Wednesday, January 18, 2012

1 Directional Derivatives and Differentiability

Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. Given a direction $\mathbf{v} \in \mathbb{R}^N$, let L be the line through $\mathbf{x}_0$ in the direction $\mathbf{v}$, that is,

$$L := \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v},\, t \in \mathbb{R}\right\},$$

and assume that $\mathbf{x}_0$ is an accumulation point of the set $E \cap L$. The *directional derivative* of f at $\mathbf{x}_0$ in the direction $\mathbf{v}$ is defined as

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) := \lim_{t\to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t},$$

provided the limit exists in $\mathbb{R}$. In the special case in which $\mathbf{v} = \mathbf{e}_i$, the directional derivative $\frac{\partial f}{\partial \mathbf{e}_i}(\mathbf{x}_0)$, if it exists, is called the *partial derivative* of f with respect to x_i and is denoted $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ or $f_{x_i}(\mathbf{x}_0)$ or $D_i f(\mathbf{x}_0)$.

Remark 1 *Let $F := \{t \in \mathbb{R} : \mathbf{x}_0 + t\mathbf{v} \in E\} \subseteq \mathbb{R}$. If we consider the function of one variable $g(t) := f(\mathbf{x}_0 + t\mathbf{v})$, $t \in F$, then $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$, when it exists, is simply the derivative of g at $t = 0$, $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = g'(0)$. If $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$ exists, then g is differentiable in $t = 0$ and so it is continuous at $t = 0$. Thus, the function f restricted to the line L is continuous at $\mathbf{x}_0$.*

Remark 2 *The previous definition continues to hold if in place of $\mathbb{R}^N$ one takes a normed space V, so that $f : E \to \mathbb{R}$ where $E \subseteq V$.*

In view of the previous remark, one would be tempted to say that if the directional derivatives at $\mathbf{x}_0$ exist and are finite in every direction, then f is continuous at $\mathbf{x}_0$. This is false in general, as the following example shows.

Example 3 *Let*

$$f(x, y) := \begin{cases} 1 & \text{if } y = x^2,\, x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given a direction $\mathbf{v} = (v_1, v_2)$, the line L through $\mathbf{0}$ in the direction $\mathbf{v}$ intersects the parabola $y = x^2$ only in $\mathbf{0}$ and in at most one point. Hence, if t is very small,

$$f(0 + tv_1, 0 + tv_2) = 0.$$

It follows that

$$\frac{\partial f}{\partial \mathbf{v}}(0, 0) = \lim_{t\to 0} \frac{f(0 + tv_1, 0 + tv_2) - f(0, 0)}{t} = \lim_{t\to 0} \frac{0 - 0}{t} = 0.$$

However, f is not continuous in $\mathbf{0}$, since $f\left(x, x^2\right) = 1 \to 1$ as $x \to 0$, while $f(x, 0) = 0 \to 0$ as $x \to 0$.

Exercise 4 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all directional derivatives of f *at* $\mathbf{0}$ *and prove that* f *is not continuous at* $\mathbf{0}$.

The previous examples show that in dimension $N \geq 2$ partial derivatives do not give the same kind of results as in the case $N = 1$. To solve this problem, we introduce a stronger notion of derivative, namely, the notion of differentiability.

We recall that a function $T : \mathbb{R}^N \to \mathbb{R}$ is *linear* if

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and

$$T(s\mathbf{x}) = sT(\mathbf{x})$$

for all $s \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$. Write $\mathbf{x} = \sum_{i=1}^N x_i \mathbf{e}_i$. Then by the linearity of T,

$$T(\mathbf{x}) = T\left(\sum_{i=1}^N x_i \mathbf{e}_i\right) = \sum_{i=1}^N x_i T(\mathbf{e}_i).$$

Define $\mathbf{b} := (T(\mathbf{e}_1), \ldots, T(\mathbf{e}_N)) \in \mathbb{R}^N$. Then the previous calculation shows that

$$T(\mathbf{x}) = \mathbf{b} \cdot \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^N.$$

Definition 5 *Let* $E \subseteq \mathbb{R}^N$*, let* $f : E \to \mathbb{R}$*, and let* $\mathbf{x}_0 \in E$ *be an accumulation point of* E*. The function* f *is* differentiable at $\mathbf{x}_0$ *if there exists a linear function* $T : \mathbb{R}^N \to \mathbb{R}$ *(depending on* f *and* $\mathbf{x}_0$*) such that*

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

provided the limit exists. The function T*, if it exists, is called the* differential *of* f *at* $\mathbf{x}_0$ *and is denoted* $df(\mathbf{x}_0)$ *or* $df_{\mathbf{x}_0}$*.*

Exercise 6 *Prove that if* $N = 1$*, then* f *is differentiable at* x_0 *if and only there exists the derivative* $f'(x_0) \in \mathbb{R}$*.*

Remark 7 *The previous definition continues to hold if in place of* $\mathbb{R}^N$ *one takes a normed space* V *, so that* $f : E \to \mathbb{R}$ *where* $E \subseteq V$*. In this case, however, we require* $T : V \to \mathbb{R}$ *to be linear and continuous. Note that in* $\mathbb{R}^N$ *every linear function is continuous.*

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Example 8 *Let* $V = \{f : [-1,1] \to \mathbb{R} : f$ *is differentiable in* $[-1,1]\}$. *Note that* V *is a normed space with the norm* $\|f\| := \max_{x\in[-1,1]} |f(x)|$. *Consider the linear function* $T : V \to \mathbb{R}$ *defined by*

$$T(f) := f'(0).$$

Then T *is linear. To prove that* T *is not continuous, consider*

$$f_n(x) := \frac{1}{n}\sin\left(n^2 x\right).$$

Then

$$\|f_n - 0\| \le \frac{1}{n} \to 0$$

but

$$f_n'(x) = n\cos\left(n^2 x\right)$$

so that

$$T(f_n) = f_n'(0) = n \to \infty$$

and so T *is not continuous, since* $T(f_n) \nrightarrow T(0) = 0$.

The next theorem shows that differentiability in dimension $N \ge 2$ plays the same role of the derivative in dimension $N = 1$.

Theorem 9 *Let* $E \subseteq \mathbb{R}^N$, *let* $f : E \to \mathbb{R}$, *and let* $\mathbf{x}_0 \in E$ *be an accumulation point of* E. *If* f *is differentiable at* $\mathbf{x}_0$, *then* f *is continuous at* $\mathbf{x}_0$.

Proof. Let T be the differential of f at $\mathbf{x}_0$. Define

$$R(\mathbf{x}) := f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0).$$

Note that by the definition of differentiability

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{R(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

Then

$$f(\mathbf{x}) - f(\mathbf{x}_0) = T(\mathbf{x} - \mathbf{x}_0) + R(\mathbf{x}) = \mathbf{b}\cdot(\mathbf{x} - \mathbf{x}_0) + R(\mathbf{x}).$$

Hence, by Cauchy's inequality for $\mathbf{x} \in E$, $\mathbf{x} \neq \mathbf{x}_0$,

$$\begin{aligned} 0 &\le |f(\mathbf{x}) - f(\mathbf{x}_0)| \le |\mathbf{b}\cdot(\mathbf{x} - \mathbf{x}_0)| + |R(\mathbf{x})| \le \|\mathbf{b}\|\,\|\mathbf{x} - \mathbf{x}_0\| + |R(\mathbf{x})| \\ &= \|\mathbf{x} - \mathbf{x}_0\|\left(\|\mathbf{b}\| + \frac{|R(\mathbf{x})|}{\|\mathbf{x} - \mathbf{x}_0\|}\right) \to 0\,(\|\mathbf{b}\| + 0) \end{aligned}$$

as $\mathbf{x} \to \mathbf{x}_0$. It follows by the squeeze theorem that f is continuous at $\mathbf{x}_0$. ■

Next we study the relation between directional derivatives and differentiability. The next theorem gives a formula for the vector $\mathbf{b}$ used in the previous proof and hence determines T. Here we need $\mathbf{x}_0$ to be an interior point of E.

Theorem 10 *Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ be differentiable at some point $\mathbf{x}_0 \in E^\circ$. Then*

(i) all the directional derivatives of f at $\mathbf{x}_0$ exist and

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = df_{\mathbf{x}_0}(\mathbf{v}),$$

(ii) for every direction $\mathbf{v}$,

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, v_i. \tag{1}$$

Proof. Since $\mathbf{x}_0$ is an interior point, there exists $B(\mathbf{x}_0, r) \subseteq E$. Let $\mathbf{v} \in \mathbb{R}^N$ be a direction and $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$. Note that for $|t| < r$, we have that

$$\|\mathbf{x} - \mathbf{x}_0\| = \|\mathbf{x}_0 + t\mathbf{v} - \mathbf{x}_0\| = \|t\mathbf{v}\| = |t|\,\|\mathbf{v}\| = |t|\,1 < r$$

and so $\mathbf{x}_0 + t\mathbf{v} \in B(\mathbf{x}_0, r) \subseteq E$. Moreover, $\mathbf{x} \to \mathbf{x}_0$ as $t \to 0$ and so, since f is differentiable at $\mathbf{x}_0$,

$$0 = \lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - T(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - T(\mathbf{x}_0 + t\mathbf{v} - \mathbf{x}_0)}{\|\mathbf{x}_0 + t\mathbf{v} - \mathbf{x}_0\|}$$
$$= \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - tT(\mathbf{v})}{|t|}.$$

Since $\frac{|t|}{t}$ is bounded by one, it follows that

$$\lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - tT(\mathbf{v})}{t} = \lim_{t \to 0} \frac{|t|}{t} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - tT(\mathbf{v})}{|t|} = 0.$$

But then

$$0 = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0) - tT(\mathbf{v})}{t} = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{v}) - f(\mathbf{x}_0)}{t} - T(\mathbf{v}),$$

which shows that there exists $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = T(\mathbf{v})$.

Part (ii) follows from the linearity of T. Indeed, writing $\mathbf{v} = \sum_{i=1}^{N} v_i \mathbf{e}_i$, by the linearity of T,

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = T(\mathbf{v}) = T\left(\sum_{i=1}^{N} v_i \mathbf{e}_i\right) = \sum_{i=1}^{N} v_i T(\mathbf{e}_i) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, v_i.$$

■

Remark 11 *If in the previous theorem $\mathbf{x}_0$ is not an interior point but for some direction $\mathbf{v} \in \mathbb{R}^N$, the point $\mathbf{x}_0$ is an accumulation point of the set $E \cap L$, where L is the line through $\mathbf{x}_0$ in the direction $\mathbf{v}$, then as in the first part of the proof we can show that there exists the directional derivative $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$ and*

$$\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = df_{\mathbf{x}_0}(\mathbf{v}).$$

If all the partial derivatives of f at $\mathbf{x}_0$ exist, the vector

$$\left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0), \ldots, \frac{\partial f}{\partial x_N}(\mathbf{x}_0)\right) \in \mathbb{R}^N$$

is called the *gradient* of f at $\mathbf{x}_0$ and is denoted by $\nabla f(\mathbf{x}_0)$ or $\operatorname{grad} f(\mathbf{x}_0)$ or $Df(\mathbf{x}_0)$. Note that part (ii) of the previous theorem shows that

$$df_{\mathbf{x}_0}(\mathbf{v}) = T(\mathbf{v}) = \nabla f(\mathbf{x}_0) \cdot \mathbf{v} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0)\, v_i. \tag{2}$$

for all directions $\mathbf{v}$. Hence, only at *interior points* of E, to check differentiability it is enough to prove that

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = 0. \tag{3}$$

Exercise 12 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at 0, that all directional derivatives of f at 0 exist but that the formula

$$\frac{\partial f}{\partial \mathbf{v}}(0,0) = \frac{\partial f}{\partial x}(0,0)\, v_1 + \frac{\partial f}{\partial y}(0,0)\, v_2$$

fails.

Exercise 13 *Let*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all directional derivatives of f at $\mathbf{0}$. Study the continuity and the differentiability of f at $\mathbf{0}$.

Theorems 9 and 10 give necessary conditions for the differentiability of f at $\mathbf{x}_0$. Let's prove that these conditions are not, however, sufficient.

Example 14 *Let*

$$f(x,y) := \begin{cases} x & \text{if } y = x^2,\ x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Given a direction $\mathbf{v} = (v_1, v_2)$, the line L through $\mathbf{0}$ in the direction $\mathbf{v}$ intersects the parabola $y = x^2$ only in $\mathbf{0}$ and in at most one point. Hence, if t is very small,

$$f(0 + tv_1, 0 + tv_2) = 0.$$

It follows that

$$\frac{\partial f}{\partial \mathbf{v}}(0,0)=\lim_{t\to 0}\frac{f(0+tv_1,0+tv_2)-f(0,0)}{t}=\lim_{t\to 0}\frac{0-0}{t}=0.$$

Thus formula (1) holds. Moreover, f is continuous in $\mathbf{0}$, since $|f(x,y)|\le|x|\to 0$ as $(x,y)\to(0,0)$. We claim that f is not differentiable at $(0,0)$. To negate differentiability, in view of the previous theorem and since $(0,0)$ is an interior point, we need to prove that the quotient

$$\frac{f(x,y)-f(0,0)-\nabla f(0,0)\cdot(x,y)}{\sqrt{(x-0)^2+(y-0)^2}}$$

does not tend to zero as $(x,y)\to(0,0)$. Take $y=x^2$. Then

$$\begin{aligned}\frac{f\left(x,x^2\right)-f(0,0)-\nabla f(0,0)\cdot\left(x,x^2\right)}{\sqrt{(x-0)^2+(x^2-0)^2}}&=\frac{x-0-0\cdot\left(x,x^2\right)}{\sqrt{(x-0)^2+(x^2-0)^2}}\\&=\frac{x}{\sqrt{x^2+x^4}}=\frac{x}{|x|\sqrt{1+x^2}}\nrightarrow 0.\end{aligned}$$

Exercise 15 *Let $f:E\to\mathbb{R}$ be Lipschitz and let $\mathbf{x}_0\in E^\circ$.*

(i) Assume that all the directional derivatives of f at $\mathbf{x}_0$ exist and that $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)=\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\mathbf{x}_0)v_i$ for every direction $\mathbf{v}$. Prove that f is differentiable at $\mathbf{x}_0$.

(ii) Assume that all the partial derivatives of f at $\mathbf{x}_0$ exist, that the directional derivatives $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$ exist for all $\mathbf{v}\in S$, where S is dense in the unit sphere, and that $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)=\sum_{i=1}^N\frac{\partial f}{\partial x_i}(\mathbf{x}_0)v_i$ for every direction $\mathbf{v}\in S$. Prove that f is differentiable at $\mathbf{x}_0$.

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The next theorem gives an important sufficient condition for differentiability at a point $\mathbf{x}_0$.

Theorem 16 *Let $E\subseteq\mathbb{R}^N$, let $f:E\to\mathbb{R}$, let $\mathbf{x}_0\in E^\circ$. Assume that there exists $r>0$ such that $B(\mathbf{x}_0,r)\subseteq E$ and the partial derivatives $\frac{\partial f}{\partial x_j}$, $j=1,\dots,N$, exist for every $\mathbf{x}\in B(\mathbf{x}_0,r)$ and are continuous at $\mathbf{x}_0$. Then f is differentiable at $\mathbf{x}_0$.*

Proof. Let $\mathbf{x}\in B(\mathbf{x}_0,r)$. Write $\mathbf{x}=(x_1,\dots,x_N)$ and $\mathbf{x}_0=(y_1,\dots,y_N)$. Then

$$\begin{aligned}f(\mathbf{x})-f(\mathbf{x}_0)=&\left(f(x_1,\dots,x_N)-f(y_1,x_2,\dots,x_N)\right)\\&+\cdots+\left(f(y_1,\dots,y_{N-1},x_N)-f(y_1,\dots,y_N)\right).\end{aligned}$$

By the mean value theorem applied to the function of one variable $f(\cdot,x_2,\dots,x_N)$,

$$f(x_1,\dots,x_N)-f(y_1,x_2,\dots,x_N)=\frac{\partial f}{\partial x_1}(\mathbf{z}_1)(x_1-y_1),$$

where $\mathbf{z}_1 := (\theta_1 x_1 + (1-\theta_1) y_1, x_2, \dots, x_N)$ for some $\theta_1 \in (0,1)$. Note that

$$\|\mathbf{z}_1 - \mathbf{x}_0\| \le \|\mathbf{x} - \mathbf{x}_0\| .$$

Similarly, for $i = 2, \dots, N$,

$$f(y_1, \dots, y_{i-1}, x_i, \dots, x_N) - f(y_1, \dots, y_{i-1}, y_i, \dots, x_N) = \frac{\partial f}{\partial x_i}(\mathbf{z}_i)(x_i - y_i),$$

where $\mathbf{z}_i := (y_1, \dots, y_{i-1}, \theta_i x_i + (1-\theta_i) y_i, x_{i+1}, \dots, x_N)$ for some $\theta_i \in (0,1)$ and

$$\|\mathbf{z}_i - \mathbf{x}_0\| \le \|\mathbf{x} - \mathbf{x}_0\| .$$

Write

$$\begin{aligned} &f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\ &\quad = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right)(x_i - y_i) . \end{aligned}$$

Then

$$\frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \le \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \frac{|x_i - y_i|}{\|\mathbf{x} - \mathbf{x}_0\|}.$$

Since $\frac{|x_i - y_i|}{\|\mathbf{x} - \mathbf{x}_0\|} \le 1$, we have that

$$0 \le \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \le \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right|. \tag{4}$$

Using the fact that $\|\mathbf{z}_i - \mathbf{x}_0\| \le \|\mathbf{x} - \mathbf{x}_0\| \to 0$ as $\mathbf{x} \to \mathbf{x}_0$, together with the continuity of $\frac{\partial f}{\partial x_i}$ at $\mathbf{x}_0$, gives

$$0 \le \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \le \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \to 0,$$

which implies the differentiability of f at $\mathbf{x}_0$. ■

The previous theorem can be significantly improved. Indeed, we have the following result.

Theorem 17 *Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$, let $\mathbf{x}_0 \in E^\circ$, and let $i \in \{1, \dots, N\}$. Assume that there exists $r > 0$ such that $B(\mathbf{x}_0, r) \subseteq E$ and for all $j \ne i$ and for all $\mathbf{x} \in B(\mathbf{x}_0, r)$ the partial derivative $\frac{\partial f}{\partial x_j}$ exists at $\mathbf{x}$ and is continuous at $\mathbf{x}_0$. Assume also that $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ exists. Then f is differentiable at $\mathbf{x}_0$.*

Proof. Without loss of generality, we may assume that $i = N$. Reasoning as before we have that

$$\begin{aligned}
&f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \\
&\quad = \sum_{i=1}^{N-1} \left(\frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right)(x_i - y_i) \\
&\qquad + \left(\frac{f(y_1, \ldots, y_{N-1}, x_N) - f(y_1, \ldots, y_N)}{x_N - y_N} - \frac{\partial f}{\partial x_N}(\mathbf{x}_0) \right)(x_N - y_N).
\end{aligned}$$

Hence, as before,

$$0 \leq \frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \leq \sum_{i=1}^{N-1} \left| \frac{\partial f}{\partial x_i}(\mathbf{z}_i) - \frac{\partial f}{\partial x_i}(\mathbf{x}_0) \right| \tag{5}$$
$$+ \left| \frac{f(y_1, \ldots, y_{N-1}, y_N + (x_N - y_N)) - f(y_1, \ldots, y_N)}{x_N - y_N} - \frac{\partial f}{\partial x_N}(\mathbf{x}_0) \right|.$$

Since $t := x_N - y_N \to 0$ as $\mathbf{x} \to \mathbf{x}_0$, we have that

$$\begin{aligned}
&\frac{f(y_1, \ldots, y_{N-1}, y_N + (x_N - y_N)) - f(y_1, \ldots, y_N)}{x_N - y_N} \\
&= \frac{f(y_1, \ldots, y_{N-1}, y_N + t) - f(y_1, \ldots, y_N)}{t} \to \frac{\partial f}{\partial x_N}(\mathbf{x}_0),
\end{aligned}$$

and so the right-hand side of (5) goes to zero as $\mathbf{x} \to \mathbf{x}_0$. ■

The next exercise shows that the previous conditions are sufficient but not necessary for differentiability.

Example 18 *Let*

$$f(x, y) := \begin{cases} (x^2 + y^2) \sin \frac{1}{x+y} & \text{if } (x, y) \neq (0, 0) \text{ or } x + y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let's prove that f is differentiable at $(0, 0)$ but the partial derivatives are not continuous at $(0, 0)$. We have

$$\frac{f(x, 0) - f(0, 0)}{x - 0} = \frac{(x^2 + 0) \sin \frac{1}{x+0} - 0}{x - 0} = x \sin \frac{1}{x} \to 0$$

as $x \to 0$, since $\sin \frac{1}{x}$ is bounded. Hence, $\frac{\partial f}{\partial x}(0, 0) = 0$. Similarly, $\frac{\partial f}{\partial y}(0, 0) = 0$. Hence,

$$\begin{aligned}
\frac{f(x, 0) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)(x - 0) - \frac{\partial f}{\partial y}(0, 0)(y - 0)}{\sqrt{(x - 0)^2 + (y - 0)^2}} &= \begin{cases} \frac{(x^2 + y^2) \sin \frac{1}{x+y}}{\sqrt{x^2 + y^2}} & \text{if } x + y \neq 0, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \sqrt{x^2 + y^2} \sin \frac{1}{x+y} & \text{if } x + y \neq 0, \\ 0 & \text{otherwise.} \end{cases} \\
&\to 0
\end{aligned}$$

as $(x,y) \to (0,0)$ *since* $\sin\frac{1}{x+y}$ *is bounded and* $\sqrt{x^2+y^2} \to 0$*. Hence,* f *is differentiable at* $(0,0)$*.*

On the other hand, if $x+y \neq 0$*,*

$$\frac{\partial f}{\partial x}(x,y) = (2x+0)\sin\frac{1}{x+y} + \left(x^2+y^2\right)\left(\cos\frac{1}{x+y}\right)\left(-\frac{1}{(x+y)^2}\right),$$
$$\frac{\partial f}{\partial y}(x,y) = (0+2y)\sin\frac{1}{x+y} + \left(x^2+y^2\right)\left(\cos\frac{1}{x+y}\right)\left(-\frac{1}{(x+y)^2}\right).$$

Taking $x = y$ *gives*

$$\frac{\partial f}{\partial x}(x,x) = 2x\sin\frac{1}{2x} - \cos\frac{1}{2x},$$
$$\frac{\partial f}{\partial y}(x,x) = 2x\sin\frac{1}{2x} - \cos\frac{1}{2x}$$

which have no limit as $x \to 0$*, thus* $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *are not continuous at* $(0,0)$*. Note that one should also check the existence of the partial derivatives at points such that* $x+y=0$*.*

Wednesday, January 25, 2012

Definition 19 *Given* $E \subseteq \mathbb{R}^N$*, we say that*

(i) E *is* disconnected *if there exist two nonempty open sets* $U, V \subseteq \mathbb{R}^N$ *such that*

$$E \subseteq U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset, \quad E \cap U \cap V = \emptyset.$$

(ii) E *is* connected *if it is not disconnected.*

We study properties of connected sets. We begin by showing that continuous functions preserve connectedness.

Proposition 20 *Let* $F \subseteq \mathbb{R}^N$ *and let* $\mathbf{f} : F \to \mathbb{R}^M$ *be a continuous function. Then* $\mathbf{f}(E)$ *is connected for every connected set* $E \subseteq F$*.*

Proof. Let $E \subseteq F$ be a connected set and assume by contradiction that $\mathbf{f}(E)$ is disconnected. Then there exist two open sets $U, V \subset \mathbb{R}^M$ such that

$$\mathbf{f}(E) \subseteq U \cup V, \quad \mathbf{f}(E) \cap U \neq \emptyset, \quad \mathbf{f}(E) \cap V \neq \emptyset, \quad \mathbf{f}(E) \cap U \cap V = \emptyset.$$

By continuity, $\mathbf{f}^{-1}(U)$ and $\mathbf{f}^{-1}(V)$ are relatively open, hence, there exist two open sets A and B in $\mathbb{R}^N$ such that $\mathbf{f}^{-1}(U) = F \cap A$ and $\mathbf{f}^{-1}(V) = F \cap B$. Hence,

$$E \subseteq A \cup B, \quad E \cap A \neq \emptyset, \quad E \cap B \neq \emptyset, \quad E \cap A \cap B = \emptyset,$$

which shows that E is disconnected. ■

Theorem 21 *A set $C \subseteq \mathbb{R}$ is connected if and only if it is convex.*

Proof. Exercise. ■

We now introduce another notion of connectedness, which is simpler to verify.

Definition 22 *A* continuous path *is a continuous function $\varphi : I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval. The set $\varphi(I) \subseteq \mathbb{R}^N$ is called the* range *of the path. If $I = [a, b]$, the points $\varphi(a)$ and $\varphi(b)$ are called* endpoints *of the path. A set $E \subseteq \mathbb{R}^N$ is called* pathwise connected *if for all $\mathbf{x}$,$y \in E$ there exists a continuous path with endpoints $\mathbf{x}$ and y and range contained in E.*

Proposition 23 *Let $E \subseteq \mathbb{R}^N$ be pathwise connected. Then E is connected.*

Proof. We claim that E is connected. If not, then there exist two nonempty open sets $U, V \subseteq \mathbb{R}^N$ such that

$$E \subseteq U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset, \quad E \cap U \cap V = \emptyset.$$

Let $\mathbf{x} \in C \cap U$ and $\mathbf{y} \in C \cap V$. By hypothesis there exists a continuous path $\varphi : [a, b] \to \mathbb{R}^N$ such that $\varphi(a) = \mathbf{x}$, $\varphi(b) = \mathbf{y}$ and $\varphi([a, b]) \subseteq E$. By Proposition 20 and Theorem 21, we have that $\varphi([a, b])$ is connected. On the other hand,

$$\varphi([a, b]) \subseteq E \subseteq U \cup V, \quad \mathbf{x} \in \varphi([a, b]) \cap U, \quad \mathbf{y} \in \varphi([a, b]) \cap V,$$

which is a contradiction. ■

Remark 24 *In particular, convex sets and star-shaped sets are connected.*

Exercise 25 *Let $E \subseteq \mathbb{R}^N$ be a connected set. Prove that $\overline{E}$ is connected.*

The next example and exercise show that in $\mathbb{R}^N$ a connected set may fail to be pathwise connected, unless the set is open.

Exercise 26 *Let $E \subset \mathbb{R}^2$ be the set given by*

$$\begin{aligned} E_1 &= \left\{(x, y) \in \mathbb{R}^2 : \, x = 0, \, -1 \leq y \leq 1\right\}, \\ E_2 &= \left\{(x, y) \in \mathbb{R}^2 : \, x > 0, \, y = \sin \frac{1}{x}\right\}, \\ E &= E_1 \cup E_2. \end{aligned}$$

Prove that E is connected but not pathwise connected.

Definition 27 *In the Euclidean space $\mathbb{R}^N$, a* polygonal path *is a continuous path $\varphi : [a, b] \to \mathbb{R}^N$ for which there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ with the property that $\varphi : [t_{i-1}, t_i] \to \mathbb{R}^N$ is affine for all $i = 1, \ldots, n$, that is,*

$$\varphi(t) = \mathbf{c}_i + t\mathbf{d}_i \quad \textit{for } t \in [t_{i-1}, t_i],$$

for some $\mathbf{c}_i, \mathbf{d}_i \in \mathbb{R}^N$.

Theorem 28 *Let $O \subseteq \mathbb{R}^N$ be open and connected. Then O is pathwise connected.*

Proof. Define the sets

$$U := \{\mathbf{x} \in O : \text{ there exists a polygonal path with endpoints } \mathbf{x} \text{ and } \mathbf{x}_0 \text{ and range contained in } O\}$$

and

$$V := \{\mathbf{x} \in O : \text{ there does not exist a polygonal path with endpoints } \mathbf{x} \text{ and } \mathbf{x}_0 \text{ and range contained in } O\}.$$

We claim that U and V are open. To see this, let $\mathbf{x} \in U$. Since O is open, there exists $B(\mathbf{x}, r) \subseteq O$. Let's show that $B(\mathbf{x}, r) \subseteq U$. Indeed, let $\mathbf{y} \in B(\mathbf{x}, r)$. Since $\mathbf{x} \in U$, there exists a polygonal path $\varphi : [a, b] \to \mathbb{R}^N$ such that $\varphi(a) = \mathbf{x}_0$, $\varphi(b) = \mathbf{x}$ and $\varphi([a, b]) \subseteq O$. Define the polygonal path $\psi : [a, b+1] \to \mathbb{R}^N$ as follows

$$\psi(t) := \begin{cases} \varphi(t) & \text{if } t \in [a, b], \\ \mathbf{x} + (t - b)\,\mathbf{y} & \text{if } t \in [b, b+1]. \end{cases}$$

Then $\psi(a) = \mathbf{x}_0$, $\psi(b+1) = \mathbf{y}$. Since $\psi([a, b+1]) = \varphi([a, b]) \cup \psi([b, b+1])$ and $\psi([b, b+1])$ is the segment joining $\mathbf{x}$ with $\mathbf{y}$, which is contained in $B(\mathbf{x}, r)$, we have that $\psi([a, b+1])$ is contained in O. Hence, $B(\mathbf{x}, r) \subseteq U$. This shows that U is open.

To prove that V is open, let $\mathbf{x} \in V$. Since O is open, there exists $B(\mathbf{x}, r) \subseteq O$. Let's show that $B(\mathbf{x}, r) \subseteq V$. Indeed, let $\mathbf{y} \in B(\mathbf{x}, r)$. Assume by contradiction that $\mathbf{y} \in U$. Then there exists a polygonal path $\varphi : [a, b] \to \mathbb{R}^N$ such that $\varphi(a) = \mathbf{x}_0$, $\varphi(b) = \mathbf{y}$ and $\varphi([a, b]) \subseteq O$. Define the polygonal path $\psi : [a, b+1] \to \mathbb{R}^N$ as follows

$$\psi(t) := \begin{cases} \varphi(t) & \text{if } t \in [a, b], \\ \mathbf{y} + (t - b)\,\mathbf{x} & \text{if } t \in [b, b+1]. \end{cases}$$

Then $\psi(a) = \mathbf{x}_0$, $\psi(b+1) = \mathbf{x}$ and $\psi([a, b+1])$ is contained in O, which shows that $\mathbf{x} \in U$, which is a contradiction. Hence, $B(\mathbf{x}, r) \subseteq V$. This shows that V is open.

Hence, U and V are open and by their definition they are disjoint and $O = U \cup V$. Moreover, U is nonempty, since $\mathbf{x}_0 \in U$. Thus, V must be empty, otherwise O would be disconnected. Hence, $O = U$. It follows that U is pathwise connected. Indeed, if $\mathbf{x}, \mathbf{y} \in O$ then there exist two polygonal paths $\varphi : [a, b] \to \mathbb{R}^N$ and $\psi : [c, d] \to \mathbb{R}^N$such that $\varphi(a) = \mathbf{x}_0$, $\varphi(b) = \mathbf{x}$ and $\varphi([a, b]) \subseteq O$, and $\psi(c) = \mathbf{x}_0$, $\psi(d) = \mathbf{y}$ and $\psi([c, d]) \subseteq O$. By "gluing" these two polygonal paths appropriately, we can construct a polygonal path joining $\mathbf{x}$ and $\mathbf{y}$ with range contained in O. ■

Exercise 29 *Prove that the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.*

Exercise 30 *Let $E_1, E_2 \subseteq \mathbb{R}^N$ be two connected sets. Prove that if there exists $\mathbf{x} \in E_1 \cap \overline{E_2}$, then $E_1 \cup E_2$ is connected.*

Next we show that if a set is not connected, we can decompose it uniquely into a disjoint union of maximal connected subsets.

Proposition 31 *Let $E \subseteq \mathbb{R}^N$. Assume that*

$$E = \bigcup_{\alpha \in \Lambda} E_\alpha,$$

where each E_α is a connected set. If $\bigcap_{\alpha \in \Lambda} E_\alpha$ is nonempty, then E is connected.

Proof. We claim that E is connected. If not, then there exist two nonempty open sets $U, V \subset \mathbb{R}^N$ such that

$$E \subseteq U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset, \quad E \cap U \cap V = \emptyset.$$

Since each E_α is connected, we must have that either $E_\alpha \subseteq U$ or $E_\alpha \subseteq V$. On the other hand, if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta$ is nonempty, while $E \cap U \cap V$ is empty. Thus, all E_α either belong to U or to V. This contradicts the fact that $E \cap U \neq \emptyset$ and that $E \cap V \neq \emptyset$. ■

Let $E \subseteq \mathbb{R}^N$. For every $\mathbf{x} \in E$, let $E_\mathbf{x}$ be the union of all the connected subsets of E that contain $\mathbf{x}$. Note that $E_\mathbf{x}$ is nonempty, since $\{\mathbf{x}\}$ is a connected subset. In view of the previous proposition, the set $E_\mathbf{x}$ is connected. Moreover, if $\mathbf{x}, \mathbf{y} \in E$ and $\mathbf{x} \neq \mathbf{y}$, then either $E_\mathbf{x} \cap E_\mathbf{y} = \emptyset$ or $E_\mathbf{x} = E_\mathbf{y}$. Indeed, if not, then again by the previous proposition the set $E_\mathbf{x} \cup E_\mathbf{y}$ would be connected, contained in E, and would contain $\mathbf{x}$ and $\mathbf{y}$, which would contradict the definition of $E_\mathbf{x}$ and of $E_\mathbf{y}$. Thus, we can partition E into a disjoint union of maximal connected subsets, called the *connected components* of E.

Exercise 32 *Let $C \subseteq \mathbb{R}^N$ be a closed set. Then the connected components of C are closed.*

Exercise 33 *Prove that if $U \subseteq \mathbb{R}^N$ is open, then the connected components of U are open.*

Example 34 *In a metric space the connected components of an open set need not be open. Consider the space $X = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ with the metric $d(x, y) := |x - y|$. We claim that the connected component X_0 containing 0 is the singleton $\{0\}$. Note that this is not open, since any ball $B(0, r)$ contains all $\frac{1}{n}$ for $n > \frac{1}{r}$. To prove the claim, assume by contradiction that X_0 contains other element of X, say, $\frac{1}{m} \in X_0$. Consider an irrational number $\frac{1}{m+1} < r < \frac{1}{m}$ and consider the open sets $U = (-\infty, r) \cap X$ and $V = (r, \infty) \cap X$. They disconnect X_0, which is a contradiction.*

Next we prove the following theorem.

Theorem 35 *Let $U \subseteq \mathbb{R}^N$ be open and let $f : U \to \mathbb{R}$ be such that for all $\mathbf{x} \in U$ and all $i = 1, \ldots, N$ there exists $\frac{\partial f}{\partial x_i}(\mathbf{x}) = 0$. Then f is constant in each connected component of U.*

The proof makes use of the mean value theorem.

Theorem 36 (Mean Value Theorem) *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$*, with* $\mathbf{x} \neq \mathbf{y}$*, let* S *be the segment of endpoints* $\mathbf{x}$ *and* $\mathbf{y}$*, that is,*

$$S = \{t\mathbf{x} + (1-t)\,\mathbf{y} : \, t \in [0,1]\},$$

and let $f : S \to \mathbb{R}$ *be such that* f *is continuous in* S *and there exists the directional derivative* $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{z})$ *for all* $\mathbf{z} \in S$ *except at most* $\mathbf{x}$ *and* $\mathbf{y}$*, where* $\mathbf{v} := \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$*. Then there exists* $\theta \in (0,1)$ *such that*

$$f(\mathbf{x}) - f(\mathbf{y}) = \frac{\partial f}{\partial \mathbf{v}}(\theta\mathbf{x} + (1-\theta)\,\mathbf{y})\,\|\mathbf{x}-\mathbf{y}\|. \tag{6}$$

Lemma 37 *Under the hypotheses of the previous theorem, the function* $g(t) := f(t\mathbf{x} + (1-t)\,\mathbf{y})$*,* $t \in [0,1]$*, is differentiable for all* $t \in (0,1)$*, with*

$$g'(t) = \frac{\partial f}{\partial \mathbf{v}}(t\mathbf{x} + (1-t)\,\mathbf{y})\,\|\mathbf{x}-\mathbf{y}\|.$$

Proof. Fix $t_0 \in (0,1)$ and consider

$$\begin{aligned}\frac{g(t)-g(t_0)}{t-t_0} &= \frac{f(t\mathbf{x}+(1-t)\,\mathbf{y}) - f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y})}{t-t_0} \\ &= \frac{f\left(t_0\mathbf{x}+(1-t_0)\,\mathbf{y}+(t-t_0)\,\|\mathbf{x}-\mathbf{y}\|\,\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}\right) - f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y})}{(t-t_0)\,\|\mathbf{x}-\mathbf{y}\|}\,\|\mathbf{x}-\mathbf{y}\| \\ &= \frac{f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y}+s\mathbf{v}) - f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y})}{s}\,\|\mathbf{x}-\mathbf{y}\|,\end{aligned}$$

where $s := (t-t_0)\,\|\mathbf{x}-\mathbf{y}\|$. Since $s \to 0$ as $t \to t_0$, it follows that there exists

$$\begin{aligned}g'(t_0) &= \lim_{t\to t_0}\frac{g(t)-g(t_0)}{t-t_0} = \lim_{s\to 0}\frac{f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y}+s\mathbf{v}) - f(t_0\mathbf{x}+(1-t_0)\,\mathbf{y})}{s}\,\|\mathbf{x}-\mathbf{y}\| \\ &= \frac{\partial f}{\partial \mathbf{v}}(t_0\mathbf{x}+(1-t_0)\,\mathbf{y})\,\|\mathbf{x}-\mathbf{y}\|.\end{aligned}$$

■

Proof of the Mean Value Theorem. Consider the function $g(t) := f(t\mathbf{x}+(1-t)\,\mathbf{y})$, $t \in [0,1]$. Since compositions of continuous functions is continuous, we have that g is continuous. By the previous lemma, we are in a position to apply the mean value theorem to the function g to find $\theta \in [0,1]$ such that

$$g(1) - g(0) = \frac{dg}{dt}(\theta)\,(1-0),$$

that is,

$$f(\mathbf{x}) - f(\mathbf{y}) = \frac{\partial f}{\partial \mathbf{v}}(\theta\mathbf{x}+(1-\theta)\,\mathbf{y})\,\|\mathbf{x}-\mathbf{y}\|.$$

■

Remark 38 *If f is defined on a larger domain E and all points of S except at most $\mathbf{x}$ and $\mathbf{y}$ are interior points of E, then by (1), we have that*

$$\frac{\partial f}{\partial \mathbf{v}}(\theta \mathbf{x}+(1-\theta)\mathbf{y})=\sum_{i=i}^{N}\frac{\partial f}{\partial x_i}(\theta \mathbf{x}+(1-\theta)\mathbf{y})\frac{(x_i-y_i)}{\|\mathbf{x}-\mathbf{y}\|},$$

and so we can rewrite (6) in the form

$$f(\mathbf{x})-f(\mathbf{y})=\sum_{i=i}^{N}\frac{\partial f}{\partial x_i}(\theta \mathbf{x}+(1-\theta)\mathbf{y})(x_i-y_i).$$

We now turn to the proof of Theorem 35.

Proof of Theorem 35. Since all the partial derivatives are zero, in particular they are continuous. It follows from Theorem 16 that f is differentiable for all $\mathbf{x}\in U$.

Step 1: Let $\mathbf{x}\in U$. Since U is open, there exists $B(\mathbf{x},r)\subseteq U$. We claim that f is constant in $B(\mathbf{x},r)$. Fix $\mathbf{y}\in B(\mathbf{x},r)$. By the mean value theorem and Remark 38, there exists $\theta\in(0,1)$ such that

$$f(\mathbf{x})-f(\mathbf{y})=\sum_{i=i}^{N}\frac{\partial f}{\partial x_i}(\theta \mathbf{x}+(1-\theta)\mathbf{y})(x_i-y_i)=0.$$

This proves the claim.

Step 2: Let U_α be a connected component of U. By the previous exercise, U_α is open. Next fix $\mathbf{x}_0\in U_\alpha$, let $c:=f(\mathbf{x}_0)$, and define the set

$$A:=\{\mathbf{x}\in U_\alpha:\ f(\mathbf{x})=c\}.$$

We claim that A is open. Indeed, if $\mathbf{x}\in A$, then by the previous step there exists $B(\mathbf{x},r)\subseteq U_\alpha$ such that f is constant in $B(\mathbf{x},r)$. Hence, $f=c$ in $B(\mathbf{x},r)$, which implies that $B(\mathbf{x},r)\subseteq A$. Thus every point of A is an interior point and so A is open. Moreover, A is nonempty since $\mathbf{x}_0\in A$. Next consider the set

$$A_1=\{\mathbf{x}\in U_\alpha:\ f(\mathbf{x})\neq c\}=f^{-1}((-\infty,c)\cup(c,\infty)).$$

Since $(-\infty,c)\cup(c,\infty)$ is open, f is continuous, and U_α is open, we have that $f^{-1}((-\infty,c)\cup(c,\infty))$ is open. Thus, we can write

$$U_\alpha=A\cup A_1,$$

where A and A_1 are open and disjoint. Since A is nonempty and U_α is connected, necessarily A_1 must be empty, since otherwise we would negate the fact that U_α is connected. Hence, $U_\alpha=A$, that is, $f(\mathbf{x})=c$ for all $\mathbf{x}\in U_\alpha$. ■

Monday, January 30, 2012

2 Higher Order Derivatives

Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. Let $i \in \{1, \ldots, N\}$ and assume that there exists the partial derivatives $\frac{\partial f}{\partial x_i}(\mathbf{x})$ for all $\mathbf{x} \in E$. If $j \in \{1, \ldots, N\}$ and $\mathbf{x}_0$ is an accumulation point of $E \cap L$, where L is the line through $\mathbf{x}_0$ in the direction $\mathbf{e}_j$, then we can consider the partial derivative of the function $\frac{\partial f}{\partial x_i}$ with respect to x_j, that is,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) =: \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Note that in general the order in which we take derivatives is important.

Example 39 *Let*

$$f(x, y) := \begin{cases} y^2 \arctan \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

If $y \neq 0$, then

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}\left(y^2 \arctan \frac{x}{y}\right) = y^2 \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{\partial}{\partial x}\left(\frac{x}{y}\right) \\ &= \frac{y^3}{x^2 + y^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}\left(y^2 \arctan \frac{x}{y}\right) = 2y \arctan \frac{x}{y} + y^2 \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \\ &= 2y \arctan \frac{x}{y} - \frac{xy^2}{x^2 + y^2}, \end{aligned}$$

while at points $(x_0, 0)$ we have:

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, 0) &= \lim_{t \to 0} \frac{f(x_0 + t, 0) - f(x_0, 0)}{t} = \lim_{t \to 0} \frac{0 - 0}{t} = 0, \\ \frac{\partial f}{\partial y}(x_0, 0) &= \lim_{t \to 0} \frac{f(x_0, 0 + t) - f(x_0, 0)}{t} = \lim_{t \to 0} \frac{t^2 \arctan \frac{x_0}{t} - 0}{t} \\ &= \lim_{t \to 0} t \arctan \frac{x_0}{t} - = 0, \end{aligned}$$

where we have used the fact that $\arctan \frac{x_0}{t}$ is bounded and $t \to 0$. Thus,

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \frac{y^3}{x^2 + y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases} \qquad \frac{\partial f}{\partial y}(x, y) = \begin{cases} 2y \arctan \frac{x}{y} - \frac{xy^2}{x^2 + y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

To find $\frac{\partial^2 f}{\partial y \partial x}(0,0)$, *we calculate*

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x}(0,0) &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{t\to 0}\frac{\frac{\partial f}{\partial x}(0,0+t) - \frac{\partial f}{\partial x}(0,0)}{t} \\ &= \lim_{t\to 0}\frac{\frac{t^3}{0+t^2} - 0}{t} = \lim_{t\to 0} 1 = 1,\end{aligned}$$

while

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(0,0) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{t\to 0}\frac{\frac{\partial f}{\partial y}(0+t,0) - \frac{\partial f}{\partial y}(0,0)}{t} \\ &= \lim_{t\to 0}\frac{0-0}{t} = \lim_{t\to 0} 0 = 0.\end{aligned}$$

Hence, $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Exercise 40 *Let*

$$f(x,y) := \begin{cases} \frac{x^3 y - x y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

We present an improved version of the Schwartz theorem.

Theorem 41 (Schwartz) *Let* $E \subseteq \mathbb{R}^N$, *let* $f : E \to \mathbb{R}$, *let* $\mathbf{x}_0 \in E^\circ$, *and let* $i, j \in \{1, \ldots, N\}$. *Assume that there exists* $r > 0$ *such that* $B(\mathbf{x}_0, r) \subseteq E$ *and for all* $\mathbf{x} \in B(\mathbf{x}_0, r)$, *the partial derivatives* $\frac{\partial f}{\partial x_i}(\mathbf{x})$, $\frac{\partial f}{\partial x_j}(\mathbf{x})$, *and* $\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x})$ *exist. Assume also that* $\frac{\partial^2 f}{\partial x_j \partial x_i}$ *is continuous at* $\mathbf{x}_0$. *Then there exists* $\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)$ *and*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

Lemma 42 *Let* $A : ((-r,r) \setminus \{0\}) \times ((-r,r) \setminus \{0\}) \to \mathbb{R}$. *Assume that the double limit* $\lim_{(s,t)\to(0,0)} A(s,t)$ *exists in* $\mathbb{R}$ *and that the limit* $\lim_{t\to 0} A(s,t)$ *exists in* $\mathbb{R}$ *for all* $s \in (-r,r) \setminus \{0\}$. *Then the iterated limit* $\lim_{s\to 0}\lim_{t\to 0} A(s,t)$ *exists and*

$$\lim_{s\to 0}\lim_{t\to 0} A(s,t) = \lim_{(s,t)\to(0,0)} A(s,t).$$

Proof. Let $\ell = \lim_{(s,t)\to(0,0)} A(s,t)$. Then for every $\varepsilon > 0$ there exists $\delta = \delta((0,0), \varepsilon) > 0$ such that

$$|A(s,t) - \ell| \leq \varepsilon$$

for all $(s,t) \in ((-r,r) \setminus \{0\}) \times ((-r,r) \setminus \{0\})$, with $\sqrt{|s-0|^2 + |t-0|^2} \leq \delta$.

Fix $s \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \setminus \{0\}$. Then for all $t \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \setminus \{0\}$,

$$|A(s,t) - \ell| \le \varepsilon$$

and so letting $t \to 0$ in the previous inequality (and using the fact that the limit $\lim_{t\to 0} A(s,t)$ exists), we get

$$\left|\lim_{t\to 0} A(s,t) - \ell\right| \le \varepsilon$$

for all $s \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \setminus \{0\}$. But this implies that there exists $\lim_{s\to 0} \lim_{t\to 0} A(s,t) = \ell$. ■

Proof of Theorem 41. Step 1: Assume that $N = 2$. Let $|t|, |s| < \frac{r}{\sqrt{2}}$. Then the points $(x_0 + s, y_0)$, $(x_0 + s, y_0 + t)$, and $(x_0, y_0 + t)$ belong to $B((x_0, y_0), r)$. Define

$$\begin{aligned} A(s,t) &:= \frac{f(x_0+s, y_0+t) - f(x_0+s, y_0) - f(x_0, y_0+t) + f(x_0, y_0)}{st}, \\ g(x) &:= f(x, y_0+t) - f(x, y_0). \end{aligned}$$

By the mean value theorem

$$A(s,t) = \frac{g(x_0+s) - g(x_0)}{st} = \frac{g'(\xi)}{t} = \frac{\frac{\partial f}{\partial x}(\xi_t, y_0+t) - \frac{\partial f}{\partial x}(\xi_t, y_0)}{t}$$

where ξ is between x_0 and $x_0 + t$. Fix t and consider the function

$$h(y) := \frac{\partial f}{\partial x}(\xi_t, y).$$

By the mean value theorem,

$$h(b) - h(a) = h'(c)(b-a) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, c)(b-a)$$

for some c between a and b. Taking $b = t$ and $a = 0$, we get

$$\frac{\partial f}{\partial x}(\xi_t, y_0+t) - \frac{\partial f}{\partial x}(\xi_t, y_0) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, \eta_t)\, t$$

where η_t is between y_0 and $y_0 + t$. Hence,

$$A(s,t) = \frac{\partial^2 f}{\partial y \partial x}(\xi_t, \eta_t) \to \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0),$$

where we have used the fact that $(\xi, \eta) \to (x_0, y_0)$ as $(s,t) \to (0,0)$ together with the continuity of $\frac{\partial^2 f}{\partial y \partial x}$ at (x_0, y_0). Note that this shows that there exists the limit

$$\lim_{(s,t)\to(0,0)} A(s,t) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0).$$

On the other hand, for all $s \neq 0$,

$$\lim_{t\to 0} A(s,t) = \frac{1}{s}\lim_{t\to 0}\left[\frac{f(x_0+s,y_0+t)-f(x_0+s,y_0)}{t} - \frac{f(x_0,y_0+t)-f(x_0,y_0)}{t}\right]$$
$$= \frac{\frac{\partial f}{\partial y}(x_0+s,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)}{s}.$$

Hence, we are in a position to apply the previous lemma to obtain

$$\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) = \lim_{(s,t)\to(0,0)} A(s,t) = \lim_{s\to 0}\lim_{t\to 0} A(s,t)$$
$$= \lim_{s\to 0}\frac{\frac{\partial f}{\partial y}(x_0+s,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)}{s} = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0).$$

Step 2: In the case $N \geq 2$ let $\mathbf{x} = (x_1,\dots,x_N)$, $\mathbf{x}_0 = (c_1,\dots,c_N)$. Assume that $1 < i < j < N$ (the cases $i = 1$ and $j = N$ are similar) and apply Step 1 to the function of two variables

$$F(x_i,x_j) := f(c_1,\dots,c_{i-1},x_i,c_{i+1},\dots,c_{j-1},x_j,c_{j+1},\dots,c_N)$$

■

Wednesday, February 1, 2012

Next we prove Taylor's formula in higher dimensions. We set $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. A *multi-index* $\boldsymbol{\alpha}$ is a vector $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_N) \in (\mathbb{N}_0)^N$. The *length* of a multi-index is defined as

$$|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_N.$$

Given a multi-index $\boldsymbol{\alpha}$, the partial derivative $\frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}$ is defined as

$$\frac{\partial^{\boldsymbol{\alpha}}}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\cdots \partial x_N^{\alpha_N}},$$

where $\mathbf{x} = (x_1,\dots,x_N)$. If $\boldsymbol{\alpha} = \mathbf{0}$, we set $\frac{\partial^{\mathbf{0}} f}{\partial \mathbf{x}^{\mathbf{0}}} := f$.

Example 43 *If $N = 3$ and $\boldsymbol{\alpha} = (2,1,0)$, then*

$$\frac{\partial^{(2,1,0)}}{\partial (x,y,z)^{(2,1,0)}} = \frac{\partial^3}{\partial x^2 \partial y}.$$

Given a multi-index $\boldsymbol{\alpha}$ and $\mathbf{x} \in \mathbb{R}^N$, we set

$$\boldsymbol{\alpha}! := \alpha_1!\cdots\alpha_N!, \quad \mathbf{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1}\cdots x_N^{\alpha_N}.$$

If $\boldsymbol{\alpha} = \mathbf{0}$, we set $\mathbf{x}^{\mathbf{0}} := 1$.

Using this notation, we can extend the binomial theorem.

Theorem 44 (Multinomial Theorem) *Let* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ *and let* $n \in \mathbb{N}$. *Then*

$$(x_1 + \cdots + x_N)^n = \sum_{\boldsymbol{\alpha} \text{ multi-index, } |\boldsymbol{\alpha}|=n} \frac{n!}{\boldsymbol{\alpha}!} \mathbf{x}^{\boldsymbol{\alpha}}.$$

Proof. Exercise. ■

Given an open set $U \subseteq \mathbb{R}^N$, for every nonnegative integer $m \in \mathbb{N}_0$, we denote by $C^m(U)$ the space of all functions that are continuous together with their partial derivatives up to order m. We set $C^\infty(U) := \bigcap_{m=0}^{\infty} C^m(U)$.

