, let

$$Y_n := \operatorname{span}\{x_1, \dots, x_n\}.$$

Then Y_n is closed by Theorem 118. Moreover, since the set $\{x_n\}$ is linearly independent, $Y_{n+1} \neq Y_n$. Hence, by Lemma 172 with $\theta = \frac{1}{2}$ for every $n \in \mathbb{N}$, $n \ge 2$, we may find $y_n \in Y_n$ such that $\|y_n\| = 1$ and

$$\operatorname{dist}(y_n, Y_{n-1}) \ge \frac{1}{2}.$$

Then for $n > m$,

$$\frac{1}{\lambda_n}T(y_n) - \frac{1}{\lambda_m}T(y_m) = -\left(y_n - \frac{1}{\lambda_n}T(y_n)\right) + \left(y_m - \frac{1}{\lambda_m}T(y_m)\right) + (y_n - y_m)$$
$$=: z_{n,m} + y_n.$$

We claim that $z_{n,m} \in Y_{n-1}$. Indeed, since $y_n \in \operatorname{span}\{x_1, \ldots, x_n\}$, we may write $y_n = \sum_{k=1}^{n} t_k x_k$. Hence,

$$y_n - \frac{1}{\lambda_n}T(y_n) = \sum_{k=1}^{n} t_k x_k - \frac{1}{\lambda_n}T\left(\sum_{k=1}^{n} t_k x_k\right) = \sum_{k=1}^{n}\left(t_k x_k - \frac{1}{\lambda_n}t_k T(x_k)\right)$$
$$= \sum_{k=1}^{n}\left(t_k - \frac{1}{\lambda_n}t_k\lambda_k\right)x_k = \sum_{k=1}^{n-1}\left(t_k - \frac{1}{\lambda_n}t_k\lambda_k\right)x_k + 0 \in \operatorname{span}\{x_1, \ldots, x_{n-1}\} = Y_{n-1}.$$

Similarly, $y_m - \frac{1}{\lambda_m}T(y_m) \in Y_{m-1}$. Since $Y_{n-1} \supset \cdots \supset Y_{m-1}$, we have that $z_{n,m} \in X_{n-1}$, and so

$$\left\|\frac{1}{\lambda_n}T(y_n) - \frac{1}{\lambda_m}T(y_m)\right\| \geq \frac{1}{2}.$$

Since $\lambda_n \to \lambda \neq 0$, this contradicts the fact that T is compact.

Hence, if the eigenvalues λ_n are all distinct, then every subsequence of $\{\lambda_n\}$ converges to zero, and so the entire sequence converges to zero, while if $\lambda_n \equiv \lambda \neq 0$, then its multiplicity is finite. ■

Friday, April 23, 2010

8.2 Adjoints

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and an operator $T : X \to Y$. The *adjoint* of T is the operator $T^* : Y^{'} \to X'$ defined by

$$(T^*(L))(x) := L(T(x)) \tag{35}$$

for all $x \in X$ and $L \in Y'$. Using the duality pair, we may rewrite (35) as $\langle L, T(x)\rangle_{Y',Y} = \langle T^*(L), x\rangle_{X',X}$.

Theorem 179 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $T : X \to Y$ be a linear operator. Then*

$$\|T^*\|_{\mathcal{L}(Y';X')} = \|T\|_{\mathcal{L}(X;Y)}.$$

In particular, $T \in \mathcal{L}(X;Y)$ if and only if $T^* \in \mathcal{L}(Y';X')$.

Proof. For every $x \in X$ and $L \in Y'$,

$$|T^*(L)(x)| = |L(T(x))| \le \|L\|_{Y'} \|T(x)\|_Y \le \|L\|_{Y'} \|T\|_{\mathcal{L}(X;Y)} \|x\|_X$$

and so

$$\|T^*(L)\|_{X'} = \sup_{x \in \overline{B_X(0,1)}} |T^*(L)(x)| \le \|L\|_{Y'} \|T\|_{\mathcal{L}(X;Y)}$$

and, in turn,

$$\|T^*\|_{\mathcal{L}(Y';X')} = \sup_{L \in Y' \setminus \{0\}} \frac{\|T^*(L)\|_{X'}}{\|L\|_{Y'}} \le \|T\|_{\mathcal{L}(X;Y)}.$$

Conversely,

$$|L(T(x))| = |T^*(L)(x)| = \|T^*(L)\|_{X'} \|x\|_X \le \|T^*\|_{\mathcal{L}(Y';X')} \|L\|_{Y'} \|x\|_X$$

and so by Corollary 102,

$$\|T(x)\|_Y = \max_{L \in \overline{B_{Y'}(0,1)}} |L(T(x))| \le \|T^*\|_{\mathcal{L}(Y';X')} \|x\|_X$$

and, in turn,

$$\|T\|_{\mathcal{L}(X;Y)} = \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} \le \|T^*\|_{\mathcal{L}(Y';X')}.$$

■

Exercise 180 *Consider the operator $\Psi : \mathcal{L}(X;Y) \to \mathcal{L}(Y';X')$ defined by*

$$\Psi(T) := T^*.$$

Prove that Ψ is linear and continuous.

Exercise 181 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces and let* $T \in \mathcal{L}(X;Y)$*. Let* $(T^*)^* : X'' \to Y''$ *be the adjoint of* T^**. Prove that*

$$T^{**}(J_X(x)) = J_Y(T(x))$$

for all $x \in X$*.*

Exercise 182 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two Banach spaces and let* $T \in \mathcal{L}(X;Y)$*. Prove that* T *is invertible if and only if* T^* *is invertible and in this case* $(T^*)^{-1} = \left(T^{-1}\right)^*$*.*

Exercise 183 *Let* $(X, \|\cdot\|_X)$*,* $(Y, \|\cdot\|_Y)$*, and* $(Z, \|\cdot\|_Z)$ *be normed spaces and let* $T \in \mathcal{L}(X;Y)$ *and* $S \in \mathcal{L}(Y;Z)$*. Prove that* $(S \circ T)^* = T^* \circ S^*$*.*

Next we discuss the compactness of the adjoint.

Theorem 184 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces and let* $T : X \to Y$ *be a linear operator. If* T *is compact, then* T^* *is compact. The converse holds provided* Y *is a Banach space.*

Proof. Assume that T is compact. To prove that T^* is compact, it is enough to show that $T^*(B_{Y'}(0,1))$ is relatively sequentially compact in X'. Let $\{L_n\} \subset B_{Y'}(0,1)$ and consider $K := \overline{T(B_X(0,1))}$. Since T is compact, the set K is a compact set. For every $n \in \mathbb{N}$ consider the function $f_n(y) := L_n(y)$, $y \in K$. Then $f_n \in C(K)$. We claim that the sequence $\{f_n\}_n$ is equibounded and equicontinuous. Indeed, for all $n \in \mathbb{N}$ and $y, y_1 \in K$,

$$\begin{aligned} |L_n(y)| &\le \|L\|_{Y'} \|y\|_Y \le 1 \|y\|_Y \le M, \\ |L_n(y) - L_n(y_1)| &= |L_n(y - y_1)| \le \|L\|_{Y'} \|y - y_1\|_Y \le \|y - y_1\|_Y, \end{aligned}$$

where we have used the fact that K is bounded in Y. It follows from the Ascoli-Arzela's theorem that there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ that converges uniformly to some function f in $C(K)$. Hence,

$$\sup_{y \in K} |L_{n_k}(y) - f(y)| \to 0.$$

In particular, since $K = \overline{T(B_X(0,1))}$, we have that

$$\sup_{x \in B_X(0,1)} |L_{n_k}(T(x)) - f(T(x))| \to 0,$$

and so

$$\begin{aligned} \lim_{k,m\to\infty} \|T^*(L_{n_k}) - T^*(L_{n_l})\|_{X'} &= \lim_{k,m\to\infty} \sup_{x \in B_X(0,1)} |(T^*(L_{n_k}))(x) - (T^*(L_{n_l}))(x)| \\ &= \lim_{k,m\to\infty} \sup_{x \in B_X(0,1)} |L_{n_k}(T(x)) - L_{n_l}(T(x))| = 0. \end{aligned}$$

Hence, $\{T^*(L_{n_k})\}$ is a Cauchy sequence in the Banach space X' and so it converges.