Theorem 45 (Taylor's Formula) *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $f \in C^m(U)$, $m \in \mathbb{N}$, *and let* $\mathbf{x}_0 \in U$. *Then for every* $\mathbf{x} \in U$,

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \text{ multi-index, } 0 \le |\boldsymbol{\alpha}| \le m} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + R_m(\mathbf{x}),$$

where

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_m(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^m} = 0.$$

Proof. Step 1: Since $\mathbf{x}_0 \in U$ and U is open, there exists $B(\mathbf{x}_0, r) \subseteq U$. We prove Taylor's formula with Lagrange's reminder, that is,

$$f(\mathbf{x}) = \sum_{0 \le |\boldsymbol{\alpha}| < m} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + \sum_{|\boldsymbol{\alpha}|=m} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0 + c(\mathbf{x} - \mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}}, \tag{7}$$

for all $\mathbf{x} \in B(\mathbf{x}_0, r)$ and where $c \in (0, 1)$. Fix $\mathbf{x} \in B(\mathbf{x}_0, r)$, let $\mathbf{h} := \mathbf{x} - \mathbf{x}_0$ and consider the function $g(t) := f(\mathbf{x}_0 + t\mathbf{h})$ defined for $t \in [0, 1]$. By Lemma 37 and Remark 38, we have that

$$\frac{dg}{dt}(t) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0 + t\mathbf{h}) h_i = (\mathbf{h} \cdot \nabla) f(\mathbf{x}_0 + t\mathbf{h})$$

with for all $t \in [0, 1]$. By repeated applications of Lemma 37 and Remark 38, we get that

$$\frac{d^{(n)} g}{dt^n}(t) = (\mathbf{h} \cdot \nabla)^n f(\mathbf{x}_0 + t\mathbf{h})$$

for all $n = 1, \ldots, m$, where $(\mathbf{h} \cdot \nabla)^n$ means that we apply the operator

$$\mathbf{h} \cdot \nabla = h_1 \frac{\partial}{\partial x_1} + \cdots + h_N \frac{\partial}{\partial x_N}$$

n times to f. By the multinomial theorem, and the fact that for functions in C^m partial derivatives commute,

$$\begin{aligned}(\mathbf{h}\cdot\nabla)^n &= \left(h_1\frac{\partial}{\partial x_1}+\cdots+h_N\frac{\partial}{\partial x_N}\right)^n\\ &= \sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\mathbf{h}^{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\alpha}}}{\partial\mathbf{x}^{\boldsymbol{\alpha}}},\end{aligned}$$

and so

$$\frac{d^{(n)}g}{dt^n}(t)=\sum_{\boldsymbol{\alpha}\text{ multi-index, }|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\mathbf{h}^{\boldsymbol{\alpha}}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+t\mathbf{h}).$$

■

Friday, February 03, 2012

Proof. Using Taylor's formula for g, we get

$$g(1)=g(0)+\sum_{n=1}^{m-1}\frac{1}{n!}\frac{d^{(n)}g}{dt^n}(0)(1-0)^n+\frac{1}{m!}\frac{d^{(m)}g}{dt^m}(c)(1-0)^m$$

for some $c\in(0,1)$. Substituting, we obtain

$$f(\mathbf{x}_0+\mathbf{h})=\sum_{0\le|\boldsymbol{\alpha}|=m-1}\frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\mathbf{h}^{\boldsymbol{\alpha}}+\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+c\mathbf{h})\mathbf{h}^{\boldsymbol{\alpha}}. \quad (8)$$

Step 2: We add and subtract $\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}$ in (7), to get

$$\begin{aligned}f(\mathbf{x})={}&\sum_{0\le|\boldsymbol{\alpha}|\le m}\frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}\\ &+\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}\left(\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+c(\mathbf{x}-\mathbf{x}_0))-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right),\end{aligned}$$

and set

$$R_m(\mathbf{x}):=\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}\left(\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+c(\mathbf{x}-\mathbf{x}_0))-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right).$$

Fix $\varepsilon>0$ and find $\delta=\delta(\mathbf{x}_0,\varepsilon)>0$ such that

$$\left|\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x})-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right|\le\varepsilon$$

for all $|\boldsymbol{\alpha}|=m$ and all $\mathbf{x}\in U$ with $\|\mathbf{x}-\mathbf{x}_0\|\le\delta$. Then, since $\left|(\mathbf{x}-\mathbf{x}_0)^{\boldsymbol{\alpha}}\right|\le\|\mathbf{x}-\mathbf{x}_0\|^m$, we have that

$$\begin{aligned}\frac{|R_m(\mathbf{x})|}{\|\mathbf{x}-\mathbf{x}_0\|^m}&\le\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}\left|\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0+c(\mathbf{x}-\mathbf{x}_0))-\frac{\partial^{\boldsymbol{\alpha}}f}{\partial\mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0)\right|\\ &\le\varepsilon\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!},\end{aligned}$$

which shows that $\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{R_m(\mathbf{x})}{\|\mathbf{x}-\mathbf{x}_0\|^m} = 0$. ■

Example 46 *Let's calculate the limit*

$$\lim_{(x,y)\to(0,0)} \frac{(1+x)^y - 1}{\sqrt{x^2+y^2}}.$$

By substituting we get $\frac{0}{0}$*. Consider the function*

$$f(x,y) = (1+x)^y - 1 = e^{\log(1+x)^y} - 1 = e^{y\log(1+x)} - 1,$$

which is defined in the set $U := \left\{(x,y) \in \mathbb{R}^2 : 1+x > 0\right\}$*. The function* f *is of class* C^∞*. Let's use Taylor's formula of order* $m = 1$ *at* $(0,0)$*,*

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) + o\left(\sqrt{x^2+y^2}\right).$$

We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(e^{y\log(1+x)} - 1\right) = e^{y\log(1+x)} y \frac{1}{1+x},$$
$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(e^{y\log(1+x)} - 1\right) = e^{y\log(1+x)} y \log(1+x),$$

and so

$$f(x,y) = 0 + 0(x-0) + 0(y-0) + o\left(\sqrt{x^2+y^2}\right),$$

which means that

$$\lim_{(x,y)\to(0,0)} \frac{(1+x)^y - 1}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{o\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} = 0.$$

Note that if we had to calculate the limit

$$\lim_{(x,y)\to(0,0)} \frac{(1+x)^y - 1}{x^2+y^2},$$

then we would need Taylor's formula of order $m = 2$ *at* $(0,0)$*,*

$$\begin{aligned} f(x,y) = f(0,0) &+ \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) \\ &+ \frac{1}{(2,0)!}\frac{\partial^2 f}{\partial x^2}(0,0)(x-0)^2 + \frac{1}{(1,1)!}\frac{\partial^2 f}{\partial x \partial y}(0,0)(x-0)(y-0) \\ &+ \frac{1}{(0,2)!}\frac{\partial^2 f}{\partial y^2}(0,0)(y-0)^2 + o\left(x^2+y^2\right). \end{aligned}$$

Another simpler method would be to use the Taylor's formulas for e^t *and* $\log(1+s)$*.*

Exercise 47 *Calculate the limit*

$$\lim_{(x,y)\to(2,0)} \frac{1-\cos[(x-2)y]}{\log\left[1+(x-2)^2+y^2\right]}.$$

3 Local Minima and Maxima

Definition 48 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}^N$, and let $\mathbf{x}_0 \in E$. We say that*

(i) f attains a local minimum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(ii) f attains a global minimum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$,*

(iii) f attains a local maximum *at $\mathbf{x}_0$ if there exists $r > 0$ such that $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E \cap B(\mathbf{x}_0, r)$,*

(iv) f attains a global maximum *at $\mathbf{x}_0$ if $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ for all $\mathbf{x} \in E$.*

Theorem 49 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}^N$. Assume that f attains a local minimum (or maximum) at some point $\mathbf{x}_0 \in E$. If there exists a direction $\mathbf{v}$ and $\delta > 0$ such that the set*

$$\{\mathbf{x}_0 + t\mathbf{v} : t \in (-\delta, \delta)\} \subseteq E \tag{9}$$

and if there exists $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0)$, then necessarily, $\frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_0) = 0$. In particular, if $\mathbf{x}_0$ is an interior point of E and f is differentiable at $\mathbf{x}_0$, then all partial derivatives and directional derivatives of f at $\mathbf{x}_0$ are zero.

Proof. Exercise. ■

Remark 50 *In view of the previous theorem, when looking for local minima and maxima, we have to search among the following:*

- *Interior points at which f is differentiable and $\nabla f(\mathbf{x}) = \mathbf{0}$, these are called* critical points*. Note that if $\nabla f(\mathbf{x}_0) = \mathbf{0}$, the function f may not attain a local minimum or maximum at $\mathbf{x}_0$. Indeed, consider the function $f(x) = x^3$. Then $f'(0) = 0$, but f is strictly increasing, and so f does not attain a local minimum or maximum at 0.*

- *Interior points at which f is not differentiable. The function $f(x) = |x|$ attains a global minimum at $x = 0$, but f is not differentiable at $x = 0$.*

- *Boundary points.*

To find sufficient conditions for a critical point to be a point of local minimum or local maximu, we study the second order derivatives of f.

Definition 51 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}^N$, and let $\mathbf{x}_0 \in E$. The Hessian matrix of f at $\mathbf{x}_0$ is the $N \times N$ matrix*

$$\begin{aligned} H_f(\mathbf{x}_0) :&= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_1}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_N}(\mathbf{x}_0) & \cdots & \frac{\partial^2 f}{\partial x_N^2}(\mathbf{x}_0) \end{pmatrix} \\ &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) \right)_{i,j=1}^{N}, \end{aligned}$$

whenever it is defined.

Remark 52 *If the hypotheses of Schwartz's theorem are satisfied for all* $i, j = 1, \ldots, N$*, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0),$$

which means that the Hessian matrix $H_f(\mathbf{x}_0)$ *is symmetric.*

Given an $N \times N$ matrix H, the *characteristic polynomial* of H is the polynomial

$$p(t) := \det(tI_N - H), \quad t \in \mathbb{R}.$$

Theorem 53 *Let* H *be an* $N \times N$ *matrix. If* H *is symmetric, then all roots of the characteristic polynomial are real.*

Theorem 54 *Given a polynomial of the form*

$$p(t) = t^N + a_{N-1}t^{N-1} + a_{N-2}t^{N-2} + \cdots + a_1 t + a_0, \quad t \in \mathbb{R},$$

where the coefficients a_i *are real for every* $i = 0, \ldots, N-1$*, assume that all roots of* p *are real. Then*

(i) all roots of p *are positive if and only if the coefficients alternate sign, that is,* $a_{N-1} < 0$*,* $a_{N-2} > 0$*,* $a_{N-3} < 0$*, etc.*

(ii) all roots of p *are negative if and only* $a_i > 0$ *for every* $i = 0, \ldots, N-1$*.*

Monday, February 06, 2012

Proof of Theorem 53. We begin by observing that if $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^N$, then $(H\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (H\mathbf{y})$. Indeed, we have

$$(H\mathbf{x}) \cdot \mathbf{y} = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} H_{ij} x_j \right) y_i = \sum_{j=1}^{N} x_j \left(\sum_{i=1}^{N} H_{ji} y_i \right) = \mathbf{x} \cdot (H\mathbf{y}).$$

Step 1: We claim that there cannot be more than N distinct eigenvalues. To see this it is enough to show that if $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to different eigenvalues λ and μ, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. Indeed, we have

$$\lambda(\mathbf{x} \cdot \mathbf{y}) = (\lambda \mathbf{x}) \cdot \mathbf{y} = (H\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (H\mathbf{y}) = \mathbf{x} \cdot (\mu \mathbf{y}) = \mu(\mathbf{x} \cdot \mathbf{y}),$$

and so

$$(\lambda - \mu)(\mathbf{x} \cdot \mathbf{y}) = 0.$$

Since $\lambda \neq \mu$, it follows that $\mathbf{x} \cdot \mathbf{y} = 0$.

Step 2: We prove the existence of one real eigenvalue λ_1. Consider the function

$$f(\mathbf{x}) := \frac{(H\mathbf{x}) \cdot \mathbf{x}}{\|\mathbf{x}\|^2} = \frac{\sum_{j=1}^{N} \sum_{i=1}^{N} H_{ij} x_i x_j}{x_1^2 + \cdots + x_N^2}$$

defined for all $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{x} \neq \mathbf{0}$. The expression $\frac{(H\mathbf{x})\cdot\mathbf{x}}{\|\mathbf{x}\|^2}$ is called the *Rayleigh quotient*. Note that f is of class C^∞ and that

$$\begin{aligned}\frac{\partial f}{\partial x_k}(\mathbf{x}) &= \frac{\left(x_1^2+\cdots+x_N^2\right)\sum_{j=1}^N\sum_{i=1}^N H_{ij}\left(\delta_{i,k}x_j + x_i\delta_{j,k}\right) - \left(\sum_{j=1}^N\sum_{i=1}^N H_{ij}x_ix_j\right)2x_k}{\left(x_1^2+\cdots+x_N^2\right)^2} \\ &= \frac{\left(x_1^2+\cdots+x_N^2\right)\left(\sum_{j=1}^N H_{kj}x_j + \sum_{i=1}^N H_{ik}x_i\right) - \left(\sum_{j=1}^N\sum_{i=1}^N H_{ij}x_ix_j\right)2x_k}{\left(x_1^2+\cdots+x_N^2\right)^2} \\ &= \frac{2\left(x_1^2+\cdots+x_N^2\right)\sum_{j=1}^N H_{kj}x_j - \left(\sum_{j=1}^N\sum_{i=1}^N H_{ij}x_ix_j\right)2x_k}{\left(x_1^2+\cdots+x_N^2\right)^2} \\ &= 2\frac{(H\mathbf{x})_k}{\|\mathbf{x}\|^2} - 2\frac{(H\mathbf{x})\cdot\mathbf{x}}{\|\mathbf{x}\|^4}x_k\end{aligned} \tag{10}$$

for all $k = 1, \ldots, N$, where in the third equality we have used the fact that H is symmetric.

Since f is a continuous function and $\partial B(\mathbf{0}, 1)$ is compact, by the Weierstrass theorem, there exists

$$\max_{\mathbf{x}\in\partial B(\mathbf{0},1)} f(\mathbf{x}) = f(\mathbf{x}_1).$$

Hence, $f(\mathbf{x}) \leq f(\mathbf{x}_1)$ for all $\mathbf{x} \in \partial B(\mathbf{0}, 1)$. If now $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{x} \neq \mathbf{0}$, then $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ belongs to $\partial B(\mathbf{0}, 1)$, and so

$$f(\mathbf{x}) = f\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \leq f(\mathbf{x}_1),$$

where we have used the fact that $f(t\mathbf{x}) = f(\mathbf{x})$ for $t > 0$. This shows that $\mathbf{x}_1$ is a point of absolute maximum for f in $\mathbb{R}^N \setminus \{\mathbf{0}\}$. Since $\mathbf{x}_1$ is an interior point and f is of class C^∞, it follows from Theorem 49 that $\mathbf{x}_1$ is a critical point of f, that is,

$$\nabla f(\mathbf{x}_1) = \mathbf{0}$$

for all $k = 1, \ldots, N$. It follows by (10) and the fact that $\|\mathbf{x}_1\| = 1$ that

$$H\mathbf{x}_1 - ((H\mathbf{x}_1)\cdot\mathbf{x}_1)\,\mathbf{x}_1 = \mathbf{0},$$

or, equivalently, that

$$H\mathbf{x}_1 = \lambda_1\mathbf{x}_1,$$

where

$$\lambda_1 := (H\mathbf{x}_1)\cdot\mathbf{x}_1.$$

This shows that λ_1 is a real eigenvalue of H and $\mathbf{x}_1$ is the corresponding eigenvector. Note that

$$\lambda_1 = f(\mathbf{x}_1) = \max\{(H\mathbf{x})\cdot\mathbf{x} : \|\mathbf{x}\| = 1\}.$$

Step 3: Let $1 \le n \le N-1$ and assume by induction that real eigenvalues $\lambda_1, \ldots, \lambda_n$ of H have been found with corresponding orthonormal eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Note that the eigenvalues may not be distinct, but $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{i,j}$ for all $i, j = 1, \ldots, n$. Consider the set

$$Y := \left\{ \mathbf{x} \in \mathbb{R}^N : \mathbf{x} \cdot \mathbf{x}_i = 0 \text{ for all } i = 1, \ldots, n \right\}.$$

Note that Y is a subspace of $\mathbb{R}^N$ of dimension $N-k$. Since $Y \cap \partial B(\mathbf{0}, 1)$ is compact, by the Weierstrass theorem, there exists

$$\max_{\mathbf{x} \in Y \cap \partial B(\mathbf{0},1)} f(\mathbf{x}) = f(\mathbf{x}_{n+1}).$$

Hence, $f(\mathbf{x}) \le f(\mathbf{x}_{n+1})$ for all $\mathbf{x} \in Y \cap \partial B(\mathbf{0}, 1)$. If now $\mathbf{x} \in Y$, $\mathbf{x} \ne \mathbf{0}$, then $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ belongs to $Y \cap \partial B(\mathbf{0}, 1)$, and so

$$f(\mathbf{x}) = f\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) \le f(\mathbf{x}_{n+1}),$$

where we have used the fact that $f(t\mathbf{x}) = f(\mathbf{x})$ for $t > 0$. In particular, if $\mathbf{v} \in Y$ is a direction, the for every $t \in \mathbb{R}$,

$$f(\mathbf{x}_{n+1} + t\mathbf{v}) \le f(\mathbf{x}_{n+1}),$$

which shows that the function $g(t) := f(\mathbf{x}_{n+1} + t\mathbf{v})$ has a maximum at $t = 0$. Hence,

$$0 = g'(0) = \frac{\partial f}{\partial \mathbf{v}}(\mathbf{x}_{n+1}).$$

Using Theorems 16 and 10, we get that

$$\nabla f(\mathbf{x}_{n+1}) \cdot \mathbf{v} = 0$$

for all vectors $\mathbf{v} \in Y$ of norm one, and, in turn, for all $\mathbf{v} \in Y$. By (10) and the fact that $\|\mathbf{x}_{n+1}\| = 1$ it follows that

$$2\left(H\mathbf{x}_{n+1} - \left((H\mathbf{x}_{n+1}) \cdot \mathbf{x}_{n+1}\right)\mathbf{x}_{n+1}\right) \cdot \mathbf{v} = \mathbf{0} \tag{11}$$

for $\mathbf{v} \in Y$. On the other hand, $H\mathbf{x}_{n+1}$ belongs to Y. Indeed, for all $i = 1, \ldots, n$

$$(H\mathbf{x}_{n+1}) \cdot \mathbf{x}_i = \mathbf{x}_{n+1} \cdot (H\mathbf{x}_i) = \mathbf{x}_{n+1} \cdot (\lambda_i \mathbf{x}_i) = \lambda_i (\mathbf{x}_{n+1} \cdot \mathbf{x}_i) = 0,$$

where we have used the fact that $\mathbf{x}_{n+1} \in Y$. Thus the vector $\nabla f(\mathbf{x}_{n+1}) = 2\left(H\mathbf{x}_{n+1} - \left((H\mathbf{x}_{n+1}) \cdot \mathbf{x}_{n+1}\right)\mathbf{x}_{n+1}\right)$ belongs to Y and so it is orthogonal to itself by (11). It follows that

$$\mathbf{0} = \nabla f(\mathbf{x}_{n+1}) = 2\left(H\mathbf{x}_{n+1} - \left((H\mathbf{x}_{n+1}) \cdot \mathbf{x}_{n+1}\right)\mathbf{x}_{n+1}\right),$$

so that

$$H\mathbf{x}_{n+1} = \lambda_{n+1}\mathbf{x}_{n+1},$$

where $\lambda_{n+1} := (H\mathbf{x}_{n+1}) \cdot \mathbf{x}_{n+1}$. This proves that λ_{n+1} is an eigenvalue and $\mathbf{x}_{n+1}$ a corresponding eigenvector. Note that λ_{n+1} may coincide with some of the previous eigenvalues, but since $\mathbf{x}_{n+1}$ belongs to Y, $\mathbf{x}_{n+1}$ is distinct from all the previous eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Moreover,

$$\lambda_{n+1} = f(\mathbf{x}_{n+1}) = \max\{(H\mathbf{x}) \cdot \mathbf{x} : \mathbf{x} \in Y \cap \partial B(\mathbf{0}, 1)\}.$$

This induction argument shows that there exist N distinct orthonormal eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_N$. Thus, the eigenvectors form an orthonormal basis in $\mathbb{R}^N$.

Step 4: Since for any $t \in \mathbb{R}$ and $i = 1, \ldots, N$,

$$(tI_N - H)\mathbf{x}_i = (t - \lambda_i)\mathbf{x}_i,$$

it follows that (see the following exercise),

$$\det(tI_N - H) = (t - \lambda_1) \cdots (t - \lambda_N).$$

■

Exercise 55 *Let H be an $N \times N$ symmetric matrix and let $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \mathbb{R}^N$ be orthonormal eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ (possibly repeated). Let A be the $N \times N$ matrix whose columns are $\mathbf{x}_1^T, \ldots, \mathbf{x}_N^T$.*

(i) Prove that $A^T A = I_N$.

(ii) Prove that $\det A = \pm 1$.

(iii) Prove that $HA = \left(\lambda_1 \mathbf{x}_1^T \cdots \lambda_N \mathbf{x}_N^T\right)$.

(iv) Prove that $\det H = \lambda_1 \ldots \lambda_N$.

Wednesday, February 08, 2012

The next theorem gives necessary and sufficient conditions for a point to be of local minimum or maximum.

Theorem 56 *Let $U \subseteq \mathbb{R}^N$ be open, let $f : U \to \mathbb{R}$ be of class $C^2(U)$ and let $\mathbf{x}_0 \in U$ be a critical point of f.*

(i) If $H_f(\mathbf{x}_0)$ is positive definite, then f attains a local minimum at $\mathbf{x}_0$,

(ii) if f attains a local minimum at $\mathbf{x}_0$, then $H_f(\mathbf{x}_0)$ is positive semidefinite,

(iii) if $H_f(\mathbf{x}_0)$ is negative definite, then f attains a local maximum at $\mathbf{x}_0$,

(iv) if f attains a local maximum at $\mathbf{x}_0$, then $H_f(\mathbf{x}_0)$ is negative semidefinite.

Proof. (i) Assume that $H_f(\mathbf{x}_0)$ is positive definite. Then by Remark **??**,

$$\sum_{j=1}^{N}\sum_{i=1}^{N}\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0)\, v_i v_j \geq m\,\|\mathbf{v}\|^2 \tag{12}$$

for all $\mathbf{v} \in \mathbb{R}^N$ and for some $m > 0$.

We now apply Taylor's formula of order two to obtain

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \sum_{|\boldsymbol{\alpha}|=2} \frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)^{\boldsymbol{\alpha}} + R_2(\mathbf{x}) \\ &= f(\mathbf{x}_0) + 0 + \sum_{j=1}^{N}\sum_{i=1}^{N}\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)_i (\mathbf{x} - \mathbf{x}_0)_j + R_2(\mathbf{x}), \end{aligned}$$

where we have used the fact that $\mathbf{x}_0$ is a critical point and where

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0.$$

Using the definition of limit with $\varepsilon = \frac{m}{2}$, we can find $\delta > 0$ such that

$$\left| \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right| \leq \frac{m}{2}$$

for all $\mathbf{x} \in E$ with $\|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. Using this property and (12), we get

$$\begin{aligned} f(\mathbf{x}) &\geq f(\mathbf{x}_0) + m\,\|\mathbf{x} - \mathbf{x}_0\|^2 + R_2(\mathbf{x}) = f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \left(m + \frac{R_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} \right) \\ &\geq f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \left(m - \frac{m}{2} \right) = f(\mathbf{x}_0) + \|\mathbf{x} - \mathbf{x}_0\|^2 \frac{m}{2} > f(\mathbf{x}_0) \end{aligned}$$

for all $\mathbf{x} \in E$ with $0 < \|\mathbf{x} - \mathbf{x}_0\| \leq \delta$. This shows that f attains a (strict) local minimum at $\mathbf{x}_0$.

(ii) Since $\mathbf{x}_0 \in U$ and U is open, there exists $B(\mathbf{x}_0, r) \subseteq U$. Let $\mathbf{v} \in \mathbb{R}^N$ and consider the function

$$g(t) := f(\mathbf{x}_0 + t\mathbf{v}), \quad t \in (-r, r).$$

Since $g(t) = f(\mathbf{x}_0 + t\mathbf{v}) \geq f(\mathbf{x}_0) = g(0)$ for all $t \in (-r, r)$, g attains a local minimum at $t = 0$. By two applications of Lemma 37 and Remark 38, we have that for $t \in (-r, r)$,

$$\begin{aligned} g'(t) &= \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}_0 + t\mathbf{v})\, v_i, \\ g''(t) &= \sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0 + t\mathbf{v})\, v_i v_j. \end{aligned}$$

Hence, $g \in C^2(-r, r)$, and so, since g attains a local minimum at $t = 0$,

$$0 \leq g''(0) = \sum_{j=1}^{N} \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}_0) v_i v_j.$$

Since this is true for every direction $\mathbf{v}$, it follows that $H_f(\mathbf{x}_0)$ is positive semidefinite. ■

Remark 57 *Note that in view of the previous theorem, if at a critical point* $\mathbf{x}_0$ *the characteristic polynomial of* $H_f(\mathbf{x}_0)$ *has one positive root and one negative root, then* f *does not admit a local minimum or a local maximum at* $\mathbf{x}_0$*.*

Theorem 56 shows that if $H_f(\mathbf{x}_0)$ is positive definite, then f admits a local minimum at $\mathbf{x}_0$. The following exercise shows that we cannot weakened this hypothesis to $H_f(\mathbf{x}_0)$ positive semidefinite.

Example 58 *Let* $f(x, y) := x^2 - y^4$*. Let's find the critical points of* f*. We have*

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}\left(x^2 - y^4\right) = 2x - 0 = 0, \\ \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}\left(x^2 - y^4\right) = 0 - 4y^3 = 0. \end{aligned}$$

Hence, $(0, 0)$ *is the only critical point. Let's find the Hessian matrix at these points. Note that the function is of class* C^∞*, so we can apply Schwartz's theorem. We have*

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}(2x-) = 2, \\ \frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y}\left(-4y^3\right) = -12y^2, \\ \frac{\partial^2 f}{\partial y \partial x}(x, y) &= \frac{\partial}{\partial y}(2x) = 0. \end{aligned}$$

Hence,

$$H_f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\begin{aligned} 0 &= \det\left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - H_f(0, 0)\right) \\ &= \det\begin{pmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 0 \end{pmatrix} = (\lambda - 2)(\lambda - 0) - 0 \\ &= (\lambda - 2)\lambda, \end{aligned}$$

and so the roots are $\lambda = 2$ or $\lambda = 0$. Hence, at $(0,0)$ we cannot have a local maximum. But it could be a local minimum. However, taking

$$f(0,y) = -y^4,$$

which has a strict maximum at $y = 0$. This shows that f does not admit a local minimum or a local maximum at $(0,0)$.

4 Vector-Valued Functions

Since we will work with the different euclidean spaces, $\mathbb{R}^M$ and $\mathbb{R}^N$, when needed and to avoid confusion, we will denote by $\|\cdot\|_M$ and $\|\cdot\|_N$ the norms in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively. We recall that a function $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^M$ is *linear* if

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and

$$\mathbf{T}(s\mathbf{x}) = s\mathbf{T}(\mathbf{x})$$

for all $s \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$.

Definition 59 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{f} : E \to \mathbb{R}^M$, and let $\mathbf{x}_0 \in E$ be an accumulation point of E. The function $\mathbf{f}$ is* differentiable at $\mathbf{x}_0$ *if there exists a linear function $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^M$ (depending on $\mathbf{f}$ and $\mathbf{x}_0$) such that*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{T}(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|_N} = \mathbf{0}. \tag{13}$$

provided the limit exists. The function T, if it exists, is called the differential *of $\mathbf{f}$ at $\mathbf{x}_0$ and is denoted $d\mathbf{f}(\mathbf{x}_0)$ or $d\mathbf{f}_{\mathbf{x}_0}$.*

Theorem 60 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{f} : E \to \mathbb{R}^M$, and let $\mathbf{x}_0 \in E$ be an accumulation point of E. Then $\mathbf{f} = (f_1, \ldots, f_M)$ is differentiable at $\mathbf{x}_0$ if and only if all its components f_j, $j = 1, \ldots, M$, are differentiable at $\mathbf{x}_0$. Moreover, $d\mathbf{f}_{\mathbf{x}_0} = \left(d(f_1)_{\mathbf{x}_0}, \ldots, d(f_M)_{\mathbf{x}_0}\right)$.*

Proof. The proof relies on Exercise **??** and is left as an exercise. ■

We study the differentiability of composite functions.

Theorem 61 (Chain Rule) *Let $F \subseteq \mathbb{R}^M$, $E \subseteq \mathbb{R}^N$, let $\mathbf{g} : F \to E$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_N)$, and let $f : E \to \mathbb{R}$. Assume that at some point $\mathbf{y}_0 \in F$ there exist the directional derivatives $\frac{\partial g_1}{\partial \mathbf{v}}(\mathbf{y}_0)$, ..., $\frac{\partial g_N}{\partial \mathbf{v}}(\mathbf{y}_0)$ for some direction $\mathbf{v}$ and f is differentiable at the point $\mathbf{g}(\mathbf{y}_0)$. Then the composite function $f \circ \mathbf{g}$ admits a directional derivative at $\mathbf{y}_0$ in the direction $\mathbf{v}$ and*

$$\frac{\partial (f \circ \mathbf{g})}{\partial \mathbf{v}}(\mathbf{y}_0) = \sum_{i=1}^{N} df_{\mathbf{g}(\mathbf{y}_0)}(\mathbf{e}_i) \frac{\partial g_i}{\partial \mathbf{v}}(\mathbf{y}_0).$$

Moreover, if $\mathbf{g}$ is differentiable at $\mathbf{y}_0$ and f is differentiable at $\mathbf{g}(\mathbf{y}_0)$, then $f \circ \mathbf{g}$ is differentiable at $\mathbf{y}_0$.

Proof. Set $\mathbf{x}_0 := \mathbf{g}(\mathbf{y}_0)$. Since f is differentiable at $\mathbf{x}_0$, we can write

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \mathbf{b} \cdot (\mathbf{x} - \mathbf{x}_0) + R(\mathbf{x}),$$

where $\mathbf{b} := \left(df_{\mathbf{g}(\mathbf{y}_0)}(\mathbf{e}_1), \ldots, df_{\mathbf{g}(\mathbf{y}_0)}(\mathbf{e}_N)\right)$ and

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|_N} = 0. \tag{14}$$

Note that

$$R(\mathbf{x}_0) = 0.$$

Take $\mathbf{x} = \mathbf{g}(\mathbf{y}_0 + t\mathbf{v})$. Then

$$f(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v})) = f(\mathbf{g}(\mathbf{y}_0)) + \mathbf{b} \cdot (\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)) + R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v})),$$

and so

$$\begin{aligned} \frac{(f \circ \mathbf{g})(\mathbf{y}_0 + t\mathbf{v}) - (f \circ \mathbf{g})(\mathbf{y}_0)}{t} &= \frac{f(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v})) - f(\mathbf{g}(\mathbf{y}_0))}{t} \\ &= \mathbf{b} \cdot \frac{\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)}{t} + \frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{t}. \end{aligned} \tag{15}$$

We claim that

$$\lim_{t \to 0} \frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{t} = 0.$$

If for some t, we have $\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) = \mathbf{g}(\mathbf{y}_0)$, then

$$R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v})) = R(\mathbf{g}(\mathbf{y}_0)) = 0. \tag{16}$$

On the other have, if $\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) \neq \mathbf{g}(\mathbf{y}_0)$, then

$$\frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{t} = \frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{\|\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)\|_N} \frac{|t|}{t} \left\| \frac{\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)}{t} \right\|_N. \tag{17}$$

Since each function g_i admits a finite directional derivative at $\mathbf{y}_0$, it follows that $\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) \to \mathbf{g}(\mathbf{y}_0)$ as $t \to 0$ (why?). Hence, by (14)

$$\lim_{t \to 0} \frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{\|\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)\|_N} = 0, \tag{18}$$

where the limit is taken over all t such that $\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) \neq \mathbf{g}(\mathbf{y}_0)$. On the other hand, since $\frac{\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)}{t} \to \frac{\partial \mathbf{g}}{\partial \mathbf{v}}(\mathbf{y}_0)$, and $\frac{|t|}{t}$ is bounded by one, we have that $\frac{|t|}{t} \left\| \frac{\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) - \mathbf{g}(\mathbf{y}_0)}{t} \right\|_N$ is bounded, which, together with (17) and (14), implies that

$$\lim_{t \to 0} \frac{R(\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}))}{t} = 0$$

where the limit is taken over all t such that $\mathbf{g}(\mathbf{y}_0 + t\mathbf{v}) \neq \mathbf{g}(\mathbf{y}_0)$ (here we are using the fact that if a function is bounded and another function goes to zero, then their product goes to zero). Together with (16), this implies that the claim holds.

Using the claim, it follows that

$$\frac{(f\circ\mathbf{g})(\mathbf{y}_0+t\mathbf{v})-(f\circ\mathbf{g})(\mathbf{y}_0)}{t}=\mathbf{b}\cdot\frac{\mathbf{g}(\mathbf{y}_0+t\mathbf{v})-\mathbf{g}(\mathbf{y}_0)}{t}+\frac{R(\mathbf{g}(\mathbf{y}_0+t\mathbf{v}))}{t}$$
$$\to\mathbf{b}\cdot\frac{\partial\mathbf{g}}{\partial\mathbf{v}}(\mathbf{y}_0)+0,$$

which proves the first part of the statement.

The second part of the statement is left as an exercise. ■

Remark 62 *If* $\mathbf{x}_0\in E^\circ$, *then*

$$\frac{\partial(f\circ\mathbf{g})}{\partial\mathbf{v}}(\mathbf{y}_0)=\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}(\mathbf{g}(\mathbf{y}_0))\frac{\partial g_i}{\partial\mathbf{v}}(\mathbf{y}_0)$$
$$=\nabla f(\mathbf{g}(\mathbf{y}_0))\cdot\frac{\partial\mathbf{g}}{\partial\mathbf{v}}(\mathbf{y}_0).$$

Exercise 63 *Prove the second part of the theorem.*

Example 64 *Consider the functions* $f:\mathbb{R}^2\to\mathbb{R}$ *and* $\mathbf{g}:\mathbb{R}^2\to\mathbb{R}^2$ *defined by*

$$f(x_1,x_2)=x_1x_2-1,\quad \mathbf{g}(y_1,y_2)=(y_1y_2-e^{y_1},y_1\sin(y_1y_2)).$$

Then

$$\frac{\partial f}{\partial x_1}(x_1,x_2)=1x_2-0,\quad \frac{\partial f}{\partial x_2}(x_1,x_2)=x_11-0,$$
$$\frac{\partial g_1}{\partial y_1}(y_1,y_2)=1y_2-e^{y_1},\quad \frac{\partial g_1}{\partial y_2}(y_1,y_2)=y_11-0,$$
$$\frac{\partial g_2}{\partial y_1}(y_1,y_2)=1\sin(y_1y_2)+y_1\cos(y_1y_2)(1y_2),\quad \frac{\partial g_2}{\partial y_2}(y_1,y_2)=y_1\cos(y_1y_2)(y_11).$$

First method: *Consider the composition*

$$(f\circ\mathbf{g})(y_1,y_2)=f(g_1(y_1,y_2),g_2(y_1,y_2)).$$

Then by the chain rule,

$$\frac{\partial(f\circ\mathbf{g})}{\partial y_1}(y_1,y_2)=\frac{\partial f}{\partial x_1}(g_1(y_1,y_2),g_2(y_1,y_2))\frac{\partial g_1}{\partial y_1}(y_1,y_2)+\frac{\partial f}{\partial x_2}(g_1(y_1,y_2),g_2(y_1,y_2))\frac{\partial g_2}{\partial y_1}(y_1,y_2)$$
$$=y_1\sin(y_1y_2)(1y_2-e^{y_1})+(y_1y_2-e^{y_1})(1\sin(y_1y_2)+y_1\cos(y_1y_2)(1y_2)),$$

while

$$\frac{\partial(f\circ\mathbf{g})}{\partial y_2}(y_1,y_2)=\frac{\partial f}{\partial x_1}(g_1(y_1,y_2),g_2(y_1,y_2))\frac{\partial g_1}{\partial y_2}(y_1,y_2)+\frac{\partial f}{\partial x_2}(g_1(y_1,y_2),g_2(y_1,y_2))\frac{\partial g_2}{\partial y_2}(y_1,y_2)$$
$$=y_1\sin(y_1y_2)(y_11-0)+(y_1y_2-e^{y_1})(y_1\cos(y_1y_2)(y_11)).$$

Second method: *Write the explicit formula for* $(f \circ \mathbf{g})$*, that is,*

$$\begin{aligned}(f \circ \mathbf{g})(y_1, y_2) &= f(g_1(y_1, y_2), g_2(y_1, y_2)) \\ &= (y_1 y_2 - e^{y_1})(y_1 \sin(y_1 y_2)) - 1.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial (f \circ \mathbf{g})}{\partial y_1}(y_1, y_2) &= \frac{\partial}{\partial y_1}\left[(y_1 y_2 - e^{y_1})(y_1 \sin(y_1 y_2)) - 1\right] \\ &= (1 y_2 - e^{y_1})(y_1 \sin(y_1 y_2)) + (y_1 y_2 - e^{y_1})(1 \sin(y_1 y_2) + y_1 \cos(y_1 y_2)(1 y_2)) - 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial (f \circ \mathbf{g})}{\partial y_2}(y_1, y_2) &= \frac{\partial}{\partial y_2}\left[(y_1 y_2 - e^{y_1})(y_1 \sin(y_1 y_2)) - 1\right] \\ &= (y_1 1 - 0)(y_1 \sin(y_1 y_2)) + (y_1 y_2 - e^{y_1})(y_1 \cos(y_1 y_2)(y_1 1)) - 0.\end{aligned}$$

Next we define the Jacobian of a vectorial function $\mathbf{f} : E \to \mathbb{R}^M$.

Definition 65 *Given a set* $E \subseteq \mathbb{R}^N$ *and a function* $\mathbf{f} : E \to \mathbb{R}^M$*, the* Jacobian matrix *of* $\mathbf{f} = (f_1, \ldots, f_M)$ *at some point* $\mathbf{x}_0 \in E$*, whenever it exists, is the* $M \times N$ *matrix*

$$J_{\mathbf{f}}(\mathbf{x}_0) := \begin{pmatrix} \nabla f_1(\mathbf{x}_0) \\ \vdots \\ \nabla f_M(\mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_N}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial f_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix}.$$

It is also denoted

$$\frac{\partial (f_1, \ldots, f_M)}{\partial (x_1, \ldots, x_N)}(\mathbf{x}_0).$$

When $M = N$*,* $J_{\mathbf{f}}(\mathbf{x}_0)$ *is an* $N \times N$ *square matrix and its determinant is called the* Jacobian determinant *of* $\mathbf{f}$ *at* $\mathbf{x}_0$*. Thus,*

$$\det J_{\mathbf{f}}(\mathbf{x}_0) = \det \left(\frac{\partial f_j}{\partial x_i}(\mathbf{x}_0) \right)_{i,j=1,\ldots,N}.$$

Remark 66 *Let* $E \subseteq \mathbb{R}^N$*, let* $\mathbf{f} : E \to \mathbb{R}^M$*, and let* $\mathbf{x}_0 \in E^\circ$*. Assume that* $\mathbf{f}$ *is differentiable at* $\mathbf{x}_0$*. Then by Theorem 60 all its components* f_j*,* $j = 1, \ldots, M$*, are differentiable at* $\mathbf{x}_0$ *with*

$$d\mathbf{f}_{\mathbf{x}_0} = \left(d(f_1)_{\mathbf{x}_0}, \ldots, d(f_M)_{\mathbf{x}_0} \right).$$

Since $\mathbf{x}_0$ *is an interior point, it follows from (2) that for every direction* $\mathbf{v}$*,*

$$d(f_j)_{\mathbf{x}_0}(\mathbf{v}) = \nabla f_j(\mathbf{x}_0) \cdot \mathbf{v} = \sum_{i=1}^{N} \frac{\partial f_j}{\partial x_i}(\mathbf{x}_0) v_i.$$

Hence,

$$\begin{aligned}d\mathbf{f}_{\mathbf{x}_0}(\mathbf{v}) &= \left(d(f_1)_{\mathbf{x}_0}(\mathbf{v}), \ldots, d(f_M)_{\mathbf{x}_0}(\mathbf{v}) \right) \\ &= J_{\mathbf{f}}(\mathbf{x}_0)\, \mathbf{v}.\end{aligned}$$

As a corollary of Theorem 61, we have the following result.

Corollary 67 *Let $F \subseteq \mathbb{R}^M$, $E \subseteq \mathbb{R}^N$, let $\mathbf{g} : F \to E$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_N)$, and let $\mathbf{f} : E \to \mathbb{R}^P$. Assume that $\mathbf{g}$ is differentiable at some point $\mathbf{y}_0 \in F^\circ$ and that $\mathbf{f}$ is differentiable at the point $\mathbf{g}(\mathbf{y}_0)$ and that $\mathbf{g}(\mathbf{y}_0) \in E^\circ$. Then the composite function $\mathbf{f} \circ \mathbf{g}$ is differentiable at $\mathbf{y}_0$ and*

$$J_{\mathbf{f} \circ \mathbf{g}}(\mathbf{y}_0) = J_{\mathbf{f}}(\mathbf{g}(\mathbf{y}_0)) J_{\mathbf{g}}(\mathbf{x}_0).$$

Definition 68 *Given an open set $U \subseteq \mathbb{R}^N$ and a function $\mathbf{f} : U \to \mathbb{R}^M$, we say that $\mathbf{f}$ is of class C^m for some nonnegative integer $m \in \mathbb{N}_0$, if all its components f_i, $i = 1, \ldots, M$, are of class C^m. The space of all functions $\mathbf{f} : U \to \mathbb{R}^M$ of class C^m is denoted $C^m(U; \mathbb{R}^M)$. We set $C^\infty(U; \mathbb{R}^M) := \bigcap_{m=0}^{\infty} C^m(U; \mathbb{R}^M)$.*

Friday, February 10, 2012

5 Implicit and Inverse Function

Given a function f of two variables $(x, y) \in \mathbb{R}^2$, consider the equation

$$f(x, y) = 0.$$

We want to solve for y, that is, we are interested in finding a function $y = g(x)$ such that

$$f(x, g(x)) = 0.$$

We will see under which conditions we can do this. The result is going to be local.

In what follows given $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{y} \in \mathbb{R}^M$ and $\mathbf{f}(\mathbf{x}, \mathbf{y})$, we write

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_1}{\partial x_N}(\mathbf{x}, \mathbf{y}) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_M}{\partial x_N}(\mathbf{x}, \mathbf{y}) \end{pmatrix}$$

and

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_1}{\partial y_M}(\mathbf{x}, \mathbf{y}) \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial y_1}(\mathbf{x}, \mathbf{y}) & \cdots & \frac{\partial f_M}{\partial y_M}(\mathbf{x}, \mathbf{y}) \end{pmatrix}.$$

Theorem 69 (Implicit Function) *Let $U \subseteq \mathbb{R}^N \times \mathbb{R}^M$ be open, let $\mathbf{f} : U \to \mathbb{R}^M$, and let $(\mathbf{a}, \mathbf{b}) \in U$. Assume that $\mathbf{f} \in C^m(U)$ for some $m \in \mathbb{N}$, that*

$$\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0} \quad \text{and} \quad \det \frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b}) \neq 0.$$

Then there exists $B_N(\mathbf{a}, r_0) \subset \mathbb{R}^N$ *and* $B_M(\mathbf{b}, r_1) \subset \mathbb{R}^M$ *such that* $B_N(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1) \subseteq U$ *and for every* $\mathbf{x} \in B_N(\mathbf{a}, r_0)$ *there exists a unique* $\mathbf{y} \in B_M(\mathbf{b}, r_1)$ *such that* $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$*. Moreover the function*

$$\begin{aligned} \mathbf{g} : B_N(\mathbf{a}, r_0) &\to B_M(\mathbf{b}, r_1) \\ \mathbf{x} &\mapsto \mathbf{y} \end{aligned}$$

satisfies $\mathbf{g}(\mathbf{a}) = \mathbf{b}$*, belongs to* $C^m(B_N(\mathbf{a}, r_0))$ *and*

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x}) = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{g}(\mathbf{x}))\right)^{-1} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{g}(\mathbf{x})).$$

Proof. We present a proof in the case $N = M = 1$.
Step 1: Existence of g. Since $\frac{\partial f}{\partial y}(a, b) \neq 0$, without loss of generality, we can assume that $\frac{\partial f}{\partial y}(a, b) > 0$ (the case $\frac{\partial f}{\partial y}(a, b) < 0$ is similar). Using the fact that $\frac{\partial f}{\partial y}$ is continuous at (a, b), we can find $r > 0$ such that

$$R := [a - r, a + r] \times [b - r, b + r] \subseteq U$$

and

$$\frac{\partial f}{\partial y}(x, y) > 0 \quad \text{for all } (x, y) \in R.$$

Consider the function $h(y) := f(a, y)$, $y \in [b - r, b + r]$. Since

$$h'(y) = \frac{\partial f}{\partial y}(a, y) > 0 \quad \text{for all } y \in [b - r, b + r],$$

we have that h is strictly increasing. Using the fact that $h(b) = f(a, b) = 0$, it follows that

$$0 > h(b - r) = f(a, b - r), \quad 0 < h(b + r) = f(a, b + r).$$

Consider the function $k_1(x) := f(x, b - r)$, $x \in [a - r, a + r]$. Since $k_1(a) < 0$ and k_1 is continuous at a, there exists $0 < \delta_1 < r$ such that

$$0 > k_1(x) = f(x, b - r) \quad \text{for all } x \in (a - \delta_1, a + \delta_1).$$

Similarly, consider the function $k_2(x) := f(x, b + r)$, $x \in [a - r, a + r]$. Since $k_2(a) > 0$ and k_2 is continuous at a, there exists $0 < \delta_2 < r$ such that

$$0 < k_2(x) = f(x, b + r) \quad \text{for all } x \in (a - \delta_2, a + \delta_2).$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Then for all $x \in (a - \delta, a + \delta)$,

$$f(x, b - r) < 0, \quad f(x, b + r) > 0.$$

Fix $x \in (a - \delta, a + \delta)$ and consider the function $k(y) := f(x, y)$, $y \in [b - r, b + r]$. Since

$$k'(y) = \frac{\partial f}{\partial y}(x, y) > 0 \quad \text{for all } y \in [b - r, b + r],$$

we have that k is strictly increasing. Using the fact that $k(b-r) = f(x, b-r) < 0$ and $k(b+r) = f(x, b+r) > 0$, it follows that there exists a unique $y \in (b-r, b+r)$ (depending on x) such that $0 = k(y) = f(x, y)$.

Thus, we have shown that for every $x \in (a-\delta, a+\delta)$ there exists a unique $y \in (b-r, b+r)$ depending on x such that $f(x, y) = 0$. We define $g(x) := y$.

Step 2: Continuity of g. Fix $x_0 \in (a-\delta, a+\delta)$. Note that $b-r < g(x_0) < b+r$. Let $\varepsilon > 0$ be so small that

$$b - r < g(x_0) - \varepsilon < g(x_0) < g(x_0) + \varepsilon < b + r.$$

Consider the function $j(y) := f(x_0, y)$, $y \in [b-r, b+r]$. Since

$$j'(y) = \frac{\partial f}{\partial y}(x_0, y) > 0 \quad \text{for all } y \in [b-r, b+r],$$

we have that j is strictly increasing. Using the fact that $j(g(x_0)) = f(x_0, g(x_0)) = 0$, it follows that

$$f(x_0, g(x_0) - \varepsilon) < 0, \quad f(x_0, g(x_0) + \varepsilon) > 0.$$

Consider the function $j_1(x) := f(x, g(x_0) - \varepsilon)$, $x \in (a-\delta, a+\delta)$. Since $j_1(x_0) < 0$ and j_1 is continuous at x_0, there exists $0 < \eta_1 < \delta$ such that

$$0 > j_1(x) = f(x, g(x_0) - \varepsilon) \quad \text{for all } x \in (x_0 - \eta_1, x_0 + \eta_1).$$

Similarly, consider the function $j_2(x) := f(x, g(x_0) + \varepsilon)$, $x \in (a-\delta, a+\delta)$. Since $j_2(x_0) > 0$ and j_2 is continuous at x_0, there exists $0 < \eta_2 < \delta$ such that

$$0 < j_2(x) = f(x, g(x_0) + \varepsilon) \quad \text{for all } x \in (x_0 - \eta_2, x_0 + \eta_2).$$

Let $\eta := \min\{\eta_1, \eta_2\}$. Then for all $x \in (a-\eta, a+\eta)$,

$$f(x, g(x_0) - \varepsilon) < 0, \quad f(x, g(x_0) + \varepsilon) > 0.$$

But $f(x, g(x)) = 0$ and $y \in [b-r, b+r] \mapsto f(x, y)$ is strictly increasing. It follows that

$$g(x_0) - \varepsilon < g(x) < g(x_0) + \varepsilon$$

and so g is continuous at x_0.

Step 3: Differentiability of g. Fix $x_0 \in (a-\delta, a+\delta)$. Consider the segment joining $(x, g(x))$ and $(x_0, g(x_0))$,

$$S = \{t(x, g(x)) + (1-t)(x_0, g(x_0)) : t \in [0, 1]\}.$$

By the mean value theorem there exists $\theta \in (0, 1)$ such that

$$\begin{aligned} 0 = f(x, g(x)) - f(x_0, g(x_0)) &= \frac{\partial f}{\partial x}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))(x - x_0) \\ &+ \frac{\partial f}{\partial y}(\theta x + (1-\theta)x_0, \theta g(x) + (1-\theta) g(x_0))(g(x) - g(x_0)). \end{aligned}$$

Hence,

$$\frac{g(x)-g(x_0)}{x-x_0}=-\frac{\frac{\partial f}{\partial x}(\theta x+(1-\theta)x_0,\theta g(x)+(1-\theta)g(x_0))}{\frac{\partial f}{\partial y}(\theta x+(1-\theta)x_0,\theta g(x)+(1-\theta)g(x_0))}$$

letting $x\to x_0$ and using the continuity of g and of $\frac{\partial f}{\partial x}$ and of $\frac{\partial f}{\partial y}$, we get

$$\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}=-\frac{\frac{\partial f}{\partial x}(x_0,g(x_0))}{\frac{\partial f}{\partial y}(x_0,g(x_0))}$$

and so

$$g'(x_0)=-\frac{\frac{\partial f}{\partial x}(x_0,g(x_0))}{\frac{\partial f}{\partial y}(x_0,g(x_0))}.$$

Since the right-hand side is continuous, it follows that g' is continuous. Thus g is of class C^1. ■

The next examples show that when $\det\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{a},\mathbf{b})=\mathbf{0}$, then anything can happen.

Example 70 *In all these examples $N=M=1$ and $\frac{\partial f}{\partial y}(x_0,y_0)=0$.*

(i) Consider the function

$$f(x,y):=(y-x)^2.$$

Then $f(0,0)=0$, $\frac{\partial f}{\partial y}(0,0)=0$ and $g(x)=x$ satisfies $f(x,g(x))=0$.

(ii) Consider the function

$$f(x,y):=x^2+y^2.$$

Then $f(0,0)=0$, $\frac{\partial f}{\partial y}(0,0)=0$ but there is no function g defined near $x=0$ such that $f(x,g(x))=0$.

(iii) Consider the function

$$f(x,y):=(xy-1)\left(x^2+y^2\right).$$

Then $f(0,0)=0$, $\frac{\partial f}{\partial y}(0,0)=0$ but

$$g(x)=\begin{cases}0 & \text{if } x=0,\\ \frac{1}{x} & \text{if } x\neq 0,\end{cases}$$

which is discontinuous.

Monday, February 13, 2012

Next we give some examples on how to apply the implicit function theorem.

Example 71 *Consider the function*

$$f(x,y) := xe^y - y.$$

Then $f(0,0) = 0$, $\frac{\partial f}{\partial y}(x,y) = xe^y - 1$ *so that* $\frac{\partial f}{\partial y}(0,0) = -1$. *By the implicit function theorem there exist* $r > 0$ *and a function* $g \in C^\infty((-r,r))$ *such that* $g(0) = 0$ *and* $f(x, g(x)) = 0$. *To find the behavior of* g *near* $x = 0$, *we can use Taylor's formula. Let's find* $g'(0)$ *and* $g''(0)$. *We have*

$$xe^{g(x)} - g(x) = 0.$$

Hence, differentiating twice

$$1e^{g(x)} + xg'(x)e^{g(x)} - g'(x) = 0,$$
$$e^{g(x)}g'(x) + g'(x)e^{g(x)} + xg''(x)e^{g(x)} + x(g'(x))^2 e^{g(x)} - g''(x) = 0.$$

Substituting $x = 0$ *and using* $g(0) = 0$ *we get*

$$1e^0 + 0 - g'(0) = 0,$$
$$e^0 g'(0) + g'(0)e^0 + 0 + 0 - g''(0) = 0.$$

So that $g'(0) = 1$ *and* $g''(0) = 2$. *Hence,*

$$\begin{aligned} g(x) &= g(0) + g'(0)(x-0) + \frac{1}{2}g''(0)(x-0)^2 + o\left((x-0)^2\right) \\ &= 0 + 1(x-0) + \frac{1}{2}2(x-0)^2 + o\left((x-0)^2\right). \end{aligned}$$

Example 72 *Consider the function*

$$\mathbf{f}(x,y,z) = \left(y\cos(xz) - x^2 + 1, y\sin(xz) - x\right).$$

Let's prove that there exist $r > 0$ *and* $\mathbf{g} : (1-r, 1+r) \to \mathbb{R}^2$ *of class* C^∞ *such that* $\mathbf{g}(1) = \left(1, \frac{\pi}{2}\right)$ *and* $\mathbf{f}(x, \mathbf{g}(x)) = 0$. *Note that* $\mathbf{f}$ *is of class* C^∞. *Here the point is* $\left(1, 1, \frac{\pi}{2}\right)$ *and*

$$\mathbf{f}\left(1, 1, \frac{\pi}{2}\right) = \left(1\cos\left(1\frac{\pi}{2}\right) - 1 + 1, 1\sin\left(1\frac{\pi}{2}\right) - 1\right) = (0,0).$$

Moreover,

$$\begin{aligned} \frac{\partial \mathbf{f}}{\partial(y,z)}(x,y,z) &= \begin{pmatrix} \frac{\partial f_1}{\partial y}(x,y,z) & \frac{\partial f_1}{\partial z}(x,y,z) \\ \frac{\partial f_2}{\partial y}(x,y,z) & \frac{\partial f_1}{\partial z}(x,y,z) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial y}\left(y\cos(xz) - x^2 + 1\right) & \frac{\partial}{\partial z}\left(y\cos(xz) - x^2 + 1\right) \\ \frac{\partial}{\partial y}\left(y\sin(xz) - x\right) & \frac{\partial}{\partial z}\left(y\sin(xz) - x\right) \end{pmatrix} \\ &= \begin{pmatrix} 1\cos(xz) - 0 - 0 & -xy\sin(xz) - 0 - 0 \\ 1\sin(xz) - 0 & xy\cos(xz) - 0 \end{pmatrix} \end{aligned}$$

and so

$$\det \frac{\partial \mathbf{f}}{\partial (y,z)} \left(1,1,\frac{\pi}{2}\right) = \det \begin{pmatrix} \cos\left(1\frac{\pi}{2}\right) & -1\sin\left(1\frac{\pi}{2}\right) \\ \sin\left(1\frac{\pi}{2}\right) & 1\cos\left(1\frac{\pi}{2}\right) \end{pmatrix} = 1 \neq 0.$$

Hence, by the implicit function theorem there exist $r > 0$ *and* $\mathbf{g} : (1-r, 1+r) \to \mathbb{R}^2$ *of class* C^∞ *such that* $\mathbf{g}(1) = \left(1, \frac{\pi}{2}\right)$ *and* $\mathbf{f}(x, \mathbf{g}(x)) = 0$ *for all* $x \in (1-r, 1+r)$*, that is,*

$$\begin{cases} g_1(x)\cos(xg_2(x)) - x^2 + 1 = 0, \\ g_1(x)\sin(xg_2(x)) - x = 0. \end{cases}$$

Reasoning as before, we can use Taylor's formula to find the behavior of g_1 *and* g_2 *near* $x = 1$*, that is,*

$$g_1(x) = g_1(1) + g_1'(1)(x-1) + o((x-1)),$$
$$g_2(x) = g_2(1) + g_2'(1)(x-1) + o((x-1)).$$

Let's differentiate the two equations. We get

$$\begin{cases} g_1'(x)\cos(xg_2(x)) - g_1(x)(1g_2(x) + xg_2'(x))\sin(xg_2(x)) - 2x + 0 = 0, \\ g_1'(x)\sin(xg_2(x)) + g_1(x)(1g_2(x) + xg_2'(x))\cos(xg_2(x)) - 1 = 0. \end{cases}$$

Taking $x = 1$ *and using the fact that* $g_1(1) = 1$ *and* $g_2(1) = \frac{\pi}{2}$*, we obtain*

$$\begin{cases} g_1'(1)\cos\left(1\frac{\pi}{2}\right) - 1\left(\frac{\pi}{2} + 1g_2'(1)\right)\sin\left(1\frac{\pi}{2}\right) - 2 = 0, \\ g_1'(1)\sin\left(1\frac{\pi}{2}\right) + 1\left(\frac{\pi}{2} + 1g_2'(1)\right)\cos\left(1\frac{\pi}{2}\right) - 1 = 0, \end{cases}$$

that is,

$$\begin{cases} 0 - 1\left(\frac{\pi}{2} + g_2'(1)\right)1 - 2 = 0, \\ g_1'(1)\,1 + 0 - 1 = 0, \end{cases}$$

and so $g_1'(1) = 1$ *and* $g_2'(1) = -2 - \frac{\pi}{2}$*. Hence,*

$$g_1(x) = 1 + 1(x-1) + o((x-1)),$$
$$g_2(x) = \frac{\pi}{2} + \left(-2 - \frac{\pi}{2}\right)(x-1) + o((x-1)).$$

Next we prove the inverse function theorem.