Conversely, assume that T^* is compact and that Y is a Banach space. Then by the first part of the proof, the adjoint $T^{**} : X'' \to Y''$ of T^* is compact. In particular, $T^{**}(J_X(B_X(0,1)))$ is relatively compact in Y''. Hence, for any sequence $\{x_n\} \subset B_X(0,1)$ there exists a subsequence $\{x_{n_k}\}$ such that $\{T^{**}(J_X(x_{n_k}))\}$ converges to some element of Y''. By Exercise 181,

$$T^{**}(J_X(x)) = J_Y(T(x))$$

for all $x \in X$. Hence, since the norms are preserved by J_Y,

$$\begin{aligned}\|T^{**}(J_X(x_{n_k})) - T^{**}(J_X(x_{n_l}))\|_{Y''} &= \|J_Y(T(x_{n_k})) - J_Y(T(x_{n_l}))\|_{Y''} \\ &= \|J_Y(T(x_{n_k}) - T(x_{n_l}))\|_{Y''} \\ &= \|T(x_{n_k}) - T(x_{n_l})\|_Y ,\end{aligned}$$

which implies that the sequence $\{T(x_{n_k})\}$ is a Cauchy sequence in the Banach space Y and so it converges. ■

The main theorem of this section is Banach's closed range theorem.

Theorem 185 (Closed range theorem) *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two Banach spaces and let* $T \in \mathcal{L}(X; Y)$*. Then* $T(X)$ *is closed in* Y *if and only if* $T^*(Y')$ *is closed in* X'*.*

To prove this theorem, we will need several auxiliary results that are of interest in themselves.

Given a normed space $(X, \|\cdot\|)$ and a subspace $W \subset X$, we define the *annihilator* of Y as

$$W^\perp := \{L \in X' : L(x) = 0 \text{ for all } x \in W\}.$$

Similarly, given a subspace $Z \subset X'$, we define the *annihilator* of Z as

$$^\perp Z := \{x \in X : L(x) = 0 \text{ for all } L \in Z\}.$$

Note that $W^\perp$ and $^\perp Z$ are closed subspaces of X' and X, respectively.

Exercise 186 *Given a normed space* $(X, \|\cdot\|)$ *and two subspaces* $W \subset X$ *and* $Z \subset X'$*. Prove that*

(i) $W \subset^\perp (W^\perp)$,

(ii) $^\perp (W^\perp) = \overline{W}$,

(iii) $Z \subset (^\perp Z)^\perp$.

Lemma 187 *Given a normed space* $(X, \|\cdot\|)$ *and a subspace* $W \subset X$*, we have that* W *is dense in* X *if and only* $W^\perp = \{0\}$*.*

Proof. Assume that $\overline{W} = X$. If $L \in (W)^\perp$, then $L(x) = 0$ for all $x \in W$ but since L is continuous, it follows that $L(x) = 0$ for all $x \in \overline{W} = X$, which implies that $L = 0$, that is, $(W)^\perp = \{0\}$. Conversely, if $(W)^\perp = \{0\}$ and $\overline{W} \neq X$, then given $x_0 \in X \setminus \overline{W}$ by the Hahn–Banach theorem we may find $L \in X'$ such that $L(x) = 0$ for all $x \in \overline{W}$, while $L(x_0) \neq 0$. This implies that $L \in (W)^\perp$ and contradicts the fact that $(W)^\perp = \{0\}$. ■

Lemma 188 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces and let* $T \in \mathcal{L}(X; Y)$*. Then*

$$\ker T^* = (T(X))^\perp, \quad \ker T =^\perp (T^*(Y')).$$

Proof. Note that $L \in Y'$ belongs to $\ker T^*$ if and only if $T^*(L) = 0$, which by (35), is equivalent to $0 = (T^*(L))(x) = L(T(x))$ for all $x \in X$. In turn, this is equivalent to $L \in (T(X))^\perp$.