Theorem 73 (Inverse Function) *Let* $U \subseteq \mathbb{R}^N$ *be open, let* $\mathbf{f} : U \to \mathbb{R}^N$*, and let* $\mathbf{a} \in U$*. Assume that* $\mathbf{f} \in C^m(U)$ *for some* $m \in \mathbb{N}$ *and that*

$$\det J_{\mathbf{f}}(\mathbf{a}) \neq 0.$$

Then there exists $B(\mathbf{a}, r_0) \subseteq U$ *such that* $\mathbf{f}(B(\mathbf{a}, r_0))$ *is open, the function*

$$\mathbf{f} : B(\mathbf{a}, r_0) \to \mathbf{f}(B(\mathbf{a}, r_0))$$

is invertible and $\mathbf{f}^{-1} \in C^m(\mathbf{f}(B(\mathbf{a}, r_0)))$*. Moreover,*

$$J_{\mathbf{f}^{-1}}(\mathbf{y}) = \left(J_{\mathbf{f}}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right)^{-1}.$$

Proof. We apply the implicit function theorem to the function $h : U \times \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$h(\mathbf{x}, \mathbf{y}) := \mathbf{f}(\mathbf{x}) - \mathbf{y}.$$

Let $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then $h(\mathbf{a}, \mathbf{b}) = 0$ and

$$\det \frac{\partial h}{\partial \mathbf{x}} (\mathbf{a}, \mathbf{b}) = \det J_{\mathbf{f}} (\mathbf{a}) \neq 0.$$

Hence, by the implicit function theorem there exists $B(\mathbf{a}, r_0) \subset \mathbb{R}^N$ and $B(\mathbf{b}, r_1) \subset \mathbb{R}^N$ such that $B(\mathbf{a}, r_0) \times B(\mathbf{b}, r_1) \subseteq U \times \mathbb{R}^N$ and a function $\mathbf{g} : B(\mathbf{b}, r_1) \to B(\mathbf{a}, r_0)$ of class C^m such that $h(\mathbf{g}(\mathbf{y}), \mathbf{y}) = 0$ for all $\mathbf{y} \in B(\mathbf{b}, r_1)$, that is,

$$\mathbf{f}(\mathbf{g}(\mathbf{y})) = \mathbf{y}$$

for all $\mathbf{y} \in B(\mathbf{b}, r_1)$. This implies that $\mathbf{g} = \mathbf{f}^{-1}$. Moreover,

$$\frac{\partial \mathbf{g}}{\partial y_k} (\mathbf{y}) = - \left(\frac{\partial h}{\partial \mathbf{x}} (\mathbf{g}(\mathbf{y}), \mathbf{y}) \right)^{-1} \frac{\partial h}{\partial y_k} (\mathbf{g}(\mathbf{y}), \mathbf{y}),$$

that is,

$$\frac{\partial \mathbf{f}^{-1}}{\partial y_k} (\mathbf{y}) = \left(J_{\mathbf{f}} \left(\mathbf{f}^{-1} (\mathbf{y}) \right) \right)^{-1} e_k.$$

■

The next exercise shows that differentiability is not enough for the inverse function theorem.

Exercise 74 *Consider the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$f_1(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \end{cases}$$
$$f_2(x, y) = y.$$

Prove that $\mathbf{f} = (f_1, f_2)$ *is differentiable in* $(0,0)$ *and* $J_{\mathbf{f}}(0,0) = 1$. *Prove that* $\mathbf{f}$ *is not one-to-one in any neighborhood of* $(0,0)$.

The next exercise shows that the existence of a local inverse at every point does not imply the existence of a global inverse.

Exercise 75 *Consider the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x, y) = (e^x \cos y, e^x \sin y).$$

Prove that $\det J_{\mathbf{f}}(x, y) \neq 0$ *for all* $(x, y) \in \mathbb{R}^2$ *but that* $\mathbf{f}$ *is not injective.*

Wednesday, February 15, 2012

Next we give a second proof of the inverse function theorem, which has the adavntage that it can be extended to the case in which $\mathbb{R}^N$ is replaced by a normed space.

Theorem 76 (Inverse Function) *Let $U \subseteq \mathbb{R}^N$ be open, let $\mathbf{f} : U \to \mathbb{R}^N$, and let $\mathbf{a} \in U$. Assume that $\mathbf{f}$ is continuous and that for all $\mathbf{x} \in U$ there exist $\frac{\partial f_j}{\partial x_i}$, $i, j = 1, \dots, N$, and that they are continuous at $\mathbf{a}$. If*

$$\det J_{\mathbf{f}}(\mathbf{a}) \neq 0,$$

then there exist $r_0 > 0$ and $r_1 > 0$ such that the function

$$\mathbf{f} : B(\mathbf{a}, r_0) \to \mathbf{f}(B(\mathbf{a}, r_0))$$

is invertible, $\mathbf{f}(B(\mathbf{a}, r_0))$ is open, and $\mathbf{f}^{-1} : \mathbf{f}(B(\mathbf{a}, r_0)) \to B(\mathbf{a}, r_0)$ is Lipschitz continuous in $\mathbf{f}(B(\mathbf{a}, r_0))$ and differentiable at $\mathbf{f}(\mathbf{a})$, with

$$J_{\mathbf{f}^{-1}}(\mathbf{f}(\mathbf{a})) = (J_{\mathbf{f}}(\mathbf{a}))^{-1}.$$

Proof. Step 1: Assume that $\mathbf{a} = \mathbf{0}$, that $f(\mathbf{0})$ =0, and that $J_{\mathbf{f}}(\mathbf{0}) = I_N$, the identity matrix. Write

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} + \mathbf{h}(\mathbf{x}).$$

Then $h(\mathbf{0})$ =0 and $J_{\mathbf{h}}(\mathbf{0}) = 0_N$. Since $\frac{\partial h_j}{\partial x_i}$ are continuous at $\mathbf{0}$, there exists $r_0 > 0$ such that $\overline{B(\mathbf{0}, r_0)} \subset U$ and

$$\|J_{\mathbf{h}}(\mathbf{x})\| < \frac{1}{2} \quad \text{for all } \mathbf{x} \in \overline{B(\mathbf{0}, r_0)}.$$

By the mean value theorem (applied how?) for all $\mathbf{x}_1, \mathbf{x}_2 \in \overline{B(\mathbf{0}, r_0)}$ and $j = 1, \dots, N$,

$$|h_j(\mathbf{x}_1) - h_j(\mathbf{x}_2)| \leq \|\mathbf{x}_1 - \mathbf{x}_2\| \sum_{i=1}^{N} \left| \frac{\partial h_j}{\partial x_i}(\mathbf{z}_{i,j}) \right|,$$

where $\|\mathbf{z}_{i,j} - \mathbf{x}_2\| \leq \|\mathbf{x}_1 - \mathbf{x}_2\|$, and so

$$\begin{aligned} &\|\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2)\| \\ &\leq \|\mathbf{x}_1 - \mathbf{x}_2\| \sqrt{\sum_{j=1}^{N} \left(\sum_{i=1}^{N} \left| \frac{\partial h_j}{\partial x_i}(\mathbf{z}_{i,j}) \right| \right)^2} < \frac{1}{2} \|\mathbf{x}_1 - \mathbf{x}_2\|. \end{aligned}$$

This shows that $\mathbf{h} : \overline{B(\mathbf{0}, r_0)} \to \mathbb{R}^N$ is a contraction. Moreover, since $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $\|\mathbf{h}(\mathbf{x})\| \leq \frac{1}{2}\|\mathbf{x}\|$ for all $\mathbf{x} \in \overline{B(\mathbf{0}, r_0)}$.

Fix $\mathbf{y} \in B\left(\mathbf{0}, \frac{1}{2} r_0\right)$ and consider the function

$$\mathbf{h}_{\mathbf{y}}(\mathbf{x}) := \mathbf{y} - \mathbf{h}(\mathbf{x}).$$

Then $\mathbf{h}_{\mathbf{y}}$ is a contraction, and for all $\mathbf{x} \in \overline{B(\mathbf{0}, r_0)}$,

$$\|\mathbf{h}_{\mathbf{y}}(\mathbf{x})\| \leq \|\mathbf{h}(\mathbf{x})\| + \|\mathbf{y}\| \leq \frac{1}{2}\|\mathbf{x}\| + \|\mathbf{y}\| \leq \frac{1}{2} r_0 + \frac{1}{2} r_0.$$

Thus, we can apply the Banach fixed point theorem to $\mathbf{h}_{\mathbf{y}} : \overline{B(\mathbf{0}, r_0)} \to \overline{B(\mathbf{0}, r_0)}$ to conclude that $\mathbf{h}_{\mathbf{y}}$ has a unique fixed point $\mathbf{z} \in \overline{B(\mathbf{0}, r_0)}$, that is,

$$\mathbf{y} - \mathbf{h}(\mathbf{z}) = \mathbf{h}_{\mathbf{y}}(\mathbf{z}) = \mathbf{z}.$$

In turn,

$$\mathbf{f}(\mathbf{z}) = \mathbf{z} + \mathbf{h}(\mathbf{z}) = \mathbf{y}.$$

Hence, we proved that for every $\mathbf{y} \in \overline{B\left(\mathbf{0}, \frac{1}{2}r_0\right)}$ there exists a unique $\mathbf{z} \in \overline{B(\mathbf{0}, r_0)}$ such that $\mathbf{f}(\mathbf{z}) = \mathbf{y}$. This means that we can define $\mathbf{f}^{-1} : \overline{B\left(\mathbf{0}, \frac{1}{2}r_0\right)} \to \overline{B(\mathbf{0}, r_0)}$.

To prove that $\mathbf{f}^{-1}$ is continuous, let $\mathbf{y}_1, \mathbf{y}_1 \in \overline{B\left(\mathbf{0}, \frac{1}{2}r_0\right)}$ and define $\mathbf{x}_1 := \mathbf{f}^{-1}(\mathbf{y}_1)$ and $\mathbf{x}_2 := \mathbf{f}^{-1}(\mathbf{y}_1)$. Then

$$\begin{aligned}\mathbf{x}_1 + \mathbf{h}(\mathbf{x}_1) &= \mathbf{y}_1, \\ \mathbf{x}_2 + \mathbf{h}(\mathbf{x}_2) &= \mathbf{y}_2,\end{aligned}$$

and since $\|\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2)\| \le \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|$, we have

$$\|\mathbf{x}_1 - \mathbf{x}_2\| \le \|\mathbf{y}_1 - \mathbf{y}_2\| + \|\mathbf{h}(\mathbf{x}_1) - \mathbf{h}(\mathbf{x}_2)\| \le \|\mathbf{y}_1 - \mathbf{y}_2\| + \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|,$$

and so

$$\frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\| \le \|\mathbf{y}_1 - \mathbf{y}_2\|,$$

which shows that

$$\left\|\mathbf{f}^{-1}(\mathbf{y}_1) - \mathbf{f}^{-1}(\mathbf{y}_2)\right\| \le 2\|\mathbf{y}_1 - \mathbf{y}_2\|.$$

Thus, $\mathbf{f}^{-1}$ is Lipschitz continuous. Taking $V = B\left(\mathbf{0}, \frac{1}{2}r_0\right)$ and $U_1 := B\left(\mathbf{0}, \frac{1}{2}r_0\right) \cap \mathbf{f}^{-1}\left(B\left(\mathbf{0}, \frac{1}{2}r_0\right)\right)$, we have that $\mathbf{f} : U_1 \to V$ is a homeomorphism.

To prove that $\mathbf{f}^{-1}$ is differentiable at $\mathbf{0}$, let $\mathbf{y} \in \overline{B\left(\mathbf{0}, \frac{1}{2}r_0\right)}$ and define $\mathbf{x} := \mathbf{f}^{-1}(\mathbf{y})$. Then

$$\mathbf{x} + \mathbf{h}(\mathbf{x}) = \mathbf{y},$$

and so

$$\mathbf{f}^{-1}(\mathbf{y}) = \mathbf{y} - \mathbf{h}\left(\mathbf{f}^{-1}(\mathbf{y})\right).$$

Since $\mathbf{h}$ is of class C^1, $h(\mathbf{0}) = 0$ and $J_{\mathbf{h}}(\mathbf{0}) = 0_N$, we have that $\mathbf{h}(\mathbf{z}) = o(\mathbf{z})$. But since $\mathbf{f}^{-1}$ is Lipschitz continuous, it follows that $\mathbf{h}\left(\mathbf{f}^{-1}(\mathbf{y})\right) = o(\mathbf{y})$. Hence, $\mathbf{f}^{-1}$ is differentiable at $\mathbf{0}$ with $J_{\mathbf{f}^{-1}}(\mathbf{0}) = I_N$.

Step 2: Up to a translation we may assume that $\mathbf{a} = \mathbf{0}$ and that $f(\mathbf{0}) = 0$. Moreover, if $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^N$ is invertible, then to prove that $\mathbf{f}$ is locally invertible, it is enough to prove that $\mathbf{T} \circ \mathbf{f}$ is locally invertible. Since $\det J_{\mathbf{f}}(\mathbf{0}) \neq 0$, the matrix $J_{\mathbf{f}}(\mathbf{0})$ is invertible. In turn the linear function $\mathbf{T}(\mathbf{y}) = (J_{\mathbf{f}}(\mathbf{0}))^{-1}\mathbf{y}$ is invertible. ■

To extend the previous theorem to the case of normed spaces, we first need to extend the mean value theorem to this setting. The proof relies on the Hahn–Banach theorem.

Friday, February 17, 2012

Given two nonempty sets X, Y, a (binary) *relation* is a subset $\mathcal{R} \subseteq X \times Y$. Usually, we associate a symbol to it, say $*$, so that $x * y$ means that $(x, y) \in \mathcal{R}$.

A *partial ordering* on a nonempty set is a relation $\mathcal{R} \subseteq X \times X$, denoted $\leq$, such that

(i) $x \leq x$ for every $x \in X$; that is $(x, x) \in \mathcal{R}$ (reflexivity).

(ii) For all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$; that is, if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$ (antisymmetry).

(iii) For all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$; that is, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$ (transitivity).

The word "partial" means that given $x, y \in X$, in general we cannot always say that $x \leq y$ or $y \leq x$.

Example 77 *Let $X = \mathcal{P}(\mathbb{R}) = \{$all subsets of $\mathbb{R}\}$. Given $E, F \in X$, we say that $E \leq F$ if $E \subseteq F$. Then $\leq$ is a partial ordering, but given the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$, one is not contained into the other.*

Given a partially ordered set $(X, \leq)$, a *totally ordered set*, or *chain*, $E \subset X$ is a set with the property that for all $x, y \in X$, either $x \leq y$ or $y \leq x$ (or both).

In the previous example $E = \{\{1, 2, 3\}, \{1, 2\}, \{2\}\}$ is a chain.

Given a partially ordered set $(X, \leq)$, and a set $E \subset X$, an *upper bound of* E is an element $x \in X$ such that $y \leq x$ for all $y \in E$. A set E may not have any upper bounds. A *maximal element* of E is an element $x \in E$ such that if $x \leq y$ for some $y \in E$, then $x = y$. A set E may not have maximal elements or it may have maximal elements that are not upper bounds (it can happen that a maximal element cannot be compared with all the elements of E).

Proposition 78 (Zorn's lemma) *Given a partially ordered set $(X, \leq)$, if every totally ordered subset of X has an upper bound, then X has a maximal element.*

Theorem 79 (Hahn-Banach) *Let X be a vector space and let $f : X \to \mathbb{R}$ be convex. Let $X_1 \subset X$ be a subspace of X and let $L_1 : X_1 \to \mathbb{R}$ be a linear function such that*

$$f(x) \geq L_1(x)$$

for all $x \in X_1$. Then there exists a linear extension $L : X \to \mathbb{R}$ of L_1 such that

$$f(x) \geq L(x) \geq -f(-x) \quad \text{for all } x \in X.$$

Proof. Step 1: If $X_1 = X$, then there is nothing to prove. Thus let $y_0 \in X \setminus X_1$, with $y_0 \neq 0$ and consider the subspace Y given by the linear span of $X_1 \cup \{y_0\}$. If $y \in Y$, then $y = x + ty_0$ where $x \in X_1$ and $t \in \mathbb{R}$ (this decomposition is unique). Define

$$\hat{L}(x + ty_0) := L_1(x) + tc,$$

where $c \in \mathbb{R}$ has to be chosen appropriately. We claim that

$$\inf_{x \in X_1,\, t>0} \frac{f(x+ty_0) - L_1(x)}{t} \geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x-sy_0)}{s}, \tag{19}$$

or, equivalently,

$$\frac{f(x_1+ty_0) - L_1(x_1)}{t} \geq \frac{L_1(x_2) - f(x_2-sy_0)}{s}$$

for all $x_1,\, x_2 \in X_1$ and for all $s,\, t > 0$. This inequality may be rewritten as

$$sf(x_1+ty_0) + tf(x_2-sy_0) \geq L_1(tx_2+sx_1).$$

Since f is convex and L_1 is linear,

$$\begin{aligned}
&sf(x_1+ty_0) + tf(x_2-sy_0) \\
&\quad = (s+t)\left[\frac{s}{s+t} f(x_1+ty_0) + \frac{t}{s+t} f(x_2-sy_0)\right] \\
&\quad \geq (s+t)\, f\left(\frac{s}{s+t}(x_1+ty_0) + \frac{t}{s+t}(x_2-sy_0)\right) \\
&\quad = (s+t)\, f\left(\frac{s}{s+t}x_1 + \frac{t}{s+t}x_2\right) \geq (s+t)\, L_1\left(\frac{s}{s+t}x_1 + \frac{t}{s+t}x_2\right) \\
&\quad = L_1(sx_1+tx_2).
\end{aligned}$$

Hence (19) holds.

Given $x_1 \in X_1$ and $t_1,\, s_1 > 0$,

$$\begin{aligned}
\infty &> \frac{f(x_1+t_1y_0) - L_1(x_1)}{t_1} \geq \inf_{x \in X_1,\, t>0} \frac{f(x+ty_0) - L_1(x)}{t} \\
&\geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x-sy_0)}{s} \geq \frac{L_1(x_1) - f(x_1-s_1y_0)}{s_1} > -\infty,
\end{aligned}$$

which shows that the numbers in (19) are real and thus we can choose a real number $c \in \mathbb{R}$ with

$$\inf_{x \in X_1,\, t>0} \frac{f(x+ty_0) - L_1(x)}{t} \geq c \geq \sup_{x \in X_1,\, s>0} \frac{L_1(x) - f(x-sy_0)}{s}.$$

By the choice of c, we have that

$$f(x+ty_0) \geq L_1(x) + tc = \hat{L}(x+ty_0)$$

for all $x \in X_1$ and all $t \in \mathbb{R}$. Hence, we have extended L_1 as a linear function $\hat{L}$ to Y in such a way that $\hat{L} \leq f$ in Y.

Step 2: Now we use Zorn's lemma. We consider all extensions $\left(\tilde{L}, \tilde{X}\right)$ of (L_1, X_1), where $\tilde{X}$ is a subspace of X containing X_1, $\tilde{L} : \tilde{X} \to \mathbb{R}$ is linear, $\tilde{L} \leq f$ in $\tilde{X}$, and $\tilde{L} = L_1$ in X_1.

We introduce a partial order. We say that

$$(L', X') \preccurlyeq (L'', X'')$$

if $X' \subseteq X''$ and $L'' = L'$ in X'. Given a chain $\{(L_\alpha, X_\alpha)\}$, define

$$X_\infty := \bigcup_\alpha X_\alpha$$

and $L_\infty(x) := L_\alpha(x)$ if $x \in X_\alpha$. Then X_∞ is a subspace of X (by Proposition **??**), $L_\infty \le f$ in X_∞, $L_\infty = L_1$ in X_1, and $(L_\alpha, X_\alpha) \preccurlyeq (L_\infty, X_\infty)$. Hence, every chain has an upper bound. It follows by Zorn's lemma that there exists a maximal element (L, Y). If $Y \neq X$, then we can proceed as in Step 1 to extend L. This would contradict the maximality of (L, Y). Hence $Y = X$ and the proof is complete. ■

Corollary 80 *Let $(X, \|\cdot\|)$ be a normed space. Then for every $x_0 \in X \setminus \{0\}$ there exists $L : X \to \mathbb{R}$ linear such that*

$$L(x_0) = \|x_0\| \quad \text{and} \quad |L(x)| \le \|x\| \text{ for every } x \in X$$

Proof. Let $Y = \operatorname{span}\{x_0\}$ and define

$$L'(tx) = t\|x\|, \quad t \in \mathbb{R}.$$

Then L' is linear , $L'(x_0) = \|x_0\|$, and $|L'(x)| \le \|x\|$ for all $x \in Y$. Take $f(x) := \|x\|$, $x \in X$. By the Hahn-Banach theorem we can extend L' to a linear function $L : X \to \mathbb{R}$ such that $L(x) \le \|x\|$ for all $x \in Y$. ■

Using the previous corollary, we can extend the mean value theorem in the infinite-dimensional case.

Definition 81 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, let $E \subset X$, and let $f : E \to Y$. Given $x_0 \in E$ and $v \in X$, let L be the line through x_0 in the direction v, that is,*

$$L := \{x \in X : x = x_0 + tv, \, t \in \mathbb{R}\},$$

and assume that x_0 is an accumulation point of the set $E \cap L$. We say that f admits a directional derivative *at x_0 in the direction v if there exists*

$$\frac{\partial f}{\partial v}(x_0) := \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \in Y.$$

Theorem 82 (Mean Value Theorem) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, let $x, y \in X$, with $x \neq y$, let S be the segment of endpoints x and y, that is,*

$$S = \{tx + (1 - t)y : t \in [0, 1]\},$$

and let $f : S \to R$ be such that f is continuous in S and there exists the directional derivative $\frac{\partial f}{\partial v}(z)$ for all $z \in S$ except at most x and y, where $v := \frac{x-y}{\|x-y\|_X}$. Then

$$\|f(x) - f(y)\|_Y \le \sup_{w \in S} \left\| \frac{\partial f}{\partial v}(w) \right\|_Y \|x - y\|_X.$$

The proof is left as an exercise.

Monday, February 20, 2012

The next exercise shows that the existence of a local inverse at every point does not imply the existence of a global inverse.

Exercise 83 *Consider the function* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{f}(x, y) = (e^x \cos y, e^x \sin y).$$

Prove that $\det J_{\mathbf{f}}(x, y) \neq 0$ *for all* $(x, y) \in \mathbb{R}^2$ *but that* $\mathbf{f}$ *is not injective.*

6 Lagrange Multipliers

In Section 3 (see Theorem 56) we have seen how to find points of local minima and maxima of a function $f : E \to \mathbb{R}$ in the interior E° of E. Now we are ready to find points of local minima and maxima of a function $f : E \to \mathbb{R}$ on the boundary ∂E of E. We assume that the boundary of E has a special form, that is, it is given by a set of the form

$$\left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{g}(\mathbf{x}) = \mathbf{0}\right\}.$$

Definition 84 *Let* $f : E \to \mathbb{R}$*, where* $E \subseteq \mathbb{R}^N$*, let* $F \subseteq E$ *and let* $\mathbf{x}_0 \in F$*. We say that*

(i) f *attains a* constrained local minimum *at* $\mathbf{x}_0$ *if there exists* $r > 0$ *such that* $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ *for all* $\mathbf{x} \in F \cap B(\mathbf{x}_0, r)$*,*

(ii) f *attains a* constrained local maximum *at* $\mathbf{x}_0$ *if there exists* $r > 0$ *such that* $f(\mathbf{x}) \leq f(\mathbf{x}_0)$ *for all* $\mathbf{x} \in F \cap B(\mathbf{x}_0, r)$*.*

The set F is called the *constraint*.

Theorem 85 (Lagrange Multipliers) *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $f : U \to \mathbb{R}$ *be a function of class* C^1 *and let* $\mathbf{g} : U \to \mathbb{R}^M$ *be a class of function* C^1*, where* $M < N$*, and let*

$$F := \{\mathbf{x} \in U : \mathbf{g}(\mathbf{x}) = \mathbf{0}\}.$$

Let $\mathbf{x}_0 \in F$ *and assume that* f *attains a constrained local minimum (or maximum) at* $\mathbf{x}_0$*. If* $J_{\mathbf{g}}(\mathbf{x}_0)$ *has maximum rank* M*, then there exist* $\lambda_1, \ldots, \lambda_M \in \mathbb{R}$ *such that*

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_M \nabla g_M(\mathbf{x}_0).$$

Proof. Assume that f attains a constrained local minimum at $\mathbf{x}_0$ (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ for all $\mathbf{x} \in U \cap B(\mathbf{x}_0, r)$ such that $\mathbf{g}(\mathbf{x}) = \mathbf{0}$. By taking $r > 0$ smaller, and since U is open, we can assume that $B(\mathbf{x}_0, r) \subseteq U$ so that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) \text{ for all } \mathbf{x} \in B(\mathbf{x}_0, r) \text{ with } \mathbf{g}(\mathbf{x}) = \mathbf{0}. \tag{20}$$

Since $J_{\mathbf{g}}(\mathbf{x}_0)$ has maximum rank M, there exists a $M \times M$ submatrix which has determinant different from zero. By relabeling the coordinates, if necessary, we may assume that $\mathbf{x} = (\mathbf{z}, \mathbf{y}) \in \mathbb{R}^{N-M} \times \mathbb{R}^M$, $\mathbf{x}_0 = (\mathbf{a}, \mathbf{b})$ and

$$\det \frac{\partial \mathbf{g}}{\partial \mathbf{y}} (\mathbf{a}, \mathbf{b}) \neq 0.$$

By the implicit function theorem applied to the function $\mathbf{g} : B(\mathbf{x}_0, r) \to \mathbb{R}^M$ there exist $r_0 > 0$, $r_1 > 0$, and a function $\mathbf{h} : B_{N-M}(\mathbf{a}, r_0) \to B_M(\mathbf{b}, r_1)$ of class C^1 such that $B_{N-M}(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1) \subset B(\mathbf{x}_0, r)$, $\mathbf{h}(\mathbf{a}) = \mathbf{b}$, and

$$\mathbf{g}(\mathbf{z}, \mathbf{h}(\mathbf{z})) = \mathbf{0} \quad \text{for all } \mathbf{z} \in B_{N-M}(\mathbf{a}, r_0).$$

Consider the function $\mathbf{k} : B_{N-M}(\mathbf{a}, r_0) \to B_{N-M}(\mathbf{a}, r_0) \times B_M(\mathbf{b}, r_1)$ defined by

$$\mathbf{k}(\mathbf{z}) := (\mathbf{z}, \mathbf{h}(\mathbf{z})).$$

Then

$$J_{\mathbf{k}}(\mathbf{a}) = \begin{pmatrix} I_{N-M} \\ \nabla h_1(\mathbf{a}) \\ \vdots \\ \nabla h_M(\mathbf{a}) \end{pmatrix},$$

which has rank $N - M$. Moreover, since $\mathbf{g}(\mathbf{k}(\mathbf{z})) = \mathbf{0}$ for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$, by Theorem 61,

$$\begin{aligned} \mathbf{0} &= J_{\mathbf{g}}(\mathbf{x}_0) J_{\mathbf{k}}(\mathbf{a}) \\ &= \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_1}{\partial x_N}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial g_M}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix} \begin{pmatrix} \frac{\partial k_1}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_1}{\partial z_{N-M}}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial k_N}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_{N-M}}(\mathbf{a}) \end{pmatrix}. \end{aligned}$$

Considering the transpose of this expression, we get

$$\begin{aligned} \mathbf{0} &= (J_{\mathbf{k}}(\mathbf{a}))^T (J_{\mathbf{g}}(\mathbf{x}_0))^T \\ &= \begin{pmatrix} \frac{\partial k_1}{\partial z_1}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_1}(\mathbf{a}) \\ \vdots & & \vdots \\ \frac{\partial k_1}{\partial z_{N-M}}(\mathbf{a}) & \cdots & \frac{\partial k_N}{\partial z_{N-M}}(\mathbf{a}) \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_1}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial g_1}{\partial x_N}(\mathbf{x}_0) & \cdots & \frac{\partial g_M}{\partial x_N}(\mathbf{x}_0) \end{pmatrix}, \end{aligned}$$

which implies that the vectors $\nabla g_i(\mathbf{x}_0)$, $i = 1, \ldots, M$, belong to the kernel of the linear transformation $\mathbf{T} : \mathbb{R}^N \to \mathbb{R}^M$ defined by

$$\mathbf{T}(\mathbf{x}) := (J_{\mathbf{k}}(\mathbf{a}))^T \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^N.$$

Hence,

$$V := \operatorname{span}\{\nabla g_1(\mathbf{x}_0), \ldots, \nabla g_M(\mathbf{x}_0)\} \subseteq \ker \mathbf{T}.$$

But $\dim V = \operatorname{rank} J_{\mathbf{g}}(\mathbf{x}_0) = M = N - \operatorname{rank}(J_{\mathbf{k}}(\mathbf{a}))^T = \dim\ker \mathbf{T}$. Hence,

$$V = \ker \mathbf{T}.$$

Consider the function

$$p(\mathbf{z}) := f(\mathbf{k}(\mathbf{z})), \quad \mathbf{z} \in B_{N-M}(\mathbf{a}, r_0).$$

Since $\mathbf{g}(\mathbf{k}(\mathbf{z})) = \mathbf{0}$ for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$, it follows from (20) that

$$p(\mathbf{z}) = f(\mathbf{k}(\mathbf{z})) \geq f(\mathbf{x}_0) = f(\mathbf{k}(\mathbf{a})) = p(\mathbf{a})$$

for all $\mathbf{z} \in B_{N-M}(\mathbf{a}, r_0)$. Hence, the function p attains a local minimum at $\mathbf{a}$. It follows from Theorems 49 and 61 that

$$\mathbf{0} = J_p(\mathbf{a}) = \nabla f(\mathbf{x}_0) J_{\mathbf{k}}(\mathbf{a}).$$

Considering the transpose of this expression, we get

$$(J_{\mathbf{k}}(\mathbf{a}))^T (\nabla f(\mathbf{x}_0))^T = \mathbf{0},$$

which implies that the vector $\nabla f(\mathbf{x}_0)$, $i = 1, \ldots, M$ belong to the kernel of T, which coincides with V. It follows from the definition of V that there exist λ_1, $\ldots$, $\lambda_M \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g_1(\mathbf{x}_0) + \cdots + \lambda_m \nabla g_M(\mathbf{x}_0).$$

■

Wednesday, February 22, 2012

Example 86 *We want to find points of local minima and maxima of the function* $f(x, y, z) := x - y + 2z$ *over the set*

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 2z^2 \leq 2\right\}.$$

Let's find critical points of f *in the interior. Since* $\frac{\partial f}{\partial x}(x, y, z) = 1 \neq 0$, *there are no critical points in the interior. Thus, points of local minima and maxima, if they exist, must be found on the boundary of* E, *that is,*

$$\partial E = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 2z^2 = 2\right\}.$$

In other words, we are looking at a constrained problem. The constraint is given by $x^2 + y^2 + 2z^2 = 2$. *Define* $g(x, y, z) := x^2 + y^2 + 2z^2 - 2$. *In this case the Jacobian matrix of* g *coincides with the gradient, that is,*

$$J_g(x, y, z) = \nabla g(x, y, z) = (2x, 2y, 4z).$$

In this case the maximum rank is one, so it is enough to check that $(2x, 2y, 4z) \neq (0, 0, 0)$ *at points in the constraint* ∂E. *But* $(2x, 2y, 4z) = (0, 0, 0)$ *only at the origin and this point does not belong to the constraint.*

To find the Lagrange multipliers, let's consider the function

$$F(x,y,z) := f(x,y,z) + \lambda g(x,y,z).$$

Let's find the critical points of F which are on ∂E. We have

$$\begin{cases} \frac{\partial F}{\partial x}(x,y,z) = \frac{\partial}{\partial x}\left(x-y+2z+\lambda\left(x^2+y^2+2z^2-2\right)\right) = 1+2x\lambda = 0, \\ \frac{\partial F}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(x-y+2z+\lambda\left(x^2+y^2+2z^2-2\right)\right) = -1+2y\lambda = 0, \\ \frac{\partial F}{\partial z}(x,y,z) = \frac{\partial}{\partial z}\left(x-y+2z+\lambda\left(x^2+y^2+2z^2-2\right)\right) = 2+4z\lambda = 0, \\ g(x,y,z) = x^2+y^2+2z^2-2 = 0. \end{cases}$$

If $\lambda = 0$, we have no solution, while if $\lambda \neq 0$, we get

$$\begin{cases} x = -\frac{1}{2\lambda}, \\ y = \frac{1}{2\lambda}, \\ z = -\frac{1}{2\lambda}, \\ x^2+y^2+2z^2-2 = 0. \end{cases}$$

Substituting gives $\left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + 2\left(-\frac{1}{2\lambda}\right)^2 - 2 = 0$, that is, $2\lambda^2 - 1 = 0$, and so $\lambda = \pm\frac{1}{\sqrt{2}}$. Thus, the only two possible points are $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Now, since E is closed and bounded (why?), it is compact. Since f is continuous, it follows by the Weierstrass theorem that there exist the minimum and the maximum of f over E. Hence, $f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2}$ is the minimum and $f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2}$ is the maximum.

Exercise 87 *Given the set*

$$E = \left\{(x,y,z) \in \mathbb{R}^3 : \, x^2 - xy + y^2 - z^2 = 1, \, x^2 + y^2 = 1\right\},$$

find the points of E having minimal distance from the origin.

7 Curves

Let $I \subseteq \mathbb{R}$ be an interval and let $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ be a function. As the *parameter* t traverses I, $\boldsymbol{\varphi}(t)$ traverses a curve in $\mathbb{R}^N$. Rather than calling $\boldsymbol{\varphi}$ a curve, it is better to regard any vector function $\mathbf{g}$ obtained from $\boldsymbol{\varphi}$ by a suitable change of parameter as representing the same curve as $\boldsymbol{\varphi}$. Thus, one should define a curve as an equivalence class of equivalent parametric representations.

Definition 88 *Given two intervals $I, J \subseteq \mathbb{R}$ and two functions $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and $\boldsymbol{\psi} : J \to \mathbb{R}^N$, we say that they are* equivalent *if there exists a continuous, bijective function $h : I \to J$ such that*

$$\boldsymbol{\varphi}(t) = \boldsymbol{\psi}(h(t))$$

for all $t \in I$. We write $\boldsymbol{\varphi} \sim \boldsymbol{\psi}$ and we call $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ parametric representations *and the function h a* parameter change*.*

Note that in view of a theorem in the Real Analysis I notes, $h^{-1}: J \to I$ is also continuous.

Exercise 89 *Prove that* $\sim$ *is an equivalence relation.*

Definition 90 *A* curve $\boldsymbol{\gamma}$ *is an equivalence class of parametric representations.*

Given a curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi}: I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval, the *multiplicity* of a point $\mathbf{y} \in \mathbb{R}^N$ is the (possibly infinite) number of points $t \in I$ such that $\boldsymbol{\varphi}(t) = \mathbf{y}$. Since every parameter change $h: I \to J$ is bijective, the multiplicity of a point does not depend on the particular parametric representation. The *range* of $\boldsymbol{\gamma}$ is the set of points of $\mathbb{R}^N$ with positive multiplicity, that is, $\boldsymbol{\varphi}(I)$. A point in the range of $\boldsymbol{\gamma}$ with multiplicity one is called a *simple point.* If every point of the range is simple, then $\boldsymbol{\gamma}$ is called a *simple arc*.

If $I = [a, b]$ and $\boldsymbol{\varphi}(a) = \boldsymbol{\varphi}(b)$, then the curve $\boldsymbol{\gamma}$ is called a *closed curve.* A closed curve is called *simple* if every point of the range is simple, with the exception of $\boldsymbol{\varphi}(a)$, which has multiplicity two.

Example 91 *The two functions*

$$\begin{aligned} \boldsymbol{\varphi}(t) &:= (\cos t, \sin t), \quad t \in [0, 2\pi], \\ \boldsymbol{\psi}(s) &:= (\cos 2s, \sin 2s), \quad s \in [0, \pi], \end{aligned}$$

are equivalent, take $h(t) = \frac{t}{2}$, *while the function*

$$\mathbf{p}(r) := (\cos r, \sin r), \quad r \in [0, 4\pi],$$

is not equivalent to the previous two, since the first curve is simple and the second no. Note that the range is the same, the unit circle.

Example 92 *Consider the curve with parametric representations* $\boldsymbol{\varphi}: \mathbb{R} \to \mathbb{R}^2$, *given by*

$$\boldsymbol{\varphi}(t) := (t(t-1), t(t-1)(2t-1)), \quad t \in \mathbb{R}.$$

Let's try to sketch the range of the curve. Set $x(t) := t(t-1)$ *and* $y(t) := t(t-1)(2t-1)$. *We have* $x(t) \geq 0$ *for* $t(t-1) \geq 0$, *that is, when* $t \geq 1$ *or* $t \leq 0$, *while* $x'(t) = 1(t-1) + t(1-0) = 2t - 1 \geq 0$ *for* $t \geq \frac{1}{2}$. *Moreover,*

$$\begin{aligned} \lim_{t \to -\infty} x(t) &= \lim_{t \to -\infty} t(t-1) = -\infty(-\infty - 1) = \infty, \\ \lim_{t \to \infty} x(t) &= \lim_{t \to \infty} t(t-1) = \infty(\infty - 1) = \infty. \end{aligned}$$

On the other hand, $y(t) \geq 0$ *for* $t(t-1)(2t-1) \geq 0$, *that is, when* $0 \leq t \leq \frac{1}{2}$ *or* $t \geq 1$, *while* $y'(t) = 6t^2 - 6t + 1 \geq 0$ *for* $t \geq \frac{3+\sqrt{3}}{6}$ *or* $t \leq \frac{3-\sqrt{3}}{6}$. *Moreover,*

$$\begin{aligned} \lim_{t \to -\infty} y(t) &= \lim_{t \to -\infty} t(t-1)(2t-1) = -\infty, \\ \lim_{t \to \infty} y(t) &= \lim_{t \to \infty} t(t-1)(2t-1) = \infty. \end{aligned}$$

To draw the range of the curve in the xy plane, note that x increasing means that we are moving to the right, x decreasing to the left, y increasing means that we are moving upwards, y decreasing downwards.

Next we start discussing the regularity of a curve.

Definition 93 *The curve $\boldsymbol{\gamma}$ is said to be* continuous *if one (and so all) of its parametric representations is continuous.*

The next result shows that the class of continuous curves is somehow too large for our intuitive idea of curve. Indeed, we construct a continuous curve that fills the unit square in $\mathbb{R}^2$. The first example of this type was given by Peano in 1890.

Theorem 94 (Peano) *There exists a continuous function $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^2$ such that $\boldsymbol{\varphi}([0,1]) = [0,1] \times [0,1]$.*

Proof. For every $n \in \mathbb{N}$ divide the interval $[0,1]$ into 4^n closed intervals $I_{k,n}$, $k = 1, \ldots, 4^n$, of length $\frac{1}{4^n}$ and the square $[0,1]^2$ into 4^n closed squares $Q_{k,n}$, $k = 1, \ldots, 4^n$, of side length $\frac{1}{2^n}$. Construct a bijective correspondence between the 4^n intervals $I_{k,n}$ and the 4^n squares $Q_{k,n}$ in such a way that (see Figure **??**)

(i) to two adjacent intervals there correspond adjacent squares,

(ii) to the four intervals of length $\frac{1}{4^{n+1}}$ contained in some interval $I_{k,n}$ there correspond the four squares of side length $\frac{1}{2^{n+1}}$ contained in the square corresponding to $I_{k,n}$.

By relabeling the squares, if necessary, we will assume that the square $Q_{k,n}$ corresponds to the interval $I_{k,n}$. Let

$$\mathcal{F} := \{I_{k,n} : n \in \mathbb{N},\, k = 1, \ldots, 4^n\},$$
$$\mathcal{G} := \{Q_{k,n} : n \in \mathbb{N},\, k = 1, \ldots, 4^n\}.$$

By the axiom of continuity of the reals, if $\{J_j\} \subset \mathcal{F}$ is any infinite sequence of intervals such that $J_{j+1} \subset J_j$ for all $j \in \mathbb{N}$ and $\{R_j\} \subset \mathcal{G}$ is the corresponding sequence of squares, then there exist unique $t \in [0,1]$ and $(x,y) \in [0,1]^2$ such that

$$\{t\} = \bigcap_{j=1}^{\infty} J_j, \quad \{(x,y)\} = \bigcap_{j=1}^{\infty} R_j.$$

We set the point t and the point (x,y) in correspondence.

We claim that this correspondence defines a continuous function $\boldsymbol{\varphi} : [0,1] \to \mathbb{R}^2$ with all the desired properties. Indeed, a point $t \in [0,1]$ that is not an endpoint of any interval determines uniquely a sequence $\{J_j\} \subset \mathcal{F}$ to which it belongs and hence a point (x,y) belonging to the corresponding sequence $\{R_j\} \subset \mathcal{G}$. The same is true for $t = 0$ and $t = 1$. A point t that is common to two different intervals $I_{k_1,m}$ and $I_{k_2,m}$ for some $m \in \mathbb{N}$ is also common to two different intervals $I_{k,n}$, $k = 1, \ldots, 4^n$, for all $n \geq m$. Hence, it belongs to two different sequences $\{J_j\}$, $\{J'_j\} \subset \mathcal{F}$. Since the squares R_j and R'_j, corresponding to J_j and J'_j, respectively, are adjacent by property (i), it follows that

$$\bigcap_{j=1}^{\infty} R_j = \bigcap_{j=1}^{\infty} R'_j.$$

Thus, to every $t \in [0,1]$ there corresponds a unique $(x,y) \in [0,1]^2$ that we denote $u(t)$.

Since every $(x,y) \in [0,1]^2$ belongs to one, two, three, or four sequences $\{R_j\} \subset \mathcal{G}$, there exists one, two, three, or four $t \in [0,1]$ such that $\boldsymbol{\varphi}(t) = (x,y)$. Hence, $\boldsymbol{\varphi}([0,1]) = [0,1]^2$.

To prove that $\boldsymbol{\varphi}$ is continuous, write $\boldsymbol{\varphi}(t) = (x(t), y(t))$, $t \in [0,1]$. By conditions (i) and (ii) we have that

$$|x(t_1) - x(t_2)| \leq 2\frac{1}{2^n}, \quad |y(t_1) - y(t_2)| \leq 2\frac{1}{2^n}$$

for all $t_1, t_2 \in [0,1]$, with $|t_1 - t_2| \leq \frac{1}{4^n}$. This proves the uniform continuity of $\boldsymbol{\varphi}$. ■

In view of the previous result, to recover the intuitive idea of a curve, we restrict the class of continuous curves to those with finite length.

In what follows, given an interval $I \subseteq \mathbb{R}$, a *partition* of I is a finite set $P := \{t_0, \ldots, t_n\} \subset I$, where

$$t_0 < t_1 < \cdots < t_n.$$

Definition 95 *Let $I \subseteq \mathbb{R}$ be an interval and let $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ be a function. The* pointwise variation *of $\boldsymbol{\varphi}$ on the interval I is*

$$\operatorname{Var}_I \boldsymbol{\varphi} := \sup \left\{ \sum_{i=1}^{n} \|\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1})\| \right\}, \tag{21}$$

where the supremum is taken over all partitions $P := \{t_0, \ldots, t_n\}$ of I, $n \in \mathbb{N}$. A function $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ has finite *or* bounded pointwise variation *if $\operatorname{Var}_I \boldsymbol{\varphi} < \infty$.*

Exercise 96 *Prove that if two parametric representations $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and $\boldsymbol{\psi} : J \to \mathbb{R}^N$ are equivalent, then $\operatorname{Var}_I \boldsymbol{\varphi} = \operatorname{Var}_J \boldsymbol{\psi}$.*

We are now ready to define the length of a curve.

Definition 97 *Given a curve $\boldsymbol{\gamma}$, let $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ be a parametric representation of $\boldsymbol{\gamma}$, where $I \subseteq \mathbb{R}$ is an interval. We define the* length *of $\boldsymbol{\gamma}$ as*

$$L(\boldsymbol{\gamma}) := \operatorname{Var}_I \boldsymbol{\varphi}.$$

We say that the curve $\boldsymbol{\gamma}$ is rectifiable *if $L(\boldsymbol{\gamma}) < \infty$.*

In view of the previous exercise, the definition makes sense, that is, the length of the curve does not depend on the particular representation of the curve.

Friday, February 24, 2012

Peano's example shows that continuous curves in general are not rectifiable, that is, they may have infinite length. Next we show that C^1 or piecewise C^1 curves are rectifiable.

Definition 98 *Given an interval $I \subseteq \mathbb{R}$, a function $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is said to be of class C^1 in I if $\boldsymbol{\varphi}$ is differentiable in I and $\boldsymbol{\varphi}'$ is continuous.*

Definition 99 *Given a function $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$, we say that f is* piecewise *C^1 if there exists a partition $P = \{t_0, \ldots, t_n\}$ of $[a,b] \in I$, with $a = t_0 < t_1 < \cdots < t_n = b$ such that $\boldsymbol{\varphi}$ is of class C^1 in each interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n$.*

Note that at each t_i, $i = 1, \ldots, n-1$, the function $\boldsymbol{\varphi}$ has a right and a left derivative, but they might be different.

Example 100 *Consider the curve with parametric representations $\boldsymbol{\varphi} : [-1,1] \to \mathbb{R}^2$, given by*

$$\boldsymbol{\varphi}(t) := (t, |t|), \quad t \in [-1,1],$$

is not a curve of class C^1 since at $t = 0$ the second component of $\boldsymbol{\varphi}$ is not differentiable, but it is piecewise C^1, since

$$\boldsymbol{\varphi}(t) = (t, -t), \quad t \in [-1,0],$$
$$\boldsymbol{\varphi}(t) = (t, t), \quad t \in [0,1],$$

are both C^1 functions. Note that the left derivative at 0 is $\boldsymbol{\varphi}'_-(0) = (1,-1)$ while the right derivative is $\boldsymbol{\varphi}'_+(0) = (1,1)$. Hence, the curve has a corner at the point $\boldsymbol{\varphi}(0) = (0,0)$.

By requiring parametric representations, parameter changes and their inverses to be differentiable, or Lipschitz, or of class C^n, $n \in \mathbb{N}_0$, etc., or piecewise C^1, we may define curves $\boldsymbol{\gamma}$ that are differentiable, or Lipschitz, or of class C^n, $n \in \mathbb{N}_0$, or piecewise C^1, respectively.

Theorem 101 *Let $\boldsymbol{\gamma}$ be a curve of class C^1, with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$. Then*

$$\int_a^b \|\boldsymbol{\varphi}'(t)\| \, dt = L(\boldsymbol{\gamma}).$$

Before proving the theorem we need to define the integral of a function $\boldsymbol{\psi} : [c,d] \to \mathbb{R}^N$.

Definition 102 *Given $\mathbf{f} : [c,d] \to \mathbb{R}^N$, we say that $\mathbf{f}$ is* Riemann integrable *over $[c,d]$ if each component f_i, $i = 1, \ldots, N$, is Riemann integrable over $[c,d]$ and we set*

$$\int_c^d \mathbf{f}(t) \, dt := \left(\int_c^d f_1(t) \, dt, \ldots, \int_c^d f_N(t) \, dt \right).$$

Hence, $\int_c^d \mathbf{f}(t)\,dt$ is a vector in $\mathbb{R}^N$.

Lemma 103 *Assume that* $\mathbf{f} : [c,d] \to \mathbb{R}^N$ *is Riemann integrable. Then* $\|\mathbf{f}\| : [c,d] \to \mathbb{R}$ *is Riemann integrable over* $[c,d]$ *and*

$$\left\| \int_c^d \mathbf{f}(t)\,dt \right\| \le \int_c^d \|\mathbf{f}(t)\|\,dt. \tag{22}$$

Proof. Since products and sums of Riemann integrable functions are Riemann integrable, it follows that the function

$$h(t) := f_1^2(t) + \cdots + f_N^2(t)$$

is Riemann integrable over $[c,d]$. Using the fact that the square root is a continuous function, we have that

$$\|\mathbf{f}(t)\| = \sqrt{h(t)}$$

is still Riemann integrable over $[c,d]$ (why?).

It remains to prove the inequality (22). Set $\mathbf{x} := \int_c^d \mathbf{f}(t)\,dt$. Then by the linearity of the integral and Cauchy's inequality

$$\begin{aligned} \|\mathbf{x}\|^2 &= \sum_{i=1}^N x_i^2 = \sum_{i=1}^N x_i \int_c^d f_i(t)\,dt = \int_c^d \sum_{i=1}^N x_i f_i(t)\,dt \\ &\le \int_c^d \|\mathbf{x}\|\,\|\mathbf{f}(t)\|\,dt = \|\mathbf{x}\| \int_c^d \|\mathbf{f}(t)\|\,dt. \end{aligned}$$

If $\mathbf{x} = \mathbf{0}$, then there is nothing to prove. If $\mathbf{x} \ne \mathbf{0}$, then we can divide both sides of the previous inequality by $\|\mathbf{x}\|$, to obtain

$$\left\| \int_c^d \mathbf{f}(t)\,dt \right\| = \|\mathbf{x}\| \le \int_c^d \|\mathbf{f}(t)\|\,dt.$$

■

The opposite inequality does not hold in general, but we have the following lemma.

Lemma 104 *Assume that* $\mathbf{f} : [c,d] \to \mathbb{R}^N$ *is Riemann integrable. Then for* $t_0 \in [c,d]$,

$$\left\| \int_c^d \mathbf{f}(t)\,dt \right\| \ge \int_c^d \|\mathbf{f}(t)\|\,dt - 2\int_c^d \|\mathbf{f}(t) - \mathbf{f}(t_0)\|\,dt.$$

Proof. We have

$$\int_c^d \mathbf{f}(t)\,dt = (d-c)\,\mathbf{f}(t_0) + \int_c^d (\mathbf{f}(t) - \mathbf{f}(t_0))\,dt,$$

and so

$$\left\| \int_c^d \mathbf{f}(t)\, dt \right\| \geq (d-c) \|\mathbf{f}(t_0)\| - \left\| \int_c^d \mathbf{f}(t) - \mathbf{f}(t_0)\, dt \right\|$$
$$\geq \int_c^d \|\mathbf{f}(t_0)\|\, dt - \int_c^d \|\mathbf{f}(t) - \mathbf{f}(t_0)\|\, dt.$$

On the other hand, $\|\mathbf{f}(t_0)\| \geq \|\mathbf{f}(t)\| - \|\mathbf{f}(t) - \mathbf{f}(t_0)\|$, and so, upon integration,

$$\int_c^d \|\mathbf{f}(t_0)\|\, dt \geq \int_c^d \|\mathbf{f}(t)\|\, dt - \int_c^d \|\mathbf{f}(t) - \mathbf{f}(t_0)\|\, dt.$$

Combining the last two inequalities gives the desired result. ■

We now turn to the proof of Theorem 101.

Proof of Theorem 101. Given a partition $P := \{t_0, \ldots, t_n\}$, by the fundamental theorem of calculus and Lemma 103, we have

$$\sum_{i=1}^n \|\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1})\| = \sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} \boldsymbol{\varphi}'(t)\, dt \right\| \leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\boldsymbol{\varphi}'(t)\|\, dt = \int_a^b \|\boldsymbol{\varphi}'(t)\|\, dt$$

and taking the supremum over all partitions gives

$$L(\boldsymbol{\gamma}) \leq \int_a^b \|\boldsymbol{\varphi}'(t)\|\, dt.$$

To prove the other inequality fix $\varepsilon > 0$. Since $\boldsymbol{\varphi}'$ is uniformly continuous, there exists $\delta > 0$ such that $\|\boldsymbol{\varphi}'(t) - \boldsymbol{\varphi}'(s)\| \leq \varepsilon$ for all $t, ys \in [a, b]$ with $|t - s| \leq \delta$. Consider a partition $P := \{t_0, \ldots, t_n\}$ with $t_i - t_{i-1} \leq \delta$. Then by Lemma 104 in $[t_{i-1}, t_i]$,

$$\|\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1})\| = \left\| \int_{t_{i-1}}^{t_i} \boldsymbol{\varphi}'(t)\, dt \right\|$$
$$\geq \int_{t_{i-1}}^{t_i} \|\boldsymbol{\varphi}'(t)\|\, dt - 2\int_{t_{i-1}}^{t_i} \|\boldsymbol{\varphi}'(t) - \boldsymbol{\varphi}'(t_{i-1})\|\, dt$$
$$\geq \int_{t_{i-1}}^{t_i} \|\boldsymbol{\varphi}'(t)\|\, dt - 2\int_{t_{i-1}}^{t_i} \varepsilon\, dt,$$

and so

$$L(\boldsymbol{\gamma}) \geq \sum_{i=1}^n \|\boldsymbol{\varphi}(t_i) - \boldsymbol{\varphi}(t_{i-1})\| \geq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\boldsymbol{\varphi}'(t)\|\, dt - 2\int_{t_{i-1}}^{t_i} \varepsilon\, dt$$
$$= \int_a^b \|\boldsymbol{\varphi}'(t)\|\, dt - 2\varepsilon(b-a).$$

It suffices to let $\varepsilon \to 0^+$. ■

Remark 105 *The previous theorem continues to hold for piecewise C^1 curves.*

Exercise 106 *Let $f : [0,1] \to \mathbb{R}$ be the Cantor function and consider the curve $\boldsymbol{\gamma}$ of parametric representation*

$$\boldsymbol{\varphi}(t) := (t, f(t)).$$

Prove that $L(\boldsymbol{\gamma}) = 2$.

Remark 107 *Note that the derivative of the Cantor function f exists and is zero except on a set of Lebesgue measure zero, hence (here we are using the Lebesgue integral)*

$$\int_0^1 \|\boldsymbol{\varphi}'(t)\| \, dt = \int_0^1 \sqrt{1+0}\, dt = 1 < L(\boldsymbol{\gamma}) = \operatorname{Var} \boldsymbol{\varphi} = 2.$$

It turns out that the class of curves for which $\int_a^b \|\boldsymbol{\varphi}'(t)\| \, dt = L(\boldsymbol{\gamma})$ (again, using the Lebesgue integral instead of the Riemann integral) is given by absolutely continuous curves.

Definition 108 *Let $I \subset \mathbb{R}$ be an interval. A function $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is said to be* absolutely continuous *on I if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{k=1}^{\ell} \|\boldsymbol{\varphi}(b_k) - \boldsymbol{\varphi}(a_k)\| \le \varepsilon \tag{23}$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

The space of all absolutely continuous functions $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ is denoted by $AC\left(I; \mathbb{R}^N\right)$. When $N = 1$ we simply write $AC(I)$.

Remark 109 *Note that since ℓ is arbitrary, we can also take $\ell = \infty$, namely, replace finite sums by series.*

Exercise 110 *Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$.*

(i) Prove that f belongs to $AC(I)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{k=1}^{\ell} (f(b_k) - f(a_k)) \right| \le \varepsilon$$

for every finite number of nonoverlapping intervals (a_k, b_k), $k = 1, \ldots, \ell$, with $[a_k, b_k] \subset I$ and

$$\sum_{k=1}^{\ell} (b_k - a_k) \le \delta.$$

(ii) Assume that for every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| \sum_{k=1}^{\ell} \left(f\left(b_k\right) - f\left(a_k\right) \right) \right| \leq \varepsilon$$

for every finite number of intervals (a_k, b_k), $k = 1, \ldots, \ell$, *with* $[a_k, b_k] \subset I$ *and*

$$\sum_{k=1}^{\ell} (b_k - a_k) \leq \delta.$$

Prove that f *is locally Lipschitz.*

Part (ii) of the previous exercise shows that in Definition 108 we cannot remove the condition that the intervals (a_k, b_k) are pairwise disjoint.

By taking $\ell = 1$ in Definition 108, it follows that an absolutely continuous function $f : I \to \mathbb{R}$ is uniformly continuous. The next exercise shows that the converse is false.

Exercise 111 *Let* $f : (0, 1] \to \mathbb{R}$ *be defined by*

$$f(x) := x^a \sin \frac{1}{x^b},$$

where $a, b \in \mathbb{R}$. *Study to see for which* a, b *the function* f *is absolutely continuous. Prove that there exist* a, b *for which* f *is uniformly continuous but not absolutely continuous.*

Exercise 112 *Let* $I \subset \mathbb{R}$ *be an interval and let* $f : I \to \mathbb{R}$ *be differentiable with bounded derivative. Prove that* f *belongs to* $AC(I)$.

Exercise 113 *Let* $f, g \in AC([a, b])$. *Prove the following.*

(i) $f \pm g \in AC([a, b])$.

(ii) $fg \in AC([a, b])$.

(iii) If $g(x) > 0$ *for all* $x \in [a, b]$, *then* $\frac{f}{g} \in AC([a, b])$.

(iv) What happens if the interval $[a, b]$ *is replaced by an arbitrary interval* $I \subset \mathbb{R}$ *(possibly unbounded)?*

Exercise 114 *Let* $I \subset \mathbb{R}$ *be an interval and let* $f : I \to \mathbb{R}$ *be uniformly continuous.*

(i) Prove that f *may be extended uniquely to* $\overline{I}$ *in such a way that the extended function is still uniformly continuous.*

(ii) Prove that if f *belongs to* $AC(I)$, *then its extension belongs to* $AC\left(\overline{I}\right)$.

(iii) Prove that there exist $A, B > 0$ such that for all $x \in I$,

$$|f(x)| \leq A + B|x|.$$

The previous exercise shows that although an absolutely continuous function may be unbounded, it cannot grow faster than linear when $|x| \to \infty$.

Exercise 115 *Let $I \subseteq \mathbb{R}$ be an interval, let $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ be a function, and let $c \in I^\circ$. Prove that*

$$\operatorname{Var}_I \boldsymbol{\varphi} = \operatorname{Var}_{I\cap(-\infty,c]} \boldsymbol{\varphi} + \operatorname{Var}_{I\cap[c,\infty)} \boldsymbol{\varphi}.$$

Next we show that absolutely continuous functions have bounded variation.

Proposition 116 *Let $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ be absolutely continuous. Then $\boldsymbol{\varphi}$ has finite variation.*

Proof. Take $\varepsilon = 1$, and let $\delta > 0$ be as in Definition 108. Let n be the integer part of $\frac{2(b-a)}{\delta}$ and partition $[a, b]$ into n intervals $[t_{i-1}, t_i]$ of equal length $\frac{b-a}{n}$,

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Since $\frac{b-a}{n} \leq \delta$, in view of (23), on each interval $[t_{i-1}, t_i]$ we have that $\operatorname{Var}_{[t_{i-1},t_i]} \boldsymbol{\varphi} \leq 1$, and so by the previous exercise

$$\operatorname{Var}_{[a,b]} \boldsymbol{\varphi} = \sum_{i=1}^{n} \operatorname{Var}_{[t_{i-1},t_i]} \boldsymbol{\varphi} \leq n \leq \frac{2(b-a)}{\delta} < \infty,$$

where we have used the fact that $\frac{b-a}{n} \geq \frac{\delta}{2}$. ■

Monday, February 27, 2012

Solutions, first midterm :(

Wednesday, February 29, 2012

Given a curve $\boldsymbol{\gamma}$, with parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$, where $I \subseteq \mathbb{R}$ is an interval. If $\boldsymbol{\varphi}$ is differentiable at a point $t_0 \in I$ and $\boldsymbol{\varphi}'(t_0) \neq 0$, the straight line parametrized by

$$\boldsymbol{\psi}(t) := \boldsymbol{\varphi}(t_0) + \boldsymbol{\varphi}'(t_0)(t - t_0), \quad t \in \mathbb{R},$$

is called a *tangent line* to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\varphi}(t_0)$, and the vector $\boldsymbol{\varphi}'(t_0)$ is called a *tangent vector* to $\boldsymbol{\gamma}$ at the point $\boldsymbol{\varphi}(t_0)$. Note that if $\boldsymbol{\gamma}$ is a simple arc, then the tangent line at a point $\mathbf{y}$ of the range Γ of $\boldsymbol{\gamma}$, if it exists, is unique, while if the curve is self-intersecting at some point $\mathbf{y} \in \Gamma$, then there could be more than one tangent line at $\mathbf{y}$.

Given a function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, the graph of f is the range of a curve parametrized by

$$\boldsymbol{\varphi}(t) = (t, f(t)).$$

If f is differentiable at some point $t_0 \in I$, then the tangent vector is given by

$$\boldsymbol{\varphi}'(t_0) = (1, f'(t_0)).$$

Definition 117 *Given a rectifiable curve* $\boldsymbol{\gamma}$*, we say that* $\boldsymbol{\gamma}$ *is parametrized by* arclength *if it admits a parametrization* $\boldsymbol{\psi} : [0, L(\gamma)] \to \mathbb{R}^N$ *such that* $\boldsymbol{\psi}$ *is differentiable for all but finitely many* $\tau \in [0, L(\gamma)]$ *with* $\left\|\boldsymbol{\psi}'(\tau)\right\| = 1$ *for all but finitely many* τ*.*

The name arclength comes from the fact that if $\boldsymbol{\gamma}$ is piecewise C^1 then for every $\tau_1 < \tau_2$, by Theorem 101 we have that the length of the portion of the curve parametrized by $\boldsymbol{\psi} : [\tau_1, \tau_2] \to \mathbb{R}^N$ is given by

$$\int_{\tau_1}^{\tau_2} \left\|\boldsymbol{\psi}'(\tau)\right\| d\tau = \int_{\tau_1}^{\tau_2} 1 \, d\tau = \tau_2 - \tau_1.$$

Definition 118 *A curve* $\boldsymbol{\gamma}$ *is* regular *if is piecewise* C^1 *and it admits a parametric representation* $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ *for which the left and right derivative are different from zero for all* $t \in [a, b]$*.*

Theorem 119 *If* $\boldsymbol{\gamma}$ *is a regular curve, then* $\boldsymbol{\gamma}$ *can be parametrized by arclength.*

Proof. Let $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ be a piecewise C^1 parametric representation of $\boldsymbol{\gamma}$ and define the *length function*

$$s(t) := \int_a^t \|\boldsymbol{\varphi}'(r)\| \, dr.$$

Note that by by Theorem 101, $s(b) = L(\boldsymbol{\gamma})$, while $s(a) = 0$. Hence, $s : [a, b] \to [0, L(\gamma)]$ is onto. We claim that s is invertible. Indeed, if $t_1 < t_2$, then since $\|\boldsymbol{\varphi}'(r)\| > 0$ for all but finitely many t in $[t_1, t_2]$, we have that

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} \|\boldsymbol{\varphi}'(r)\| \, dr > 0$$

(why?). Hence, $s : [a, b] \to [0, L(\gamma)]$ is invertible. By a theorem in the Real Analysis notes, $s^{-1} : [0, L(\gamma)] \to [a, b]$ is continuous. By the fundamental theorem of calculus, $s'(t) = \|\boldsymbol{\varphi}'(t)\| > 0$ for all but finitely many t. Hence, s is piecewise C^1. In turn, by a theorem in the Real Analysis notes, for any such t,

$$\frac{d}{d\tau}\left(s^{-1}\right)(s(t)) = \frac{1}{s'(t)}.$$

Hence, there exists

$$\frac{d}{d\tau}\left(s^{-1}\right)(\tau) = \frac{1}{s'(s^{-1}(\tau))} = \frac{1}{\|\boldsymbol{\varphi}'(s^{-1}(\tau))\|} \tag{24}$$

for all but finitely many τ. Since $\boldsymbol{\varphi}$ is piecewise C^1 and the left and right derivatives of $\boldsymbol{\varphi}$ are always different from zero, it follows from (24) and the continuity of s^{-1} that s^{-1} is piecewise C^1. Thus, the function $\boldsymbol{\psi} : [0, L(\gamma)] \to \mathbb{R}^N$, defined by

$$\boldsymbol{\psi}(\tau) := \boldsymbol{\varphi}\left(s^{-1}(\tau)\right), \quad \tau \in [0, L(\gamma)], \tag{25}$$

is equivalent to $\boldsymbol{\varphi}$. Moreover, by the chain rule and (24), for all but finitely many τ,

$$\boldsymbol{\psi}'(\tau) = \boldsymbol{\varphi}'\left(s^{-1}(\tau)\right)\frac{d}{d\tau}\left(s^{-1}\right)(\tau) = \frac{\boldsymbol{\varphi}'\left(s^{-1}(\tau)\right)}{\|\boldsymbol{\varphi}'\left(s^{-1}(\tau)\right)\|}$$

and so $\left\|\boldsymbol{\psi}'(\tau)\right\| = 1$. ■

Example 120 *Consider the curve with parametric representations* $\boldsymbol{\varphi} : [0, 2\pi] \to \mathbb{R}^3$*, given by*

$$\boldsymbol{\varphi}(t) := (R\cos t, R\sin t, 2t), \quad t \in [0, 2\pi].$$

Since

$$\boldsymbol{\varphi}'(t) = (-R\sin t, R\cot st, 2) \neq (0, 0, 0),$$

the curve is regular. Let's parametrize it using arclength. Set

$$\begin{aligned} s(t) &:= \int_0^t \|\boldsymbol{\varphi}'(r)\|\, dr = \int_0^t \sqrt{R^2\sin^2 t + R^2\cos^2 t + 4}\, dr \\ &= \sqrt{R^2 + 4t}. \end{aligned}$$

In particular, $s(2\pi) = 2\pi\sqrt{R^2+4}$ *is the length of the curve. The inverse of* s *is* $t(s) := \frac{s}{\sqrt{R^2+4}}$, $s \in \left[0, 2\pi\sqrt{R^2+4}\right]$. *Hence,*

$$\boldsymbol{\psi}(s) := \boldsymbol{\varphi}(t(s)) = \left(R\cos\frac{s}{\sqrt{R^2+4}}, R\sin\frac{s}{\sqrt{R^2+4}}, \frac{2s}{\sqrt{R^2+4}}\right),$$

where $s \in \left[0, 2\pi\sqrt{R^2+4}\right]$*, is the parametrization by arclength.*

8 Curve Integrals

In this section we define the integral of a function along a curve.

Definition 121 *Given a piecewise* C^1 *curve* $\boldsymbol{\gamma}$ *with parametric representation* $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ *and a bounded function* $f : E \to \mathbb{R}$*, where* $\boldsymbol{\varphi}([a, b]) \subseteq E$*, we define the* curve *(or* line*)* integral *of* f *along the curve* $\boldsymbol{\gamma}$ *as the number*

$$\int_{\boldsymbol{\gamma}} f\, ds := \int_a^b f(\boldsymbol{\varphi}(t))\, \|\boldsymbol{\varphi}'(t)\|\, dt,$$

whenever the function $t \in [a, b] \mapsto f(\boldsymbol{\varphi}(t))\, \|\boldsymbol{\varphi}'(t)\|$ *is Riemann integrable.*

Note that $\int_{\boldsymbol{\gamma}} f\, ds$ is always defined if f is continuous.

Next we show that the curve integral does not depend on the particular parametric representation.

Proposition 122 *Let $\boldsymbol{\gamma}$ be a piecewise C^1 curve and let $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ and $\boldsymbol{\psi} : [c,d] \to \mathbb{R}^N$ be two parametric representations. Given a continuous function $f : E \to \mathbb{R}$, where E contains the range of $\boldsymbol{\gamma}$,*

$$\int_a^b f(\boldsymbol{\varphi}(t)) \|\boldsymbol{\varphi}'(t)\| \, dt = \int_c^d f(\boldsymbol{\psi}(\tau)) \left\|\boldsymbol{\psi}'(\tau)\right\| \, d\tau.$$

Proof. Exercise ∎

Remark 123 *In particular, if $\boldsymbol{\gamma}$ is a regular curve and $\boldsymbol{\psi} : [0, L(\boldsymbol{\gamma})] \to \mathbb{R}^N$ is the parametric representation obtained using arclength, then $\left\|\boldsymbol{\psi}'(\tau)\right\| = 1$ for all but finitely many τ. Hence,*

$$\int_{\boldsymbol{\gamma}} f \, ds = \int_0^{L(\boldsymbol{\gamma})} f(\boldsymbol{\psi}(\tau)) \, d\tau.$$

The following properties are left as an exercise.

Proposition 124 *Let $\boldsymbol{\gamma}$ be a piecewise C^1 curve and let $f, g : E \to \mathbb{R}$ be two continuous functions, where E contains the range of $\boldsymbol{\gamma}$. Then*

(i) for all $a, b \in \mathbb{R}$,

$$\int_{\boldsymbol{\gamma}} (af + bg) \, ds = a \int_{\boldsymbol{\gamma}} f \, ds + b \int_{\boldsymbol{\gamma}} g \, ds,$$

(ii) if $f \le g$, then $\int_{\boldsymbol{\gamma}} f \, ds \le \int_{\boldsymbol{\gamma}} g \, ds$,

(iii) $\left| \int_{\boldsymbol{\gamma}} f \, ds \right| \le \int_{\boldsymbol{\gamma}} |f| \, ds \le L(\gamma) \max_\Sigma |f|$, where Σ is the range of $\boldsymbol{\gamma}$,

(iv) if $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ is a parametric representation of $\boldsymbol{\gamma}$, $c \in (a,b)$, and $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are the curves of parametric representations $\boldsymbol{\varphi}_1 : [a,c] \to \mathbb{R}^N$ and $\boldsymbol{\varphi}_2 : [c,b] \to \mathbb{R}^N$, then

$$\int_{\boldsymbol{\gamma}} f \, ds = \int_{\boldsymbol{\gamma}_1} f \, ds + \int_{\boldsymbol{\gamma}_2} f \, ds.$$

Next we introduce the notion of an oriented curve.

Definition 125 *Given a curve $\boldsymbol{\gamma}$ with parametric representations $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and $\boldsymbol{\psi} : J \to \mathbb{R}^N$, we say that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the* same orientation *if the parameter change $h : I \to J$ is increasing and* opposite orientation *if the parameter change $h : I \to J$ is decreasing. If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ have the same orientation, we write $\boldsymbol{\varphi} \overset{*}{\sim} \boldsymbol{\psi}$.*

Exercise 126 *Prove that $\overset{*}{\sim}$ is an equivalence relation.*

Definition 127 *An* oriented curve $\boldsymbol{\gamma}$ *is an equivalence class of parametric representations with the same orientation.*

Note that any curve $\boldsymbol{\gamma}$ gives rise to two oriented curves. Indeed, it is enough to fix a parametric representation $\boldsymbol{\varphi} : I \to \mathbb{R}^N$ and considering the equivalence class $\boldsymbol{\gamma}^+$ of parametric representations with the same orientation of $\boldsymbol{\varphi}$ and the equivalence class $\boldsymbol{\gamma}^-$ of parametric representations with the opposite orientation of $\boldsymbol{\varphi}$.

Let $E \subseteq \mathbb{R}^N$ and let $\mathbf{g} : E \to \mathbb{R}^N$ be a continuous function. Given a piecewise C^1 oriented curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}([a,b]) \subseteq E$, we define

$$\int_{\boldsymbol{\gamma}} \mathbf{g} := \int_a^b \sum_{i=1}^N g_i(\boldsymbol{\varphi}(t))\, \varphi_i'(t)\, dt.$$

Definition 128 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$. We say that $\mathbf{g}$ is* conservative vector field *if there exists a differentiable function $f : U \to \mathbb{R}$ such that*

$$\nabla f(\mathbf{x}) = \mathbf{g}(\mathbf{x})$$

for all $\mathbf{x} \in U$. The function f is called a scalar potential *for $\mathbf{g}$.*

Theorem 129 (Fundamental Theorem of Calculus for Curves) *Let $U \subseteq \mathbb{R}^N$ be an open set, let $f \in C^1(U)$, let $\mathbf{x}$, $\mathbf{y} \in U$ and let $\boldsymbol{\gamma}$ a piecewise C^1 oriented curve with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}(b) = \mathbf{x}$, $\boldsymbol{\varphi}(a) = \mathbf{y}$, and $\boldsymbol{\varphi}([a,b]) \subset U$. Then*

$$\int_{\boldsymbol{\gamma}} \nabla f = f(\mathbf{x}) - f(\mathbf{y}).$$

Proof. Define $p(t) := f(\boldsymbol{\varphi}(t))$ and observe that by Theorem 61, p is piecewise C^1 with

$$p'(t) = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(\boldsymbol{\varphi}(t))\, \varphi_i'(t)$$

for all but finitely many t. Hence,

$$\int_{\boldsymbol{\gamma}} \nabla f = \int_a^b \sum_{i=1}^N \frac{\partial f}{\partial x_i}(\boldsymbol{\varphi}(t))\, \varphi_i'(t)\, dt = \int_a^b p'(t)\, dt = p(b) - p(a) = f(\mathbf{x}) - f(\mathbf{y}),$$

where we have used the fundamental theorem of calculus. ■

The previous theorem shows that if a conservative vector field is continuous, then its integral along a curve joining two points depends only on the value at the two points and not on the particular curve. If U is connected, then this condition turns out to be equivalent to the vector field being conservative.

Theorem 130 *Let $U \subseteq \mathbb{R}^N$ be an open connected set and let $\mathbf{g} : U \to \mathbb{R}^N$ be a continuous function. Then the following conditions are equivalent.*

(i) $\mathbf{g}$ *is a conservative vector field,*

(ii) *for every* $\mathbf{x}$, $\mathbf{y} \in U$ *and for every two piecewise* C^1 *oriented curves* $\boldsymbol{\gamma}_1$ *and* $\boldsymbol{\gamma}_2$ *with parametric representations* $\varphi_1 : [a,b] \to \mathbb{R}^N$ *and* $\varphi_2 : [c,d] \to \mathbb{R}^N$, *respectively, such that* $\varphi_1(b) = \varphi_2(d) = \mathbf{x}$, $\varphi_1(a) = \varphi_2(c) = \mathbf{y}$, *and* $\varphi_1([a,b]), \varphi_2([c,d]) \subset U$,

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} = \int_{\boldsymbol{\gamma}_2} \mathbf{g}.$$

(iii) *for every piecewise* C^1 *closed oriented curve* $\boldsymbol{\gamma}$ *with range contained in* U,

$$\int_{\boldsymbol{\gamma}} \mathbf{g} = \mathbf{0}.$$

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Proof. We prove that (i) implies (ii). Assume that $\mathbf{g}$ is a conservative vector field with scalar potential $f : U \to \mathbb{R}$, let $\mathbf{x}$, $\mathbf{y} \in U$ and let $\varphi_1 : [a,b] \to \mathbb{R}^N$ and $\varphi_2 : [c,d] \to \mathbb{R}^N$ be as in (ii). Then by the previous theorem

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} = \int_{\boldsymbol{\gamma}_1} \nabla f = f(\mathbf{x}) - f(\mathbf{y}) = \int_{\boldsymbol{\gamma}_2} \nabla f = \int_{\boldsymbol{\gamma}_2} \mathbf{g}.$$

Conversely assume that (ii) holds. We need to find a scalar potential for $\mathbf{g}$. Fix a point $\mathbf{x}_0 \in U$ and for every $\mathbf{x} \in U$ define

$$f(\mathbf{x}) := \int_{\boldsymbol{\gamma}} \mathbf{g},$$

where $\boldsymbol{\gamma}$ a piecewise C^1 oriented curve with parametric representation $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}(b) = \mathbf{x}$, $\boldsymbol{\varphi}(a) = \mathbf{x}_0$, and $\boldsymbol{\varphi}([a,b]) \subset U$. We claim that there exist

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = g_i(\mathbf{x}).$$

Since U is open and $\mathbf{x} \in U$, there exists $B(\mathbf{x}, r) \subseteq U$. Fix $|h| < r$, then the segment joining the point $\mathbf{x} + h\mathbf{e}_i$ with $\mathbf{x}$ is contained in $B(\mathbf{x}, r)$. Define the curve $\psi : [a, b+1] \to \mathbb{R}^N$ as follows

$$\psi(t) := \begin{cases} \varphi(t) & \text{if } t \in [a,b], \\ \mathbf{x} + (t-b)h\mathbf{e}_i & \text{if } t \in [b, b+1]. \end{cases}$$

Using (ii), we have that

$$\begin{aligned} f(\mathbf{x} + h\mathbf{e}_i) = \int_{\varphi} \mathbf{g} &= f(\mathbf{x}) + \int_b^{b+1} \sum_{j=1}^N g_j(\mathbf{x} + (t-b)h\mathbf{e}_i)\, h\delta_{ij}\, dt \\ &= f(\mathbf{x}) + \int_b^{b+1} g_i(\mathbf{x} + (t-b)h\mathbf{e}_i)\, h\, dt = \\ &= f(\mathbf{x}) + \int_0^h g_i(\mathbf{x} + s\mathbf{e}_i)\, ds, \end{aligned}$$

where in the last equality we have used the change of variable $s = (t - b)\,h$. It follows by the mean value theorem that

$$\frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h} = \frac{1}{h}\int_0^h g_i(\mathbf{x} + s\mathbf{e}_i)\,ds = g_i(\mathbf{x} + s_h\mathbf{e}_i)\,,$$

where s_h is between 0 and h. As $h \to 0$, we have that $s_h \to 0$ and so $\mathbf{x}+s_h\mathbf{e}_i \to \mathbf{x}$. Using the continuity of g_i, we have that there exists

$$\lim_{h\to 0}\frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h} = \lim_{h\to 0} g_i(\mathbf{x} + s_h\mathbf{e}_i) = g_i(\mathbf{x})\,,$$

which proves the claim.

The equivalence between (ii) and (iii) is left as an exercise. ■

Next we give a simple necessary condition for a field $\mathbf{g}$ to be conservative.

Definition 131 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be differentiable. We say that $\mathbf{g}$ is an* irrotational vector field *or* a curl-free vector field *if*

$$\frac{\partial g_i}{\partial x_j}(\mathbf{x}) = \frac{\partial g_j}{\partial x_i}(\mathbf{x})$$

for all $i, j = 1, \ldots, N$ and all $\mathbf{x} \in U$.

Theorem 132 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be a conservative vector field of class C^1. Then $\mathbf{g}$ is irrotational.*

Proof. Since $\mathbf{g}$ is a conservative vector field, there exists a a scalar potential $f : U \to \mathbb{R}$ with $\nabla f = \mathbf{g}$ in U. But since $\mathbf{g}$ is of class C^1, we have that f is of class C^2. Hence, we are in a position to apply the Schwartz theorem to conclude that

$$\frac{\partial g_i}{\partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial g_j}{\partial x_i}(\mathbf{x})$$

for all $i, j = 1, \ldots, N$ and all $\mathbf{x} \in U$. ■

The next example shows that there exist irrotational vector fields that are not conservative.

Example 133 *Let $U := \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the function*

$$\mathbf{g}(x,y) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right).$$

Note that

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2},$$
$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{-x^2+y^2}{\left(x^2+y^2\right)^2},$$

and so $\mathbf{g}$ *is irrotational. However, if we consider the closed curve* $\boldsymbol{\gamma}$ *of parametric representation* $\varphi(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, *we have that*

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \mathbf{g} &= \int_0^{2\pi} \left(-\frac{\sin t}{\cos^2 t + \sin^2 t} (\cos t)' + \frac{\cos t}{\cos^2 t + \sin^2 t} (\sin t)' \right) dt \\ &= \int_0^{2\pi} \left(\sin^2 t + \cos^2 t \right) dt = 2\pi \neq 0,\end{aligned}$$

and so $\mathbf{g}$ *cannot be conservative.*

The problem here is the fact that the domain has a hole.

Definition 134 *A set* $E \subseteq \mathbb{R}^N$ *is* starshaped *with respect to a point* $\mathbf{x}_0 \in \mathbb{R}^N$ *if for every* $\mathbf{x} \in E$, *the segment joining* $\mathbf{x}$ *and* $\mathbf{x}_0$ *is contained in* E.

Theorem 135 (Poincaré's Lemma) *Let* $U \subseteq \mathbb{R}^N$ *be an open set starshaped with respect to a point* $\mathbf{x}_0$ *and let* $\mathbf{g} : U \to \mathbb{R}^N$ *be an irrotational vector field of class* C^1. *Then* $\mathbf{g}$ *is a conservative vector field.*

Lemma 136 (Differentiation under the Integral Sign) *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $R \subset \mathbb{R}^M$ *be a closed rectangle and let* $f : U \times R \to \mathbb{R}$ *be a continuous function, let* $i = 1, \ldots, N$ *and assume that there exists* $\frac{\partial f}{\partial x_i}(\mathbf{x}, \mathbf{y})$ *for every* $\mathbf{x} \in U$ *and* $\mathbf{y} \in R$ *and that* $\frac{\partial f}{\partial x_i}$ *is continuous in* $U \times R$. *Consider the function*

$$g(\mathbf{x}) := \int_R f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}, \quad \mathbf{x} \in U.$$

Then g *is continuous and for every* $\mathbf{x} \in U$ *there exists*

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = \int_R \frac{\partial f}{\partial x_i}(\mathbf{x}, \mathbf{y}) \, d\mathbf{y},$$

and $\frac{\partial g}{\partial x_i}$ *is continuous in* U.

Proof. Exercise. ■

We now turn to the proof of Theorem 135.

Proof of Theorem 135. For every $\mathbf{x} \in U$ define

$$f(\mathbf{x}) := \int_{\boldsymbol{\gamma}} \mathbf{g},$$

where $\boldsymbol{\gamma}$ is the curve given by the parametric representation $\varphi : [0, 1] \to \mathbb{R}^N$ is defined by

$$\varphi(t) := \mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0).$$

Note that

$$f(\mathbf{x}) = \int_0^1 \sum_{j=1}^N g_j(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))(x_j - x_{0j}) \, dt.$$

By the previous lemma, we have that

$$\begin{aligned}
\frac{\partial f}{\partial x_i}(\mathbf{x}) &= \int_0^1 \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N g_j(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))(x_j - x_{0j})\right) dt \\
&= \int_0^1 \left(\sum_{j=1}^N \frac{\partial g_j}{\partial x_i}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\, t\,(x_j - x_{0j}) + g_i(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\, 1\right) dt \\
&= \int_0^1 \left(\sum_{j=1}^N \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\, t\,(x_j - x_{0j}) + g_i(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\, 1\right) dt,
\end{aligned}$$

where we have used the fact that $\mathbf{g}$ is an irrotational vector field. Define

$$h(t) := t g_i(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)).$$

By Theorem 61,

$$h'(t) = \sum_{j=1}^N \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\, t\,(x_j - x_{0j}) + g_i(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)).$$

Hence, by the fundamental theorem of calculus,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \int_0^1 h'(t)\, dt = h(1) - h(0) = 1 g_i(\mathbf{x}) - 0,$$

which completes the proof. ■

Monday, March 05, 2012

Next we prove that the Poincaré lemma continues to hold in a very important class of sets, namely *simply connected sets*.

Definition 137 *Given a set $E \subseteq \mathbb{R}^N$, two continuous closed curves, with parametric representations $\varphi : [a,b] \to \mathbb{R}^N$ and $\psi : [a,b] \to \mathbb{R}^N$ such that $\varphi([a,b]) \subseteq E$ and $\psi([a,b]) \subseteq E$, are* homotopic *in E if there exists a continuous function $\mathbf{h} : [a,b] \times [0,1] \to \mathbb{R}^N$ such that $\mathbf{h}([a,b] \times [0,1]) \subseteq E$,*

$$\mathbf{h}(t,0) = \varphi(t) \text{ for all } t \in [a,b], \quad \mathbf{h}(t,1) = \psi(t) \text{ for all } t \in [a,b],$$
$$\mathbf{h}(a,s) = \mathbf{h}(b,s) \text{ for all } s \in [0,1].$$

The function $\mathbf{h}$ is called a homotopy in E *between the two curves.*

Roughly speaking, two curves are homotopic in E if it is possible to deform the first continuously until it becomes the second *without leaving* the set E.

Definition 138 *A set $E \subseteq \mathbb{R}^N$ is* simply connected *if it is pathwise connected and if every continuous closed curve with range in E is homotopic in E to a point in E (that is, to a curve with parametric representation a constant function).*

Example 139 *A star-shaped set is simply connected. Indeed, let* $E \subseteq \mathbb{R}^N$ *be star-shaped with respect to some point* $\mathbf{x}_0 \in E$ *and consider a continuous closed curve* $\boldsymbol{\gamma}$ *with parametric representation* $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ *such that* $\boldsymbol{\varphi}([a, b]) \subseteq E$. *Then the function*

$$\mathbf{h}(t, s) := s\boldsymbol{\varphi}(t) + (1 - s)\mathbf{x}_0$$

is an homotopy between $\boldsymbol{\gamma}$ *and the point* $\mathbf{x}_0$.

Remark 140 *It can be shown that* $\mathbb{R}^2 \setminus$ *a point is not simply connected, while* $\mathbb{R}^3 \setminus$ *a point is. On the other hand,* $\mathbb{R}^3 \setminus$ *a line is not simply connected.*

Using the Poincaré lemma, we can define the integral of $\mathbf{g} : U \to \mathbb{R}^N$ be an irrotational vector field of class C^1 over a curve that is only continuous. More precisely, let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be an irrotational vector field of class C^1.

Consider a continuous oriented curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ such that $\boldsymbol{\varphi}([a, b]) \subseteq U$. Since U is open, for every $\mathbf{x} \in \boldsymbol{\varphi}([a, b])$ we may find $B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$. Using the continuity of $\boldsymbol{\varphi}$, we have that $\boldsymbol{\varphi}([a, b])$ is a compact set. Thus, there exists a finite number of balls $B_1, \ldots, B_n$ such that

$$\boldsymbol{\varphi}([a, b]) \subset \bigcup_{i=1}^{n} B_i.$$

By increasing the number n and by counting some of the balls more than once, we can construct a partition P of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that $\boldsymbol{\varphi}([t_{i-1}, t_i]) \subset B_i$ for all $i = 1, \ldots, n$. Since each ball is convex, by Poincaré lemma $\mathbf{g} : B_i \to \mathbb{R}^N$ is conservative, and so there exists a function $f_i : B_i \to \mathbb{R}$ of class C^1 such that $\nabla f_i = \mathbf{g}$ in B_i. Let $\boldsymbol{\gamma}_i$ be the curve with parametric representation $\boldsymbol{\varphi} : [t_{i-1}, t_i] \to \mathbb{R}^N$. Assuming for a moment that $\boldsymbol{\gamma}$ is piecewise C^1, it follows by the fundamental theorem of calculus for curves that

$$\int_{\boldsymbol{\gamma}} \mathbf{g} = \sum_{i=1}^{n} \int_{\boldsymbol{\gamma}_i} \mathbf{g} = \sum_{i=1}^{n} \int_{\boldsymbol{\gamma}_i} \nabla f_i = \sum_{i=1}^{n} \left(f_i(\boldsymbol{\varphi}(t_i)) - f_i(\boldsymbol{\varphi}(t_{i-1}))\right).$$

If instead $\boldsymbol{\gamma}$ is only continuous, we could use the right-hand side of the previous equality to define $\int_{\boldsymbol{\gamma}} \mathbf{g}$, that is, we would define

$$\int_{\boldsymbol{\gamma}} \mathbf{g} := \sum_{i=1}^{n} \left(f_i(\boldsymbol{\varphi}(t_i)) - f_i(\boldsymbol{\varphi}(t_{i-1}))\right). \tag{26}$$

Proposition 141 *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $\mathbf{g} : U \to \mathbb{R}^N$ *be an irrotational vector field of class* C^1*, and let* $\boldsymbol{\gamma}$ *be a continuous oriented curve with parametric representation* $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^N$ *such that* $\boldsymbol{\varphi}([a, b]) \subset U$. *Then the number* $\int_{\boldsymbol{\gamma}} \mathbf{g}$ *defined in (26) does not depend on the particular partition* P.

Proof. Step 1: Let's start by showing that (26) does not change if we add a point to the partition. Let $i \in \{1, \ldots, n\}$ and fix $c \in (t_{i-1}, t_i)$. Consider the partition $P' := P \cup \{c\}$. Since $\boldsymbol{\varphi}(c) \in B_i$, we have that

$$(f_i(\boldsymbol{\varphi}(t_i)) - f_i(\boldsymbol{\varphi}(c))) + (f_i(\boldsymbol{\varphi}(c)) - f_i(\boldsymbol{\varphi}(t_{i-1}))) = (f_i(\boldsymbol{\varphi}(t_i)) - f_i(\boldsymbol{\varphi}(t_{i-1}))).$$

Hence, the number $\int_{\boldsymbol{\gamma}} \mathbf{g}$ does not change if we add c to the partition P. More generally, if we consider any refinement of P, the number $\int_{\boldsymbol{\gamma}} \mathbf{g}$ does not change.
Step 2: Assume now that Q is another partition with corresponding balls B'_1, $\ldots$, B'_m and with function $h_j : B'_j \to \mathbb{R}$ of class C^1 such that $\nabla h_j = \mathbf{g}$ in B'_j. Consider a partition $P' = \{s_1, \ldots, s_\ell\}$, which is a common refinement of both P and Q. For every $k = 1, \ldots, \ell$, we have that $[s_{k-1}, s_k]$ is contained in some interval $[t_{i-1}, t_i]$ for some $i = 1, \ldots, n$ and so $\boldsymbol{\varphi}([s_{k-1}, s_k]) \subset B_i$. Similarly, $\boldsymbol{\varphi}([s_{k-1}, s_k]) \subset B'_j$ for some $j = 1, \ldots, m$. Note that the set $B_i \cap B'_j$ and on this set we have that $\nabla f_i = \mathbf{g} = \nabla h_j$. Hence, $\nabla(f_i - h_j) = \mathbf{0}$ in $B_i \cap B'_j$ and since this set is convex (and in particular connected), it follows from Theorem **??** that $f_i(\mathbf{x}) - h_j(\mathbf{x}) = c_{ij}$ for all $\mathbf{x} \in B_i \cap B'_j$. Since $\boldsymbol{\varphi}([s_{k-1}, s_k]) \subset B_i \cap B'_j$, it follows that

$$f_i(\boldsymbol{\varphi}(s_k)) - f_i(\boldsymbol{\varphi}(s_{k-1})) = (h_j(\boldsymbol{\varphi}(s_k)) + c_{ij}) - (h_j(\boldsymbol{\varphi}(s_{k-1})) + c_{ij}) = h_j(\boldsymbol{\varphi}(s_k)) - h_j(\boldsymbol{\varphi}(s_{k-1})).$$

Thus, the two sums have the same elements, which shows that $\int_{\boldsymbol{\gamma}} \mathbf{g}$ as defined in (26) does not depend on the particular partition P. ■

Remark 142 *Note that if the curve $\boldsymbol{\gamma}$ is closed and $\boldsymbol{\varphi}([a, b])$ is contained in one ball $B \subseteq U$, then by Poincaré lemma there exists a function $f : B \to \mathbb{R}$ of class C^1 such that $\nabla f = \mathbf{g}$ in B, and so by considering the partition $t_0 := a < t_1 := b$, we get*

$$\int_{\boldsymbol{\gamma}} \mathbf{g} = f(\boldsymbol{\varphi}(b)) - f(\boldsymbol{\varphi}(a)) = 0.$$

Next we prove that the integral of an irrotational vector field over homotopic curves does not change.

Theorem 143 *Let $U \subseteq \mathbb{R}^N$ be an open set, let $\mathbf{g} : U \to \mathbb{R}^N$ be an irrotational vector field of class C^1, and let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be two continuous, closed, oriented curves that are homotopic in U. Then*

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} = \int_{\boldsymbol{\gamma}_2} \mathbf{g},$$

where $\int_{\boldsymbol{\gamma}_1} \mathbf{g}$ and $\int_{\boldsymbol{\gamma}_2} \mathbf{g}$ are defined in (26).

In view of the previous theorem, we can generalize the Poincaré lemma to simply connected domains.

Theorem 144 (Poincaré's Lemma, II) *Let $U \subseteq \mathbb{R}^N$ be an open simply connected set and let $\mathbf{g} : U \to \mathbb{R}^N$ be an irrotational vector field of class C^1. Then $\mathbf{g}$ is a conservative vector field.*

Proof. In view of Theorem 130, it suffices to show that

$$\int_{\boldsymbol{\gamma}} \mathbf{g} = \mathbf{0}$$

for every piecewise C^1 closed oriented curve $\boldsymbol{\gamma}$ with range contained in U. Fix such a curve $\boldsymbol{\gamma}_1$. Since U is simply connected, we have that $\boldsymbol{\gamma}_2$ is homotopic to a point of U, that is, to a curve $\boldsymbol{\gamma}_2$ with constant parametric representation. By the previous theorem

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} = \int_{\boldsymbol{\gamma}_2} \mathbf{g},$$

but since $\boldsymbol{\gamma}_2$ has a constant parametric representation, we have that $\int_{\boldsymbol{\gamma}_2} \mathbf{g} = 0$, and the proof is completed. ■

Wednesday, March 07, 2012

To prove Theorem 130 we begin with a useful lemma.

Lemma 145 (Lebesgue's Number) *Let $K \subset \mathbb{R}^N$ be a compact set and let $\{U_\alpha\}_\alpha$ be a family of open sets covering K. Then there exists a number $\delta > 0$ (called* Lebesgue's number*) such that if $E \subseteq K$ has diameter less than δ, then E is contained in one of the U_α.*

Proof. If one of the U_α is $\mathbb{R}^N$, then any $\delta > 0$ will do. Thus, assume that none of the U_α is the entire space. Since K is compact, there exist a finite subfamily of $\{U_\alpha\}_\alpha$ that still covers K, say, $U_1, \ldots, U_n$. Consider the closed set $C_i := \mathbb{R}^N \setminus U_i$ and define

$$f(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} \operatorname{dist}(\mathbf{x}, C_i).$$

We claim that $f > 0$ in K. Indeed, let $\mathbf{x} \in K$. Then there exists $i \in \{1, \ldots, n\}$ such that $\mathbf{x} \in U_i$. Since U_i is open, there is $B(\mathbf{x}, r) \subseteq U_i$, and so $\operatorname{dist}(\mathbf{x}, C_i) \geq r$, which shows that $f(\mathbf{x}) \geq \frac{r}{n} > 0$. This prove the claim.

Since f is continuous (exercise) and K is compact, by the Weierstrass theorem there exists

$$\min_{\mathbf{x} \in K} f(\mathbf{x}).$$

Let $\delta := \min_{\mathbf{x} \in K} f(\mathbf{x})$ and not that $\delta > 0$ by what we proved before. Consider a set $E \subseteq K$ with diameter less than δ, and let $\mathbf{x}_0 \in E$. Then

$$\delta \leq f(\mathbf{x}_0) = \frac{1}{n} \sum_{i=1}^{n} \operatorname{dist}(\mathbf{x}_0, C_i) \leq \frac{1}{n} \sum_{i=1}^{n} \operatorname{dist}(\mathbf{x}_0, C_m) = \operatorname{dist}(\mathbf{x}_0, C_m),$$

where $\operatorname{dist}(\mathbf{x}_0, C_m) := \max\{\operatorname{dist}(\mathbf{x}_0, C_i) : i = 1, \ldots, n\}$. Since $\operatorname{dist}(\mathbf{x}_0, C_m) \geq \delta$, it follows that $B(\mathbf{x}_0, \delta) \subseteq U_m$. But E has diameter less than δ, and so $E \subseteq B(\mathbf{x}_0, \delta) \subseteq U_m$ and the proof is completed. ■

We now turn to the proof of Theorem 143.

Proof of Theorem 143.. Let $\varphi : [a,b] \to \mathbb{R}^N$ and $\psi : [a,b] \to \mathbb{R}^N$ be parametric representations of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. Since $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are homotopic in U, there exists a continuous function $\mathbf{h} : [a,b] \times [0,1] \to \mathbb{R}^N$ such that $\mathbf{h}([a,b] \times [0,1]) \subset U$,

$$\mathbf{h}(t,0) = \varphi(t) \text{ for all } t \in [a,b], \quad \mathbf{h}(t,1) = \psi(t) \text{ for all } t \in [a,b].$$

Since U is open, for every $\mathbf{x} \in \mathbf{h}([a,b] \times [0,1])$ we may find $B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$. Since $[a,b] \times [0,1]$ is compact and $\mathbf{h}$ is continuous, the set $\mathbf{h}([a,b] \times [0,1])$ is compact. The family of balls $\{B(\mathbf{x}, r_{\mathbf{x}}) : \mathbf{x} \in \mathbf{h}([a,b] \times [0,1])\}$ covers the compact set $\mathbf{h}([a,b] \times [0,1])$. Thus, there exists a finite number of balls $B(\mathbf{x}_1, r_1)$, $\dots$, $B(\mathbf{x}_n, r_n)$ such that

$$\mathbf{h}([a,b] \times [0,1]) \subset \bigcup_{i=1}^{n} B(\mathbf{x}_i, r_i).$$

Since the function $\mathbf{h}$ is continuous and $B(\mathbf{x}_i, r_i)$ is open, we have that $\mathbf{h}^{-1}(B(\mathbf{x}_i, r_i))$ is relatively open in $[a,b] \times [0,1]$, that is, it is given by the intersection of an open set W_i with $[a,b] \times [0,1]$. Since the sets $W_1, \dots, W_n$ cover the compact set $[a,b] \times [0,1]$, by the Lebesgue's number lemma there exists $\delta > 0$ such that any subset of $[a,b] \times [0,1]$ with diameter less than δ is contained in some of the W_i.

Consider a partition $a = t_0 < t_1 < \cdots < t_\ell = b$ of $[a,b]$ and a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of $[0,1]$ such that each rectangle $R_{j,k} := [t_{j-1}, t_j] \times [s_{k-1}, s_k]$ has diameter less than δ. By the Lebesgue number lemma, there exists $i \in \{1, \dots, n\}$ such that $R_{j,k} \subset \mathbf{h}^{-1}(B(\mathbf{x}_i, r_i))$. Hence, $\mathbf{h}(\partial R_{j,k}) \subset B(\mathbf{x}_i, r_i)$. If we move counterclockwise along the boundary of $R_{j,k}$ starting from the vertex (t_{j-1}, s_{k-1}), we have a closed curve. By composing a parametric representation of this curve with $\mathbf{h}$ we obtain a closed continuous curve $\boldsymbol{\gamma}_{j,k}$ with range $\mathbf{h}(\partial R_{j,k}) \subset B(\mathbf{x}_i, r_i)$. It follows by Remark 142 that

$$\int_{\boldsymbol{\gamma}_{j,k}} \mathbf{g} = 0.$$

Hence,

$$\sum_{j=1}^{\ell} \sum_{k=1}^{m} \int_{\boldsymbol{\gamma}_{j,k}} \mathbf{g} = 0.$$

Now each $\boldsymbol{\gamma}_{j,k}$ can be decomposed into four curves corresponding to the four sides of the rectangle $R_{j,k}$. Moreover, if one of the side of $R_{j,k}$ is in the interior of $[a,b] \times [0,1]$ (except for its endpoints), then the corresponding curve will appear twice with opposite orientation. Hence, after cancelling out all the interior curves, we are left with four curves corresponding to the boundary of $[a,b] \times [0,1]$, precisely,

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} - \int_{\boldsymbol{\gamma}_3} \mathbf{g} + \int_{\boldsymbol{\gamma}_4} \mathbf{g} - \int_{\boldsymbol{\gamma}_2} \mathbf{g} = 0,$$

where $\boldsymbol{\gamma}_3$ and $\boldsymbol{\gamma}_3$ are the curves parametrized by $\mathbf{h}(a,s)$ and $\mathbf{h}(b,s)$, $s \in [0,1]$, respectively. But since $\mathbf{h}(a,s) = \mathbf{h}(b,s)$ for all $s \in [0,1]$, it follows that

$$\int_{\boldsymbol{\gamma}_1} \mathbf{g} - \int_{\boldsymbol{\gamma}_2} \mathbf{g} = 0,$$

and the proof is completed. ■

Example 146 *Consider the function* $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by*

$$\mathbf{g}(x, y) := (ye^x, e^x - \cos y).$$

We have,

$$\frac{\partial}{\partial y}(ye^x) = 1e^x,$$
$$\frac{\partial}{\partial x}(e^x - \cos y) = e^x - 0,$$

and so $\mathbf{g}$ *is irrotational. Since* $\mathbb{R}^2$ *is convex, it is starshaped with respect to every point. Hence, by the previous theorem,* $\mathbf{g}$ *is conservative. To find a scalar potential, note that*

$$\frac{\partial f}{\partial x}(x, y) = ye^x, \quad \frac{\partial f}{\partial y}(x, y) = e^x - \cos y$$

and so

$$f(x, y) - f(0, y) = \int_0^x \frac{\partial f}{\partial x}(s, y)\ ds = \int_0^x ye^s\, ds$$
$$= y(e^x - 1).$$

Hence,

$$f(x, y) = f(0, y) + y(e^x - 1).$$

Differentiating with respect to y*, we get*

$$e^x - \cos y = \frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y}(0, y) + e^x - 1$$

and so

$$\frac{\partial f}{\partial y}(0, y) = -\cos y + 1.$$

Hence,

$$f(0, y) - f(0, 0) = \int_0^y \frac{\partial f}{\partial y}(0, t)\ dt = \int_0^y (-\cos t + 1)\ dt$$
$$= y - \sin y$$

and so

$$f(x, y) = f(0, y) + y(e^x - 1) =$$
$$= f(0, 0) + y - \sin y + ye^x - y = f(0, 0) - \sin y + ye^x.$$

Given an open set $U \subseteq \mathbb{R}^N$ and a differentiable function $\mathbf{g} : U \to \mathbb{R}^N$, a differentiable function $h : U \to \mathbb{R}$ is called an *integrating factor* for $\mathbf{g}$ if the function

$$h\mathbf{g} = (hg_1, \ldots, hg_N)$$

is irrotational.

Exercise 147 *Assume that $N = 2$ and that $\mathbf{g} : U \to \mathbb{R}^2$ is of class C^2. Prove that if*

$$\frac{\partial g_1}{\partial y}(x, y) - \frac{\partial g_2}{\partial x}(x, y) = \alpha(x) g_2(x, y) - \beta(x) g_1(x, y)$$

for some continuous functions $\alpha : I \to \mathbb{R}$ and $\beta : J \to \mathbb{R}$, where I and J are intervals, then an integrating factor is given by

$$u(x, y) = e^{\int_{x_0}^{x} a(t)\,dt} e^{\int_{y_0}^{y} b(s)\,ds},$$

where $x_0 \in I$ and $y_0 \in J$ are some fixed points.

Exercise 148 *Given the function $\mathbf{g} : U \to \mathbb{R}^2$ defined by*

$$\mathbf{g}(x, y) := \left(2\ln(xy) + 1, -\frac{x}{y}\right),$$

1. *Find the domain U of $\mathbf{g}$ and prove that is $\mathbf{g}$ is not irrotational in U.*
2. *Find an integrating factor h for the function $\mathbf{g}$.*
3. *Prove that the function $h\mathbf{g}$ is conservative. (Do not use part (4)).*
4. *Find a potential f of the function $h\mathbf{g}$.*

Remark 149 *Conservative fields are useful is solving differential equations. Consider the differential equation*

$$y'(x) = -\frac{g_1(x, y(x))}{g_2(x, y(x))},$$

where

$$\mathbf{g}(x, y) = (g_1(x, y), g_2(x, y))$$

is of class C^1. If $\mathbf{g}$ conservative, then there exists a function $f : U \to \mathbb{R}$ such that $\nabla f = \mathbf{g}$, so that

$$y'(x) = -\frac{\frac{\partial f}{\partial x}(x, y(x))}{\frac{\partial f}{\partial y}(x, y(x))},$$

which implies that

$$\frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))\, y'(x) = 0.$$

Consider the function

$$h(x) := f(x, y(x)).$$

By the chain rule, we have that

$$h'(x) = \frac{\partial f}{\partial x}(x, y(x)) + \frac{\partial f}{\partial y}(x, y(x))\, y'(x) = 0.$$

Hence, h is constant in each connected domain of its domain. Assume that h is defined in some interval I, then

$$f(x, y(x)) = \textit{constant}$$

for all $x \in I$. Thus, we have solved implicitly our differential equation.

If $\mathbf{g}$ is not irrotational, but it admits an integrating factor $h \neq 0$, then

$$y'(x) = -\frac{g_1(x, y(x))}{g_2(x, y(x))} = -\frac{h(x, y(x))\, g_1(x, y(x))}{h(x, y(x))\, g_2(x, y(x))},$$

and if the function $h\mathbf{g}$ is conservative with $\nabla p = h\mathbf{g}$, then we can conclude as before that

$$p(x, y(x)) = \textit{constant}$$

for all $x \in I$, so we have solved implicitly our differential equation.