Similarly, $x \in X$ belongs to $\ker T$ if and only if $T(x) = 0$, which by (35), is equivalent to $0 = L(0) = L(T(x)) = (T^*(L))(x)$ for all $L \in Y'$. In turn, this is equivalent to $x \in^\perp (T^*(Y'))$. ■

Corollary 189 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces and let* $T \in \mathcal{L}(X;Y)$. *Then* $T(X)$ *is dense in* Y *if and only if* T^* *is one-to-one.*

Proof. By Lemma 187, $T(X)$ is dense in Y if and only $(T(X))^{\perp} = \{0\}$ and by the previous lemma this is equivalent to requiring that T^* is one-to-one. ■

Lemma 190 *Let* $(X, \|\cdot\|_X)$ *be a Banach space, let* $(Y, \|\cdot\|_Y)$ *be a normed space, and let* $T \in \mathcal{L}(X;Y)$. *Assume that there exists* $c > 0$ *such that*

$$c \|L\|_{Y'} \leq \|T^*(L)\|_{X'} \tag{36}$$

for all $L \in Y'$. *Then* T *is onto.*

Proof. **Step 1:** We claim that $\overline{T(B_X(0,1))} \supset \overline{B_Y(0,c)}$. To see this, let $y_0 \notin \overline{T(B_X(0,1))}$. By Corollary 85 there exists a continuous linear functional $L : Y \to \mathbb{R}$ such that

$$L(y) \leq 1 \quad \text{for all } y \in \overline{T(B_X(0,1))} \quad \text{and } L(y_0) > 1.$$

Since $-y \in \overline{T(B_X(0,1))}$ whenever $y \in \overline{T(B_X(0,1))}$, it follows that $|L(y)| \leq 1$ for all $y \in \overline{T(B_X(0,1))}$. If $x \in B_X(0,1)$, then by (35),

$$|(T^*(L))(x)| = |L(T(x))| \leq 1$$

and so $\|T^*(L)\|_{X'} \leq 1$. It now follows from (36) that

$$c < cL(y_0) \leq c \|L\|_{Y'} \|y_0\|_Y \leq \|T^*(L)\|_{X'} \|y_0\|_Y \leq \|y_0\|_Y .$$

This proves the claim.
Step 2: We claim that

$$T(B_X(0,1)) \supset B_Y(0,c).$$

This follows exactly as in Step 2 of the proof of the open mapping theorem. ■

Wednesday, April 28, 2010

We now turn to the proof of Theorem 185.

Proof of Theorem 185. Assume that $T(X)$ is closed in Y. To prove that $T^*(Y')$ is closed, it is enough to show that

$$T^*(Y') = (\ker T)^{\perp}.$$

Fix $L \in (\ker T)^{\perp}$ and define the linear functional $S : T(X) \to \mathbb{R}$ as follows. If $y \in T(X)$, let $x \in X$ be such that $T(x) = y$ and define $S(y) := L(x)$. Is this well-defined? If $x_1, x_2 \in X$ are such that $T(x_1) = T(x_2) = y$, then $x_1 - x_2 \in \ker T$ and, since $L \in (\ker T)^{\perp}$, it follows that $L(x_1 - x_2) = 0$, which shows that S is well-defined. Since L is linear, we have that S is linear. Next we claim that S is continuous. By Corollary 112 applied to $T : X \to T(X)$ there exists a constant $C > 0$ such that for every $y \in T(X)$ there exists an $x \in X$ with $T(x) = y$ and

$$\|x\|_X \le C \|y\|_Y .$$

Hence,

$$|S(y)| = |L(x)| \le \|L\|_{X'} \|x\|_X \le \|L\|_{X'} C \|y\|_Y$$

for all $y \in T(X)$. By the Hahn–Banach theorem we may extend S to a continuous linear functional $S_1 : Y \to \mathbb{R}$. Hence, for $x \in X$,

$$S_1(T(x)) = S(T(x)) = L(x),$$

which by (35) implies that $L = T^*(S_1)$. This shows that $L \in T^*(Y')$ and, in turn, that $(\ker T)^{\perp} \subset T^*(Y')$. By Exercise 186 with $Z = T^*(Y')$ and Lemma 188, we have

$$T^*(Y') \subset \left({}^{\perp}(T^*(Y'))\right)^{\perp} = (\ker T)^{\perp} \subset T^*(Y'),$$

which implies that $T^*(Y') = (\ker T)^{\perp}$. Since $(\ker T)^{\perp}$ is closed subspace, it follows that $T^*(Y')$ is closed.