Friday, March 09, 2012

midsemester break, no classes

Monday, March 12–Friday, March 16, 2012

Spring break, no classes

Monday, March 19, 2012

In $\mathbb{R}^2$ there are important characterizations of simply connected domains. Indeed, we have the following result.

Theorem 150 *Let $U \subseteq \mathbb{R}^2$ be an open bounded connected set. Then the following are equivalent:*

1. *U is homeomorphic to the unit ball $B((0,0), 1)$.*
2. *U is simply connected.*
3. *$\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = 0$ for every continuous closed oriented curve $\boldsymbol{\gamma}$ with range contained in U and for every $\mathbf{x} \in \mathbb{R}^2 \setminus U$.*
4. *$\mathbb{R}^2 \setminus U$ is connected.*
5. *$\int_{\boldsymbol{\gamma}} \mathbf{g} = 0$ for every irrotational vector field $\mathbf{g} : U \to \mathbb{R}^2$ of class C^1 and for every continuous closed oriented curve $\boldsymbol{\gamma}$ with range contained in U.*

Proof. Let's prove that (1) implies (2). Assume that there exists an invertible function $\Psi : U \to B((0,0),1)$, which is continuous together with its inverse and consider a continuous closed curve, with parametric representation $\varphi : [a,b] \to \mathbb{R}^2$ such that $\varphi([a,b]) \subseteq U$. Define the function $\mathbf{h} : [a,b] \times [0,1] \to \mathbb{R}^2$ by

$$\mathbf{h}(t,s) = \Psi^{-1}(s\Psi(\varphi(t))).$$

Then $\mathbf{h}([a,b] \times [0,1]) \subset U$,

$$\mathbf{h}(t,0) = \Psi^{-1}(0) \text{ for all } t \in [a,b], \quad \mathbf{h}(t,1) = \varphi(t) \text{ for all } t \in [a,b],$$
$$\mathbf{h}(a,s) = \Psi^{-1}(s\Psi(\varphi(a))) = \Psi^{-1}(s\Psi(\varphi(b))) = \mathbf{h}(b,s) \text{ for all } s \in [0,1].$$

Hence, U is simply connected. ■

Remark 151 *If U is unbounded, then in place of (4) one has that $\mathbb{S}^2 \setminus U$ is connected, where $\mathbb{S}^2$ is the* Riemann sphere*. Precisely, we consider an element ∞ not in $\mathbb{R}^2$, and we take $\mathbb{S}^2 := \mathbb{R}^2 \cup \{\infty\}$. Given $r > 0$, we define the open ball centered at ∞ and radius r, as the set*

$$B(\infty; r) := \left(\mathbb{R}^2 \setminus \overline{B((0,0),r)}\right) \cup \{\infty\}$$

and we give a topology $\mathbb{S}^2$ be declaring a set $U \subseteq \mathbb{S}^2$ to be open if it can be written as union of open balls centered at points of $\mathbb{S}^2$ and of positive radius. The name comes from the fact that $\mathbb{S}^2$ is homeomorphic to the unit sphere in $\mathbb{R}^3$. The homeomorphism is given by

$$\Psi(r\cos\theta, r\sin\theta) := \left(\frac{2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{r^2-1}{r^2+1}\right), \quad \Psi(\infty) := (0,0,1).$$

Note that saying that $\mathbb{S}^2 \setminus U$ is connected is not equivalent to saying that $\mathbb{R}^2 \setminus U$ is connected. Indeed, consider the set $\mathbb{R} \times (0,1)$. Then its complement is not connected in $\mathbb{R}^2 \setminus U$ but it is connected in $\mathbb{S}^2 \setminus U$.

Next we define the winding number of a closed curve around a point not on its range. Let $\mathbf{x}_0 \in \mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a continuous oriented curve $\boldsymbol{\gamma}$ with range not containing $\mathbf{x}_0$ and parametric representation $\varphi : [a,b] \to \mathbb{R}^2$. Then for every $t \in [a,b]$, the vector $\varphi(t) - \mathbf{x}_0$ is different from zero and so we can write it in polar coordinates, namely,

$$\frac{\varphi(t) - \mathbf{x}_0}{\|\varphi(t) - \mathbf{x}_0\|} = (\cos\theta(t), \sin\theta(t)).$$

Now for each t, there are infinitely many possible $\theta(t)$, which differ by multiples of 2π. We want to prove that it is possible to chose $\theta(t)$ in such a way that θ is a continuous function.

Theorem 152 *Let $\mathbf{x}_0 \in \mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a continuous oriented curve $\boldsymbol{\gamma}$ with range not containing $\mathbf{x}_0$ and parametric representation $\varphi : [a,b] \to \mathbb{R}^2$, and let θ_0 be such that*

$$\frac{\varphi(a) - \mathbf{x}_0}{\|\varphi(a) - \mathbf{x}_0\|} = (\cos\theta_0, \sin\theta_0).$$

Then there exists a unique continuous function $\theta : [a,b] \to \mathbb{R}$ such that $\theta(a) = \theta_0$ and

$$\frac{\varphi(t) - \mathbf{x}_0}{\|\varphi(t) - \mathbf{x}_0\|} = (\cos\theta(t), \sin\theta(t)) \tag{27}$$

for all $t \in [a,b]$. Moreover, the number $\theta(b) - \theta(a)$ does not depend on the particular choice of θ_0 and on the parametric representation φ.

Proof. Step 1: Assume first that $(\varphi(t) - \mathbf{x}_0)\cdot(\varphi(a) - \mathbf{x}_0) > 0$ for all $t \in [a,b]$. Let

$$\mathbf{u} := (\cos\theta_0, \sin\theta_0), \quad \mathbf{v} := (-\sin\theta_0, \cos\theta_0)$$

and observe that $\{\mathbf{u}, \mathbf{v}\}$ is an orthonormal basis in $\mathbb{R}^2$. Define the function

$$\mathbf{f}(t) := \frac{\varphi(t) - \mathbf{x}_0}{\|\varphi(t) - \mathbf{x}_0\|}.$$

Then $\mathbf{f}(t)\cdot\mathbf{u} > 0$ for all $t \in [a,b]$ and so we may define the continuous function $\theta : [a,b] \to \mathbb{R}$ given by

$$\theta(t) := \theta_0 + \arctan\frac{\mathbf{f}(t)\cdot\mathbf{v}}{\mathbf{f}(t)\cdot\mathbf{u}}.$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$,

$$\theta(a) = \theta_0 + \arctan\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}} = \theta_0.$$

It remains to show that (27) holds. Since $|\theta(t) - \theta_0| < \frac{\pi}{2}$, we have that $\cos(\theta(t) - \theta_0) > 0$ for all $t \in [a,b]$, and so

$$\begin{aligned}\cos(\theta(t) - \theta_0) &= \frac{1}{\sqrt{1 + \tan^2(\theta(t) - \theta_0)}} = \frac{1}{\sqrt{1 + \left(\frac{\mathbf{f}(t)\cdot\mathbf{v}}{\mathbf{f}(t)\cdot\mathbf{u}}\right)^2}} \\ &= \frac{\mathbf{f}(t)\cdot\mathbf{u}}{\sqrt{(\mathbf{f}(t)\cdot\mathbf{u})^2 + (\mathbf{f}(t)\cdot\mathbf{v})^2}} = \frac{\mathbf{f}(t)\cdot\mathbf{u}}{\|\mathbf{f}(t)\|} = \mathbf{f}(t)\cdot\mathbf{u},\end{aligned}$$

where we have used the facts that $\{\mathbf{u}, \mathbf{v}\}$ is an orthonormal basis in $\mathbb{R}^2$, that $\mathbf{f}(t)\cdot\mathbf{u} > 0$, and that $\|\mathbf{f}(t)\| = 1$. In turn,

$$\begin{aligned}\sin(\theta(t) - \theta_0) &= \cos(\theta(t) - \theta_0)\tan(\theta(t) - \theta_0) \\ &= (\mathbf{f}(t)\cdot\mathbf{u})\frac{\mathbf{f}(t)\cdot\mathbf{v}}{\mathbf{f}(t)\cdot\mathbf{u}} = \mathbf{f}(t)\cdot\mathbf{v}.\end{aligned}$$

Hence, since $\{\mathbf{u},\mathbf{v}\}$ is an orthonormal basis in $\mathbb{R}^2$,

$$\begin{aligned}\mathbf{f}(t) &= (\mathbf{f}(t)\cdot\mathbf{u})\,\mathbf{u} + (\mathbf{f}(t)\cdot\mathbf{v})\,\mathbf{v}\\ &= \cos(\theta(t)-\theta_0)(\cos\theta_0,\sin\theta_0) + \sin(\theta(t)-\theta_0)(-\sin\theta_0,\cos\theta_0)\\ &= (\cos(\theta(t)-\theta_0)\cos\theta_0 - \sin(\theta(t)-\theta_0)\sin\theta_0, \cos(\theta(t)-\theta_0)\sin\theta_0 + \sin(\theta(t)-\theta_0)\cos\theta_0)\\ &= (\cos\theta(t), \sin\theta(t)).\end{aligned}$$

■

Wednesday, March 21, 2012

Proof. Step 2: By the Weierstrass theorem, there exists

$$\varepsilon_0 := \min_{t\in[a,b]} \|\varphi(t) - \mathbf{x}_0\| > 0.$$

Since φ is continuous on the compact set $[a,b]$, it is uniformly continuous, and so there exists $\delta = \delta(\varepsilon_0) > 0$ such that

$$\|\varphi(t) - \varphi(s)\| \le \frac{\varepsilon_0}{2}$$

for all $s,t\in[a,b]$ with $|s-t|\le\delta$. Consider a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

with $t_i - t_{i-1} \le \delta$. For $t \in [t_{i-1}, t_i]$, by Cauchy's inequality we have that

$$\begin{aligned}(\varphi(t) - \mathbf{x}_0)\cdot(\varphi(t_{i-1}) - \mathbf{x}_0) &= (\varphi(t) - \varphi(t_{i-1}) + \varphi(t_{i-1}) - \mathbf{x}_0)\cdot(\varphi(t_{i-1}) - \mathbf{x}_0)\\ &= (\varphi(t) - \varphi(t_{i-1}))\cdot(\varphi(t_{i-1}) - \mathbf{x}_0) + \|\varphi(t_{i-1}) - \mathbf{x}_0\|^2\\ &\ge -\|\varphi(t) - \varphi(t_{i-1})\|\,\|\varphi(t_{i-1}) - \mathbf{x}_0\| + \|\varphi(t_{i-1}) - \mathbf{x}_0\|^2\\ &\ge \left(-\frac{\varepsilon_0}{2} + \|\varphi(t_{i-1}) - \mathbf{x}_0\|\right)\|\varphi(t_{i-1}) - \mathbf{x}_0\| > 0.\end{aligned}$$

Hence, we can apply Step 1 in the interval $[t_0, t_1]$ to construct a continuous function $\theta : [t_0, t_i] \to \mathbb{R}$ such that $\theta(t_0) = \theta_0$ and (27) holds for all $t \in [t_0, t_1]$. Inductively, assuming that we have constructed a continuous function $\theta : [t_0, t_{i-1}] \to \mathbb{R}$ such that $\theta(t_0) = \theta_0$, we set $\theta_{i-1} := \theta(t_{i-1})$ and apply Step 1 in the interval $[t_{i-1}, t_i]$ to construct a continuous function $\theta : [t_{i-1}, t_i] \to \mathbb{R}$ such that $\theta(t_{i-1}) = \theta_{i-1}$ and (27) holds for all $t \in [t_{i-1}, t_i]$. This gives the desired function θ.

Step 3: We prove that θ is unique. Assume that there exist two continuous function $\theta_1 : [a,b] \to \mathbb{R}$ and $\theta_2 : [a,b] \to \mathbb{R}$ such that $\theta_1(a) = \theta_2(a) = \theta_0$ and (27) holds (with θ_1 and θ_2 in place of θ) for all $t \in [a,b]$.

Since $(\cos\theta_1(t), \sin\theta_1(t)) = (\cos\theta_2(t), \sin\theta_2(t))$ for all $t \in [a,b]$, we have that for every $t \in [a,b]$ there exists $k_t \in \mathbb{Z}$ such that $\theta_1(t) - \theta_2(t) = 2\pi k_t$. But since θ_1 and θ_2 are continuous, we have that k_t has to be the same for all t. Hence, $\theta_1(t) - \theta_2(t) = 2\pi k_0$ for all $t \in [a,b]$ and for some $k_0 \in \mathbb{Z}$. But, $\theta_1(a) = \theta_2(a) = \theta_0$, and so $k_0 = 0$.

Step 4: We show that the number $\theta(b)-\theta(a)$ does not depend on the particular choice of θ_0. Let θ_{01} and θ_{02} be such that

$$\frac{\varphi(a)-\mathbf{x}_0}{\|\varphi(a)-\mathbf{x}_0\|}=(\cos\theta_{01},\sin\theta_{01})=(\cos\theta_{02},\sin\theta_{02})$$

and let θ_1 and θ_2 be the corresponding functions constructed in Step 2. Then $\theta_{01}-\theta_{02}=2\pi k$ for some $k\in\mathbb{Z}$, and so the function

$$\theta_3(t):=\theta_1(t)-\theta_{01}+\theta_{02}$$

satisfies (27) and $\theta_3(a)=\theta_{02}$. Hence, by uniqueness, $\theta_3=\theta_2$, that is, $\theta_1(t)-\theta_{01}=\theta_2(t)-\theta_{02}$ for all $t\in[a,b]$. Taking $t=b$ gives

$$\theta_1(b)-\theta_1(a)=\theta_2(b)-\theta_2(a).$$

Step 5: Finally we prove that the number $\theta(b)-\theta(a)$ does not depend on the parametric representation φ. Let $\boldsymbol{\psi}:[c,d]\to\mathbb{R}^2$ be equivalent to φ. Then there exists a continuous, bijective increasing function $h:[a,b]\to[c,d]$ such that

$$\boldsymbol{\varphi}(t)=\boldsymbol{\psi}(h(t))$$

for all $t\in[a,b]$. Let θ and $\overline{\theta}$ be the functions corresponding to $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$. Then

$$\begin{aligned}\varphi(t)-\mathbf{x}_0=\boldsymbol{\psi}(h(t))-\mathbf{x}_0&=\|\boldsymbol{\psi}(h(t))-\mathbf{x}_0\|\frac{\boldsymbol{\psi}(h(t))-\mathbf{x}_0}{\|\boldsymbol{\psi}(h(t))-\mathbf{x}_0\|}\\&=\|\boldsymbol{\psi}(h(t))-\mathbf{x}_0\|\left(\cos\overline{\theta}(h(t)),\sin\overline{\theta}(h(t))\right).\end{aligned}$$

In turn, $\|\varphi(t)-\mathbf{x}_0\|=\|\boldsymbol{\psi}(h(t))-\mathbf{x}_0\|$ and so

$$\frac{\varphi(t)-\mathbf{x}_0}{\|\varphi(t)-\mathbf{x}_0\|}=\left(\cos\overline{\theta}(h(t)),\sin\overline{\theta}(h(t))\right).$$

This shows that the function $\overline{\theta}\circ h$ is admissible for φ and so by uniqueness,

$$\theta(b)-\theta(a)=\overline{\theta}(h(b))-\overline{\theta}(h(a))=\overline{\theta}(d)-\overline{\theta}(c),$$

where we have used the fact that h is increasing. This completes the proof. ■

Remark 153 *Note that if the parametrization φ is of class C^1 or piecewise C^1, then the function θ is of class C^1.*

In particular, if $\boldsymbol{\gamma}$ is a continuous closed oriented curve with range not containing $\mathbf{x}_0$ and with parametric representation $\varphi:[a,b]\to\mathbb{R}^2$, then

$$(\cos\theta(a),\sin\theta(a))=\frac{\varphi(a)-\mathbf{x}_0}{\|\varphi(a)-\mathbf{x}_0\|}=\frac{\varphi(b)-\mathbf{x}_0}{\|\varphi(b)-\mathbf{x}_0\|}=(\cos\theta(b),\sin\theta(b)),$$

and so the number $\theta(b)-\theta(a)$ is a multiple of 2π. We define the *winding number* of $\boldsymbol{\gamma}$ around $\mathbf{x}_0$ to be the integer

$$\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}_0)=\frac{\theta(b)-\theta(a)}{2\pi}. \tag{28}$$

Note that $\text{ind}_{\boldsymbol{\gamma}}(\mathbf{x}_0)$ gives the total number of times that curve $\boldsymbol{\gamma}$ travels counterclockwise around the point $\mathbf{x}_0$. Let's find an explicit formula for $\text{ind}_{\boldsymbol{\gamma}}(\mathbf{x}_0)$ in the case in which the curve is piecewise C^1.

Theorem 154 *Let $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ and let $\boldsymbol{\gamma}$ be a piecewise C^1 closed oriented curve with range not containing $\mathbf{x}_0$. Then*

$$\text{ind}_{\boldsymbol{\gamma}}(\mathbf{x}_0) = \frac{1}{2\pi}\int_{\boldsymbol{\gamma}} \mathbf{g},$$

where $\mathbf{g} : \mathbb{R}^2 \setminus \{\mathbf{x}_0\} \to \mathbb{R}^2$ is the the irrotational vector field

$$\mathbf{g}(x,y) := \left(-\frac{y-y_0}{(x-x_0)^2+(y-y_0)^2}, \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2}\right). \tag{29}$$

Proof. Let $\varphi : [a,b] \to \mathbb{R}^2$ be a parametric representation of $\boldsymbol{\gamma}$. Define $r(t) := \|\varphi(t) - \mathbf{x}_0\|$, then by (27),

$$\varphi(t) = \mathbf{x}_0 + r(t)(\cos\theta(t), \sin\theta(t)), \quad t \in [a,b].$$

Hence,

$$\varphi'(t) = r'(t)(\cos\theta(t), \sin\theta(t)) + r(t)(-\theta'(t)\sin\theta(t), \theta'(t)\cos\theta(t)), \quad t \in [a,b].$$

It follows that

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \mathbf{g} &= \int_a^b \left[-\frac{r(t)\sin\theta(t)}{r^2(t)}\left(r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t)\right)\right.\\ &\quad \left.+\frac{r(t)\cos\theta(t)}{r^2(t)}\left(r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t)\right)\right] dt\\ &= \int_a^b \left[\sin^2\theta(t) + \cos^2\theta(t)\right]\theta'(t)\, dt = \theta(b) - \theta(a).\end{aligned}$$

This concludes the proof. ■

Next we prove that the winding number is constant on connected components of the complement of the range of $\boldsymbol{\gamma}$.

Theorem 155 *Let $\boldsymbol{\gamma}$ be a piecewise C^1 closed oriented curve with range Γ. Then the function not containing $\mathbf{x}_0$. Then*

$$\mathbf{x} \in \mathbb{R}^2 \setminus \Gamma \mapsto \text{ind}_{\boldsymbol{\gamma}}(\mathbf{x})$$

is constant on each connected component of the open set $\mathbb{R}^2 \setminus \Gamma$ and it is zero on the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$.

Proof. Let $\varphi : [a,b] \to \mathbb{R}^2$ be a parametric representation of $\boldsymbol{\gamma}$. By the previous theorem,

$$\text{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = \frac{1}{2\pi}\int_a^b \frac{-(\varphi_2(t) - y)\varphi_1'(t) + (\varphi_1(t) - x)\varphi_2'(t)}{\|\varphi(t) - \mathbf{x}\|^2}\, dt.$$

Since $\varphi : [a, b] \to \mathbb{R}^2$ is piecewise C^1, there exists a partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

such that the function $\varphi : [t_{i-1}, t_i] \to \mathbb{R}^2$ is C^1 for every $i = 1, \ldots, n$. Hence, the function

$$g(t, x, y) := \frac{-(\varphi_2(t) - y)\varphi_1'(t) + (\varphi_1(t) - x)\varphi_2'(t)}{\|\varphi(t) - \mathbf{x}\|^2}$$

is uniformly continuous in $[t_{i-1}, t_i] \times \mathbb{R}^2$. In turn, the function

$$(x, y) \mapsto \frac{1}{2\pi} \int_{t_{i-1}}^{t_i} g(t, x, y)\, dt$$

is continuous. Writing,

$$\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = \frac{1}{2\pi} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} g(t, x, y)\, dt,$$

we have that the function $\operatorname{ind}_{\boldsymbol{\gamma}}$ is continuous in $\mathbb{R}^2 \setminus \Gamma$. Since it takes only integer values, it follows that it must be constant on each connected component of the open set $\mathbb{R}^2 \setminus \Gamma$.

It remains to show that it is zero in the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$. Since $\varphi : [a, b] \to \mathbb{R}^2$ is piecewise C^1, we can find $M > 0$ such that $\|\varphi(t)\| \le M$ for all $t \in [a, b]$, and $\|\varphi'(t)\| \le M$ for all but finitely many $t \in [a, b]$. Hence, for $\|\mathbf{x}\| > R > M$, we have that

$$\|\varphi(t) - \mathbf{x}\| \ge \|\mathbf{x}\| - \|\varphi(t)\| \ge \|\mathbf{x}\| - M > 0,$$

and so

$$|g(t, x, y)| \le \frac{(M + \|\mathbf{x}\|) M + (M + \|\mathbf{x}\|) M}{(\|\mathbf{x}\| - M)^2} < \frac{4\pi}{b - a}$$

provided R is sufficiently large. It follows that for $\|\mathbf{x}\| > R$,

$$|\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x})| \le \frac{1}{2},$$

and since $\operatorname{ind}_{\boldsymbol{\gamma}}$ takes only integer values, $\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = 0$. ■

Friday, March 23, 2012

Another important application of Theorem 154 is the following.

Theorem 156 *Let $U \subseteq \mathbb{R}^2$ be an open set and let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be two continuous, closed, oriented curves that are homotopic in U. Then*

$$\operatorname{ind}_{\boldsymbol{\gamma}_1}(\mathbf{x}) = \operatorname{ind}_{\boldsymbol{\gamma}_2}(\mathbf{x})$$

for all $\mathbf{x} \in \mathbb{R}^2 \setminus U$. In particular, if U is simply connected, then $\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = 0$ for every continuous closed oriented curve $\boldsymbol{\gamma}$ with range contained in U and for every $\mathbf{x} \in \mathbb{R}^2 \setminus U$.

Proof. Fix $\mathbf{x}_0 \in \mathbb{R}^2 \setminus U$ and let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be as in the statement. Since the the vector field (29) is irrotational, it follows by Theorem 143, that $\int_{\boldsymbol{\gamma}_1} \mathbf{g} = \int_{\boldsymbol{\gamma}_2} \mathbf{g}$, and so $\operatorname{ind}_{\boldsymbol{\gamma}_1}(\mathbf{x}_0) = \operatorname{ind}_{\boldsymbol{\gamma}_2}(\mathbf{x}_0)$. On the other hand, if U is simply connected, then every continuous closed oriented curve $\boldsymbol{\gamma}_1$ is homotopic to a point. But for a curve $\boldsymbol{\gamma}_2$ with constant parametric representation we have that $\int_{\boldsymbol{\gamma}_2} \mathbf{g} = 0$, and so by the first part of the theorem, $\operatorname{ind}_{\boldsymbol{\gamma}_1}(\mathbf{x}_0) = 0$. ■

Remark 157 *The previous theorem proves the implication (2)⟹(3) in Theorem 150.*

To prove that (3)⟹(4) in Theorem 150 we need the following result. Given n closed continuous oriented curves $\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_N$, the family $\Xi := \{\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_N\}$ is called a *cycle*. The *range* of Ξ is given by the union of the ranges of $\boldsymbol{\gamma}_1$, $\ldots$, $\boldsymbol{\gamma}_N$. Given a point $\mathbf{x} \in \mathbb{R}^2$ not contained in the range of Ξ, we define the *winding number* of Ξ around $\mathbf{x}$ to be the integer

$$\operatorname{ind}_{\Xi}(\mathbf{x}) := \sum_{k=1}^{n} \operatorname{ind}_{\boldsymbol{\gamma}_k}(\mathbf{x}).$$

Theorem 158 *Let $U \subseteq \mathbb{R}^2$ be an open set and let $K \subset U$ be a compact set. Then there exists a cycle Ξ with range contained in $U \setminus K$ such that*

$$\operatorname{ind}_{\Xi}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in K, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus U. \end{cases}$$

Proof. Exercise. ■

We prove that (3)⟹(4) in Theorem 150.

Proof of Theorem 150, continued. Assume that condition (3) in Theorem 150 is satisfied but that $\mathbb{R}^2 \setminus U$ is not connected. Since $\mathbb{R}^2 \setminus U$ is closed, its connected components are also closed. Hence, we can find two disjoint nonempty closed sets C and K such that

$$\mathbb{R}^2 \setminus U = C \cup K.$$

Moreover, since U is bounded, we may assume that there is only one unbounded connected component contained in C, so that K is compact.

Let $V := \mathbb{R}^2 \setminus C$. Then V is open and contains K. By the previous theorem there exists a cycle Ξ with range contained in $V \setminus K$ such that

$$\operatorname{ind}_{\Xi}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in K, \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus V. \end{cases}$$

But $V \setminus K = \left(\mathbb{R}^2 \setminus C\right) \setminus K = \mathbb{R}^2 \setminus (C \cup K) = \mathbb{R}^2 \setminus \left(\mathbb{R}^2 \setminus U\right) = U$. Hence, the range of Ξ is contained in U but $\operatorname{ind}_{\Gamma}(\mathbf{x}) = 1$ for all $\mathbf{x} \in K \subset \mathbb{R}^2 \setminus U$, which contradicts hypothesis (3) in Theorem 150,since the winding number of each closed curve in the cycle should be zero.

Next assume that (4) holds, that is, that $\mathbb{R}^2 \setminus U$ is connected. Consider a continuous closed oriented curve $\boldsymbol{\gamma}$ with range contained in U and let Γ be its range. By Theorem 155, the function $\operatorname{ind}_{\boldsymbol{\gamma}}$ is constant on connected components of $\mathbb{R}^2 \setminus \Gamma$. But $\mathbb{R}^2 \setminus U \subset \mathbb{R}^2 \setminus \Gamma$ and each connected component of the smaller set is contained in a connected component of the larger set. Hence, the function $\operatorname{ind}_{\boldsymbol{\gamma}}$ is constant on connected components of $\mathbb{R}^2 \setminus U$. Since $\mathbb{R}^2 \setminus U$ is connected and unbounded, it follows that it must be contained in the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma$, hence by Theorem 155, $\operatorname{ind}_{\boldsymbol{\gamma}}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^2 \setminus U$. This proves that (3) holds. ■

Exercise 159 *Prove that when U is open and connected (but not necessarily bounded) then condition (3) in Theorem 150 is equivalent to the fact that $\mathbb{S}^2 \setminus U$ is connected.*

Monday, March 26, 2012

Solutions second midterm.

Wednesday, March 28, 2012

9 Integration

The next theorem is very important in exercises. It allows to calculate triple, double, etc.. integrals by integrating one variable at a time.

Theorem 160 (Repeated Integration) *Let $S \subset \mathbb{R}^N$ and $T \subset \mathbb{R}^M$ be rectangles, let $f : S \times T \to \mathbb{R}$ be Riemann integrable and assume that for every $\mathbf{x} \in S$, the function $\mathbf{y} \in T \mapsto f(\mathbf{x}, \mathbf{y})$ is Riemann integrable. Then the function $\mathbf{x} \in S \mapsto \int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ is Riemann integrable and*

$$\int_{S \times T} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \int_S \left(\int_T f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}. \tag{30}$$

Similarly, if for every $\mathbf{y} \in T$, the function $\mathbf{x} \in S \mapsto f(\mathbf{x}, \mathbf{y})$ is Riemann integrable, then the function $\mathbf{y} \in T \mapsto \int_S f(\mathbf{x}, \mathbf{y})\, d\mathbf{x}$ is Riemann integrable and

$$\int_{S \times T} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) = \int_T \left(\int_S f(\mathbf{x}, \mathbf{y})\, d\mathbf{x} \right) d\mathbf{y}.$$

Proof. Let $R := S \times T$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of R. Construct a refinement (exercise) $\mathcal{P}' = \{R_1, \ldots, R_n\}$ of $\mathcal{P}$ and $\mathcal{Q}$ with the property that each rectangle R_k can be written as $R_k = S_i \times T_j$, where $\mathcal{P}_N = \{S_1, \ldots, S_m\}$ and $\mathcal{P}_M = \{T_1, \ldots, T_\ell\}$ are partitions of S and T, respectively. Using the fact that

$$\operatorname{meas}_{N+M} R_k = \operatorname{meas}_{N+M} (S_i \times T_j) = \operatorname{meas}_N S_i \operatorname{meas}_M T_j,$$

we have

$$\begin{aligned}
L(f,\mathcal{P}) \le L(f,\mathcal{P}') &= \sum_{i=1}^{m}\sum_{j=1}^{\ell} \operatorname{meas}_N S_i \operatorname{meas}_M T_j \inf_{\mathbf{x}\in S_i}\inf_{\mathbf{y}\in T_j} f(\mathbf{x},\mathbf{y}) \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \inf_{\mathbf{x}\in S_i} \sum_{j=1}^{\ell} \operatorname{meas}_M T_j \inf_{\mathbf{y}\in T_j} f(\mathbf{x},\mathbf{y}) \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \inf_{\mathbf{x}\in S_i} \int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \le \sum_{i=1}^{m} \operatorname{meas}_N S_i \sup_{\mathbf{x}\in S_i} \int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \\
&\le \sum_{i=1}^{m} \operatorname{meas}_N S_i \sup_{\mathbf{x}\in S_i} \sum_{j=1}^{\ell} \operatorname{meas}_M T_j \sup_{\mathbf{y}\in T_j} f(\mathbf{x},\mathbf{y}) \\
&\le \sum_{i=1}^{m}\sum_{j=1}^{\ell} \operatorname{meas}_N S_i \operatorname{meas}_M T_j \sup_{\mathbf{x}\in S_i}\sup_{\mathbf{y}\in T_j} f(\mathbf{x},\mathbf{y}) \\
&= U(f,\mathcal{P}') \le U(f,\mathcal{Q}),
\end{aligned}$$

which shows that

$$L(f,\mathcal{P}) \le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \le U(f,\mathcal{Q}).$$

Taking the supremum over all partitions $\mathcal{P}$ of R, we get

$$\begin{aligned}
\underline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) &\le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \le U(f,\mathcal{Q}).
\end{aligned}$$

Taking the infimum over all partitions $\mathcal{Q}$ of R, we get

$$\begin{aligned}
\underline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) &\le \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \\
&\le \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} \le \overline{\int_{S\times T}} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Since f is Riemann integrable, it follows that

$$\int_{S\times T} f(\mathbf{x},\mathbf{y})\, d(\mathbf{x},\mathbf{y}) = \underline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} = \overline{\int_S} \left(\int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x},$$

which implies that the function $\mathbf{x} \in S \mapsto \int_T f(\mathbf{x},\mathbf{y})\, d\mathbf{y}$ is Riemann integrable and that (30). ■

Remark 161 *Note that if in Theorem 160 we remove the hypothesis that for every* $\mathbf{x} \in S$*, the function* $\mathbf{y} \in T \mapsto f(\mathbf{x}, \mathbf{y})$ *is Riemann integrable, we can still prove that the functions* $\mathbf{x} \in S \mapsto \underline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ *and* $\mathbf{x} \in S \mapsto \overline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y}$ *are Riemann integrable with*

$$\begin{aligned}\int_{S\times T} f(\mathbf{x}, \mathbf{y})\, d(\mathbf{x}, \mathbf{y}) &= \int_S \left(\underline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} \\ &= \int_S \left(\overline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x}.\end{aligned}$$

The proof is similar to the one of Theorem 160 and we leave it as an exercise. In turn,

$$\int_S \left(\overline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \right) d\mathbf{x} = 0.$$

Since $\overline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \geq 0$*, it follows from Exercise* **??** *that there exists a set* $E \subseteq S$ *of Lebesgue measure zero such that*

$$\overline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} - \underline{\int_T} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} = 0$$

for all $\mathbf{x} \in S \setminus E$*. Thus, for every* $\mathbf{x} \in S \setminus E$*, the function* $\mathbf{y} \in T \mapsto f(\mathbf{x}, \mathbf{y})$ *is Riemann integrable.*

The following example shows that without assuming that f is Riemann integrable, the previous theorem fails.

Example 162 *Let* $f : [0, 1] \times [0, 1] \to \mathbb{R}$ *be defined by*

$$f(x, y) := \begin{cases} 1 & \text{if there exist } p \geq 2 \text{ prime and } m, n \in \mathbb{N} \\ & \text{such that } (x, y) = \left(\frac{m}{p}, \frac{n}{p} \right), \\ 0 & \text{otherwise.} \end{cases}$$

Let E *be the set of all* $(x, y) \in [0, 1] \times [0, 1]$ *for which there exist* $p \geq 2$ *prime and* $m, n \in \mathbb{N}$ *such that* $(x, y) = \left(\frac{m}{p}, \frac{n}{p} \right)$*. Using the density of the rationals and of the irrationals, it can be shown that both* E *and* $([0, 1] \times [0, 1]) \setminus E$ *are dense in* $[0, 1] \times [0, 1]$*. Hence, the set of discontinuity points of* f *is* $[0, 1] \times [0, 1]$*. Thus,* f *is not Riemann integrable. On the other hand, if we fix* $x \in [0, 1]$ *and we consider the function* $f(x, \cdot)$*, then we have the following two cases. If* $x = \frac{m}{p}$ *for some* $p \geq 2$ *prime and some* $m \in \mathbb{N}$*, then*

$$f(x, y) = \begin{cases} 1 & \text{if } y \in \left\{ \frac{1}{p}, \ldots, \frac{p-1}{p}, \frac{p}{p} \right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x,\cdot)$ is only discontinuous at a finite number of points and so it is Riemann integrable in $[0,1]$ with

$$\int_0^1 f(x,y)\ dy = 0.$$

In the second case, x cannot be written in the form $\frac{m}{p}$ for some $p \geq 2$ prime and some $m \in \mathbb{N}$. In this case $f(x,y) = 0$ for all $y \in [0,1]$ and so again $\int_0^1 f(x,y)\ dy = 0$, which shows that

$$\int_0^1 \left(\int_0^1 f(x,y)\ dy \right) dy = \int_0^1 0\, dy = 0$$

and similarly,

$$\int_0^1 \left(\int_0^1 f(x,y)\ dx \right) dx = \int_0^1 0\, dx = 0.$$

So the iterated integrals exist and are equal, but the integral $\int_{[0,1]\times[0,1]} f(x,y)\ d(x,y)$ does not exist.

The following example shows that the fact that $f : S \times T \to \mathbb{R}$ be Riemann integrable does not imply that for every $\mathbf{x} \in S$, the function $\mathbf{y} \in T \mapsto f(\mathbf{x},\mathbf{y})$ is Riemann integrable.

Example 163 *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by*

$$f(x,y) := \begin{cases} 1 & \text{if } y \in [0,1] \cap \mathbb{Q} \text{ and } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The function f is discontinuous only on the segment $\left\{\frac{1}{2}\right\} \times [0,1]$, which has Lebesgue measure zero. Hence, f is Riemann integrable in $[0,1] \times [0,1]$. On the other hand, if we fix $x = \frac{1}{2}$ and we consider the function $g(y) = f\left(\frac{1}{2}, y\right)$, $y \in [0,1]$, we have that g is discontinuous at every $y \in [0,1]$, and so g is not Riemann integrable in $[0,1]$.

Exercise 164 *Calculate $\int_{[0,1]\times[0,1]} f(x,y)\ d(x,y)$ in two different ways, where*

$$f(x,y) := x \sin(x+y).$$

10 Peano–Jordan Measure

Given a bounded set $E \subset \mathbb{R}^N$, let R be a rectangle containing E. We say that E is *Peano–Jordan measurable* if the function χ_E is Riemann integrable over R. In this case the *Peano–Jordan measure* of E is given by

$$\operatorname{meas} E := \int_R \chi_E(\mathbf{x})\ d\mathbf{x}.$$

In dimension $N = 2$, the Peano–Jordan of a set is called area of the set and in dimension $N = 3$ it is called volume of a set.

Exercise 165 *Prove that the previous definition does not depend on the choice of the rectangle R containing E.*

Remark 166 *Note that a rectangle is Peano–Jordan measurable and its Peano–Jordan measure coincides with its elementary measure (see Remark **??**).*

Remark 167 *If $E, F \subset \mathbb{R}^N$ are two Peano–Jordan measurable sets, with $E \subseteq F$, then $\chi_E \leq \chi_F$, and so by Proposition **??**,*

$$\operatorname{meas} E \leq \operatorname{meas} F.$$

Exercise 168 *Prove that if $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^N$ are Peano–Jordan measurable and R is a rectangle containing E, then $E \cup F$, $E \cap F$, $R \setminus E$ are Peano–Jordan measurable.*

Exercise 169 *Prove that if $E_1, \ldots, E_n \subset \mathbb{R}^N$ are Peano–Jordan measurable and pairwise disjoint, then*

$$\operatorname{meas}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \operatorname{meas} E_i.$$

Exercise 170 *Prove that if $E_1, \ldots, E_n \subset \mathbb{R}^N$ are Peano–Jordan measurable and $E \subseteq \bigcup_{i=1}^{n} E_i$ is Peano–Jordan measurable, then*

$$\operatorname{meas} E \leq \sum_{i=1}^{n} \operatorname{meas} E_i.$$

Friday, March 30, 2012

Definition 171 *A set $E \subset \mathbb{R}^N$ is called a* pluri-rectangle *if it can be written as a finite union of rectangles.*

Exercise 172 *Prove that a pluri-rectangle can be written as a finite union of disjoint rectangles.*

Remark 173 *In view of the previous exercise and of Exercise 169, it follows that every pluri-rectangle is Peano–Jordan measurable.*

Given a bounded set $E \subset \mathbb{R}^N$, the *Peano–Jordan inner measure* of E is given by

$$\operatorname{meas}_i E := \sup \{\operatorname{meas} P : P \text{ pluri-rectangle}, E \supseteq P\},$$

while the *Peano–Jordan outer measure* of E is given by

$$\operatorname{meas}_o E := \inf \{\operatorname{meas} P : P \text{ pluri-rectangle}, E \subseteq P\}.$$

The following theorem characterizes Peano–Jordan measurability.

Theorem 174 *Given a bounded set $E \subset \mathbb{R}^N$,*

$$\operatorname{meas}_i E = \underline{\int_R} \chi_E(\mathbf{x})\, d\mathbf{x}, \tag{31}$$

$$\operatorname{meas}_o E = \overline{\int_R} \chi_E(\mathbf{x})\, d\mathbf{x}. \tag{32}$$

In particular, $\operatorname{meas}_i E \le \operatorname{meas}_o E$ *and E is is Peano–Jordan measurable if and only* $\operatorname{meas}_i E = \operatorname{meas}_o E$*, in which case* $\operatorname{meas} E = \operatorname{meas}_i E = \operatorname{meas}_o E$*.*

Proof. Exercise. ■

Remark 175 *In view of the previous theorem, if a bounded set $E \subset \mathbb{R}^N$ has Peano–Jordan measure zero, then for every $\varepsilon > 0$ there exists a pluri-rectangle P is containing E such that*

$$\operatorname{meas} P \le \varepsilon.$$

By writing P as a union of disjoint rectangles, $P = \bigcup_{i=1}^n R_i$, we have that

$$\sum_{i=1}^n \operatorname{meas} R_i \le \varepsilon.$$

This implies that E has Lebesgue measure zero. However, the opposite is not true. For example the set $E := [0,1] \cap \mathbb{Q}$ has Lebesgue measure zero (since it is countable), but it is not Peano Jordan measurable and its outer measure is actually one. Indeed, its characteristic function is the Dirichlet function.

Exercise 176 *Let $E \subseteq \mathbb{R}^N$, let $\mathbf{x} \in E$, let $\mathbf{y} \in \mathbb{R}^N \setminus E$. Prove that the segment S joining $\mathbf{x}$ and $\mathbf{y}$ intersects ∂E.*

Theorem 177 *A bounded set $E \subset \mathbb{R}^N$ is Peano–Jordan measurable if and only if its boundary is Peano–Jordan measurable and it has Peano–Jordan measure zero.*

Proof. We begin by observing that if P is a pluri-rectangle, then $\operatorname{meas} \partial P = 0$ (why?), and so

$$\operatorname{meas} P = \operatorname{meas} \overline{P} = \operatorname{meas} P^\circ.$$

Step 1: Assume that $E \subset \mathbb{R}^N$ is Peano–Jordan measurable and let R be a rectangle containing E. By the previous theorem, for every $\varepsilon > 0$ there exist a pluri-rectangle P_1 contained in E and and a pluri-rectangle P_2 is containing E such that

$$0 \le \operatorname{meas} P_2 - \operatorname{meas} P_1 \le \varepsilon.$$

Hence,

$$\begin{aligned} \operatorname{meas}\left(\overline{P_2} \setminus P_1^\circ\right) &= \operatorname{meas} \overline{P_2} - \operatorname{meas} P_1^\circ \\ &= \operatorname{meas} P_2 - \operatorname{meas} P_1 \le \varepsilon. \end{aligned}$$

Note that $\overline{P_2} \setminus P_1^\circ$ is still a pluri-rectangle (exercise) and since

$$\overline{E} \subseteq \overline{P_2}, \quad P_1^\circ \subseteq E^\circ,$$

we have that

$$\partial E = \overline{E} \setminus E^\circ \subseteq \overline{P_2} \setminus P_1^\circ.$$

Hence,

$$\begin{aligned} 0 &\leq \sup \{\operatorname{meas} P : \; P \text{ pluri-rectangle}, \partial E \supseteq P\} \\ &\leq \inf \{\operatorname{meas} P : \; P \text{ pluri-rectangle}, \; \partial E \subseteq P\} \leq \operatorname{meas} \left(\overline{P_2} \setminus P_1^\circ\right) \leq \varepsilon, \end{aligned}$$

which, by letting $\varepsilon \to 0^+$, implies that

$$\begin{aligned} 0 &= \sup \{\operatorname{meas} P : \; P \text{ pluri-rectangle}, \partial E \supseteq P\} \\ &= \inf \{\operatorname{meas} P : \; P \text{ pluri-rectangle}, \; \partial E \subseteq P\} = 0. \end{aligned}$$

It follows by the previous theorem that ∂E is Peano–Jordan measurable with measure zero.

Step 2: Assume that $\partial E \subset \mathbb{R}^N$ is Peano–Jordan measurable with measure zero. Since E is bounded, so is $\overline{E}$ and so there exists a rectangle R containing $\overline{E}$. Since $\operatorname{meas} \partial E = 0$ by the previous theorem there exists a a pluri-rectangle P containing ∂E such that

$$\operatorname{meas} P \leq \varepsilon.$$

The set $R \setminus P$ is a pluri-rectangle and thus we can write it as disjoint unions of rectangles,

$$R \setminus P = \bigcup_{i=1}^{n} R_i.$$

Let P_1 be the pluri-rectangle given by the union of all the rectangles R_i that are contained in E, so that $P_1 \subseteq E$. Let $P_2 := P \cup P_1$. We claim that

$$E \subseteq P_2.$$

Fix $\mathbf{x} \in E$. If $\mathbf{x}$ does not belong to P_2, then in particular it cannot belong to P and so it belongs to $R \setminus P$. Hence, there exists R_i such that $\mathbf{x} \in R_i$. But then R_i must be contained in E. Indeed, if not, then there exists $\mathbf{y} \in R_i \cap (R \setminus E)$. It follows by Exercise 176 that the segment S joining $\mathbf{x}$ and $\mathbf{y}$ must contained a point on the boundary of ∂E, which is a contradiction since the segment S is contained in R_i and R_i does not intersect $P \supset \partial E$. This proves the claim.

Since the claim holds, for every $\varepsilon > 0$ we have found a pluri-rectangle P_1 contained in E and and a pluri-rectangle P_2 is containing E such that

$$0 \leq \operatorname{meas} P_2 - \operatorname{meas} P_1 \leq \varepsilon.$$

It follows by the previous theorem that E is Peano–Jordan measurable. ■

The next theorem shows that the integral of a nonnegative function is given by the volume of the subgraph.

Theorem 178 *Let $R \subset \mathbb{R}^N$ be a rectangle and let $f : R \to [0, \infty)$ be a bounded function. Then f is Riemann integrable over R if and only if the set*

$$S_f := \{(\mathbf{x}, y) \in R \times [0, \infty) : 0 \leq y \leq f(\mathbf{x})\}$$

is Peano–Jordan measurable in $\mathbb{R}^{N+1}$ and in this case

$$\operatorname{meas}_{N+1} S_f = \int_R f(\mathbf{x})\, d\mathbf{x}.$$

Proof. Step 1: Assume that f is Riemann integrable over R. Given $\varepsilon > 0$, by Theorem **??** there exists a partition $\mathcal{P}^\varepsilon$ of R such that

$$0 \leq U(f, \mathcal{P}^\varepsilon) - L(f, \mathcal{P}^\varepsilon) \leq \varepsilon.$$

Write $\mathcal{P}^\varepsilon = \{R_1, \ldots, R_n\}$ and set

$$T_i := R_i \times \left[0, \inf_{R_i} f\right], \quad U_i := R_i \times \left[0, \sup_{R_i} f\right]$$

and

$$P_1 := \bigcup_{i=1}^n T_i, \quad P_2 := \bigcup_{i=1}^n U_i.$$

Then P_1 and P_2 are pluri-rectangles, $P_1 \subseteq S_f \subseteq P_2$ and

$$\begin{aligned}
\operatorname{meas}_{N+1} P_2 - \operatorname{meas}_{N+1} P_1 &= \sum_{i=1}^n \operatorname{meas}_{N+1} U_i - \sum_{i=1}^n \operatorname{meas}_{N+1} T_i \\
&= \sum_{i=1}^n \left(\sup_{R_i} f - 0\right) \operatorname{meas}_N R_i - \sum_{i=1}^n \left(\inf_{R_i} f - 0\right) \operatorname{meas}_N R_i \\
&= U(f, \mathcal{P}^\varepsilon) - L(f, \mathcal{P}^\varepsilon) \leq \varepsilon.
\end{aligned}$$

It follows from Theorem 174 that the set S_f is Peano–Jordan measurable with

$$\operatorname{meas}_{N+1} S_f = \int_R f(\mathbf{x})\, d\mathbf{x}.$$

Conversely, assume that the set S_f is Peano–Jordan measurable. Let $R \times [0, M]$ be a rectangle containing S_f. Then χ_{S_f} is Riemann-integrable over $R \times [0, M]$ with

$$\operatorname{meas}_{N+1} S_f = \int_R \chi_{S_f}(\mathbf{x}, y)\, d(\mathbf{x}, y).$$

For every $\mathbf{x} \in R$, the function

$$y \in [0, M] \mapsto \chi_{S_f}(\mathbf{x}, y) = \chi_{[0, f(\mathbf{x})]}(y)$$

is Riemann integrable. Hence, by Theorem 160, the function

$$\begin{aligned}\mathbf{x} \in R \mapsto \int_{[0,M]} \chi_{S_f}(\mathbf{x}, y)\, dy &= \int_{[0,M]} \chi_{[0,f(\mathbf{x})]}(y)\, dy \\ &= \int_0^{f(\mathbf{x})} 1\, dy = f(\mathbf{x})\end{aligned}$$

is Riemann integrable and

$$\begin{aligned}\operatorname{meas}_{N+1} S_f &= \int_{R\times[0,M]} \chi_{S_f}(\mathbf{x}, y)\, d(\mathbf{x}, y) = \int_R \left(\int_{[0,M]} \chi_{S_f}(\mathbf{x}, y)\, dy \right) d\mathbf{x} \\ &= \int_R \left(\int_{[0,M]} \chi_{[0,f(\mathbf{x})]}(y)\, dy \right) d\mathbf{x} = \int_R \left(\int_0^{f(\mathbf{x})} 1\, dy \right) d\mathbf{x} \\ &= \int_R f(\mathbf{x})\, d\mathbf{x}.\end{aligned}$$

This completes the proof. ■

Remark 179 *With a similar proof one can show that if $R \subset \mathbb{R}^N$ is a rectangle and $f : R \to \mathbb{R}$ is a bounded function, with $f(\mathbf{x}) \geq c$ for all $\mathbf{x} \in R$. Then f is Riemann integrable over R if and only if the set*

$$T_f := \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y \leq f(\mathbf{x})\}$$

is Peano–Jordan measurable in $\mathbb{R}^{N+1}$ and in this case

$$\operatorname{meas}_{N+1} T_f = \int_R (f(\mathbf{x}) - c)\, d\mathbf{x}.$$

Monday, April 02, 2012

Corollary 180 *Let $R \subset \mathbb{R}^N$ be a rectangle and $f : R \to \mathbb{R}$ be a Riemann integrable function, with $f(\mathbf{x}) \geq c$ for all $\mathbf{x} \in R$. Then ∂T_f has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$. In particular, the graph of f,*

$$\operatorname{Gr} f := \{(\mathbf{x}, y) \in R \times \mathbb{R} : y = f(\mathbf{x})\}$$

has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$.

Exercise 181 *Given two sets $E, F \subseteq \mathbb{R}^N$, prove that*

$$\partial(E \setminus F) \subseteq \partial E \cup \partial F$$

Corollary 182 *Let $R \subset \mathbb{R}^N$ be a rectangle, let $\alpha : R \to \mathbb{R}$ and $\beta : R \to \mathbb{R}$ be two Riemann integrable functions, with $\alpha(\mathbf{x}) \leq \beta(\mathbf{x})$ for all $\mathbf{x} \in R$, let*

$$E := \{(\mathbf{x}, y) \in R \times \mathbb{R} : \alpha(\mathbf{x}) \leq y \leq \beta(\mathbf{x})\},$$

and let $f : E \to \mathbb{R}$ be a bounded continuous function. Then f is Riemann integrable over E and

$$\int_E f(\mathbf{x}, y)\ d(\mathbf{x}, y) = \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x}, y)\ dy \right) d\mathbf{x}.$$

Proof. Consider a rectangle $R \times [a, b]$ containing E and let

$$g(\mathbf{x}, y) := \begin{cases} f(\mathbf{x}, y) & \text{if } (\mathbf{x}, y) \in E, \\ 0 & \text{if } (\mathbf{x}, y) \in (R \times [a, b]) \setminus E. \end{cases}$$

We need to show that g is Riemann integrable over $R \times [a, b]$. Hence, we need to look at the set of discontinuity points of g. Since g is continuous in E, we have that the discontinuity points of g are on the boundary of E. Thus, we need to show that ∂E has Lebesgue measure zero in $\mathbb{R}^{N+1}$. We will show more, namely that it has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$. Let $c \in \mathbb{R}$ be such that $\alpha(\mathbf{x}) \geq c$ for all $\mathbf{x} \in R$. Then

$$\begin{aligned} E &= \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y \leq \beta(\mathbf{x})\} \setminus \{(\mathbf{x}, y) \in R \times [c, \infty) : c \leq y < \alpha(\mathbf{x})\} \\ &= T_\beta \setminus T_\alpha. \end{aligned}$$

By the previous exercise, $\partial E \subseteq \partial T_\alpha \cup \partial T_\beta$ and since ∂T_α and ∂T_β have Peano–Jordan measure zero in $\mathbb{R}^{N+1}$ in view of the previous corollary, it follows that ∂E has Peano–Jordan measure zero in $\mathbb{R}^{N+1}$. This shows that g is Riemann integrable over $R \times [a, b]$.

For every $\mathbf{x} \in R$, the function

$$y \in [a, b] \mapsto g(\mathbf{x}, y) = \begin{cases} 0 & \text{if } y > \beta(\mathbf{x}) \\ f(\mathbf{x}, y) & \text{if } \alpha(\mathbf{x}) \leq y \leq \beta(\mathbf{x}) \\ 0 & \text{if } y < \alpha(\mathbf{x}) \end{cases}$$

is Riemann integrable since it is discontinuous at most at the two points $y = \alpha(\mathbf{x})$ and $y = \beta(\mathbf{x})$. Hence, by Theorem 160, the function $\mathbf{x} \in R \mapsto \int_{[a,b]} g(\mathbf{x}, y)\ dy$ is Riemann integrable and

$$\begin{aligned} \int_{R \times [a,b]} g(\mathbf{x}, y)\ d(\mathbf{x}, y) &= \int_R \left(\int_{[a,b]} g(\mathbf{x}, y)\ dy \right) d\mathbf{x} \\ &= \int_R \left(\int_{\alpha(\mathbf{x})}^{\beta(\mathbf{x})} f(\mathbf{x}, y)\ dy \right) d\mathbf{x}. \end{aligned}$$

This completes the proof. ■

Remark 183 *If α and β are continuous, then the set of discontinuity points of g is given by the union of the graphs of α and β, but if α and β are discontinuous, then the set of discontinuity points of g is larger (why?).*

Given a rectangle $R \subset \mathbb{R}^N$, a Peano–Jordan measurable set $E \subseteq R$, and a Riemann integrable function $f : R \to \mathbb{R}$, we define the integral of f over E as

$$\int_E f(\mathbf{x})\, d\mathbf{x} := \int_R \chi_E(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x}.$$

Note that this integral is well-defined since the product of Riemann integrable functions is still Riemann integrable.

Exercise 184 (Mean Value Theorem) *Let $E \subset \mathbb{R}^N$ be a Peano–Jordan measurable set and let $f : E \to \mathbb{R}$ be a bounded continuous function.*

(i) Prove that f is Riemann integrable over E.

(ii) Prove that

$$\operatorname{meas} E \inf_{\mathbf{x} \in E} f(\mathbf{x}) \leq \int_E f(\mathbf{x})\, d\mathbf{x} \leq \operatorname{meas} E \sup_{\mathbf{x} \in E} f(\mathbf{x}).$$

(iii) Prove that if E is connected and has positive measure, then there exists $\mathbf{c} \in E$ such that

$$\frac{1}{\operatorname{meas} E} \int_E f(\mathbf{x})\, d\mathbf{x} = f(\mathbf{c}).$$

Example 185 *Let's calculate the integral*

$$\iint_E x(1-y)\, dxdy,$$

where

$$E := \left\{(x,y) \in \mathbb{R}^2 : \, y \leq x,\, x^2 + y^2 \leq 1,\, x \geq 0,\, y \geq 0\right\}.$$

We can rewrite E as follows,

$$E = \left\{(x,y) \in \mathbb{R}^2 : \, 0 \leq y \leq \frac{\sqrt{2}}{2},\, y \leq x \leq \sqrt{1-y^2}\right\}$$

and since the function $f(x,y) := x(1-y)$ is bounded and continuous in E and the functions $\alpha(y) := y$ and $\beta(y) := \sqrt{1-y^2}$ are continuous, we can apply the previous corollary to conclude that

$$\begin{aligned}\iint_E x(1-y)\, dxdy &= \int_0^{\frac{\sqrt{2}}{2}} \left(\int_y^{\sqrt{1-y^2}} x(1-y)\, dx\right) dy = \int_0^{\frac{\sqrt{2}}{2}} (1-y) \left(\left[\frac{x^2}{2}\right]_{x=y}^{x=\sqrt{1-y^2}}\right) dy \\ &= \int_0^{\frac{\sqrt{2}}{2}} (1-y) \left(\frac{1-y^2}{2} - \frac{y^2}{2}\right) dy \\ &= \frac{1}{6}\sqrt{2} - \frac{1}{16}.\end{aligned}$$

Exercise 186 *Calculate the integral*

$$\iiint_E (x+z)\ dxdydz,$$

where

$$E := \left\{(x,y,z) \in \mathbb{R}^3 : x+y+z \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\right\}.$$

11 Improper Integrals

In this section we study integrability for functions that are either unbounded or are defined on unbounded sets. If $f : [a, \infty) \to \mathbb{R}$ is a continuous function, we have seen that the improper or generalized Riemann integral of f over $\mathbb{R}$ is given by

$$\int_a^\infty f(x)\ dx := \lim_{r\to\infty} \int_a^r f(x)\ dx,$$

provided the limit exists. If now $f : E \to \mathbb{R}$ is continuous, where, say

$$E := \left\{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0\right\}$$

we would be tempted to follow a similar procedure, that is, to define

$$\iint_E f(x,y)\ dxdy := \lim_{r\to\infty} \iint_{[0,r]\times[0,r]} f(x,y)\ dxdy,$$

or

$$\iint_E f(x,y)\ dxdy := \lim_{r\to\infty} \iint_{E\cap B((0,0),r)} f(x,y)\ dxdy.$$

Unfortunately, in general, it could happen than one limit exist but the other does not.

Example 187 *Buck. Consider the function*

$$f(x,y) := \sin\left(x^2+y^2\right)$$

defined in the set

$$E := \left\{(x,y) \in \mathbb{R}^2 : x \ge 0,\ y \ge 0\right\}.$$

By Theorem 160,

$$\begin{aligned}\iint_{[0,r]\times[0,r]} \sin\left(x^2+y^2\right)\ dxdy &= \iint_{[0,r]\times[0,r]} \left(\sin x^2 \cos y^2 + \cos x^2 \sin y^2\right)\ dxdy \\ &= \int_0^r \left(\int_0^r \left(\sin x^2 \cos y^2 + \cos x^2 \sin y^2\right)\ dx\right)\ dy \\ &= 2\int_0^r \sin x^2\, dx \int_0^r \cos y^2\, dy \\ &\to 2\int_0^\infty \sin x^2\, dx \int_0^\infty \cos y^2\, dy = \frac{\pi}{4}\end{aligned}$$

as $r \to \infty$ (Fresnel integrals). On the other hand, using polar coordinates,

$$\begin{aligned}\iint_{E\cap B((0,0),r)} \sin\left(x^2+y^2\right)\,dxdy &= \int_0^{\frac{\pi}{2}} \left(\int_0^r \rho\sin\rho^2\,d\rho\right)\,d\theta \\ &= \frac{\pi}{2}\left[-\frac{1}{2}\cos\rho^2\right]_{\rho=0}^{\rho=r} \\ &= \frac{\pi}{2}\left[\frac{1}{2}-\frac{1}{2}\cos r^2\right]\end{aligned}$$

and so the limit as $r \to \infty$ does not exist.

The problem is that while in $\mathbb{R}$ there is only one "natural" way to approach $[a,\infty)$ with compact sets, namely, using $[0,r]$, $r>0$, in higher dimensions there is much more freedom to approach an unbounded set with bounded sets. We would like a notion of integral that does not depend on the particular sequence of approximating sets.

Given a set $E \subseteq \mathbb{R}^N$, a sequence $\{E_n\}$ of Peano–Jordan subsets of E is called an *exhausting sequence* if $E_n \subseteq E_{n+1}$ for all n and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Definition 188 *Let $E \subseteq \mathbb{R}^N$, let $f: E \to \mathbb{R}$, and assume that there exists at least one exhausting sequence of E such that f is Riemann integrable over each set of the sequence. We say that f is* Riemann integrable in the improper sense *over E if there is an extended real number $\ell \in [-\infty,\infty]$ with the property that there exists*

$$\lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\,d\mathbf{x} = \ell$$

for every exhausting sequence $\{E_n\}$ of E such that f is Riemann integrable over each E_n. The number ℓ is called the improper Riemann integral *of f over E and is denoted $\int_E f(\mathbf{x})\,d\mathbf{x}$.*

Remark 189 *Note that we do not test the existence of the limit along every exhausting sequence but only over those exhausting sequences $\{E_n\}$ for which f is Riemann integrable over each E_n.*

We begin by showing that if f is Riemann integrable over a set, then f is Riemann integrable in the improper sense and the two notions of integrals coincide.

Theorem 190 *Let $E \subseteq \mathbb{R}^N$ be a Peano–Jordan measurable set and let $f: E \to \mathbb{R}$ be Riemann integrable. Then f is Riemann integrable in the improper sense and the Riemann integral of f over E coincides with the improper Riemann integral of f over E.*

Proof. Exercise. ■

The definition of improper Riemann integral is not practical, since one needs to check all exhausting sequences of E. The next result shows that for nonnegative functions (and similarly for nonpositive) it is enough to consider only one exhausting sequence.

Theorem 191 *Let $E \subseteq \mathbb{R}^N$ and let $f : E \to [0, \infty)$. If there exists* one *exhausting sequence $\{E_n\}$ such that f is Riemann integrable over each E_n and the limit*

$$\lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\, d\mathbf{x}$$

exists, then f is Riemann integrable in the improper sense.

Proof. Let $\{F_k\}$ be an exhausting sequence of E such that f is Riemann integrable over each F_k. Then for each fixed k, the sets $\{E_n \cap F_k\}_n$ is an exhausting sequence of F_k and since f is Riemann integrable over each F_k, it follows from the previous theorem that

$$\int_{F_k} f(\mathbf{x})\, d\mathbf{x} = \lim_{n\to\infty} \int_{E_n \cap F_k} f(\mathbf{x})\, d\mathbf{x} \le \lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\, d\mathbf{x},$$

where in the last inequality we have used the fact that $f \ge 0$. Letting $k \to \infty$ and using again the fact that $f \ge 0$ (so that the sequence $\left\{\int_{F_k} f(\mathbf{x})\, d\mathbf{x}\right\}$ is increasing, we obtain

$$\lim_{k\to\infty} \int_{F_k} f(\mathbf{x})\, d\mathbf{x} \le \lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\, d\mathbf{x}.$$

By repeating the previous argument with E_n and F_k interchanged, we obtain the opposite inequality. This shows that the limit does not depend on the particular exhausting sequence and proves the theorem. ■

Wednesday, April 04, 2012

The previous theorem fails if f is allowed to change sign as the previous example shows.

Example 192 *Let's compute*

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

We calculate

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dxdy.$$

Consider the exhausting sequence $\{B((0,0), n)\}$. Since $f(x,y) = e^{-x^2-y^2}$ is continuous and bounded over each ball, it is Riemann integrable over each ball.

Using polar coordinates, we have

$$\iint_{B((0,0),n)} e^{-x^2-y^2}\,dxdy = \int_0^n \int_0^{2\pi} e^{-r^2} r\,d\theta dr = 2\pi \int_0^n e^{-r^2} r\,dr$$
$$= 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=n} = 2\pi\left[-\frac{1}{2}e^{-n^2}+\frac{1}{2}\right].$$

Hence,

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy = \lim_{n\to\infty}\iint_{B((0,0),n)} e^{-x^2-y^2}\,dxdy$$
$$= \lim_{n\to\infty} 2\pi\left[-\frac{1}{2}e^{-n^2}+\frac{1}{2}\right] = \pi.$$

On the other hand, using the exhausting sequence $\{[-n,n]\times[-n,n]\}$*, by Theorem 160 we get*

$$\iint_{[-n,n]\times[-n,n]} e^{-x^2-y^2}\,dxdy = \int_{-n}^n \int_n^{-n} e^{-x^2}e^{-y^2}\,dxdy = \left(\int_{-n}^n e^{-t^2}\,dt\right)^2$$

and so

$$\pi = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy = \lim_{n\to\infty}\iint_{[-n,n]\times[-n,n]} e^{-x^2-y^2}\,dxdy$$
$$= \lim_{n\to\infty}\left(\int_{-n}^n e^{-t^2}\,dt\right)^2 = \left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2,$$

which implies that

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

Next we prove a comparison result that will be useful in applications.

Theorem 193 (Comparison for Improper Riemann Integrals) *Let* $E \subseteq \mathbb{R}^N$ *and let* $f : E \to \mathbb{R}$ *and* $g : E \to \mathbb{R}$*. Assume that* f *is Riemann integrable over every subset* $F \subseteq E$ *over which* g *is Riemann integrable.*

(i) If

$$|f(\mathbf{x})| \le g(\mathbf{x})$$

and g *is Riemann integrable in the improper sense over* E *with* $\int_E g(\mathbf{x})\,d\mathbf{x} < \infty$*, then* f *and* $|f|$ *are Riemann integrable in the improper sense over* E *and*

$$\left|\int_E f(\mathbf{x})\,d\mathbf{x}\right| \le \int_E |f(\mathbf{x})|\,d\mathbf{x} \le \int_E g(\mathbf{x})\,d\mathbf{x}.$$

(ii) If

$$f(\mathbf{x}) \geq g(\mathbf{x}) \geq 0$$

and g is Riemann integrable in the improper sense over E with $\int_E g(\mathbf{x})\, d\mathbf{x} = \infty$, then f is Riemann integrable in the improper sense over E and

$$\int_E f(\mathbf{x})\, d\mathbf{x} = \infty.$$

Proof. (i) Let $\{E_n\}$ be an exhausting sequence of E such that g is Riemann integrable over each E_n. Since $|f| \geq 0$, the sequence $\left\{\int_{E_n} |f(\mathbf{x})|\, d\mathbf{x}\right\}$ is increasing, and so there exists the limit

$$\lim_{n\to\infty} \int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} = \ell \in [0, \infty].$$

On the other hand, since $\int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} \leq \int_{E_n} g(\mathbf{x})\, d\mathbf{x}$ by Proposition **??**, it follows that

$$\lim_{n\to\infty} \int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} \leq \lim_{n\to\infty} \int_{E_n} g(\mathbf{x})\, d\mathbf{x} = \int_E g(\mathbf{x})\, d\mathbf{x} < \infty.$$

In view of Theorem 191, it follows that $|f|$ is Riemann integrable in the improper sense over E.

Similarly, since $|f| - f \geq 0$, the sequence $\left\{\int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x}\right\}$ is increasing, and so there exists the limit

$$\lim_{n\to\infty} \int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x} = \ell_1 \in [0, \infty].$$

On the other hand, since $\int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x} \leq 2\int_{E_n} |f(\mathbf{x})|\, d\mathbf{x}$ by Proposition **??**, it follows that

$$\lim_{n\to\infty} \int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x} \leq \lim_{n\to\infty} 2\int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} = 2\int_E |f(\mathbf{x})|\, d\mathbf{x} < \infty.$$

In view of Theorem 191, it follows that $|f| - f$ is Riemann integrable in the improper sense over E.

Using the fact that $f = |f| - (|f| - f)$, it follows that

$$\int_{E_n} f(\mathbf{x})\, d\mathbf{x} = \int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} - \int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x}.$$

Letting $n \to \infty$, we have that the right-hand side has a limit, and so there exists

$$\begin{aligned}\lim_{n\to\infty} \int_{E_n} f(\mathbf{x})\, d\mathbf{x} &= \lim_{n\to\infty} \int_{E_n} |f(\mathbf{x})|\, d\mathbf{x} - \lim_{n\to\infty} \int_{E_n} (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x} \\ &= \int_E |f(\mathbf{x})|\, d\mathbf{x} - \int_E (|f(\mathbf{x})| - f(\mathbf{x}))\, d\mathbf{x}.\end{aligned}$$

This shows that for *every* exhausting sequence $\{E_n\}$ of E such that f is Riemann integrable over each E_n, there exists the limit

$$\lim_{n\to\infty}\int_{E_n} f(\mathbf{x})\,d\mathbf{x} = \int_E |f(\mathbf{x})|\,d\mathbf{x} - \int_E (|f(\mathbf{x})| - f(\mathbf{x}))\,d\mathbf{x}.$$

Since the limit is the same for every exhausting sequence $\{E_n\}$, we have that f is Riemann integrable in the improper sense over E.

(ii) Let $\{E_n\}$ be an exhausting sequence of E such that g is Riemann integrable over each E_n. By hypothesis f is Riemann integrable over each E_n, and so by Proposition **??**, it follows that

$$\int_{E_n} f(\mathbf{x})\,d\mathbf{x} \geq \int_{E_n} g(\mathbf{x})\,d\mathbf{x}.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty}\int_{E_n} f(\mathbf{x})\,d\mathbf{x} \geq \lim_{n\to\infty}\int_{E_n} g(\mathbf{x})\,d\mathbf{x} = \int_E g(\mathbf{x})\,d\mathbf{x} = \infty.$$

It follows from Theorem 191 that f is Riemann integrable in the improper sense over E with $\int_E f(\mathbf{x})\,d\mathbf{x} = \infty$. ■

Remark 194 *An important function g in the previous theorem is given by*

$$g(\mathbf{x}) := \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|^a},$$

where $\mathbf{x}_0$ is a fixed point of $\mathbb{R}^N$ and $a > 0$.

Exercise 195 *Consider the function*

$$f(x, y) := \frac{1}{(x^2 + y^2)^2}$$

defined in the set

$$E := \left\{(x, y) \in \mathbb{R}^2 :\, y \leq x^4,\, x^2 + y^2 \leq 2,\, x > 0,\, y > 0\right\}.$$

Prove that f is Riemann integrable in the improper sense over E and that the improper Riemann integral is finite.

Exercise 196 *Consider the function*

$$f(x, y) := \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

defined in the open square Q of vertices $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$. Prove that f is not Riemann integrable in the improper sense over Q.

The next result shows that if f is Riemann integrable in the improper sense over E with finite improper Riemann integral, then the same is true for $|f|$. This is in sharp contrast with the 1-dimensional case. Hence, the definition of improper integral in $\mathbb{R}^N$ is *not* an extension of the one given for $N = 1$.

Theorem 197 *Let $E \subseteq \mathbb{R}^N$ and let $f : E \to \mathbb{R}$ be Riemann integrable in the improper sense over E with finite improper Riemann integral. Then $|f|$ is Riemann integrable in the improper sense over E with finite improper Riemann integral.*

Proof. Step 1: Let $F \subseteq E$ be Peano–Jordan measurable and assume that f is Riemann integrable over F. Then there exists a Peano–Jordan measurable set $G \subseteq F$ such that

$$\int_F |f(\mathbf{x})| \, d\mathbf{x} \leq 3 \left| \int_G f(\mathbf{x}) \, d\mathbf{x} \right|.$$

Let $s := \int_F |f(\mathbf{x})| \, d\mathbf{x}$. If $s = 0$, take $G := \emptyset$. If $s > 0$, then writing

$$s := \int_F |f(\mathbf{x})| \, d\mathbf{x} = \int_F f^+(\mathbf{x}) \, d\mathbf{x} + \int_F f^-(\mathbf{x}) \, d\mathbf{x},$$

where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$, we have that $\int_F f^+(\mathbf{x}) \, d\mathbf{x} \geq \frac{s}{2}$ or $\int_F f^-(\mathbf{x}) \, d\mathbf{x} \geq \frac{s}{2}$. Assume that $\int_F f^+(\mathbf{x}) \, d\mathbf{x} \geq \frac{s}{2}$ (the other case is similar). Let R be a rectangle containing F. Since

$$\int_R \chi_F(\mathbf{x}) f^+(\mathbf{x}) \, d\mathbf{x} = \int_F f^+(\mathbf{x}) \, d\mathbf{x} \geq \frac{s}{2} > \frac{s}{3},$$

by the definition of lower integral there exists a partition $\mathcal{P} = \{R_1, \ldots, R_n\}$ of R such that

$$L\left(\chi_F f^+, \mathcal{P}\right) := \sum_{i=1}^n \operatorname{meas} R_i \inf_{\mathbf{x} \in R_i} \chi_F(\mathbf{x}) f^+(\mathbf{x}) > \frac{s}{3}.$$

Let G be given by the union of all rectangles R_i such that $\inf_{\mathbf{x} \in R_i} \chi_F(\mathbf{x}) f^+(\mathbf{x}) > 0$. Then G is a pluri-rectangle, and so it is Peano–Jordan measurable. Moreover, if $\mathbf{x} \in G$, then $\mathbf{x} \in R_i$ for some i. In turn, $\chi_F(\mathbf{x}) f^+(\mathbf{x}) > 0$, which implies that $\mathbf{x} \in F$. Thus, $G \subseteq F$. Moreover, $f(\mathbf{x}) = f^+(\mathbf{x})$ for all $\mathbf{x} \in G$. Hence,

$$\begin{aligned} \left| \int_G f(\mathbf{x}) \, d\mathbf{x} \right| &= \int_G f^+(\mathbf{x}) \, d\mathbf{x} \\ &\geq L\left(\chi_F f^+, \mathcal{P}\right) > \frac{s}{3} = \frac{1}{3} \int_F |f(\mathbf{x})| \, d\mathbf{x}. \end{aligned}$$

■

Friday, April 06, 2012

Proof. Step 2: We claim that there exists $L > 0$ such that

$$\left| \int_F f(\mathbf{x}) \, d\mathbf{x} \right| \leq L$$

for every Peano–Jordan measurable set $F \subseteq E$ such that f is Riemann integrable over F. Assume by contradiction that this is not the case and let $\{E_n\}$ be an exhausting sequence of E such that f is Riemann integrable over each E_n. Define $G_1 := E_1$. Since f is Riemann integrable over G_1, we have that $|f|$ is also Riemann integrable over G_1, so that $\int_{G_1} |f(\mathbf{x})| \, d\mathbf{x} < \infty$. By the contradiction hypothesis, there exists a Peano–Jordan measurable set $F_1 \subseteq E$ such that f is Riemann integrable over F_1 and

$$\left| \int_{F_1} f(\mathbf{x}) \, d\mathbf{x} \right| \geq 1 + \int_{G_1} |f(\mathbf{x})| \, d\mathbf{x},$$

since otherwise we could take $L := 1 + \int_{G_1} |f(\mathbf{x})| \, d\mathbf{x}$. Inductively, for every $n \geq 2$ we can find a Peano–Jordan measurable set $F_n \subseteq E$ such that f is Riemann integrable over F_n and

$$\left| \int_{F_n} f(\mathbf{x}) \, d\mathbf{x} \right| \geq n + \int_{G_n} |f(\mathbf{x})| \, d\mathbf{x},$$

where $G_n := E_n \cup F_1 \cup \cdots \cup F_{n-1}$.

Note that the sequence $H_n := F_n \cup G_n$ is an admissible exhausting sequence of E. Moreover, using the fact that $H_n \setminus F_n \subseteq G_n$, we have that

$$\begin{aligned}
\left| \int_{H_n} f(\mathbf{x}) \, d\mathbf{x} \right| &= \left| \int_{F_n} f(\mathbf{x}) \, d\mathbf{x} + \int_{H_n \setminus F_n} f(\mathbf{x}) \, d\mathbf{x} \right| \\
&\geq \left| \int_{F_n} f(\mathbf{x}) \, d\mathbf{x} \right| - \left| \int_{H_n \setminus F_n} f(\mathbf{x}) \, d\mathbf{x} \right| \\
&\geq \left| \int_{F_n} f(\mathbf{x}) \, d\mathbf{x} \right| - \int_{H_n \setminus F_n} |f(\mathbf{x})| \, d\mathbf{x} \\
&\geq \left| \int_{F_n} f(\mathbf{x}) \, d\mathbf{x} \right| - \int_{G_n} |f(\mathbf{x})| \, d\mathbf{x} \geq n,
\end{aligned}$$

which implies that

$$\left| \int_{E} f(\mathbf{x}) \, d\mathbf{x} \right| = \lim_{n \to \infty} \left| \int_{H_n} f(\mathbf{x}) \, d\mathbf{x} \right| = \infty.$$

This contradicts the fact that $\left| \int_E f(\mathbf{x}) \, d\mathbf{x} \right| < \infty$.

Step 3: We claim that

$$\int_F |f(\mathbf{x})| \, d\mathbf{x} \leq 3L$$

for every Peano–Jordan measurable set $F \subseteq E$ such that f is Riemann integrable over F. This follows from Step 2 by applying Step 1.

Step 4: In view of the previous step, for any $\{E_n\}$ exhausting sequence of E such that $|f|$ is Riemann integrable over each E_n we have that

$$\int_{E_n} |f(\mathbf{x})| \, d\mathbf{x} \leq 3L.$$

Since $|f| \geq 0$, letting $n \to \infty$ and using Theorem 191, we get that

$$\int_E |f(\mathbf{x})| \, d\mathbf{x} \leq 3L.$$

This concludes the proof. ■

Example 198 *Consider the function $f : [\pi, \infty) \to \mathbb{R}$ defined by*

$$f(x) = \frac{\sin x}{x}.$$

Then the limit

$$\lim_{r \to \infty} \int_\pi^r \frac{\sin x}{x} \, dx$$

exists and is finite. To see this, we integrate by parts,

$$\begin{aligned} \int_\pi^r \frac{\sin x}{x} \, dx &= \left[-\frac{\cos x}{x} \right]_\pi^r - \int_\pi^r -\frac{\cos x}{x^2} \, dx \\ &= \frac{1}{\pi} - \frac{\cos r}{r} + \int_\pi^r \frac{\cos x}{x^2} \, dx. \end{aligned}$$

Since $\left| \frac{\cos x}{x^2} \right| \leq \frac{1}{x^2}$ and $\int_\pi^\infty \frac{1}{x^2} \, dx < \infty$, it follows that there exists

$$\lim_{r \to \infty} \int_\pi^r \frac{\sin x}{x} \, dx = \frac{1}{\pi} - 0 + \lim_{r \to \infty} \int_\pi^r \frac{\cos x}{x^2} \, dx =: \ell \in \mathbb{R}. \tag{33}$$

Thus, f is Riemann integrable in the improper sense according to the definition of Real Analysis I. However, f is not Riemann integrable in the improper sense according to Definition 188. We show this in two two ways.

We show that $\int_\pi^\infty \left| \frac{\sin x}{x} \right| dx = \infty$. Indeed, for $n \geq 2$,

$$\begin{aligned} \int_\pi^{n\pi} \left| \frac{\sin x}{x} \right| dx &= \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \left| \frac{\sin x}{x} \right| dx \geq \sum_{k=2}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x| \, dx \\ &= \sum_{k=2}^n \frac{1}{k\pi} \int_0^\pi |\sin x| \, dx = \sum_{k=2}^n \frac{2}{k\pi}. \end{aligned}$$

Hence,

$$\int_\pi^\infty \left| \frac{\sin x}{x} \right| dx = \lim_{n \to \infty} \int_\pi^{n\pi} \left| \frac{\sin x}{x} \right| dx \geq \lim_{n \to \infty} \sum_{k=2}^n \frac{2}{k\pi} = \infty.$$

In view of the previous theorem and of (33), f cannot be Riemann integrable in the improper sense according to Definition 188. A more direct way to see this is to consider the exhausting sequence

$$E_n := [\pi, (2n-1)\pi] \cup \bigcup_{k=n}^{2n} [2k\pi, (2k+1)\pi].$$

We have

$$\begin{aligned}\int_{E_n} \frac{\sin x}{x}\,dx &= \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \sum_{k=n}^{2n} \int_{2k\pi}^{(2k+1)\pi} \frac{\sin x}{x}\,dx \\ &\geq \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \sum_{k=n}^{2n} \frac{1}{(2k+1)\,\pi} \int_{2k\pi}^{(2k+1)\pi} \sin x\,dx \\ &= \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \sum_{k=n}^{2n} \frac{2}{(2k+1)\,\pi} \\ &\geq \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \sum_{k=n}^{2n} \frac{2}{(4n+1)\,\pi} \\ &= \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \frac{2n+2}{\pi + 4\pi n}.\end{aligned}$$

Letting $n \to \infty$ and using (33) gives

$$\liminf_{n\to\infty} \int_{E_n} \frac{\sin x}{x}\,dx \geq \lim_{n\to\infty} \int_{\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx + \lim_{n\to\infty} \frac{2n+2}{\pi + 4\pi n} = \ell + \frac{1}{2\pi} > \ell,$$

which shows that f cannot be Riemann integrable in the improper sense according to Definition 188, since along two exhausting sequences we obtain different limits.

12 Change of Variables

We show that Lipschitz transformations map sets of measure zero into sets of measure zero.

Theorem 199 *Let $E \subset \mathbb{R}^N$ be a Peano–Jordan measurable and let $\mathbf{g} : E \to \mathbb{R}^M$ be a Lipschitz function, with $N \leq M$. Moreover, if $N = M$, assume that E has measure zero. Then $\mathbf{g}(E)$ is Peano–Jordan measurable with measure zero.*

Proof. Exercise. ■

Next we show that Peano–Jordan measurability is preserved under C^1 transformations.

Theorem 200 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^M$ be a function of class C^1. Let $E \subset \mathbb{R}^N$ be a Peano–Jordan measurable set with $\overline{E} \subseteq U$. Moreover, if $N = M$, assume that E has measure zero. Then $\mathbf{g}(E)$ is Peano–Jordan measurable with measure zero.*

Proof. Exercise. ■

Remark 201 *In particular, it* $\mathbf{g}$ *is the parametrization of a curve, then* $N = 1$ *so that by the previous theorem it follows that the range of a curve has measure zero for all* $M > 1$*. Hence, sets of the type*

$$E_1 := \left\{(x, y) \in \mathbb{R}^2 : \, 4x^2 + 9y^2 \leq 16\right\},$$
$$E_2 := \left\{(x, y) \in \mathbb{R}^2 : \, x^2 + y^2 \leq 1\right\},$$

are Peano-Jordan measurable, since their boundary is the range of a curve. Consider the parametrization $\left(\cos t, \frac{4}{3}\sin t\right)$, $t \in [0, 2\pi]$, *for* ∂E_1 *and* $(\cos t, \sin t)$, $t \in [0, 2\pi]$, *for* ∂E_2.

We will see that a similar result holds in higher dimensions. In this case to describe the boundary of a set we will need to talk about surfaces.

An alternative proof would be to show that $\mathbf{g} : \overline{E} \to \mathbb{R}^M$ is Lipschitz (see the following exercise).

Exercise 202 *Let* $U \subseteq \mathbb{R}^N$ *be an open set, let* $K \subset U$ *be a compact set, and let* $f : U \to \mathbb{R}$ *be a function of class* C^1.

1. *Prove that* $R := \operatorname{dist}(K, \partial U) > 0$.
2. *For* $0 < \delta < R$ *and consider the set*

$$V := \{\mathbf{x} \in U : \, \operatorname{dist}(\mathbf{x}, K) < \delta\}.$$

   Prove that V *is open, bounded and* $K \subset V \subset \overline{V} \subset U$.

3. *Prove that* $f : K \to \mathbb{R}$ *is Lipschitz continuous.*

Corollary 203 *Let* $U \subseteq \mathbb{R}^N$ *be an open set and let* $\mathbf{g} : U \to \mathbb{R}^N$ *be a function of class* C^1*. Let* $E \subset \mathbb{R}^N$ *be a Peano–Jordan measurable set with* $\overline{E} \subseteq U$*. Assume that* $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in E^\circ$*. Then* $\mathbf{g}(E)$ *is Peano–Jordan measurable.*

Proof. Exercise. ■

Using the previous theorems, we can show the change of variables theorem.

Theorem 204 (Change of variables for multiple integrals) *Let* $U \subseteq \mathbb{R}^N$ *be an open set and let* $\mathbf{g} : U \to \mathbb{R}^N$ *be a one-to-one function of class* C^1 *such that* $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ *for all* $\mathbf{x} \in U$*. Let* $E \subset \mathbb{R}^N$ *be Peano–Jordan measurable with* $\overline{E} \subseteq U$ *and let* $f : \mathbf{g}(E) \to \mathbb{R}$ *be Riemann integrable. Then the function* $\mathbf{x} \in E \mapsto f(\mathbf{g}(\mathbf{x})) |\det J_{\mathbf{g}}(\mathbf{x})|$ *is Riemann integrable and*

$$\int_{\mathbf{g}(E)} f(\mathbf{y}) \, d\mathbf{y} = \int_E f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| \, d\mathbf{x}.$$

Example 205 *Let's calculate the integral*

$$\iint_F (x + y) \, dxdy,$$

where

$$F := \left\{ (x,y) \in \mathbb{R}^2 : \, 0 < x < y < 2x, \, 1 < xy < 2 \right\}.$$

Consider the change of variables $u = xy$ *and* $v = \frac{y}{x}$. *Let's solve for* x *and* y. *We have* $x = \sqrt{\frac{u}{v}}$ *and* $y = \sqrt{uv}$. *Define* $\mathbf{g} : U \to \mathbb{R}^2$ *as follows*

$$\mathbf{g}(u,v) := \left(\sqrt{\frac{u}{v}}, \sqrt{uv} \right),$$

where

$$U := \left\{ (u,v) \in \mathbb{R}^2 : \, u > 0, \, v > 0 \right\}.$$

Then $\mathbf{g}$ *is one-to-one and its inverse is given by* $\mathbf{g}^{-1}(x,y) := \left(xy, \frac{y}{x} \right)$. *(Note that in general it is much simpler to check that* $\mathbf{g}$ *is one-to-one than finding the inverse). Moreover,*

$$\begin{aligned} \det J_{\mathbf{g}}(u,v) &= \det \begin{pmatrix} \frac{\partial g_1}{\partial u}(u,v) & \frac{\partial g_1}{\partial v}(u,v) \\ \frac{\partial g_2}{\partial u}(u,v) & \frac{\partial g_2}{\partial v}(u,v) \end{pmatrix} \\ &= \det \begin{pmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v^{3/2}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{pmatrix} = \frac{1}{2v}. \end{aligned}$$

Set $E = (1,2) \times (1,2)$ *and note* $\mathbf{g}(E) = F$. *By the change of variables theorem,*

$$\begin{aligned} \iint_F (x+y)\, dxdy &= \iint_E (g_1(u,v) + g_2(u,v)) \left| \det J_{\mathbf{g}}(u,v) \right| dudv \\ &= \iint_E \left(\sqrt{\frac{u}{v}} + \sqrt{uv} \right) \frac{1}{2v}\, dudv \\ &= \frac{1}{2} \iint_E \left(\frac{\sqrt{u}}{v^{3/2}} + \frac{\sqrt{u}}{v} \right) dudv. \end{aligned}$$

It follows that

$$\begin{aligned} \iint_F (x+y)\, dxdy &= \frac{1}{2} \int_1^2 \left(\int_1^2 \left(\frac{\sqrt{u}}{v^{3/2}} + \frac{\sqrt{u}}{v} \right) dv \right) du = \frac{1}{2} \int_1^2 \left(\left[-2\frac{\sqrt{u}}{v^{1/2}} + \sqrt{u} \log v \right]_{v=1}^{v=2} \right) du \\ &= \frac{1}{2} \int_1^2 \sqrt{u} \ln 2\, du = \frac{1}{3} (\ln 2) \left(2\sqrt{2} - 1 \right). \end{aligned}$$

Monday, April 09, 2012

Given an open set $U \subseteq \mathbb{R}^N$, a function $\mathbf{g} : U \to \mathbb{R}^N$ is called *elementary* or *primitive* if there exists $m \in \{1, \ldots, N\}$ such that for all $i = 1, \ldots, N$, with $i \neq m$, and all $\mathbf{x} \in U$,

$$g_i(\mathbf{x}) = x_i.$$

In other words, except for the m-th component, $\mathbf{g}$ is the identity. Note that if $\mathbf{g}$ is differentiable, then

$$\nabla g_i(\mathbf{x}) = \mathbf{e}_i$$

for all $i = 1, \ldots, N$, with $i \neq m$, and all $\mathbf{x} \in U$, so that

$$\det J_{\mathbf{g}}(\mathbf{x}) = \frac{\partial g_m}{\partial x_m}(\mathbf{x}). \tag{34}$$

A function $\mathbf{g} : U \to \mathbb{R}^N$ is called a *flip* if there exist $m, n \in \{1, \ldots, N\}$ such that for all $i = 1, \ldots, N$, with $i \neq m, n$, and all $\mathbf{x} \in U$,

$$g_i(\mathbf{x}) = x_i,$$

while

$$g_m(\mathbf{x}) = x_n, \quad g_n(\mathbf{x}) = x_m.$$

Theorem 206 *Let $U \subseteq \mathbb{R}^N$ be an open set and let $\mathbf{g} : U \to \mathbb{R}^N$ be a function of class C^1 such that $\det J_{\mathbf{g}}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U$. Given $\mathbf{x}_0 \in U$, there exist $B(\mathbf{x}_0, r) \subseteq U$ and n bounded functions of class C^1, $\mathbf{g}_i : U_i \to \mathbb{R}^N$, $i = 1, \ldots, n$, such that $U_i \subset \mathbb{R}^N$ is open and bounded, $U_n = B(\mathbf{x}_0, r)$, $\det J_{\mathbf{g}_i}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U_i$, $\mathbf{g}_i$ is either elementary or a flip, $\mathbf{g}_i(U_i) \subseteq U_{i-1}$ for all $i = 2, \ldots, n$, and*

$$\mathbf{g} = \mathbf{g}_1 \circ \cdots \circ \mathbf{g}_n \quad \text{in } B(\mathbf{x}_0, r)$$

Lemma 207 *Let $g : [a, b] \to \mathbb{R}$ be a one-to-one differentiable function with continuous derivative g' such that $g'(x) \neq 0$ for all $x \in [a, b]$, and let $f : g([a, b]) \to \mathbb{R}$ be a bounded function Then*

$$\overline{\int_{g([a,b])}} f(y)\, dy = \overline{\int_a^b} f(g(x))\, |g'(x)|\, dx$$

Proof. Assume that $g' \geq 0$ (the case $g' \leq 0$ is similar). Since $[a, b]$ is compact and $g' : [a, b] \to \mathbb{R}$ is continuous, g' is uniformly continuous, and so given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|g'(x) - g'(z)| \leq \varepsilon \tag{35}$$

for all $x, z \in [a, b]$ with $|x - z| \leq \delta$.

Consider a partition P of $[a, b]$ and find a refinement $P_\varepsilon = \{x_0, \ldots, x_n\}$ with the property that $(x_i - x_{i-1}) \leq \delta$ for all $i = 1, \ldots, n$. Since $g : [a, b] \to \mathbb{R}$ is one-to-one and continuous, the set $g([a, b])$ is an interval and, setting $y_i := g(x_i)$, the set $Q_\varepsilon = \{y_0, \ldots, y_n\}$ is a partition of $g([a, b])$. Let J_i be the closed interval of endpoints y_{i-1} and y_i. By the definition of upper integral

$$\overline{\int_{g([a,b])}} f(y)\, dy \leq U(f, Q_\varepsilon) = \sum_{i=1}^n (y_i - y_{i-1}) \sup_{y \in J_i} f(y). \tag{36}$$

By the mean value theorem, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$\begin{aligned}(y_i - y_{i-1}) &= (g(x_i) - g(x_{i-1})) = g'(c_i)(x_i - x_{i-1}) \\ &\leq (x_i - x_{i-1}) \sup_{x \in J_i} g'(x).\end{aligned}$$

Moreover, since $g([x_{i-1}, x_i]) = J_i$,

$$\sup_{y \in J_i} f(y) = \sup_{x \in [x_{i-1}, x_i]} f(g(x)).$$

Hence,

$$\sum_{i=1}^{n} (y_i - y_{i-1}) \sup_{y \in J_i} f(y) \le \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(g(x)) \sup_{z \in [x_{i-1}, x_i]} g'(z). \tag{37}$$

Given $x, z \in [x_{i-1}, x_i]$, we have that

$$\begin{aligned} f(g(x))\, g'(z) &= f(g(x))\,(g'(x) + g'(z) - g'(x)) \\ &\le f(g(x))\, g'(x) + |f(g(x))|\, |g'(z) - g'(x)| \\ &\le f(g(x))\, g'(x) + \varepsilon M, \end{aligned} \tag{38}$$

where we have used (35), and where $M \ge 0$ is a bound for $|f|$. It follows that

$$\sup_{x \in [x_{i-1}, x_i]} f(g(x)) \sup_{z \in [x_{i-1}, x_i]} g'(z) \le \sup_{x \in [x_{i-1}, x_i]} (f(g(x))\, g'(x)) + \varepsilon M,$$

and so

$$\begin{aligned} &\sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(g(x)) \sup_{z \in [x_{i-1}, x_i]} g'(z) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(g(x))\, g'(x) + \sum_{i=1}^{n} (x_i - x_{i-1})\, \varepsilon M \\ &= U((f \circ g)\, g', P_\varepsilon) + \varepsilon M (b - a) \le U((f \circ g)\, g', P) + \varepsilon M (b - a). \end{aligned} \tag{39}$$

By combining the inequalities (36)-(39), we get

$$\overline{\int_{g([a,b])}} f(y)\, dy \le U((f \circ g)\, g', P) + \varepsilon M (b - a).$$

Taking the infimum over all partitions P of $[a, b]$, we get

$$\overline{\int_{g([a,b])}} f(y)\, dy \le \overline{\int_a^b} f(g(x))\, g'(x)\, dx + \varepsilon M (b - a).$$

Letting $\varepsilon \to 0$ gives

$$\overline{\int_{g([a,b])}} f(y)\, dy \le \overline{\int_{[a,b]}} f(g(x))\, g'(x)\, dx.$$

To prove the opposite inequality, consider the function $f_1(x) := f(g(x))\, g'(x)$, let $g([a, b]) = [c, d]$ and apply what we just proved to the nonnegative bounded function f_1, using g^{-1} in place of g, to get

$$\overline{\int_a^b} f(g(x))\, g'(x)\, dx = \overline{\int_{g^{-1}([c,d])}} f_1(x)\, dx \leq \overline{\int_c^d} f_1\left(g^{-1}(y)\right) \left(g^{-1}\right)'(y)\, dy$$
$$= \overline{\int_{g([a,b])}} f\left(g\left(g^{-1}(y)\right)\right) g'\left(g^{-1}(y)\right) \left(g^{-1}\right)'(y)\, dy = \overline{\int_{g([a,b])}} f(y)\, dy,$$

where we have used the fact that

$$\left(g^{-1}\right)'(y) = \frac{1}{g'\left(g^{-1}(y)\right)}$$

for all $y \in [a, b]$ (see Theorem ??). ■

The previous lemma holds under significantly weaker hypotheses.

In the proof of Theorem 204 we will use the following facts about sections. If $V \subseteq \mathbb{R}^N$ is open, writing $\mathbf{x} = (\mathbf{x}', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, we have that for every $\mathbf{x}' \in \mathbb{R}^{N-1}$ the section

$$V_{\mathbf{x}'} = \{x_N \in \mathbb{R} : (\mathbf{x}', x_N) \in V\}$$

is open in $\mathbb{R}$. In turn, if $C \subseteq \mathbb{R}^N$ is closed, we have that for every $\mathbf{x}' \in \mathbb{R}^{N-1}$ the section

$$C_{\mathbf{x}'} = \{x_N \in \mathbb{R} : (\mathbf{x}', x_N) \in C\}$$

is closed. Indeed, $\mathbb{R} \setminus C_{\mathbf{x}'} = \{x_N \in \mathbb{R} : (\mathbf{x}', x_N) \notin C\} = \left(\mathbb{R}^N \setminus C\right)_{\mathbf{x}'}$, which is open.

We now turn to the proof of Theorem 204.

Proof of Theorem 204. Step 1: Let's prove that the theorem holds if U is open and bounded and $\mathbf{g}$ is elementary and bounded. Without loss of generality, we may assume that the only component that is not the identity is the last one, so that for all $i = 1, \ldots, N-1$ and all $\mathbf{x} \in U$,

$$g_i(\mathbf{x}) = x_i.$$

In this case, by (34),

$$|\det J_{\mathbf{g}}(\mathbf{x})| = \left|\frac{\partial g_N}{\partial x_N}(\mathbf{x})\right|.$$

Consider a rectangle R containing $\mathbf{g}(U)$ and define

$$h(\mathbf{y}) := \begin{cases} f(\mathbf{y}) & \text{if } \mathbf{y} \in R, \\ 0 & \text{if } \mathbf{y} \in R \setminus \mathbf{g}(E). \end{cases} \tag{40}$$

Then by the definition of Riemann integral of f over E, we have that

$$\int_{\mathbf{g}(E)} f(\mathbf{y})\, d\mathbf{y} = \int_R h(\mathbf{y})\, d\mathbf{y}. \tag{41}$$

Let

$$R = I_1 \times \cdots \times I_N.$$

Write $\mathbf{y} = (\mathbf{y}', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and let $R = R' \times I_N$, where $R' := I_1 \times \cdots \times I_{N-1}$. By Theorem 160 and Remark 161,

$$\int_R h(\mathbf{y})\, d\mathbf{y} = \int_{R'} \left(\overline{\int_{I_N}} h(\mathbf{y}', y_N)\, dy_N \right) d\mathbf{y}'. \tag{42}$$

Fix $\mathbf{y}' \in R'$. By the inverse function theorem, the set $V := \mathbf{g}(U)$ is open and so its section $V_{\mathbf{y}'}$. Moreover, since $\overline{E}$ is compact and $\mathbf{g}$ is continuous, it follows that the set $K := \mathbf{g}(\overline{E})$ is compact. In turn, its section $K_{\mathbf{y}'}$ is also compact. Since $V_{\mathbf{y}'}$ is open in $\mathbb{R}$, it can be written as a countable union of disjoint open intervals, that is,

$$V_{\mathbf{y}'} = \bigcup_n J_n.$$

Then

$$K_{\mathbf{y}'} \subset V_{\mathbf{y}'} = \bigcup_n J_n \subseteq I_N,$$

and so by compactness there exists $\ell \in \mathbb{N}$ such that $K_{\mathbf{y}'} \subset \bigcup_{n=1}^{\ell} J_n \subseteq I_N$. Using the fact that $h(\mathbf{y}', y_N) = 0$ for all $y_N \notin (\mathbf{g}(E))_{\mathbf{y}'} \subseteq K_{\mathbf{y}'}$, we have that (see Proposition 242 in the Real Analysis I notes)

$$\overline{\int_{I_N}} h(\mathbf{y}', y_N)\, dy_N = \sum_{n=1}^{\ell} \overline{\int_{J_n}} h(\mathbf{y}', y_N)\, dy_N. \tag{43}$$

Consider the the change of variables $y_N = g_N(\mathbf{y}', x_N)$. Since the function $k(x_N) := g_N(\mathbf{y}', x_N)$, $x_N \in U_{\mathbf{y}'}$, is one-to-one and of class C^1 and $J_n \subseteq V_{\mathbf{y}'} = k(U_{\mathbf{y}'})$, we have that $k^{-1}(J_n)$ is an interval, and so we are in a position to apply the previous lemma to obtain

$$\sum_{n=1}^{\ell} \overline{\int_{J_n}} h(\mathbf{y}', y_N)\, dy_N = \sum_{n=1}^{\ell} \overline{\int_{k^{-1}(J_n)}} h(\mathbf{y}', g_N(\mathbf{y}', x_N)) \left| \frac{\partial g_N}{\partial x_N}(\mathbf{y}', x_N) \right| dx_N. \tag{44}$$

Since $g_i(\mathbf{y}', x_N) = y_i$ for $i \neq N$, $k^{-1}(J_n) \subseteq U_{\mathbf{y}'}$, and $h(\mathbf{y}', g_N(\mathbf{y}', x_N)) = 0$ for $x_N \notin E_{\mathbf{y}'} \subseteq U_{\mathbf{y}'}$, if we consider an interval $R_1 := R' \times J$ containing U and define

$$h_1(\mathbf{x}) := \begin{cases} f(\mathbf{g}(\mathbf{x})) \left| \frac{\partial g_N}{\partial x_N}(\mathbf{x}) \right| & \text{if } \mathbf{x} \in E, \\ 0 & \text{if } \mathbf{y} \in R_1 \setminus E, \end{cases}$$

then by (40) and Proposition 242 in the Real Analysis I notes

$$\sum_{n=1}^{\ell} \overline{\int_{k^{-1}(J_n)}} h(\mathbf{y}', g_N(\mathbf{y}', x_N)) \left| \frac{\partial g_N}{\partial x_N}(\mathbf{y}', x_N) \right| dx_N = \sum_{n=1}^{\ell} \overline{\int_{k^{-1}(J_n)}} h_1(\mathbf{y}', x_N)\, dx_N \tag{45}$$

$$= \overline{\int_J} h_1(\mathbf{y}', x_N)\, dx_N.$$

Hence, by (41)-(45), Theorem 160 and Remark 161,

$$\int_{\mathbf{g}(E)} f(\mathbf{y})\, d\mathbf{y} = \int_{R'} \left(\overline{\int_J} h_1(\mathbf{y}', x_N)\, dx_N \right) d\mathbf{y}'$$
$$= \int_{R_1} h_1(\mathbf{x})\, d\mathbf{x}$$
$$= \int_E f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| d\mathbf{x}.$$

Step 2: If $\mathbf{g}$ is a flip, then $J_{\mathbf{g}}(\mathbf{x})$ is obtained by interchanging two rows in the identity matrix I_N and so $\left|\det J_{\mathbf{g}}(\mathbf{x})\right| = 1$. The results follows by applying Theorem 160. ■

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Proof. Step 3: We prove that if the theorem holds for $\mathbf{g}_1 : U_1 \to \mathbb{R}^N$ and for $\mathbf{g}_2 : U_2 \to \mathbb{R}^N$, where $\mathbf{g}_2(U_2) \subseteq U_1$, then it holds for $\mathbf{g}_1 \circ \mathbf{g}_2$. More precisely, we are assuming that $U_i \subset \mathbb{R}^N$ are open bounded sets, that $\mathbf{g}_i : U_i \to \mathbb{R}^N$ are one-to-one bounded functions of class C^1 such that $\det J_{\mathbf{g}_i}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in U_i$, and that for all $E_i \subset \mathbb{R}^N$ Peano–Jordan measurable sets with $\overline{E_i} \subset U_i$ and for all $f_i : \mathbf{g}_i(E_i) \to \mathbb{R}$ Riemann integrable, the functions $\mathbf{x} \in E_i \mapsto f_i(\mathbf{g}_i(\mathbf{x})) \left|\det J_{\mathbf{g}_i}(\mathbf{x})\right|$ are Riemann integrable and

$$\int_{\mathbf{g}_i(E_i)} f_i(\mathbf{y})\, d\mathbf{y} = \int_{E_i} f_i(\mathbf{g}_i(\mathbf{x})) \left|\det J_{\mathbf{g}_i}(\mathbf{x})\right| d\mathbf{x},$$

where $i = 1, 2$.

Consider the function $\mathbf{g} := \mathbf{g}_1 \circ \mathbf{g}_2 : U_2 \to \mathbb{R}^N$, let $E \subset \mathbb{R}^N$ be a Peano–Jordan measurable set with $\overline{E} \subset U_2$ and let $f : \mathbf{g}(E) \to \mathbb{R}$ be Riemann integrable. Then

$$\int_{\mathbf{g}(E)} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathbf{g}_1(\mathbf{g}_2(E))} f(\mathbf{y})\, d\mathbf{y},$$

and so, by applying the change of variable $\mathbf{y} = \mathbf{g}_1(\mathbf{z})$ (with $E_1 := \mathbf{g}_2(E)$ and $f_1 := f$), we get

$$\int_{\mathbf{g}_1(\mathbf{g}_2(E))} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathbf{g}_2(E)} f(\mathbf{g}_1(\mathbf{z})) \left|\det J_{\mathbf{g}_1}(\mathbf{z})\right| d\mathbf{z}.$$

On the other hand, applying the change of variable $\mathbf{z} = \mathbf{g}_2(\mathbf{x})$ (with $f_2(\mathbf{z}) := f(\mathbf{g}_1(\mathbf{z})) \left|\det J_{\mathbf{g}_1}(\mathbf{z})\right|$ and $E_2 := E$), we get

$$\int_{\mathbf{g}_2(E)} f(\mathbf{g}_1(\mathbf{z})) \left|\det J_{\mathbf{g}_1}(\mathbf{z})\right| d\mathbf{z} = \int_E f(\mathbf{g}_1(\mathbf{g}_2(\mathbf{x}))) \left|\det J_{\mathbf{g}_1}(\mathbf{g}_2(\mathbf{x}))\right| \left|\det J_{\mathbf{g}_2}(\mathbf{x})\right| d\mathbf{x}$$
$$= \int_E f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| d\mathbf{x},$$

where we have used the fact that

$$J_{\mathbf{g}}(\mathbf{x}) = J_{\mathbf{g}_1}(\mathbf{g}_2(\mathbf{x}))\, J_{\mathbf{g}_2}(\mathbf{x})$$

by Corollary 67, so that

$$\det J_{\mathbf{g}}(\mathbf{x}) = \det J_{\mathbf{g}_1}(\mathbf{g}_2(\mathbf{x})) \det J_{\mathbf{g}_2}(\mathbf{x}) \neq 0.$$

Step 4: By Theorem 206 for every $\mathbf{x} \in U$ there exists $B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$ such that $\mathbf{g}$ restricted to $B(\mathbf{x}, r_{\mathbf{x}})$ is given by the union of finite many elementary diffeomorphisms and flips. Since $\overline{E} \subset U$, the family of balls $\left\{B\left(\mathbf{x}, \frac{1}{2}r_{\mathbf{x}}\right)\right\}_{\mathbf{x}\in U}$ covers the compact set $\overline{E}$, and so by compactness there exist $\mathbf{x}_1, \ldots, \mathbf{x}_n \in U$ such that

$$\overline{E} \subset \bigcup_{i=1}^{n} B\left(\mathbf{x}_i, \frac{1}{2}r_{\mathbf{x}_i}\right). \tag{46}$$

Let $\delta := \min\left\{\frac{1}{2}r_{\mathbf{x}_1}, \ldots, \frac{1}{2}r_{\mathbf{x}_n}\right\} > 0$. Consider a rectangle P containing E and define

$$h_2(\mathbf{x}) := \begin{cases} f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| & \text{if } \mathbf{x} \in E, \\ 0 & \text{if } \mathbf{y} \in P \setminus E, \end{cases}$$

Partition P into a finite number of rectangles of diameter less than δ, say,

$$P = \bigcup_{j=1}^{m} R_j.$$

Then

$$\begin{aligned} \int_E f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| \, d\mathbf{x} &= \int_P h_2(\mathbf{x}) \, d\mathbf{x} = \sum_{j=1}^{m} \int_{R_j} h_2(\mathbf{x}) \, d\mathbf{x} \\ &= \sum_{j \text{ such that } R_j \cap E \neq \emptyset} \int_{R_j \cap E} f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| \, d\mathbf{x}, \end{aligned}$$

where in the last equality we have used the fact that if R_j does not intersect the set E, then $\int_{R_j} h_2(\mathbf{x}) \, d\mathbf{x} = 0$.

Fix j such that $R_j \cap E \neq \emptyset$. Then by (46), there exists $i \in \{1, \ldots, m\}$ such that R_j intersects $B\left(\mathbf{x}_i, \frac{1}{2}r_{\mathbf{x}_i}\right)$. On the other hand, the diameter of R_j is less than $\delta \leq \frac{1}{2}r_{\mathbf{x}_i}$, and so R_j is contained in $B(\mathbf{x}_i, r_{\mathbf{x}_i})$.