Conversely, assume that $T^*(Y')$ is closed and let $Z = \overline{T(X)} \subset Y$. Consider the operator $\Psi : X \to Z$ defined by $\Psi(x) := T(x)$ for every $x \in X$. Since $\Psi(X) = T(X)$ is dense in Z, by Corollary 189, $\Psi^* : Z' \to X'$ is one-to-one.

We claim that $\Psi^*(Z') = T^*(Y')$. Indeed, if $L \in Z'$, then by Corollary 100 we can extend L to $L_1 \in Y'$. Then for all $x \in X$, we have that $T(x) \in Z$, and so

$$(T^*(L_1))(x) = L_1(T(x)) = L(T(x)) = (\Psi^*(L))(x). \tag{37}$$

This implies that $T^*(L_1) = \Psi^*(L)$. On the other hand, if $L_1 \in Y'$, then its restriction L to Z satisfies (37). This proves the claim. In particular, since by hypothesis $T^*(Y')$ is closed in X', it follows that $\Psi^*(Z')$ is closed in X'. By Corollary 112 and the fact that Ψ^* is one-to-one, there exists $C > 0$ such that

$$\|L\|_{Z'} \le C \|\Psi^*(L)\|_{X'}$$

for all $L \in Z'$.

By Lemma 190, this implies that $\Psi : X \to Z$ onto, that is, $T(X) = \Psi(X) = Z = \overline{T(X)}$. Hence, $T(X)$ is closed. ■

9 Differentiability

Definition 191 *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces, let* $E \subset X$*, and let* $f : U \to Y$*. Given* $x_0 \in E^\circ$ *and* $v \in X$*, we say that* f *admits a* directional derivative *at* x_0 *in the direction* v *if there exists*

$$\frac{\partial f}{\partial v}(x_0) := \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \in Y.$$

If f admits directional derivatives at x_0 in all directions v, we can consider the operator

$$\begin{aligned} T_{x_0} &: X \to Y \\ v &\mapsto T_{x_0}(v) := \frac{\partial f}{\partial v}(x_0) \end{aligned}$$

We say that the function f is *Gâteaux differentiable* at x_0 if the operator T is linear and continuous.

Remark 192 *Taking $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we have that*

$$T_{x_0}(\lambda v) = \lim_{t \to 0} \frac{f(x_0 + t\lambda v) - f(x_0)}{t} = \lambda \lim_{t \to 0} \frac{f(x_0 + t\lambda v) - f(x_0)}{\lambda t}$$
$$= \lambda \lim_{s \to 0} \frac{f(x_0 + sv) - f(x_0)}{s} = \lambda T_{x_0}(v).$$

However, T_{x_0} need not be linear or continuous.

Example 193 *Let $X = \mathbb{R}^2$ and define*

$$f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Given $v = (v_1, v_2) \in \mathbb{R}^2$, $v \neq 0$, we have that

$$\frac{\partial f}{\partial v}((0,0)) = \lim_{t \to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{t^3 v_1^3}{t^2 v_1^2 + t^2 v_2^2} - 0}{t} = \frac{v_1^3}{v_1^2 + v_2^2},$$

which is not linear in v.

If X is infinite dimensional, then T_{x_0} could be linear, but not continuous.

Example 194 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $L : X \to Y$ be linear and discontinuous. Then for every $x_0 \in X$ and $v \in X$, by linearity,*

$$\frac{\partial L}{\partial v}(x_0) = \lim_{t \to 0} \frac{L(x_0 + tv) - L(x_0)}{t} = L(v),$$

and so T_{x_0} is linear but not continuous. Thus L is nowhere Gâteaux differentiable.