By construction $\mathbf{g}$ restricted to $B(\mathbf{x}_i, r_{\mathbf{x}_i})$ is given by the union of finitely many elementary diffeomorphisms and flips. Hence, by applying Steps 1, 2, and 3 (with U replaced by $B(\mathbf{x}_i, r_{\mathbf{x}_i})$ and with E replaced by $R_j \cap E$), we conclude that

$$\int_{R_j \cap E} f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| \, d\mathbf{x} = \int_{\mathbf{g}(R_j \cap E)} f(\mathbf{y}) \, d\mathbf{y},$$

and by summing over all R_j that intersect the set E, we get

$$\begin{aligned} \int_E f(\mathbf{g}(\mathbf{x})) \left|\det J_{\mathbf{g}}(\mathbf{x})\right| \, d\mathbf{x} &= \sum_{j \text{ such that } R_j \cap E \neq \emptyset} \int_{\mathbf{g}(R_j \cap E)} f(\mathbf{y}) \, d\mathbf{y} \\ &= \int_{\mathbf{g}(E)} f(\mathbf{y}) \, d\mathbf{y}, \end{aligned}$$

where we have used the fact that, since $\mathbf{g}$ is one-to-one, the sets $\mathbf{g}(R_j \cap E)$ are disjoint, so that

$$\bigcup_{j \text{ such that } R_j \cap E \neq \emptyset} \mathbf{g}(R_j \cap E) = \mathbf{g}\left(\bigcup_{j \text{ such that } R_j \cap E \neq \emptyset} (R_j \cap E)\right)$$
$$= \mathbf{g}\left(\bigcup_{j=1}^{m} (R_j \cap E)\right) = \mathbf{g}(P \cap E) = \mathbf{g}(E).$$

This concludes the proof. ■

13 Spherical Coordinates in $\mathbb{R}^N$

The most important applications of the previous theorem is spherical coordinates in $\mathbb{R}^N$. We introduce them by induction. For $N = 2$ we use polar coordinates. More precisely, we define

$$\mathbf{g}(r, \theta) := (r \cos \theta, r \sin \theta), \quad (\theta, r) \in [0, \infty) \times [0, 2\pi).$$

Note that partial derivatives of any order exist in $\mathbb{R}^2$ and are continuous. The Jacobian of $\mathbf{g}$ is given by

$$\det J_{\mathbf{g}}(r, \theta) = \det \begin{pmatrix} \frac{\partial g_1}{\partial r}(r, \theta) & \frac{\partial g_1}{\partial \theta}(r, \theta) \\ \frac{\partial g_2}{\partial r}(r, \theta) & \frac{\partial g_2}{\partial \theta}(r, \theta) \end{pmatrix}$$
$$= \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r \cos^2 \theta + r \sin^2 \theta = r,$$

which is zero at $r = 0$. We observe that $\mathbf{g}$ is onto but $\mathbf{g}$ is one-to-one only in the set where $(0, \infty) \times [0, 2\pi)$. However, this is not open. Consider instead, the open sets $U := (0, \infty) \times (0, 2\pi)$ and $V := \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ and let's prove that $\mathbf{g} : U \to V$ is invertible. For every $(x, y) \in V$, we have that $r = \sqrt{x^2 + y^2}$, while θ is the angle from $(0, 0)$ to the half-line through (x, y), with $0 < \theta < 2\pi$. To find θ in terms of x and y, we use the fact that

$$\cot \frac{\theta}{2} = \frac{\sin \theta}{1 - \cos \theta},$$

so that

$$\theta = 2\left(\frac{\theta}{2}\right) = 2 \operatorname{arccot}\left(\cot \frac{\theta}{2}\right) = 2 \operatorname{arccot}\left(\frac{\sin \theta}{1 - \cos \theta}\right).$$

Hence, $\mathbf{g}^{-1}(x, y) := \left(\sqrt{x^2 + y^2}, 2 \operatorname{arccot} \frac{y}{\sqrt{x^2+y^2} - x}\right)$. Note that we cannot apply Theorem 204 to $\mathbf{g}$ but we can apply Corollary **??**, since if W is, say, a bounded set $W \subset [0, \infty) \times [0, 2\pi)$ the "bad" set F is given by

$$F := \{(r, 0) : 0 \leq r \leq R\} \cup \{(0, \theta) : \theta \in [0, 2\pi)\}$$

for some large $R > 0$. Moreover $\mathbf{g}(F)$ is a bounded subset of the x-axis. Hence, $\operatorname{meas}(F) = \operatorname{meas}(\mathbf{g}(F)) = 0$.

Exercise 208 *Find the integral*

$$\iint_F \left(x + y^2\right) \, dxdy,$$

where

$$F := \left\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4, \, x > 0, \, y > 0\right\}.$$

Assume that spherical coordinates have been given in $\mathbb{R}^{N-1}$. Given $\mathbf{x} \in \mathbb{R}^N$, let $r = \|\mathbf{x}\|$ and let θ_1 be the angle from the positive x_1-axis to $\mathbf{x}$, precisely,

$$\theta_1 := \cos^{-1}\left(\frac{x_1}{r}\right), \quad 0 < \theta_1 < \pi.$$

Then $x_1 = r\cos\theta_1$. Let $(\rho, \theta_2, \ldots, \theta_{N-1})$ be the spherical coordinates of $\mathbf{x}' = (x_2, \ldots, x_N)$. Then

$$\rho = \|\mathbf{x}'\|_{N-1} = \sqrt{\|\mathbf{x}\|^2 - x_1^2} = \sqrt{r^2 - r^2\cos^2\theta_1} = r\sqrt{\sin^2\theta_1} = r\,|\sin\theta_1| = r\sin\theta_1.$$

Hence, by induction we get

$$\begin{aligned}
x_1 &= r\cos\theta_1, \\
x_2 &= r\sin\theta_1\cos\theta_2, \\
&\vdots \\
x_{N-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\cos\theta_{N-1}, \\
x_N &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\theta_{N-1}.
\end{aligned}$$

Consider the transformation

$$\mathbf{g}(r, \theta_1, \ldots, \theta_{N-1}) := (r\cos\theta_1, \cdots, r\sin\theta_1\sin\theta_2\cdots\sin\theta_{N-2}\sin\theta_{N-1})$$

where $r \geq 0$, $0 \leq \theta_{N-1} < 2\pi$, $0 \leq \theta_i < \pi$ for all $i = 1, \ldots, N-2$. Note that partial derivatives of any order exist in $\mathbb{R}^N$ and are continuous. Moreover, by induction it can be shown that

$$|\det J_{\mathbf{g}}(r, \theta_1, \ldots, \theta_{N-1})| = r^{N-1}\sin^{N-2}\theta_1\sin^{N-3}\theta_2\cdots\sin\theta_{N-2}.$$

Consider the open sets

$$\begin{aligned}
U &:= (0, \infty) \times (0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi), \\
V &:= \mathbf{g}(U).
\end{aligned}$$

Then $\mathbf{g} : U \to V$ is invertible and the inverse is given by

$$\begin{aligned} r &= \|\mathbf{x}\|, \\ \theta_1 &= \operatorname{arccot} \frac{x_1}{\sqrt{x_2^2 + \cdots + x_N^2}}, \\ &\vdots \\ \theta_{N-2} &= \operatorname{arccot} \frac{x_{N-2}}{\sqrt{x_{N-1}^2 + x_N^2} - x_{N-1}}, \\ \theta_{N-1} &= 2 \operatorname{arccot} \frac{x_N}{\sqrt{x_{N-1}^2 + x_N^2} - x_{N-1}}. \end{aligned}$$

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If f is radial, that is, if $f(\mathbf{x}) = h(\|\mathbf{x}\|)$, where $h : [r_1, r_2] \to [0, \infty)$ is a continuous function, then

$$\int_{B(\mathbf{0},r_2)\setminus B(\mathbf{0},r_1)} f(\mathbf{x})\, d\mathbf{x} = \beta_N \int_{r_1}^{r_2} h(r)\, r^{N-1}\, dr,$$

where β_N does not depend on h, r_1, r_2. To find β_N, take $h = 1$ and $r_1 = 0$ and $r_2 = 1$. Then

$$\operatorname{meas}(B(\mathbf{0}, 1)) = \beta_N \int_0^1 r^{N-1}\, dr = \frac{\beta_N (1 - 0)}{N},$$

and so $\beta_N = N \operatorname{meas}(B(\mathbf{0}, 1))$.

Note that by taking $h = 1$ and $r_1 = 0$ and $r_2 = R$ we get

$$\operatorname{meas}(B(\mathbf{0}, R)) = \frac{\beta_N}{N} R^N = \operatorname{meas}(B(\mathbf{0}, 1))\, R^N. \tag{47}$$

To find $\operatorname{meas}(B(\mathbf{0}, 1))$, we have the following:

Theorem 209 *Let $N \geq 1$. Then the measure of $B(\mathbf{x}_0, r)$ is*

$$\frac{\pi^{N/2}}{\Gamma(1 + N/2)} r^N.$$

Proof. Since $B(\mathbf{x}_0, r) = \mathbf{x}_0 + rB(\mathbf{0}, 1)$, by homogeneity, we have that the volume of $B(\mathbf{x}_0, r)$ is given by r^N times the volume of the unit ball $B(\mathbf{0}, 1)$.

Write points of $\mathbb{R}^{N+1}$ as $(\mathbf{x}, y)$ with $\mathbf{x} \in \mathbb{R}^N$ and $y \in \mathbb{R}$ and consider the set

$$D := \left\{ (\mathbf{x}, y) \in \mathbb{R}^{N+1} : \|\mathbf{x}\|_N^2 < y \right\}.$$

We now consider the integral

$$\int_D e^{-y}\, d\mathbf{x} dy.$$

We now integrate into two different ways.

$$
\begin{aligned}
\int_D e^{-y}\,d\mathbf{x}dy &= \int_0^\infty \left(\int_{B\left(\mathbf{0},y^{1/2}\right)} e^{-y}\,d\mathbf{x}\right) dy \\
&= \int_0^\infty e^{-y} \left(\int_{B\left(\mathbf{0},y^{1/2}\right)} 1\,d\mathbf{x}\right) dy \\
&= \int_0^\infty e^{-y}\operatorname{meas}_N B\left(\mathbf{0},y^{1/2}\right)\,dy \\
&= \int_0^\infty e^{-y}y^{N/2}\operatorname{meas}_N B\left(\mathbf{0},1\right)\,dy = \operatorname{meas}_N B\left(\mathbf{0},1\right)\int_0^\infty e^{-y}y^{N/2}\,dy \\
&= \Gamma\left(1+N/2\right)\operatorname{meas}_N B\left(\mathbf{0},1\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_D e^{-y}\,d\mathbf{x}dy &= \int_{\mathbb{R}^N}\left(\int_{\|\mathbf{x}\|_N^2}^\infty e^{-y}\,dy\right) d\mathbf{x} \\
&= \int_{\mathbb{R}^N}\left[-e^{-y}\right]_{y=\|\mathbf{x}\|_N^2}^{y=\infty}\,d\mathbf{x} \\
&= \int_{\mathbb{R}^N} e^{-\|\mathbf{x}\|_N^2}\,d\mathbf{x} = \int_{\mathbb{R}^N} e^{-x_1^2-x_2^2-\cdots-x_N^2}\,dx_1dx_2\cdots dx_N \\
&= \int_{\mathbb{R}^N} e^{-x_1^2}e^{-x_2^2}\cdots e^{-x_N^2}\,dx_1dx_2\cdots dx_N \\
&= \left(\int_{\mathbb{R}} e^{-x_1^2}\,dx_1\right)\left(\int_{\mathbb{R}} e^{-x_2^2}\,dx_2\right)\cdots\left(\int_{\mathbb{R}} e^{-x_N^2}\,dx_N\right) = I^N,
\end{aligned}
$$

where

$$I := \int_{\mathbb{R}} e^{-t^2}\,dt.$$

This shows that

$$\Gamma\left(1+N/2\right)\operatorname{meas}_N B\left(\mathbf{0},1\right) = I^N.$$

Now for $N=2$, we know that $\operatorname{meas}_2 B\left(\mathbf{0},1\right)=\pi$, while $\Gamma\left(1+2/2\right)=\Gamma\left(2\right)=1$. Hence,

$$\pi = I^2,$$

which shows that $I=\pi^{1/2}$. This completes the proof. ■

Example 210 *An important function g in Theorem 193 is the radial function given by*

$$g\left(\mathbf{x}\right) := \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a},$$

where $\mathbf{x}_0$ is a fixed point of $\mathbb{R}^N$ and $a>0$. Note that g is continuous in its domain $\mathbb{R}^N\setminus\{\mathbf{x}_0\}$. Consider the domain $E=B\left(\mathbf{x}_0,R\right)\setminus\{\mathbf{x}_0\}$. An exhausting sequence $\{E_n\}$ is given by $E_n := B\left(\mathbf{x}_0,R\right)\setminus B\left(\mathbf{x}_0,\frac{1}{n}\right)$. Using spherical *coordinates, we have that for all* $n > \frac{1}{R}$*,*

$$\begin{aligned}
\int_{E_n} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} &= \beta_N \int_{1/n}^{R} \frac{r^{N-1}}{r^a}\,dr = \beta_N \int_{1/n}^{R} r^{N-a-1}\,dr \\
&= \beta_N \begin{cases} [\log r]_{r=1/n}^{r=R} & \text{if } a = N, \\ \left[\frac{r^{N-a}}{N-a}\right]_{r=1/n}^{r=R} & \text{if } a \neq N \end{cases} \\
&= \beta_N \begin{cases} \log R + \log n & \text{if } a = N, \\ \frac{1}{N-a}\left(R^{N-a} - \frac{1}{n^{N-a}}\right) & \text{if } a \neq N. \end{cases}
\end{aligned}$$

Letting $n \to \infty$ *and using Theorem 191, we obtain*

$$\begin{aligned}
\int_{B(\mathbf{x}_0,R)\setminus\{\mathbf{x}_0\}} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} &= \lim_{n\to\infty} \int_{E_n} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} \\
&= \beta_N \begin{cases} \infty & \text{if } a \geq N, \\ \frac{1}{N-a} R^{N-a} & \text{if } a < N. \end{cases}
\end{aligned}$$

On the other hand, if we consider the domain $F = \mathbb{R}^N \setminus B(\mathbf{x}_0, R)$*. An exhausting sequence* $\{F_n\}$ *is given by* $F_n = B(\mathbf{x}_0, n) \setminus B(\mathbf{x}_0, R)$*. Using spherical coordinates, we have that for all* $n > R$*,*

$$\begin{aligned}
\int_{F_n} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} &= \beta_N \int_{R}^{n} \frac{r^{N-1}}{r^a}\,dr \\
&= \beta_N \begin{cases} \log n - \log R & \text{if } a = N, \\ \frac{1}{N-a}\left(n^{N-a} - R^{N-a}\right) & \text{if } a \neq N. \end{cases}
\end{aligned}$$

Letting $n \to \infty$ *and using Theorem 191, we obtain*

$$\begin{aligned}
\int_{B(\mathbf{x}_0,R)\setminus\{\mathbf{x}_0\}} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} &= \lim_{n\to\infty} \int_{E_n} \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|^a}\,d\mathbf{x} \\
&= \beta_N \begin{cases} \infty & \text{if } a \leq N, \\ \frac{1}{a-N} R^{N-a} & \text{if } a > N. \end{cases}
\end{aligned}$$

Corollary 211 *Let* $E \subseteq \mathbb{R}^N$*, let* $f : E \to \mathbb{R}$ *be continuous and let* $\mathbf{x}_0 \in \mathbb{R}^N \setminus E$*. Assume that there exist* $a > 0$ *and* $C > 0$ *such that*

$$|f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x}-\mathbf{x}_0\|^a} \quad \text{for all } \mathbf{x} \in E.$$

(i) If E *is a (bounded) Peano–Jordan measurable set and* $a < N$*, then* f *is Riemann integrable in the improper sense over* E *with finite improper Riemann integral.*

(ii) If $E \subseteq \mathbb{R}^N \setminus B(\mathbf{x}_0, R)$ *admits an exhausting sequence and* $a > N$*, then* f *is Riemann integrable in the improper sense over* E *with finite improper Riemann integral.*

Proof. Since f is continuous and $|f(\mathbf{x})| \leq g(\mathbf{x}) := \frac{C}{\|\mathbf{x}-\mathbf{x}_0\|^a}$, we have that f is bounded whenever g is bounded. Hence, f is Riemann integrable over every subset $F \subseteq E$ over which g is Riemann integrable. Thus, we are in a position to apply the Theorem 193. ■

Corollary 212 *Let $E \subseteq \mathbb{R}^N$, let $f : E \to \mathbb{R}$ be continuous and let $\mathbf{x}_0 \in \mathbb{R}^N \setminus E$. Assume that there exist $a > 0$ and $C > 0$ such that*

$$f(\mathbf{x}) \geq \frac{C}{\|\mathbf{x}-\mathbf{x}_0\|^a} \quad \textit{for all } \mathbf{x} \in E.$$

(i) If E is a (bounded) Peano–Jordan measurable set containing $B(\mathbf{x}_0, R) \setminus \{\mathbf{x}_0\}$, $a \leq N$, and f is bounded on each $E_n := E \setminus B\left(\mathbf{x}_0, \frac{1}{n}\right)$, then f is Riemann integrable in the improper sense over E with infinite improper Riemann integral.

(ii) If E admits an exhausting sequence $\{E_n\}$ such that f is bounded on each E_n, E contains $\mathbb{R}^N \setminus B(\mathbf{x}_0, R)$, and $a > N$, then f is Riemann integrable in the improper sense over E with infinite improper Riemann integral.

Proof. Since f is continuous, and by hypothesis f is bounded on each E_n, where E_n is given by either part (i) or (ii), it follows that f is Riemann integrable over E_n. We can now continue as in the proof of part (ii) of Theorem 193 to conclude that f is Riemann integrable in the improper sense over E and

$$\int_E f(\mathbf{x})\, d\mathbf{x} = \infty.$$

■

Note that if in part (i) the set E does not contain a punctured ball, then the integral could be finite.

Exercise 213 *Consider the function*

$$f(x, y) := \frac{1}{(x^2+y^2)^2}$$

defined in the set

$$E := \left\{(x, y) \in \mathbb{R}^2 : \, y \leq x^4, \, x^2 + y^2 \leq 2, \, x > 0, \, y > 0\right\}.$$

Prove that f is Riemann integrable in the improper sense over E and that the improper Riemann integral is finite.

Monday, April 16, 2012

14 Differential Surfaces

Definition 214 *Given $1 \leq k < N$, a nonempty set $M \subseteq \mathbb{R}^N$ is called a* k-dimensional differential surface *or* manifold *if for every $\mathbf{x}_0 \in M$ there exist an open set U containing $\mathbf{x}_0$ and a differentiable function $\varphi : W \to \mathbb{R}^N$, where $W \subseteq \mathbb{R}^k$ is an open set such that*

(i) $\varphi : W \to M \cap U$ is a homeomorphism, that is, it is invertible and continuous together with its inverse $\varphi^{-1} : M \cap U \to W$,

(ii) $J_\varphi(\mathbf{y})$ has maximum rank k for all $\mathbf{y} \in W$.

The function φ is called a local chart *or a* system of local coordinates *or a* local parametrization *around $\mathbf{x}_0$. We say that M is of class C^m, $m \in \mathbb{N}$, (respectively, C^∞) if all local charts are of class C^m (respectively, C^∞).*

Roughly speaking a set $M \subset \mathbb{R}^N$ is a k-dimensional differential surface if for every point $\mathbf{x}_0 \in M$ we can "cut" a piece of M around $\mathbf{x}_0$ and deform it/flatten it in a smooth way to get, say, a ball of $\mathbb{R}^k$. Another way to say this is that locally M looks like $\mathbb{R}^k$. Thus the range of a simple differentiable curve is a 1-dimensional surface since locally it looks like $\mathbb{R}$, while the range of a self-intersecting curve is not, because at self-intersection point the range of the curve does not look like $\mathbb{R}$. Similarly, a sphere in $\mathbb{R}^3$ is a 2-dimensional surface because locally it looks like $\mathbb{R}^2$, while a cone is not because near the tip it does not look like $\mathbb{R}^2$.

A simple way to construct k-dimensional differential surface is to start with a set of $\mathbb{R}^k$ and then deform it in a smooth way.

Remark 215 *The difference between curves and surfaces, it's that a surface is a set of $\mathbb{R}^N$, while a curve is an equivalence class of functions. Also for surfaces we do not allow self-intersections. Also, a curve can be described by only one parametrization, while a surface usually needs more than one.*

Remark 216 *Note that Theorem 200 implies that if a bounded k-dimensional differential surface is Peano–Jordan measurable, then its measure must be zero.*

Example 217 *Consider the hyperbola*

$$M := \left\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\right\}.$$

Note that the set M is not the range of a curve, because it is not connected. To cover M we need at least two local charts, precisely, we can take the open sets

$$V := \left\{(x, y) \in \mathbb{R}^2 : x > 0\right\}, \quad W := \left\{(x, y) \in \mathbb{R}^2 : x < 0\right\},$$

and the functions $\varphi : \mathbb{R} \to M \cap V$ and $\psi : \mathbb{R} \to M \cap W$ defined by

$$\varphi(t) := \left(\sqrt{1 + t^2}, t\right), \quad \psi(t) := \left(-\sqrt{1 + t^2}, t\right), \quad t \in \mathbb{R}.$$

Note that both φ and ψ are of class C^∞ (the argument inside the square roots is never zero). Moreover, $\varphi'(t) = \left(\frac{t}{\sqrt{1+t^2}}, 1\right)$ and $\psi'(t) := \left(-\frac{t}{\sqrt{1+t^2}}, 1\right)$, and so the rank of $J_\varphi(t)$ and of $J_\psi(t)$ is one. Finally, $\varphi^{-1} : M \cap V \to \mathbb{R}$ and $\psi^{-1} : M \cap W \to \mathbb{R}$ are given by

$$\varphi^{-1}(x, y) = y, \quad \psi^{-1}(x, y) = y,$$

which are continuous. Thus, M is a 1-dimensional surface of class C^∞.

Example 218 *Given an open set $V \subseteq \mathbb{R}^k$ and a differential function $h : V \to \mathbb{R}$, consider the graph of h,*

$$\operatorname{Gr} h := \{(\mathbf{y}, t) \in V \times \mathbb{R} : \, t = h(\mathbf{y})\}.$$

A chart is given by the function $\varphi : V \to \mathbb{R}^{k+1}$ given by $\varphi(\mathbf{y}) := (\mathbf{y}, h(\mathbf{y}))$. Then,

$$J_\varphi(\mathbf{y}) = \begin{pmatrix} I_k \\ \nabla f(\mathbf{y}) \end{pmatrix},$$

which has rank N. Note that φ is one-to-one and that $\varphi(V) = \operatorname{Gr} h$. Hence, there exists $\varphi^{-1} : \operatorname{Gr} f \to V$. Moreover, φ^{-1} is continuous, since the projection

$$\begin{aligned} \Pi : \mathbb{R}^{k+1} &\to \mathbb{R}^k \\ (\mathbf{y}, t) &\mapsto \mathbf{y} \end{aligned}$$

is of class C^∞ and φ^{-1} is given by the restriction of Π to $\operatorname{Gr} h$. Hence, $\operatorname{Gr} h$ is an k-dimensional differential surface.

The next proposition gives an equivalent definition of surfaces, which is very useful for examples.

Proposition 219 *Given $1 \leq k < N$, a nonempty set $M \subseteq \mathbb{R}^N$, and $m \in \mathbb{N}$, then the following are equivalent:*

(i) M is a k-dimensional surface of class C^m.

(ii) For every $\mathbf{x}_0 \in M$ there exist an open set $U \subseteq \mathbb{R}^N$ containing $\mathbf{x}_0$ and a function $\mathbf{g} : U \to \mathbb{R}^{N-k}$ of class C^m, such that

$$M \cap U = \{\mathbf{x} \in U : \, \mathbf{g}(\mathbf{x}) = \mathbf{0}\}$$

and $J_{\mathbf{g}}(\mathbf{x})$ has maximum rank $N - k$ for all $\mathbf{x} \in M \cap U$.

Example 220 (Torus) *A* torus *is a 2-dimensional surface M obtained from a rectangle of $\mathbb{R}^2$ by identifying opposite sides. Consider the chart $\varphi : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^2$ given by*

$$\varphi(u, v) := ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u), \tag{48}$$

where $a > 0$ and $r > 0$. Define $M := \varphi((0,2\pi) \times (0,2\pi)) \subset \mathbb{R}^3$.
First method. *Let's prove that M is a 2-dimensional surface. Note that φ is of class C^1 and*

$$J_\varphi(u,v) = \begin{pmatrix} -r\sin u\cos v & -(r\cos u + a)\sin v \\ -r\sin u \sin v & (r\cos u + a)\cos v \\ r\cos u & 0 \end{pmatrix}.$$

Observe that if $r\cos u + a = 0$, then

$$J_\varphi(u,v) = \begin{pmatrix} -r\sin u\cos v & 0 \\ -r\sin u\sin v & 0 \\ r\cos u & 0 \end{pmatrix}$$

and so $J_\varphi(u,v)$ cannot have rank 2. Thus, for M to be a 2-dimensional surface we need to assume that $r\cos u + a \neq 0$ for all $u \in (0,2\pi)$. In what follows we assume that

$$a > r.$$

The submatrix $\begin{pmatrix} -r\sin u\cos v & -(r\cos u + a)\sin v \\ -r\sin u\sin v & (r\cos u + a)\cos v \end{pmatrix}$ has determinant $-r\sin u(r\cos u + a)$. Then $r\cos u + a > 0$. If $\sin u \neq 0$, then the determinant is different from zero and so so $J_\varphi(u,v)$ has rank 2. On the other hand, when $\sin u = 0$, that is, when $u = \pi$, then

$$J_\varphi(\pi,v) = \begin{pmatrix} 0 & -(-r + a)\sin v \\ 0 & (-r + a)\cos v \\ -r & 0 \end{pmatrix}.$$

For any $v \in (0,2\pi)$, either $\cos v \neq 0$ or $\sin v \neq 0$ (or both) and so either the submatrix $\begin{pmatrix} 0 & (-r + a)\cos v \\ -r & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & -(-r + a)\sin v \\ -r & 0 \end{pmatrix}$ have determinant different from zero. Thus, we have shown that when $a > r$, $J_\varphi(u,v)$ has rank 2 for all $(u,v) \in (0,2\pi) \times (0,2\pi)$.

To see that φ is one-to-one, consider $(x,y,z) \in M$. We want to find a unique pair (u,v) such that $\varphi(u,v) = (x,y,z)$. Note that to find u we can use $\sin u = \frac{z}{r}$. The problem is that there are two values of u, one in $\left(0,\frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and one in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. If $x^2 + y^2 \leq a^2$, then

$$(r\cos u + a)^2 \cos^2 v + (r\cos u + a)^2 \sin^2 v = (r\cos u + a)^2 \leq a^2,$$

that is $\cos u(r\cos u + 2a) \leq 0$, so that $\cos u \leq 0$, that is, $\frac{\pi}{2} \leq u \leq \frac{3\pi}{2}$. So in this case this determines uniquely u in terms of (x,y,z). On the other hand, if $x^2 + y^2 > a^2$, then $\cos u > 0$, and so either $0 < u < \frac{\pi}{2}$ or $\frac{3\pi}{2} \leq u < 2\pi$. Again, this determines uniquely u in terms of (x,y,z). In turn,

$$\cos v = \frac{x}{(r\cos u + a)}, \quad \sin v = \frac{y}{(r\cos u + a)},$$

which determine uniquely v. Thus, φ is one-to-one. We leave as an exercise to check that φ^{-1} is continuous.

Second method: *Let's write down* M *explicitly.*

$$M = \left\{ (x, y, z) \in \mathbb{R}^3 : \, z^2 = r^2 - \left(\sqrt{x^2 + y^2} - a \right)^2 \right\}.$$

Note that $r^2 - \left(\sqrt{x^2 + y^2} - a \right)^2 \geq 0$*, that is*

$$r \geq \left| \sqrt{x^2 + y^2} - a \right|.$$

Since $a > r$*, if* $\sqrt{x^2 + y^2} - a \leq 0$*, we would get* $r \geq a - \sqrt{x^2 + y^2}$*, which is a contradiction. Hence,* $\sqrt{x^2 + y^2} > a$*. Consider the function*

$$g(x, y, z) := z^2 + \left(\sqrt{x^2 + y^2} - a \right)^2 - r^2.$$

Then

$$\begin{aligned} J_g(x, y, z) &= \nabla g(x, y, z) \\ &= \left(2 \left(\sqrt{x^2 + y^2} - a \right) \frac{2x}{2\sqrt{x^2 + y^2}}, 2 \left(\sqrt{x^2 + y^2} - a \right) \frac{2y}{2\sqrt{x^2 + y^2}}, 2z \right). \end{aligned}$$

Since $\sqrt{x^2 + y^2} > a$*,* x *and* y *cannot be both zero, so* $\frac{\partial g}{\partial x}(x, y, z)$ *or* $\frac{\partial g}{\partial y}(x, y, z)$ *are different from zero. Thus,* $J_g(x, y, z)$ *has maximum rank* 1*. This shows that* M *is a* 2*-dimensional surface of class* C^∞*.*

Note that M *is obtained by taking the circle of radius* r *and center* $(0, a, 0)$*, namely,*

$$(y - a)^2 + z^2 = r^2$$

and revolving it about the z*.*

The *Klein bottle* is obtained by starting from a rectangle of $\mathbb{R}^2$, reflecting one of its sides across its center and then performing the identification.

Exercise 221 (Klein Bottle) *Consider the function* $\varphi : (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^4$ *given by*

$$\varphi(u, v) := \left((r \cos v + a) \cos u, (r \cos v + a) \sin u, r \sin v \cos \frac{u}{2}, r \sin v \sin \frac{u}{2} \right).$$

Then Klein bottle is the set $M := \varphi((0, 2\pi) \times (0, 2\pi)) \subset \mathbb{R}^4$*. Prove that* M *is a* 2*-dimensional surface of class* C^∞*.*

Example 222 *Consider the unit sphere in* $\mathbb{R}^N$*,* $N \geq 2$*,*

$$M := \left\{ \mathbf{x} \in \mathbb{R}^N : \, \|\mathbf{x}\| = 1 \right\}.$$

Given $\mathbf{x}_0 \in M$ *consider the function* $g(\mathbf{x}) := \|\mathbf{x}\|^2 - 1$ *(here* $N - k = 1$ *and so* $k = N - 1$*). Then* $J_g(\mathbf{x}) = \nabla g(\mathbf{x}) = 2\mathbf{x}$*. By taking a ball* $B(\mathbf{x}_0, r)$ *so small that it does not intersect the origin, we have that* $2\mathbf{x} \neq \mathbf{0}$ *for all* $\mathbf{x} \in M \cap B(\mathbf{x}_0, r)$ *and so* M *is a* $(N - 1)$*-dimensional surface of class* C^∞*.*

Example 223 *Given an open set $U \subseteq \mathbb{R}^N$ and a function $f : U \to \mathbb{R}$ of class C^1, for every $c \in \mathbb{R}$ consider the level set*

$$B_c := \{\mathbf{x} \in U : f(\mathbf{x}) = c\}.$$

Given $\mathbf{x}_0 \in B_c$ consider the function $g(\mathbf{x}) := f(\mathbf{x}) - c$ (here $N - k = 1$ and so $k = N - 1$). Then $J_g(\mathbf{x}) = \nabla f(\mathbf{x})$. Hence, we have a problem if B_c contains some critical points. Thus, let

$$M := \{\mathbf{x} \in U : f(\mathbf{x}) = c\} \setminus \{\mathbf{x} \in U : \nabla f(\mathbf{x}) = \mathbf{0}\}.$$

Given $\mathbf{x}_0 \in M$, since $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$ and f is of class C^1, by taking a small ball $B(\mathbf{x}_0, r)$ we have that $\nabla f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in M \cap B(\mathbf{x}_0, r)$ and so M is an $(N-1)$-dimensional surface of class C^1.

Wednesday, April 18, 2012

Proof of Proposition 219. Step 1: We prove that (i) implies (ii). Given $\mathbf{x}_0 \in M$, let U, W, and φ be as in Definition 214. Let $\mathbf{y}_0 \in W$ be such that $\varphi(\mathbf{y}_0) = \mathbf{x}_0$. Since $J_\varphi(\mathbf{y}_0)$ has maximum rank k, there is an $k \times k$ submatrix of $J_\varphi(\mathbf{y}_0)$, which has determinant different from zero. By changing the coordinates axes of $\mathbb{R}^N$, if necessary, without loss of generality, we may assume for simplicity assume that

$$\det \begin{pmatrix} \frac{\partial \varphi_1}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_1}{\partial y_k}(\mathbf{y}_0) \\ \vdots & \ldots & \vdots \\ \frac{\partial \varphi_k}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_k}{\partial y_1}(\mathbf{y}_0) \end{pmatrix} \neq 0.$$

Consider the projection

$$\Pi : \mathbb{R}^N \to \mathbb{R}^k$$

defined by

$$\Pi(\mathbf{x}) := (x_1, \ldots, x_k)$$

and let $\mathbf{f} : W \to \mathbb{R}^k$ be defined by

$$\mathbf{f}(\mathbf{y}) := \Pi(\varphi(\mathbf{y})) = (\varphi_1(\mathbf{y}), \ldots, \varphi_k(\mathbf{y})). \tag{49}$$

Then

$$\det J_{\mathbf{f}}(\mathbf{y}_0) = \det \begin{pmatrix} \frac{\partial \varphi_1}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_1}{\partial y_k}(\mathbf{y}_0) \\ \vdots & \ldots & \vdots \\ \frac{\partial \varphi_k}{\partial y_1}(\mathbf{y}_0) & \cdots & \frac{\partial \varphi_k}{\partial y_1}(\mathbf{y}_0) \end{pmatrix} \neq 0,$$

and so by the inverse function theorem there exists $r > 0$ such that $B_k(\mathbf{y}_0, r) \subseteq W$, $\mathbf{f}(B_k(\mathbf{y}_0, r))$ is open, and $\mathbf{f} : B_k(\mathbf{y}_0, r) \to \mathbf{f}(B_k(\mathbf{y}_0, r))$ is invertible, with inverse $\mathbf{f}^{-1} : \mathbf{f}(B_k(\mathbf{y}_0, r)) \to B_k(\mathbf{y}_0, r)$ of class C^m. Let $\mathbf{z} := (x_1, \ldots, x_k)$, then we have shown that we can write $\mathbf{y}$ as a function of $\mathbf{z}$, $\mathbf{y} = \mathbf{f}^{-1}(\mathbf{z})$. Let $\mathbf{w} := (x_{k+1}, \ldots, x_N)$ so that $\mathbf{x} = (\mathbf{z}, \mathbf{w})$.

Since φ is a homeomorphism, the set $\varphi(B_k(\mathbf{y}_0, r))$ is relatively open in M, that is, it can be written as

$$\varphi(B_k(\mathbf{y}_0, r)) = M \cap U_1$$

for some open set $U_1 \subseteq \mathbb{R}^N$. Then

$$\begin{aligned} M \cap U_1 &= \{\varphi(\mathbf{y}) : \mathbf{y} \in B_k(\mathbf{y}_0, r)\}. \\ &= \left\{\left(\mathbf{z}, \varphi_{k+1}\left(\mathbf{f}^{-1}(\mathbf{z})\right), \ldots, \varphi_N\left(\mathbf{f}^{-1}(\mathbf{z})\right)\right) : \mathbf{z} \in U_1\right\}. \end{aligned}$$

This shows that $M \cap U_1$ is given by the graph of the function $\mathbf{z} \in U_1 \mapsto \left(\varphi_{k+1}\left(\mathbf{f}^{-1}(\mathbf{z})\right), \ldots, \varphi_N\left(\mathbf{f}^{-1}(\mathbf{z})\right)\right)$. Consider the function $\mathbf{g} : U_1 \to \mathbb{R}^{N-k}$ of class C^m defined by

$$\mathbf{g}(\mathbf{x}) := \left(x_{k+1} - \varphi_{k+1}\left(\mathbf{f}^{-1}(x_1, \ldots, x_k)\right), \ldots, x_N - \varphi_N\left(\mathbf{f}^{-1}(x_1, \ldots, x_k)\right)\right).$$

Then

$$M \cap U_1 = \{\mathbf{x} \in U_1 : \mathbf{g}(\mathbf{x}) = \mathbf{0}\}.$$

Moreover, $J_{\mathbf{g}}(\mathbf{x})$ contains the submatrix I_{N-k}, since for $i, j \geq k+1$,

$$\frac{\partial g_i}{\partial x_j}(\mathbf{x}) = \frac{\partial}{\partial x_j}\left(x_i - \varphi_i\left(\mathbf{f}^{-1}(x_1, \ldots, x_k)\right)\right) = \delta_{i,j} - 0.$$

Hence, $J_{\mathbf{g}}(\mathbf{x})$ has maximum rank $N-k$ for all $\mathbf{x} \in U_1$.
Step 2: We prove that (ii) implies (i). Exercise. ■

15 Surface Integrals

To define the integral of a function over a surface, we use local charts. Given $1 \leq k < N$ we define

$$\Lambda_{N,k} := \left\{\boldsymbol{\alpha} \in \mathbb{N}_0^k : 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq N\right\}.$$

Given a k-dimensional surface M of class C^m, $m \in \mathbb{N}$, consider a local chart $\varphi : V \to M$, let $E \subseteq \varphi(V)$ be such that $\varphi^{-1}(E)$ is Peano–Jordan measurable and let $f : E \to \mathbb{R}$ be a bounded function. The *surface integral* of f over E is defined as

$$\int_E f \, d\mathcal{H}^k := \int_{\varphi^{-1}(E)} f(\varphi(\mathbf{y})) \sqrt{\sum_{\boldsymbol{\alpha} \in \Lambda_{N,k}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_k})}{\partial(y_1, \ldots, y_k)}(\mathbf{y})\right]^2} \, d\mathbf{y}, \quad (50)$$

provided the function $\mathbf{y} \in \varphi^{-1}(E) \mapsto f(\varphi(\mathbf{y})) \sqrt{\sum_{\boldsymbol{\alpha} \in \Lambda_{N,k}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_k})}{\partial(y_1, \ldots, y_k)}(\mathbf{y})\right]^2}$ is Riemann integrable. Note that $\int_E f \, d\mathcal{H}^k$ exists if f is bounded and continuous. In particular, if $f = 1$, the number

$$\mathcal{H}^k(E) := \int_E 1 \, d\sigma = \int_{\varphi^{-1}(E)} \sqrt{\sum_{\boldsymbol{\alpha} \in \Lambda_{N,k}} \left[\det \frac{\partial(\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_k})}{\partial(y_1, \ldots, y_k)}(\mathbf{y})\right]^2} \, d\mathbf{y}$$

is called the *k-dimensional surface measure* of E.

Remark 224 *To understand the expression under the square root, observe that the Jacobian matrix of* φ*,*

$$J_{\varphi}(\mathbf{y}) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial y_1}(\mathbf{y}) & \cdots & \frac{\partial \varphi_1}{\partial y_k}(\mathbf{y}) \\ \vdots & \cdots & \vdots \\ \frac{\partial \varphi_N}{\partial y_1}(\mathbf{y}) & \cdots & \frac{\partial \varphi_N}{\partial y_k}(\mathbf{y}) \end{pmatrix}$$

is an $N \times k$ *matrix, where* $1 \le k < N$*. The expression under square root is obtained by considering the sum of the squares of the determinant of all* $k \times k$ *submatrices of* $J_{\varphi}(\mathbf{y})$*.*

Remark 225 *It can be shown that the number* $\int_E f \, d\mathcal{H}^k$ *does not depend on the particular local chart* φ*.*

Example 226 *Consider the set*

$$M := \left\{ (x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = 1, \; z^2 + w^2 = 1, \; x, z > 0 \right\}.$$

Let's prove that it is a 2*-dimensional surface. A (local) chart* $\varphi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to M$ *is given by*

$$\varphi(u, v) := (\cos u, \sin u, \cos v, \sin v),$$

so that

$$J_{\varphi}(u, v) = \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{pmatrix}.$$

Since $\cos u, \cos v > 0$*, the submatrix* $\begin{pmatrix} \cos u & 0 \\ 0 & -\cos v \end{pmatrix}$ *has determinant* $-\cos u \cos v < 0$*, and so* $J_{\varphi}(u, v)$ *has rank* 2*. Moreover* φ *is one-to-one with continuous inverse and of class* C^1*. Hence,*

$$\begin{aligned} \sum_{\boldsymbol{\alpha} \in \Lambda_{4,2}} \left(\det \frac{\partial (\varphi_{\alpha_1}, \varphi_{\alpha_2})}{\partial (u, v)} (u, v) \right)^2 &= \left(\det \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \end{pmatrix} \right)^2 + \left(\det \begin{pmatrix} -\sin u & 0 \\ 0 & -\sin v \end{pmatrix} \right)^2 \\ &\quad + \left(\det \begin{pmatrix} -\sin u & 0 \\ 0 & \cos v \end{pmatrix} \right)^2 + \left(\det \begin{pmatrix} \cos u & 0 \\ 0 & -\sin v \end{pmatrix} \right)^2 \\ &\quad + \left(\det \begin{pmatrix} \cos u & 0 \\ 0 & \cos v \end{pmatrix} \right)^2 + \left(\det \begin{pmatrix} 0 & -\sin v \\ 0 & \sin v \end{pmatrix} \right)^2 \\ &= 0 + \sin^2 u \sin^2 v + \sin^2 u \cos^2 v + \cos^2 u \sin^2 v + \cos^2 u \cos^2 v + 0 \\ &= 1. \end{aligned}$$

Let's now calculate the surface integral $\int_M \left(x^2+z^2\right)\, d\mathcal{H}^2$. *We have*

$$\begin{aligned}\int_M \left(x^2+z^2\right)\, d\mathcal{H}^2 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2 u + \cos^2 v\right) \sqrt{1}\, du dv \\ &= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\, du + \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 v\, dv = \pi^2.\end{aligned}$$

Friday, April 21, 2012

Carnival, no classes.

Monday, April 23, 2012

16 Divergence Theorem

Definition 227 *An open and bounded set* $U \subset \mathbb{R}^N$ *is* regular *if there exist a function* $g \in C^1\left(\mathbb{R}^N\right)$, *with* $\nabla g(\mathbf{x}) \neq \mathbf{0}$ *for all* $\mathbf{x} \in \partial U$, *such that*

$$\begin{aligned} U &= \left\{\mathbf{x} \in \mathbb{R}^N : \, g(\mathbf{x}) < 0\right\}, \\ \partial U &= \left\{\mathbf{x} \in \mathbb{R}^N : \, g(\mathbf{x}) = 0\right\}. \end{aligned}$$

Since $\nabla g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \partial U$, it follows that ∂U is an $(N-1)$-dimensional surface of class C^1.

Definition 228 *Given an open, bounded, regular set* $U \subset \mathbb{R}^N$, *a point* $\mathbf{x}_0 \in \partial U$, *the* tangent space *to* ∂U *at* $\mathbf{x}_0$ *is given by*

$$T_{\mathbf{x}_0} = \left\{\mathbf{x} \in \mathbb{R}^N : \, \nabla g(\mathbf{x}_0) \cdot \mathbf{x} = 0\right\}.$$

A vector $\nu \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ *is called a normal vector to* ∂U *at* $\mathbf{x}_0$ *if it is orthogonal to all vectors in* $T_{\mathbf{x}_0}$, *that is,* $\boldsymbol{\nu} \cdot \mathbf{x} = 0$ *for all* $\mathbf{x} \in T_{\mathbf{x}_0}$. *A unit normal vector* $\boldsymbol{\nu}$ *to* ∂U *at* $\mathbf{x}_0$ *is called a* unit outward normal *to* U *at* $\mathbf{x}_0$ *if there exists* $\delta > 0$ *such that* $\mathbf{x}_0 - t\boldsymbol{\nu} \in U$ *and* $\mathbf{x}_0 + t\boldsymbol{\nu} \in \mathbb{R}^N \setminus \overline{U}$ *for all* $0 < t < \delta$.

Exercise 229 *Prove that if* $U \subset \mathbb{R}^N$ *is an open, bounded, regular set, and* g *is as in Definition 227, then for every* $\mathbf{x}_0 \in \partial U$, *the unit outward normal to* U *at* $\mathbf{x}_0$ *is the unit vector*

$$\boldsymbol{\nu}(\mathbf{x}_0) = \frac{\nabla g(\mathbf{x}_0)}{\|\nabla g(\mathbf{x}_0)\|}.$$

Hence, $\boldsymbol{\nu} : \partial U \to \mathbb{R}^N$ *is a continuous function.*

We are ready to prove the divergence theorem.

Theorem 230 (Divergence Theorem) *Let* $U \subset \mathbb{R}^N$ *be an open, bounded, regular set and let* $\mathbf{f} : \overline{U} \to \mathbb{R}^N$ *be such that* $\mathbf{f}$ *is bounded and continuous in* $\overline{U}$ *and there exist the partial derivatives of* $\mathbf{f}$ *in* $\mathbb{R}^N$ *at all* $\mathbf{x} \in U$ *and they are continuous and bounded. Then*

$$\int_U \operatorname{div} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_{\partial U} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, d\mathcal{H}^{N-1}(\mathbf{x}),$$

where

$$\operatorname{div}\mathbf{f} := \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i}.$$

Proof. Step 1: We first prove that the divergence theorem holds when U is replaced by a rectangle

$$R := (a_1, b_1) \times \cdots \times (a_N, b_N).$$

Write $R' := (a_2, b_2) \times \cdots \times (a_N, b_N)$ and let $\mathbf{x}' := (x_2, \ldots, x_N)$. Then by Theorem 160,

$$\begin{aligned}
\int_R \frac{\partial f_1}{\partial x_1}(\mathbf{x})\, d\mathbf{x} &= \int_{R'} \left(\int_{a_1}^{b_1} \frac{\partial f_1}{\partial x_1}(x_1, \mathbf{x}')\, dx_1 \right) d\mathbf{x}' \\
&= \int_{R'} \left(f_1(b_1, \mathbf{x}') - f_1(a_1, \mathbf{x}')\right) d\mathbf{x}' \\
&= \int_{R'} \mathbf{f}(b_1, \mathbf{x}') \cdot \mathbf{e}_1\, d\mathbf{x}' + \int_{R'} \mathbf{f}(a_1, \mathbf{x}') \cdot (-\mathbf{e}_1)\, d\mathbf{x}' \\
&= \int_{\{b_1\} \times R'} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\nu}\, d\mathcal{H}^{N-1}(\mathbf{x}) + \int_{\{a_1\} \times R'} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\nu}\, d\mathcal{H}^{N-1}(\mathbf{x}),
\end{aligned}$$

where in the third equality we have used the fundamental theorem of calculus applied to the function of one variable $x_1 \in [a_1, b_1] \mapsto f_1(x_1, \mathbf{x}')$ with $\mathbf{x}'$ fixed, and in the last equality we have used formula (**??**). Repeating this argument for all the integrals $\int_R \frac{\partial f_i}{\partial x_i}(\mathbf{x})\, d\mathbf{x}$ and then summing the resulting identities gives the desired result.

Step 2: Next we prove that the divergence theorem holds when U is replaced by a set of the form

$$V := \left\{ (x_1, \mathbf{x}') \in \mathbb{R} \times \mathbb{R}^{N-1} : \, h(\mathbf{x}') < x_1 < b_1,\ \mathbf{x}' \in R' \right\},$$

where $R' := (a_2, b_2) \times \cdots \times (a_N, b_N)$ and $h : \mathbb{R}^{N-1} \to \mathbb{R}$ is a function of class C^1 and $\mathbf{f} = \mathbf{0}$ in an open neighborhood of $\partial V \setminus \operatorname{graph} h$. The change of variables

$$y_1 := x_1 - h(\mathbf{x}'), \quad \mathbf{y}' := \mathbf{x}'$$

maps the set V into the set

$$W := \left\{ (y_1, \mathbf{y}') \in \mathbb{R} \times \mathbb{R}^{N-1} : \, 0 < y_1 < b_1 - h(\mathbf{y}'),\ \mathbf{x}' \in R' \right\}.$$

Let $R := (0, b_1) \times R'$ and consider the function

$$\mathbf{g}(\mathbf{y}) := \begin{cases} \mathbf{f}(y_1 + h(\mathbf{y}'), \mathbf{y}') & \text{if } \mathbf{y} \in W, \\ \mathbf{0} & \text{if } \mathbf{y} \in R \setminus W. \end{cases}$$

Since $\mathbf{f} = \mathbf{0}$ near $x_1 = b_1$, it follows that $\mathbf{g}$ is continuous in $\overline{R}$ with bounded continuous partial derivatives in R. Hence, by Step 1,

$$\int_W \operatorname{div}\mathbf{g}(\mathbf{y})\, d\mathbf{y} = \int_R \operatorname{div}\mathbf{g}(\mathbf{y})\, d\mathbf{y} = \int_{\partial R} \mathbf{g}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y})\, d\mathcal{H}^{N-1}(\mathbf{y}) = -\int_{R'} g_1(0, \mathbf{y}')\, d\mathbf{y}'.$$

By Theorem 61, if $\mathbf{y} \in W$,

$$\begin{aligned}\operatorname{div}\mathbf{g}(\mathbf{y}) &= \frac{\partial f_1}{\partial x_1}(y_1 + h(\mathbf{y}'), \mathbf{y}') \\ &\quad + \sum_{i=2}^{N}\left(\frac{\partial f_i}{\partial x_1}(y_1 + h(\mathbf{y}'), \mathbf{y}')\frac{\partial h}{\partial y_i}(\mathbf{y}') + \frac{\partial f_i}{\partial x_i}(y_1 + h(\mathbf{y}'), \mathbf{y}')\right) \\ &= \operatorname{div}\mathbf{f}(y_1 + h(\mathbf{y}'), \mathbf{y}') + \sum_{i=2}^{N}\frac{\partial f_i}{\partial x_1}(y_1 + h(\mathbf{y}'), \mathbf{y}')\frac{\partial h}{\partial y_i}(\mathbf{y}').\end{aligned}$$

■

Wednesday, April 25, 2012

Proof. Hence,

$$\begin{aligned}\int_W \operatorname{div}\mathbf{f}(y_1 + h(\mathbf{y}'), \mathbf{y}')\, d\mathbf{y} &= -\int_{R'} f_1(h(\mathbf{y}'), \mathbf{y}')\, d\mathbf{y}' \\ &\quad - \sum_{i=2}^{N}\int_W \frac{\partial f_i}{\partial x_1}(y_1 + h(\mathbf{y}'), \mathbf{y}')\frac{\partial h}{\partial y_i}(\mathbf{y}')\, d\mathbf{y}.\end{aligned}$$

Consider the change of variables

$$\begin{aligned}\mathbf{k} : \mathbb{R}^N &\to \mathbb{R}^N, \\ \mathbf{y} &\mapsto (y_1 + h(\mathbf{y}'), \mathbf{y}').\end{aligned}$$

Note that $\mathbf{k}$ is invertible, with inverse given by

$$\begin{aligned}\mathbf{k}^{-1} : \mathbb{R}^N &\to \mathbb{R}^N \\ \mathbf{x} &\mapsto (x_1 - h(\mathbf{x}'), \mathbf{x}').\end{aligned}$$

Moreover, we have

$$J_{\mathbf{k}}(\mathbf{y}) = \begin{pmatrix} 1 & \frac{\partial h}{\partial y_2}(\mathbf{y}') & \cdots & \frac{\partial h}{\partial y_N}(\mathbf{y}') \\ 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \end{pmatrix},$$

which implies that $\det J_{\mathbf{k}}(\mathbf{y}) = 1$. Hence, by Theorem 204,

$$\int_V \operatorname{div}\mathbf{f}(\mathbf{x})\, d\mathbf{x} = -\sum_{i=2}^{N}\int_V \frac{\partial f_i}{\partial x_1}(\mathbf{x})\frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x} - \int_{R'} f_1(h(\mathbf{x}'), \mathbf{x}')\, d\mathbf{x}'.$$

By the fundamental theorem of calculus,

$$\begin{aligned}\int_V \frac{\partial f_i}{\partial x_1}(\mathbf{x})\frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x} &= \int_{R'}\left(\int_{h(\mathbf{x}')}^{b_1}\frac{\partial f_i}{\partial x_1}(x_1, \mathbf{x}')\, dx_1\right)\frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x}' \\ &= \int_{R'}(f_i(b_1, \mathbf{x}') - f_i(h(\mathbf{x}'), \mathbf{x}'))\frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x}' \\ &= \int_{R'}(0 - f_i(h(\mathbf{x}'), \mathbf{x}'))\frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x},\end{aligned}$$

and so

$$\int_V \operatorname{div} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = -\int_{R'} f_1(h(\mathbf{x}'), \mathbf{x}')\, d\mathbf{x}' + \sum_{i=2}^N \int_{R'} f_i(h(\mathbf{x}'), \mathbf{x}') \frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x}'$$

$$= -\int_{R'} f_1(h(\mathbf{x}'), \mathbf{x}')\, d\mathbf{x}' + \sum_{i=2}^N \int_{R'} f_i(h(\mathbf{x}'), \mathbf{x}') \frac{\partial h}{\partial x_i}(\mathbf{x}')\, d\mathbf{x}'$$

$$= \int_{R'} \mathbf{f}(h(\mathbf{x}'), \mathbf{x}') \cdot (-1, \nabla h(\mathbf{x}')) \frac{\sqrt{1 + \|\nabla h(\mathbf{x}')\|_{N-1}^2}}{\sqrt{1 + \|\nabla h(\mathbf{x}')\|_{N-1}^2}}\, d\mathbf{x}'$$

$$= \int_{\partial V} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, d\mathcal{H}^{N-1}(\mathbf{x}),$$

where in the last equality we have used formula (**??**). This concludes the proof in this case.

Note that a similar proof works when the role of the variable x_1 is played by another variable x_i, so that

$$V := \left\{\mathbf{x} \in \mathbb{R}^{N-1} : h(\mathbf{x}_i) < x_i < b_i,\, \mathbf{x}_i \in R'\right\}, \tag{51}$$

where $\mathbf{x}_i \in \mathbb{R}^{N-1}$ is the vector obtained from $\mathbf{x}$ by removing the i-th component, and also when V has the form

$$V := \left\{\mathbf{x} \in \mathbb{R}^{N-1} : a_i < x_i < h(\mathbf{x}_i),\, \mathbf{x}_i \in R'\right\}, \tag{52}$$

Step 3: We prove that in a neighborhood of every point $\mathbf{x} \in \partial U$ on the boundary there exists an open cube $Q(\mathbf{x}, r_{\mathbf{x}})$ centered at $\mathbf{x}$ and of side-length $r_{\mathbf{x}}$ such that $U \cap Q(\mathbf{x}, r_{\mathbf{x}})$ is either of the form (51) or (52).