Definition 195 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, let $E \subset X$, and let $f : U \to Y$. Given $x_0 \in E^\circ$, we say that the function f is* Fréchet differentiable *at x_0 if there exists $T_{x_0} \in \mathcal{L}(X; Y)$ such that*

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T_{x_0}(x - x_0)\|_Y}{\|x - x_0\|_Y} = 0.$$

The element T_{x_0} is called the Fréchet differential *of f at x_0 and is denoted by $df(x_0)$.*

Remark 196 *Fréchet differentiability implies Gâteaux differentiability, but the converse is false. Let $X = \mathbb{R}^2$ and define*

$$f(x,y) := \begin{cases} x & \text{if } y = x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\frac{\partial f}{\partial v}(0,0) = 0$$

and so $T_0 = 0$*, which is linear and continuous. However,* f *is not differentiable at* $(0,0)$*. Indeed,*

$$\frac{f\left(x,x^2\right) - f(0,0) - 0}{\sqrt{(x-0)^2 + \left(x^2-0\right)^2}} = \frac{x}{|x|\sqrt{1+x^2}} \nrightarrow 0.$$

Given a normed space $(X, \|\cdot\|)$, it turns out that the differentiability of the norm has important consequences. Note that $\|\cdot\|$ does not have any directional derivatives at 0, since if $v \neq 0$, we have that

$$\frac{\partial \|\cdot\|}{\partial v}(0) = \lim_{t\to 0} \frac{\|0+tv\| - \|0\|}{t} = \lim_{t\to 0} \|v\| \frac{|t|}{t}$$

and this limit never exists. Thus, we are only interested in the differentiability of $\|\cdot\|$ at $x_0 \neq 0$.

Moreover, if $\frac{\partial\|\cdot\|}{\partial v}(x_0)$ exists, then taking $\lambda > 0$, we have that

$$\begin{aligned}\frac{\partial \|\cdot\|}{\partial v}(\lambda x_0) &= \lim_{t\to 0} \frac{\|\lambda x_0 + tv\| - \|\lambda x_0\|}{t} = \lambda \lim_{t\to 0} \frac{\left\|x_0 + \frac{t}{\lambda} v\right\| - \|x_0\|}{t} \\ &= \lim_{s\to 0} \frac{\|x_0 + sv\| - \|x_0\|}{s} = \frac{\partial \|\cdot\|}{\partial v}(x_0).\end{aligned}$$

Thus, it is enough to study differentiability at points $x_0 \in X$ such that $\|x_0\| = 1$.

Example 197 *Given a nonempty set* Z*, consider* $X = \ell^1(Z)$*. If* Z *is countable, then* $\|\cdot\|_{\ell^1}$ *is Gâteaux differentiable at those* $f : Z \to \mathbb{R}$ *such that* $f(z) \neq 0$ *for all* $z \in Z$*, while if* Z *is uncountable, then* $\|\cdot\|_{\ell^1}$ *is nowhere Gâteaux differentiable. To see this, let* $f \in \ell^1(Z)$ *and assume that* $f(z_0) = 0$ *for some* $z_0 \in Z$*. Consider the direction*

$$g(z) := \begin{cases} 1 & \text{if } z = z_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in \ell^1(Z)$*, and*

$$\begin{aligned}\frac{\partial \|\cdot\|_{\ell^1}}{\partial g}(f) &= \lim_{t\to 0} \frac{\|f+tg\|_{\ell^1} - \|f\|_{\ell^1}}{t} = \lim_{t\to 0} \frac{\sum_{z\in Z} \left(|f(z) + tg(z)| - |f(z)|\right)}{t} \\ &= \lim_{t\to 0} \frac{|f(z_0) + tg(z_0)| - |f(z_0)|}{t} = \lim_{t\to 0} \frac{|t|}{t},\end{aligned}$$

which does not exists. On the other hand, assume that Z *is countable, say* $Z = \{z_n\}$ *and let* $f \in \ell^1(Z)$ *be such that* $f(z_n) \neq 0$ *for all* n*. Then* $\|\cdot\|_{\ell^1}$ *is Gâteaux differentiable at* f*. The proof is left as an exercise.*

Theorem 198 *Given a Banach space* $(X, \|\cdot\|)$*, if the norm in* X' *is Fréchet differentiable at all* $L \in X'$*,* $L \neq 0$*, then* X *is reflexive.*

The proof is based on a theorem of James.

Theorem 199 (James) *A Banach space* $(X, \|\cdot\|)$ *is reflexive if and only if for every* $L \in X'$, $L \neq 0$,

$$\|L\|_{X'} = \max_{x \in \partial B(0,1)} |L(x)| .$$

Note that in general

$$\|L\|_{X'} = \sup_{x \in \partial B(0,1)} |L(x)| ,$$

so James' theorem says that if the supremum is always attained, then X is reflexive.