Fix $\mathbf{x}_0 \in \partial U$. Since $\nabla g(\mathbf{x}) \neq \mathbf{0}$, there exists i (which depends on $\mathbf{x}_0$) such that $\frac{\partial g}{\partial x_i}(\mathbf{x}_0) \neq 0$. By relabelling the axes, for simplicity we can assume that $i = 1$ and that $\frac{\partial g}{\partial x_1}(\mathbf{x}_0) < 0$ (the case $\frac{\partial g}{\partial x_1}(\mathbf{x}_0) > 0$ is similar). By continuity of $\frac{\partial g}{\partial x_1}$ we may find $s > 0$ such that $\frac{\partial g}{\partial x_1} < 0$ in $B(\mathbf{x}_0, s)$. It follows that if $\mathbf{x} = (x_1, \mathbf{x}') \in B(\mathbf{x}_0, s) \cap \partial U$, so that $g(x_1, \mathbf{x}') = 0$, then $g(t, \mathbf{x}') < 0$ for all $t > x_1$ with $(t, \mathbf{x}') \in B(\mathbf{x}_0, s)$, which implies that $(t, \mathbf{x}') \in U$, while $g(t, \mathbf{x}') > 0$ for all $t < x_1$ with $(t, \mathbf{x}') \in B(\mathbf{x}_0, s)$, which implies that $(t, \mathbf{x}') \in \mathbb{R}^N \setminus \overline{U}$.

Apply the implicit function theorem to the function $g : B(\mathbf{x}_0, s) \to \mathbb{R}$ at the point $\mathbf{x}_0 = (a, \mathbf{b})$ to find $B_{N-1}(\mathbf{b}, r)$, an interval $(a - \delta, a + \delta)$, with $(a - \delta, a + \delta) \times B_{N-1}(\mathbf{b}, r) \subset B(\mathbf{x}_0, s)$, and a function $h : B_{N-1}(\mathbf{b}, r) \to (a - \delta, a + \delta)$ of class C^1 such that $g(h(\mathbf{x}'), \mathbf{x}') = 0$ for all $\mathbf{x}' \in B_{N-1}(\mathbf{b}, r)$. Thus $\partial U \cap ((a - \delta, a + \delta) \times B_{N-1}(\mathbf{b}, r))$ is given by the graph of the function h. Finally, define $r_{\mathbf{x}_0} := \min\left\{\delta, \frac{r}{\sqrt{N-1}}\right\}$. Then

$$Q(\mathbf{x}_0, r_{\mathbf{x}_0}) = \left(a - \frac{1}{2} r_{\mathbf{x}_0}, a + \frac{1}{2} r_{\mathbf{x}_0}\right) \times Q_{N-1}(\mathbf{a}, r_{\mathbf{x}_0}) \subset (a - \delta, a + \delta) \times B_{N-1}(\mathbf{b}, r)$$

and by the previous discussion $U \cap Q(\mathbf{x}_0, r_{\mathbf{x}_0})$ has the form (51).

Step 4: If $\mathbf{x} \in \partial U$, there exists an open cube $Q(\mathbf{x}, r_{\mathbf{x}})$ centered at $\mathbf{x}$ and of side-length $r_{\mathbf{x}}$ such that $U \cap Q(\mathbf{x}, r_{\mathbf{x}})$ is either of the form (51) or (52). On the other hand, if $\mathbf{x} \in U$, which is open, then there exists $Q(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$.

For every $\mathbf{x} \in \overline{U}$ construct a nonnegative function $\varphi_{\mathbf{x}} \in C^1(\mathbb{R}^N)$ such that $\varphi_{\mathbf{x}} > 0$ in $Q\left(\mathbf{x}, \frac{1}{2}r_{\mathbf{x}}\right)$ and $\varphi_{\mathbf{x}} = 0$ outside$Q(\mathbf{x}, r_{\mathbf{x}})$ (exercise). Since $\overline{U}$ is closed and bounded, it is compact. Since the family of open sets $\left\{Q\left(\mathbf{x}, \frac{1}{2}r_{\mathbf{x}}\right)\right\}_{\mathbf{x}\in\overline{U}}$ cover $\overline{U}$, there exist a finite number $Q\left(\mathbf{x}_1, \frac{1}{2}r_{\mathbf{x}_1}\right)$, ..., $Q\left(\mathbf{x}_n, \frac{1}{2}r_{\mathbf{x}_n}\right)$ that still cover $\overline{U}$. Note that $Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ is either contained in U or $U \cap Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ is of the form (51) or (52) and that

$$\varphi := \sum_{k=1}^{n} \varphi_k > 0 \quad \text{in } \overline{U},$$

since $Q\left(\mathbf{x}_1, \frac{1}{2}r_{\mathbf{x}_1}\right), \ldots, Q\left(\mathbf{x}_n, \frac{1}{2}r_{\mathbf{x}_n}\right)$ that still cover $\overline{U}$ and $\varphi_k > 0$ in W_k. Define

$$\psi_k(\mathbf{x}) := \begin{cases} \frac{\varphi_k(\mathbf{x})}{\varphi(\mathbf{x})} & \text{if } \varphi_k(\mathbf{x}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi_k \in C^1(\mathbb{R}^N)$, $\psi_k = 0$ outside $Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ and $\sum_{k=1}^{n} \psi_k = 1$ in $\overline{U}$. The family $\{\psi_k\}$ is called a *partition of unity subordinated to the family of open sets* $Q(\mathbf{x}_1, r_{\mathbf{x}_1})$, ..., $Q(\mathbf{x}_n, r_{\mathbf{x}_n})$. Hence,

$$0 = \frac{\partial}{\partial x_i}(1) = \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} \psi_k\right) = \sum_{k=1}^{n} \frac{\partial \psi_k}{\partial x_i}$$

in U. For every $k = 1, \ldots, n$, we have that the function $\psi_k \mathbf{f}$ is zero outside $Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ and since $Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ is either contained in U or $U \cap Q(\mathbf{x}_k, r_{\mathbf{x}_k})$ is of the form (51) or (52), we can apply either Step 1 or Step 2 in V_k to conclude that

$$\int_U \operatorname{div}(\psi_k \mathbf{f})\, d\mathbf{x} = \int_{U\cap Q\left(\mathbf{x}_k, r_{\mathbf{x}_k}\right)} \operatorname{div}(\psi_k \mathbf{f})\, d\mathbf{x} = \int_{\partial\left(U\cap Q\left(\mathbf{x}_k, r_{\mathbf{x}_k}\right)\right)} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1}$$
$$= \int_{\partial U} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1}, \tag{53}$$

where we have used the fact that $\psi_k \mathbf{f}$ is zero outside $Q(\mathbf{x}_k, r_{\mathbf{x}_k})$. Summing (53) over k and using the fact that $\sum_{k=1}^{n} \psi_k = 1$ in $\overline{U}$, we have

$$\int_U \operatorname{div}\mathbf{f}\, d\mathbf{x} = \int_U \operatorname{div}\left(\sum_{k=1}^{n} \psi_k \mathbf{f}\right) d\mathbf{x} = \sum_{k=1}^{n} \int_U \operatorname{div}(\psi_k \mathbf{f})\, d\mathbf{x}$$
$$= \sum_{k=1}^{n} \int_{\partial U} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1} = \int_{\partial U} \sum_{k=1}^{n} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1} = \int_{\partial U} \mathbf{f}\cdot\boldsymbol{\nu}\, d\mathcal{H}^{N-1},$$

which is what we wanted. ■

Remark 231 *In physics* $\int_{\partial U} \mathbf{f}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, d\mathcal{H}^{N-1}(\mathbf{x})$ *represents the* outward flux *of a vector field* $\mathbf{f}$ *across the boundary of a region* U*.*

Friday, April 27, 2012

If $E \subseteq \mathbb{R}^N$ and $\mathbf{f} : E \to \mathbb{R}^N$ is differentiable, then $\mathbf{f}$ is called a *divergence-free field* or *solenoidal field* if

$$\operatorname{div} \mathbf{f} = 0.$$

Thus for a smooth solenoidal field, the outward flux across the boundary of a regular set U is zero.

Corollary 232 *The theorem continues to hold if* $U \subset \mathbb{R}^N$ *is open, bounded, and its boundary consists of two sets* E_1 *and* E_2*, where* E_1 *is a closed set contained in the finite union of compact surfaces of class* C^1 *and dimension less than* $N-1$*, while for every* $\mathbf{x}_0 \in E_2$ *there exist a ball* $B(\mathbf{x}_0, r)$ *and a function* $g \in C^1(B(\mathbf{x}_0, r))$ *such that with* $\nabla g(\mathbf{x}) \neq \mathbf{0}$ *for all* $\mathbf{x} \in B(\mathbf{x}_0, r) \cap \partial U$*, such that*

$$\begin{aligned} B(\mathbf{x}_0, r) \cap U &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) < 0\}, \\ B(\mathbf{x}_0, r) \setminus \overline{U} &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) > 0\}, \\ B(\mathbf{x}_0, r) \cap \partial U &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) = 0\}. \end{aligned}$$

Note that the radius of the ball and the function g depend on $\mathbf{x}_0$.

Example 233 *Let's calculate the outward flux of the function*

$$\mathbf{f}(x, y, z) := (0, yz, x)$$

across the boundary of the region

$$U := \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < z^2,\ x^2 + y^2 + z^2 < 2y,\ z > 0\right\}.$$

Note that U *is not a regular open set (why?) but it satisfies the hypotheses of the previous corollary (why?).*

We have

$$\operatorname{div} \mathbf{f}(x, y, z) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(x) = 0 + 1z + 0,$$

and so by the divergence theorem

$$\int_{\partial U} \mathbf{f} \cdot \boldsymbol{\nu}\, d\mathcal{H}^2 = \int_U z\, dx dy dz.$$

Using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, *we get* $r^2\cos^2\theta + r^2\sin^2\theta < z^2$, *that is,* $r^2 < z^2$, $r^2\cos^2\theta + r^2\sin^2\theta + z^2 < 2r\sin\theta$, *that is,* $r^2 + z^2 < 2r\sin\theta$, *and* $z > 0$, *hence,* $r^2 < z^2 < 2r\sin\theta - r^2$, *which implies that* $r^2 < 2r\sin\theta - r^2$, *or equivalently,* $r < \sin\theta$. *In turn,* $\sin\theta$ *should be positive, and so* $\theta \in (0, \pi)$.

$$W := \left\{(r, \theta, z) \in \mathbb{R}^3 : 0 < \theta < \pi,\ r < \sin\theta,\ r < z < \sqrt{2r\sin\theta - r^2}\right\}.$$

Hence, by changing variables

$$\begin{aligned}
\int_U z\,dxdydz &= \int_0^\pi \left(\int_0^{\sin\theta}\left(\int_r^{\sqrt{2r\sin\theta-r^2}} z\,dz\right) r\,dr\right) d\theta \\
&= \int_0^\pi \left(\int_0^{\sin\theta}\left[\frac{z^2}{2}\right]_{z=r}^{z=\sqrt{2r\sin\theta-r^2}} r\,dr\right) d\theta \\
&= \int_0^\pi \left(\int_0^{\sin\theta}\left(r^2\sin\theta-r^3\right)\,dr\right) d\theta \\
&= \int_0^\pi \left[\frac{r^3}{3}\sin\theta-\frac{r^4}{4}\right]_{r=0}^{r=\sin\theta} d\theta = \int_0^\pi \left[\frac{1}{3}\sin^4\theta-\frac{1}{4}\sin^4\theta\right] d\theta \\
&= \int_0^\pi \frac{1}{12}\sin^4\theta\,d\theta = \frac{1}{32}\pi.
\end{aligned}$$

Corollary 234 (Integration by Parts) *Let $U\subset\mathbb{R}^N$ be an open, bounded, regular set and let $f:\overline{U}\to\mathbb{R}$ and $g:\overline{U}\to\mathbb{R}$ be such that f and g are bounded and continuous in $\overline{U}$ and there exist the partial derivatives of f and g in $\mathbb{R}$ at all $\mathbf{x}\in U$ and they are continuous and bounded. Then for every $i=1,\ldots,N$,*

$$\int_U f(\mathbf{x})\frac{\partial g}{\partial x_i}(\mathbf{x})\,d\mathbf{x} = -\int_U g(\mathbf{x})\frac{\partial f}{\partial x_i}(\mathbf{x})\,d\mathbf{x} + \int_{\partial U} f(\mathbf{x})\,g(\mathbf{x})\,\nu_i(\mathbf{x})\,d\mathcal{H}^{N-1}(\mathbf{x}).$$

Proof. Fix $i\in\{1,\ldots,N\}$. We apply the divergence theorem to the function $\mathbf{f}:\overline{U}\to\mathbb{R}^N$ defined by

$$f_j(\mathbf{x}) := \begin{cases} f(\mathbf{x})\,g(\mathbf{x}) & \text{if } j=i, \\ 0 & \text{if } j\neq i. \end{cases}$$

Then

$$\operatorname{div}\mathbf{f} = \sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = \frac{\partial(fg)}{\partial x_i} = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i},$$

and so

$$\int_U \left(f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}\right) d\mathbf{x} = \int_U \operatorname{div}\mathbf{f}\,d\mathbf{x} = \int_{\partial U} \mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1} = \int_{\partial U} fg\nu_i\,d\mathcal{H}^{N-1}.$$

■

Definition 235 *Given an open set $U\subseteq\mathbb{R}^N$ and an integer $m\in\mathbb{N}$, we say that a function $f:\overline{U}\to\mathbb{R}$ is of class $C^m\left(\overline{U}\right)$ if f can be extended to a function of class $C^m(V)$, where V is an open set containing V.*

Another important application is given by the area's formulas in $\mathbb{R}^2$. Let $U\subset\mathbb{R}^2$ be an open, bounded set and assume that its boundary ∂U is the range of a closed, simple, regular curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\varphi}:[a,b]\to$

$\mathbb{R}^2$. Then the hypotheses of Corollary 232 are satisfied (why?). Given $t \in [a,b]$, the vector $\boldsymbol{\varphi}'(t)$ is tangent to the curve at the point $\boldsymbol{\varphi}(t) \in \partial U$. Hence, if $\boldsymbol{\varphi}(t) = (x(t), y(t))$, the outer normal to ∂U at the point $\boldsymbol{\varphi}(t)$ is either

$$\frac{(-y'(t), x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}} \quad \text{or} \quad -\frac{(-y'(t), x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}},$$

depending on the orientation of the curve. We say that $\boldsymbol{\gamma}$ has a *positive orientation* for ∂U if as we traverse the curve starting from $t = a$ we find U on our left. In this case, the outward normal at the point $\boldsymbol{\varphi}(t)$ is is given by

$$\boldsymbol{\nu}(\boldsymbol{\varphi}(t)) = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}.$$

Hence, if $f, g : \overline{U} \to \mathbb{R}$ are bounded and continuous in $\overline{U}$ and there exist the partial derivatives of f and g at all $(x,y) \in U$ and they are continuous and bounded, then applying the divergence theorem to the function $\mathbf{f}(x,y) := (f(x,y), g(x,y))$

$$\begin{aligned}
\int_U \left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y) \right) dxdy &= \int_{\partial U} (f,g) \cdot \boldsymbol{\nu}\, d\mathcal{H}^1 \\
&= \int_a^b f(x(t), y(t)) \frac{y'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&\quad + \int_a^b g(x(t), y(t)) \frac{-x'(t)}{\sqrt{(x'(t))^2 + (y'(t))^2}} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt, \\
&= \int_a^b \left[f(x(t), y(t))\, y'(t) - g(x(t), y(t))\, x'(t) \right] dt \\
&=: \int_{\partial U} f\, dy - \int_{\partial U} g\, dx.
\end{aligned}$$

Taking $g = 0$ or $f = 0$ we get the *Gauss-Green formulas*

$$\int_U \frac{\partial f}{\partial x}(x,y)\, dxdy = \int_{\partial U} f\, dy. \tag{54}$$

$$\int_U \frac{\partial g}{\partial y}(x,y)\, dxdy = -\int_{\partial U} g\, dx. \tag{55}$$

Note that by subtracting these two identities, we get

$$\int_U \left(\frac{\partial f}{\partial x}(x,y) - \frac{\partial g}{\partial y}(x,y) \right) dxdy = \int_{\partial U} f\, dy + \int_{\partial U} g\, dx. \tag{56}$$

We will use this formula in the proof of Stokes' theorem.

Taking $f(x,y) = x$ in (54), we get the *second area's formula*

$$\operatorname{meas} U = \int_U 1\, dxdy = \int_a^b x(t)\, y'(t)\ dt = \int_{\partial U} x\, dy,$$

while taking $g(x,y) = y$ in (55) we get the *second area's formula*

$$\operatorname{meas} U = \int_U 1\, dxdy = -\int_a^b y(t)\, x'(t)\ dt = -\int_{\partial U} y\, dx.$$

By adding these two identities, we get

$$\operatorname{meas} U = \frac{1}{2}\int_{\partial U} \left[-y(t)\, x'(t) + x(t)\, y'(t)\right]\, dt = \frac{1}{2}\left[\int_{\partial U} x\, dy - \int_{\partial U} y\, dx\right].$$

Example 236 *Let's find the area of the region U enclosed by the curve of parametric representation*

$$\begin{cases} x(\theta) = (1-\cos\theta)\cos\theta, \\ y(\theta) = (1-\cos\theta)\sin\theta, \end{cases} \qquad \theta \in [0, 2\pi].$$

To see that it is simple, note that for $0 < \theta < 2\pi$, $(1-\cos\theta) > 0$. The curve starts from the origin and goes around clockwise only once. Note that

$$\begin{aligned} x'(\theta) &= -\sin\theta + 2\sin\theta\cos\theta = \sin\theta\,(-1 + 2\cos\theta), \\ y'(\theta) &= \cos\theta + \sin^2\theta - \cos^2\theta = -2\cos^2\theta + \cos\theta + 1. \end{aligned}$$

Thus, $x'(\theta) = 0$ only for $\sin\theta = 0$, that is $\theta = 0, \pi, 2\pi$,and $\cos\theta = \frac{1}{2}$, that is, for $\theta = \frac{1}{3}\pi, \frac{5}{3}\pi$. At those values,

$$\begin{aligned} y'(0) = y'(2\pi) &= -2+1+1 = 0, \quad y'(\pi) = -2-1+1, \\ y'\left(\frac{1}{3}\pi\right) &= -2\cos^2\frac{1}{3}\pi + \cos\frac{1}{3}\pi + 1 = 1, \\ y'\left(\frac{5}{3}\pi\right) &= -2\cos^2\frac{5}{3}\pi + \cos\frac{5}{3}\pi + 1 = 1. \end{aligned}$$

Hence, the curve is not regular, but since these are only a finite number of bad points, we can apply Corollary 232. Using the third area formula, we get

$$\begin{aligned} \operatorname{meas} U = &-\frac{1}{2}\int_0^{2\pi} (1-\cos\theta)\sin\theta\left[\sin\theta\cos\theta - (1-\cos\theta)\sin\theta\right]\, d\theta \\ &+\frac{1}{2}\int_0^{2\pi} (1-\cos\theta)\cos\theta\left[\sin\theta\sin\theta + (1-\cos\theta)\cos\theta\right]\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\left[\cos^2\theta + \sin^2\theta\right]\, d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta = \frac{3}{2}\pi. \end{aligned}$$

Monday, April 30, 2012

Proposition 237 *Given two open sets* $A, D \subset \mathbb{R}^N$, *with* D *bounded and* $\operatorname{dist}(A, D) \geq 3d > 0$ *for some* $d > 0$, *there exists a function* $f \in C^1(\mathbb{R}^N)$ *such that,* $0 \leq f \leq 1$, $f(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in A$, $f(\mathbf{x}) = 1$ *for all* $\mathbf{x} \in D$, *and*

$$\|\nabla f(\mathbf{x})\| \leq \frac{C}{d}$$

for all $\mathbf{x} \in \mathbb{R}^N$, *where* $C > 0$ *depends only on* N.

Proof. Construct a nonnegative function $g \in C^1(\mathbb{R}^N)$ such that $g = 0$ in $\mathbb{R}^N \setminus B(\mathbf{0}, 1)$, $g > 0$ in $B\left(\mathbf{0}, \frac{1}{2}\right)$ and

$$\int_{B(\mathbf{0},1)} g(\mathbf{x})\, d\mathbf{x} = 1.$$

Since D is bounded, so is the set $E := \left\{\mathbf{x} \in \mathbb{R}^N : \operatorname{dist}(\mathbf{x}, D) < d\right\}$. Moreover, $\operatorname{dist}\left(A, \overline{E}\right) \geq 2d$. For every $\mathbf{x} \in \overline{E}$ consider the ball $B(\mathbf{x}, d)$. Since $\overline{E}$ is compact, we may find a finite number of balls that covers $\overline{E}$. Let W be the union of these balls. Then W is Peano–Jordan measurable, since the boundary is contained in the union of the sphere, which has measure zero, and

$$\left\{\mathbf{x} \in \mathbb{R}^N : \operatorname{dist}(\mathbf{x}, D) < d\right\} \subset \overline{E} \subset W \subseteq \left\{\mathbf{x} \in \mathbb{R}^N : \operatorname{dist}(\mathbf{x}, D) < 2d\right\}.$$

Define

$$f(\mathbf{x}) := \frac{1}{d^N} \int_{\mathbb{R}^N} g\left(\frac{\mathbf{x} - \mathbf{y}}{d}\right) \chi_W(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^N.$$

Let's prove that f has the right properties. Since $g = 0$ in $\mathbb{R}^N \setminus B(\mathbf{0}, 1)$, we have that $g\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right) = 0$ for all $\left|\frac{\mathbf{x}-\mathbf{y}}{d}\right| > 1$, and so

$$f(\mathbf{x}) = \frac{1}{d^N} \int_{B(\mathbf{x},d) \cap W} g\left(\frac{\mathbf{x} - \mathbf{y}}{d}\right) d\mathbf{y}.$$

If $\mathbf{x} \in A$, then since, $\operatorname{dist}(A, D) \geq 3d$, and $W \subseteq \left\{\mathbf{x} \in \mathbb{R}^N : \operatorname{dist}(\mathbf{x}, D) < 2d\right\}$, we have that $B(\mathbf{x}, d) \cap W = \emptyset$, and so $f(\mathbf{x}) = 1$. On the other hand, if $\mathbf{x} \in D$, then $B(\mathbf{x}, d) \subset \overline{E} \subset W$ and so

$$f(\mathbf{x}) = \frac{1}{d^N} \int_{B(\mathbf{x},d)} g\left(\frac{\mathbf{x} - \mathbf{y}}{d}\right) d\mathbf{y} = \int_{B(\mathbf{0},1)} g(\mathbf{z})\, d\mathbf{z} = 1,$$

where we have made the change of variables $\frac{\mathbf{x}-\mathbf{y}}{d} = \mathbf{z}$.

To prove that f is of class C^1, let $\mathbf{e}_i$, $i=1,\ldots,N$, be an element of the canonical basis of $\mathbb{R}^N$ and for every $h\in\mathbb{R}$, with $0<|h|\leq\eta$, consider

$$\begin{aligned}
&\frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x})}{h}-\frac{1}{d^N}\int_{\mathbb{R}^N}\frac{1}{d}\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\chi_W(\mathbf{y})\,d\mathbf{y}\\
&=\frac{1}{d^N}\int_{\mathbb{R}^N}\left[\frac{g\left(\frac{\mathbf{x}+h\mathbf{e}_i-\mathbf{y}}{d}\right)-g\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)}{h}-\frac{1}{d}\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\right]\chi_W(\mathbf{y})\,d\mathbf{y}\\
&=\frac{1}{d^{N+1}}\int_{\mathbb{R}^N}\left(\frac{1}{h}\int_0^h\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}+t\mathbf{e}_i}{d}\right)dt-\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\right)\chi_W(\mathbf{y})\,d\mathbf{y}\\
&=\frac{1}{d^{N+1}}\frac{1}{h}\int_0^h\left(\int_{B(\mathbf{x},d+\eta d)}\left[\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}+t\mathbf{e}_i}{d}\right)-\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\right]\chi_W(\mathbf{y})\,d\mathbf{y}\right)dt,
\end{aligned}$$

where we have used Fubini's theorem and the fact that $\operatorname{supp} g\subset\overline{B(0,1)}$.

Since $g\in C_c^1(\mathbb{R}^N)$ its partial derivatives are uniformly continuous. Hence for every $\rho>0$ there exists $\delta=\delta(\eta,\mathbf{x},\rho)>0$ such that

$$\left|\frac{\partial g}{\partial x_i}(z)-\frac{\partial g}{\partial x_i}(w)\right|\leq\rho$$

for all $z,y\in B(\mathbf{x},1+\eta)$, with $|z-w|\leq\delta$. Then for $0<|h|\leq\min\{\eta,\delta\}$ we have

$$\left|\frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x})}{h}-\frac{1}{d^{N+1}}\int_{\mathbb{R}^N}\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\chi_W(\mathbf{y})\,d\mathbf{y}\right|\leq\rho,$$

which shows that

$$\frac{\partial f}{\partial x_i}(\mathbf{x})=\frac{1}{d^{N+1}}\int_{\mathbb{R}^N}\frac{\partial g}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{d}\right)\chi_W(\mathbf{y})\,d\mathbf{y}.$$

■

Wednesday, May 2, 2012

Definition 238 *Given $1\leq k\leq N$, a bounded set $E\subseteq\mathbb{R}^N$ has k-*Peano–Jordan measure zero *if for every $\varepsilon>0$ there exists $\delta_\varepsilon>0$ with the property that for every $0<\delta\leq\delta_\varepsilon$ there exists a finite family $\{Q_n\}$ of cubes of side-length δ such that $E\subseteq\bigcup_n Q_n$, and*

$$\sum_n(\operatorname{diam}Q_n)^k\leq\varepsilon.$$

Exercise 239 *Prove that if $k=N$, then a bounded set $E\subseteq\mathbb{R}^N$ has N-Peano–Jordan measure zero if and only if it is Peano–Jordan measurable and it has Peano–Jordan measure zero.*

Theorem 240 *Let $E\subset\mathbb{R}^N$ be a bounded set with k-Peano–Jordan measure zero, $1\leq k\leq N$, and let $\mathbf{g}:E\to\mathbb{R}^M$ be a Lipschitz function, with $k\leq M$. Then $\mathbf{g}(E)$ has k-Peano–Jordan measure zero.*

Proof. The proof is very similar to the one of Theorem 199 and is left as an exercise. ■

Using the previous theorem we can show that if two k-dimensional surfaces intersect "transversally", then their intersection has k-Peano–Jordan measure zero.

Corollary 241 *Given a k-dimensional surface M of class C^m, $m \in \mathbb{N}$, consider a local chart $\varphi : V \to M$, let $K \subset V$ be a compact Peano–Jordan measurable set. Then $\varphi(\partial K)$ has k-Peano–Jordan measure zero.*

Proof. Since K is Peano–Jordan measurable in $\mathbb{R}^k$, its boundary ∂K has Peano–Jordan measure zero. In turn, by the previous exercise, ∂K has k-Peano–Jordan measure zero. It follows from the previous theorem that $\varphi(\partial F)$ has k-Peano–Jordan measure zero. ■

We will prove the following result.

Theorem 242 *The divergence theorem continues to hold if $U \subset \mathbb{R}^N$ is open, bounded, and its boundary consists of two sets E_1 and E_2, where E_1 has $(N-1)$-Peano–Jordan measure zero, while for every $\mathbf{x}_0 \in E_2$ there exist a ball $B(\mathbf{x}_0, r)$ and a function $g \in C^1(B(\mathbf{x}_0, r))$ such that with $\nabla g(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B(\mathbf{x}_0, r) \cap \partial U$ and*

$$\begin{aligned} B(\mathbf{x}_0, r) \cap U &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) < 0\}, \\ B(\mathbf{x}_0, r) \setminus \overline{U} &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) > 0\}, \\ B(\mathbf{x}_0, r) \cap \partial U &= \{\mathbf{x} \in B(\mathbf{x}_0, r) : g(\mathbf{x}) = 0\}. \end{aligned}$$

Using again the fact that partial derivatives of g are uniformly continuous, we can prove that the partial derivatives of f are continuous, so that f is of class C^1.

Proof of Theorem 242. Step 1: Assume that $\mathbf{f}$ is zero on an open neighborhood D of E_1. Then for every $\mathbf{x} \in \partial U \setminus D$, there exists an open cube $Q(\mathbf{x}, r_{\mathbf{x}})$ centered at $\mathbf{x}$ and of side-length $r_{\mathbf{x}}$ such that $U \cap Q(\mathbf{x}, r_{\mathbf{x}})$ is either of the form (51) or (52). On the other hand, if $\mathbf{x} \in U$, which is open, then there exists $Q(\mathbf{x}, r_{\mathbf{x}}) \subseteq U$. Since $\overline{U} \setminus D$ is closed and bounded, it is compact. Since the family of open sets $\left\{Q\left(\mathbf{x}, \frac{1}{2} r_{\mathbf{x}}\right)\right\}_{\mathbf{x} \in \overline{U} \setminus D}$ cover $\overline{U} \setminus D$, there exist a finite number $Q\left(\mathbf{x}_1, \frac{1}{2} r_{\mathbf{x}_1}\right), \ldots, Q\left(\mathbf{x}_n, \frac{1}{2} r_{\mathbf{x}_n}\right)$ that still cover $\overline{U} \setminus D$. As in Step 4 of Theorem 230 construct a partition of unity $\{\psi_k\}$ subordinated to the family of open sets $Q(\mathbf{x}_1, r_{\mathbf{x}_1}), \ldots, Q(\mathbf{x}_n, r_{\mathbf{x}_n})$. For $\mathbf{x} \in U \setminus D$, as in Step 4 of Theorem 230, we have $\sum_{k=1}^n \operatorname{div}(\psi_k \mathbf{f})(\mathbf{x}) = \operatorname{div} \mathbf{f}(\mathbf{x})$,

$$\int_{U \setminus D} \operatorname{div}(\psi_k \mathbf{f}) \, d\mathbf{x} = \int_{\partial U \setminus D} \psi_k \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{N-1} = \int_{\partial U} \psi_k \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{N-1},$$

and so, using the fact that $\mathbf{f} = \mathbf{0}$ in D,

$$\begin{aligned}\int_U \operatorname{div}\mathbf{f}\,d\mathbf{x} &= \int_{U\setminus D} \operatorname{div}\mathbf{f}\,d\mathbf{x} = \int_{U\setminus D} \sum_{k=1}^{n} \operatorname{div}(\psi_k \mathbf{f})\,d\mathbf{x} = \sum_{k=1}^{n} \int_{U\setminus D} \operatorname{div}(\psi_k \mathbf{f})\,d\mathbf{x} \\ &= \sum_{k=1}^{n} \int_{\partial U\setminus D} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1} = \int_{\partial U} \sum_{k=1}^{n} \psi_k \mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1} = \int_{\partial U} \mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1},\end{aligned}$$

which is what we wanted.
Step 2: Let

$$\begin{aligned}A_n &:= \left\{\mathbf{x}\in\mathbb{R}^N : \operatorname{dist}(\mathbf{x},E_1) < \frac{1}{2^n}\right\}, \\ D_n &:= \left\{\mathbf{x}\in\mathbb{R}^N : \operatorname{dist}(\mathbf{x},E_1) > \frac{3}{2^n}\right\}.\end{aligned}$$

Then $\operatorname{dist}(A_n, D_n) \geq \frac{1}{2^n}$. By the previous proposition there exists $\varphi_n \in C^1(\mathbb{R}^N)$ such that $\varphi_n(\mathbf{x}) = 0$ for all $\mathbf{x}\in A_n$, $\varphi_n(\mathbf{x}) = 1$ for all $\mathbf{x}\in D_n$, and

$$\|\nabla\varphi_n(\mathbf{x})\| \leq C2^n$$

for all $\mathbf{x}\in\mathbb{R}^N$, where $C>0$ depends only on N.

Then the function $\varphi_n\mathbf{f}$ satisfies all the hypotheses of Step 1 and so

$$\int_U \operatorname{div}(\varphi_n\mathbf{f})\,d\mathbf{x} = \int_{\partial U} \varphi_n\mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1}.$$

Now

$$\begin{aligned}\operatorname{div}(\varphi_n\mathbf{f}) &= \sum_{i=1}^{N} \frac{\partial(\varphi_n f_i)}{\partial x_i} = \sum_{i=1}^{N} f_i \frac{\partial\varphi_n}{\partial x_i} + \varphi_n \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \\ &= \nabla\mathbf{f}\cdot\nabla\varphi_n + \varphi_n \operatorname{div}\mathbf{f},\end{aligned}$$

and so

$$\int_U \operatorname{div}(\varphi_n\mathbf{f})\,d\mathbf{x} = \int_U \nabla\mathbf{f}\cdot\nabla\varphi_n\,d\mathbf{x} + \int_U \varphi_n \operatorname{div}\mathbf{f}\,d\mathbf{x}.$$

Since $\varphi_n(\mathbf{x}) = 1$ for all $\mathbf{x}\in D_n$, $0\leq\varphi_n\leq 1$, and the partial derivatives of $\mathbf{f}$ are bounded,

$$\begin{aligned}\left|\int_U 1\operatorname{div}\mathbf{f}\,d\mathbf{x} - \int_U \varphi_n\operatorname{div}\mathbf{f}\,d\mathbf{x}\right| &= \left|\int_U (1-\varphi_n)\operatorname{div}\mathbf{f}\,d\mathbf{x}\right| \\ &= \left|\int_{U\setminus D_n} (1-\varphi_n)\operatorname{div}\mathbf{f}\,d\mathbf{x}\right| \\ &\leq M\operatorname{meas}_o\left(\left\{\mathbf{x}\in U : \operatorname{dist}(\mathbf{x},E_1) \leq \frac{3}{2^n}\right\}\right) \to 0\end{aligned}$$

as $n\to\infty$ by Lemma 243. Moreover, since $\nabla\varphi_n(\mathbf{x})=\mathbf{0}$ if $\mathbf{x}\in A_n\cup D_n$, and $\|\nabla\varphi_n(\mathbf{x})\|\le C2^n$ otherwise,

$$\left|\int_U \nabla\mathbf{f}\cdot\nabla\varphi_n\,d\mathbf{x}\right| = \left|\int_{\left\{\mathbf{y}\in U:\,\frac{1}{2^n}\le\operatorname{dist}(\mathbf{y},E_0)\le\frac{3}{2^n}\right\}}\nabla\mathbf{f}\cdot\nabla\varphi_n\,d\mathbf{x}\right|$$
$$\le CM2^n\operatorname{meas}_o\left(\left\{\mathbf{x}\in U:\,\operatorname{dist}(\mathbf{x},E_1)\le\frac{3}{2^n}\right\}\right)\to 0$$

as $n\to\infty$ again by Lemma 243.

Finally, since $\varphi_n(\mathbf{x})=1$ for all $\mathbf{x}\in D_n$ and $\mathbf{f}$ is bounded,

$$\left|\int_{\partial U}1\mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1}-\int_{\partial U}\varphi_n\mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1}\right| = \left|\int_{\partial U}(1-\varphi_n)\,\mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^{N-1}\right|$$
$$\le M\int_{\partial U}(1-\varphi_n)\,d\mathcal{H}^{N-1}\le M\mathcal{H}^{N-1}(\partial U\setminus D_n).$$

Using improper surface integrals, which we have note done..., one can show that $\mathcal{H}^{N-1}(\partial U\setminus D_n)\to 0$. ■

Although it was not needed in the proof, it actually turn out the sets D_n are Peano–Jordan measurable.

Friday, May 04, 2012

Lemma 243 *Let $K\subset\mathbb{R}^N$ be a compact set with $(N-1)$-Peano–Jordan measure zero and for every $r>0$ consider the set*

$$A_r:=\left\{\mathbf{x}\in\mathbb{R}^N:\,\operatorname{dist}(\mathbf{x},K)<r\right\}.$$

Then

$$\lim_{r\to 0^+}\frac{1}{r}\operatorname{meas}_o(A_r)=0.$$

Proof. Since K has $(N-1)$-Peano–Jordan measure zero for every $\varepsilon>0$ there exists $r_\varepsilon>0$ with the property that for every $0<r\le r_\varepsilon$ there exists a finite family $\{Q(\mathbf{x}_{n,r},r)\}_n$ of open cubes such that $K\subset\bigcup_n Q(\mathbf{x}_{n,r},r)$ and

$$\sum_n\left(\sqrt{N}r\right)^{N-1}=\sum_n(\operatorname{diam}Q(\mathbf{x}_{n,r},r))^{N-1}\le\frac{N^{(N-1)/2}}{2^N}\varepsilon.$$

If $\mathbf{x}\in A_r$, then there is $\mathbf{y}\in K$ such that $\|\mathbf{x}-\mathbf{y}\|<r$. Since, $K\subset\bigcup_n Q(\mathbf{x}_{n,r},r)$, there is a cube $Q(\mathbf{x}_{n,r},r)$ such that $\mathbf{x}\in Q(\mathbf{x}_{n,r},r)$. It follows that $\mathbf{y}\in Q(\mathbf{x}_{n,r},2r)$ and so $A_r\subseteq\bigcup_n Q(\mathbf{x}_{n,r},2r)$. In turn,

$$\frac{1}{r}\operatorname{meas}_o(A_r)\le\frac{1}{r}\sum_n\operatorname{meas}(Q(\mathbf{x}_{n,r},2r))=\frac{1}{r}\sum_n(2r)^N$$
$$=\frac{2^N}{N^{(N-1)/2}}\sum_n^n\left(\sqrt{N}r\right)^{N-1}\le\varepsilon.$$

This shows that

$$\lim_{r\to 0}\frac{1}{r}\operatorname{meas}_o(A_r)=0.$$

■

17 Stokes Theorem

We recall that given two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ of $\mathbb{R}^3$, their *cross-product* or *vector-product* is defined by

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &:= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \\ &= \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.\end{aligned}$$

Let $U \subseteq \mathbb{R}^3$ be an open set and let $\mathbf{f} : U \to \mathbb{R}^3$ be a differentiable function. The *curl* of $\mathbf{f}$ is the function $\operatorname{curl} \mathbf{f} : U \to \mathbb{R}^3$ defined by

$$\begin{aligned}\operatorname{curl} \mathbf{f}\,(x, y, z) &:= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &= \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{pmatrix} = \nabla \times \mathbf{f}.\end{aligned}$$

We will present Stokes' theorem in a very special setting. The general case relies on the notion of oriented manifolds with boundaries. Let $W \subseteq \mathbb{R}^2$ be an open set and let $\varphi : W \to \mathbb{R}^3$ be a one-to-one function of class C^2 such that $J_\varphi (u, v)$ has maximum rank 2 for all $(u, v) \in W$. Consider a bounded regular open set $V \subset \mathbb{R}^2$ with $\overline{V} \subset W$ and and assume that its boundary ∂V is the range of a closed, simple, regular curve $\boldsymbol{\gamma}$ with parametric representation $\boldsymbol{\psi} : [a, b] \to \mathbb{R}^2$.

Let $M := \varphi (V)$. Note that M is a 2-dimensional differential surface. We will call the *boundary* of M to be the set

$$\partial^* M := \varphi (\partial V).$$

The boundary of M is the range of the curve Γ of parametric representation $\varphi \circ \boldsymbol{\psi} : [a, b] \to \mathbb{R}^3$. If $\boldsymbol{\gamma}$ has a positive orientation for ∂V, we will say that Γ has a *positive orientation* for ∂M. In what follows we will assume that $\boldsymbol{\gamma}$ has a positive orientation for ∂V.

Finally, for every point $(x, y, z) \in M$, let $(u, v) \in W$ be such that $(x, y, z) = \varphi (u, v)$ (note that (u, v) is unique, since φ is one-to-one). We define the *normal* $\boldsymbol{\nu}$ to M at (x, y, z) as the vector

$$\begin{aligned}\boldsymbol{\nu} (x, y, z) &= \frac{\frac{\partial \varphi}{\partial u} (u, v) \times \frac{\partial \varphi}{\partial v} (u, v)}{\left\| \frac{\partial \varphi}{\partial u} (u, v) \times \frac{\partial \varphi}{\partial v} (u, v) \right\|} \\ &= \frac{1}{\left\| \frac{\partial \varphi}{\partial u} (u, v) \times \frac{\partial \varphi}{\partial v} (u, v) \right\|} \\ &\quad \cdot \left(\frac{\partial \varphi_2}{\partial u} \frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_2}{\partial v}, \frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_1}{\partial v} - \frac{\partial \varphi_1}{\partial u} \frac{\partial \varphi_3}{\partial v}, \frac{\partial \varphi_1}{\partial u} \frac{\partial \varphi_2}{\partial v} - \frac{\partial \varphi_2}{\partial u} \frac{\partial \varphi_2}{\partial v} \right).\end{aligned}$$

Theorem 244 (Stokes' Theorem) *Let $U \subseteq \mathbb{R}^3$ be an open set and let $\mathbf{f} : U \to \mathbb{R}^3$ be a function of class C^1. Let M and Γ be as above with $M \cup \partial^* M \subseteq U$. Then*

$$\int_M \operatorname{curl} \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^2 = \int_\Gamma \mathbf{f}.$$

Proof. Let's write explicitly the left-hand side,

$$\begin{aligned}
&\int_M \operatorname{curl} \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^2 \\
&= \iint_V \left(\frac{\partial f_3}{\partial y} \circ \varphi - \frac{\partial f_2}{\partial z} \circ \varphi \right) \left(\frac{\partial \varphi_2}{\partial u} \frac{\partial \varphi_3}{\partial v} - \frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_2}{\partial v} \right) \frac{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} dudv \\
&\quad + \iint_V \left(\frac{\partial f_1}{\partial z} \circ \varphi - \frac{\partial f_3}{\partial x} \circ \varphi \right) \left(\frac{\partial \varphi_3}{\partial u} \frac{\partial \varphi_1}{\partial v} - \frac{\partial \varphi_1}{\partial u} \frac{\partial \varphi_3}{\partial v} \right) \frac{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} dudv \\
&\quad + \iint_V \left(\frac{\partial f_2}{\partial x} \circ \varphi - \frac{\partial f_1}{\partial y} \circ \varphi \right) \left(\frac{\partial \varphi_1}{\partial u} \frac{\partial \varphi_2}{\partial v} - \frac{\partial \varphi_2}{\partial u} \frac{\partial \varphi_2}{\partial v} \right) \frac{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} dudv.
\end{aligned}$$

After some tedious calculations, we obtain that the right-hand side of the previous equality reduces to

$$\iint_V \left(\frac{\partial h}{\partial u} - \frac{\partial g}{\partial v} \right) dudv,$$

where

$$\begin{aligned}
g(u,v) &:= f_1(\varphi(u,v)) \frac{\partial \varphi_1}{\partial u}(u,v) + f_2(\varphi(u,v)) \frac{\partial \varphi_2}{\partial u}(u,v) + f_3(\varphi(u,v)) \frac{\partial \varphi_3}{\partial u}(u,v), \\
h(u,v) &:= f_1(\varphi(u,v)) \frac{\partial \varphi_1}{\partial v}(u,v) + f_2(\varphi(u,v)) \frac{\partial \varphi_2}{\partial v}(u,v) + f_3(\varphi(u,v)) \frac{\partial \varphi_3}{\partial v}(u,v).
\end{aligned}$$

Indeed, by the chain rule,

$$\begin{aligned}
\frac{\partial h}{\partial u}-\frac{\partial g}{\partial v} &= \frac{\partial f_1}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_1}{\partial v}+\frac{\partial f_1}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial u}\frac{\partial\varphi_1}{\partial v}+\frac{\partial f_1}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial u}\frac{\partial\varphi_1}{\partial v}\\
&\quad+\frac{\partial f_2}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_2}{\partial v}+\frac{\partial f_2}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial u}\frac{\partial\varphi_2}{\partial v}+\frac{\partial f_2}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial u}\frac{\partial\varphi_2}{\partial v}\\
&\quad+\frac{\partial f_3}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_3}{\partial v}+\frac{\partial f_3}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial u}\frac{\partial\varphi_3}{\partial v}+\frac{\partial f_3}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial u}\frac{\partial\varphi_3}{\partial v}\\
&\quad-\frac{\partial f_1}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_1}{\partial u}-\frac{\partial f_1}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial v}\frac{\partial\varphi_1}{\partial u}-\frac{\partial f_1}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial v}\frac{\partial\varphi_1}{\partial u}\\
&\quad-\frac{\partial f_2}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_2}{\partial u}-\frac{\partial f_2}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial v}\frac{\partial\varphi_2}{\partial u}-\frac{\partial f_2}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial v}\frac{\partial\varphi_2}{\partial u}\\
&\quad-\frac{\partial f_3}{\partial x}\circ\varphi\frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_3}{\partial u}-\frac{\partial f_3}{\partial y}\circ\varphi\frac{\partial\varphi_2}{\partial v}\frac{\partial\varphi_3}{\partial u}-\frac{\partial f_3}{\partial z}\circ\varphi\frac{\partial\varphi_3}{\partial v}\frac{\partial\varphi_3}{\partial u}\\
&= \frac{\partial f_1}{\partial y}\circ\varphi\left(\frac{\partial\varphi_2}{\partial u}\frac{\partial\varphi_1}{\partial v}-\frac{\partial\varphi_2}{\partial v}\frac{\partial\varphi_1}{\partial u}\right)+\frac{\partial f_1}{\partial z}\circ\varphi\left(\frac{\partial\varphi_3}{\partial u}\frac{\partial\varphi_1}{\partial v}-\frac{\partial\varphi_3}{\partial v}\frac{\partial\varphi_1}{\partial u}\right)\\
&\quad+\frac{\partial f_2}{\partial x}\circ\varphi\left(\frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_2}{\partial v}-\frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_2}{\partial u}\right)+\frac{\partial f_2}{\partial z}\circ\varphi\left(\frac{\partial\varphi_3}{\partial u}\frac{\partial\varphi_2}{\partial v}-\frac{\partial\varphi_3}{\partial v}\frac{\partial\varphi_2}{\partial u}\right)\\
&\quad+\frac{\partial f_3}{\partial x}\circ\varphi\left(\frac{\partial\varphi_1}{\partial u}\frac{\partial\varphi_3}{\partial v}-\frac{\partial\varphi_1}{\partial v}\frac{\partial\varphi_3}{\partial u}\right)+\frac{\partial f_3}{\partial y}\circ\varphi\left(\frac{\partial\varphi_2}{\partial u}\frac{\partial\varphi_3}{\partial v}-\frac{\partial\varphi_2}{\partial v}\frac{\partial\varphi_3}{\partial u}\right).
\end{aligned}$$

Hence, by the Gauss–Green formula (56),

$$\int_M \operatorname{curl}\mathbf{f}\cdot\boldsymbol{\nu}\,d\mathcal{H}^2=\iint_V\left(\frac{\partial h}{\partial u}-\frac{\partial g}{\partial v}\right)dudv=\int_{\partial V} g\,dx+\int_{\partial V} h\,dy.$$

On the other hand,

$$\begin{aligned}
\int_{\partial V} g\,dx+\int_{\partial V} h\,dy &= \int_a^b \left(g\left(\boldsymbol{\psi}\left(t\right)\right)\psi_1'\left(t\right)+h\left(\boldsymbol{\psi}\left(t\right)\right)\psi_2'\left(t\right)\right)\,dt\\
&= \int_a^b f_1\circ\varphi\circ\boldsymbol{\psi}\left(\frac{\partial\varphi_1}{\partial u}\circ\boldsymbol{\psi}\psi_1'+\frac{\partial\varphi_1}{\partial v}\circ\boldsymbol{\psi}\psi_2'\right)\,dt\\
&\quad+\int_a^b f_2\circ\varphi\circ\boldsymbol{\psi}\left(\frac{\partial\varphi_2}{\partial u}\circ\boldsymbol{\psi}\psi_1'+\frac{\partial\varphi_2}{\partial v}\circ\boldsymbol{\psi}\psi_2'\right)\,dt\\
&\quad+\int_a^b f_3\circ\varphi\circ\boldsymbol{\psi}\left(\frac{\partial\varphi_3}{\partial u}\circ\boldsymbol{\psi}\psi_1'+\frac{\partial\varphi_3}{\partial v}\circ\boldsymbol{\psi}\psi_2'\right)\,dt\\
&= \int_a^b \left(f_1\circ\boldsymbol{\phi}\phi_1'+f_2\circ\boldsymbol{\phi}\phi_2'+f_3\circ\boldsymbol{\phi}\phi_3'\right)\,dt=\int_\Gamma \mathbf{f},
\end{aligned}$$

where $\boldsymbol{\phi}:=\varphi\circ\boldsymbol{\psi}:[a,b]\to\mathbb{R}^3$ is a parametric representation of Γ. ■

Example 245 *Given the surface M of parametric representation*

$$\varphi\left(u,v\right):=\left(u-v,u,u^2+v^2\right),\quad (u,v)\in V,$$

where $V := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$*, let's calculate*

$$\int_M \operatorname{curl} \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^2,$$

where

$$\mathbf{f}(x, y, z) := \left(x^2 z, y, yz\right), \quad (x, y, z) \in \mathbb{R}^3.$$

Note that φ *is defined in* $\mathbb{R}^2$ *so we can take* $W = \mathbb{R}^2$*. To see that* φ *is one-to-one, we find the inverse, we have* $x = u - v$*,* $y = u$*,* $z = u^2 + v^2$ *and so,* $u = z$*,* $v = u - x = z - x$*. Thus,*

$$\varphi^{-1}(x, y, z) = (z, z - x),$$

which is continuous. Hence, $\varphi : V \to M$ *is a homeomorphism. Moreover,*

$$J_\varphi(u, v) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 2u & 2v \end{pmatrix}.$$

The submatrix $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ *has determinant different from zero, and so* $J_\varphi(u, v)$ *has always maximal rank two. Thus, we are in a position to apply Stokes' theorem. The boundary of* V *is the unit circle. Thus a parametric representation of* $\boldsymbol{\gamma}$ *is* $\boldsymbol{\psi} : [0, 2\pi] \to \mathbb{R}^2$*, where* $\boldsymbol{\psi}(t) := (\cos t, \sin t)$*. In turn, a parametric representation for* Γ *is given by* $\phi := \varphi \circ \boldsymbol{\psi} : [0, 2\pi] \to \mathbb{R}^3$*, where*

$$\phi(t) = \left(\cos t - \sin t, \cos t, \cos^2 t + \sin^2 t\right).$$

In turn,

$$\phi'(t) = (-\sin t - \cos t, -\sin t, 0).$$

Hence, by Stokes' theorem

$$\begin{aligned}
\int_M \operatorname{curl} \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^2 &= \int_\Gamma \mathbf{f} \\
&= \int_0^{2\pi} \left[(\cos t - \sin t)^2 1 (-\sin t - \cos t) + (\cos t)(-\sin t) + (\cos t)(1) 0\right] dt \\
&= -\int_0^{2\pi} \left[\left(\cos^2 t - \sin^2 t\right)(\cos t - \sin t) + \cos t \sin t\right] dt \\
&= -\int_0^{2\pi} \left[\left(1 - 2\sin^2 t\right)\cos t - \left(-1 + 2\cos^2 t\right)\sin t + \cos t \sin t\right] dt \\
&= -\left[\sin t - \frac{2}{3}\sin^3 t + \cos t + \frac{2}{3}\cos^3 t + \frac{1}{2}\sin^2 t\right]_{t=0}^{t=2\pi} = 0.
\end{aligned}$$

Exercise 246 *In the previous example calculate* $\int_M \operatorname{curl} \mathbf{f} \cdot \boldsymbol{\nu} \, d\mathcal{H}^2$ *directly, without using Stokes's theorem.*

Remark 247 *Stokes' theorem continues to hold for oriented manifolds with boundaries. Roughly speaking, a* 2*-dimensional manifold* M *in* $\mathbb{R}^3$ *is orientable if it is possible to define a normal* $\boldsymbol{\nu}$ *as a function of* (x, y, z) *in a continuous way. The Klein bottle or the Möbius strip are a typical examples of manifolds that not orientable. Indeed, the normal as a function of* (x, y, z) *is discontinuous (but it is continuous if you look at it as a function of* (u, v)*). If you think of the normal as a person walking on the manifold, after going around once, that person would find himself/herself upside down, which means that there is a discontinuity.*

Saturday, May 05, 2012

I will not ask this part because it was not covered in class, but since you asked, here it is.

18 More on Surface Integrals

Next we consider the case in which the domain of the function to be integrated cannot be covered by only one chart. For this we need to use partitions of unity. Given a k-dimensional surface M of class C^m, for every $\mathbf{x} \in M$ there exist an open set $U_{\mathbf{x}}$ containing $\mathbf{x}$ and a local chart $\varphi_{\mathbf{x}} : W_{\mathbf{x}} \to M \cap U_{\mathbf{x}}$. Since $U_{\mathbf{x}}$ is open, there exists $B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U_{\mathbf{x}}$. Construct a nonnegative function $g_{\mathbf{x}} \in C^1(\mathbb{R}^N)$ such that $g_{\mathbf{x}} > 0$ in $A_{\mathbf{x}} := B\left(\mathbf{x}, \frac{1}{2} r_{\mathbf{x}}\right)$ and $g_{\mathbf{x}} = 0$ outside $Y_{\mathbf{x}} := B(\mathbf{x}, r_{\mathbf{x}}) \subseteq U_{\mathbf{x}}$.

Assume that the set M is compact. Since the family of open sets $\{A_{\mathbf{x}}\}_{\mathbf{x} \in M}$ covers M, there exist $\mathbf{x}_1, \ldots, \mathbf{x}_n \in M$ such that $A_{\mathbf{x}_1}, \ldots, A_{\mathbf{x}_n}$ still covers M. For simplicity in the notation, we set $g_i := g_{\mathbf{x}_i}$, $\varphi_i := \varphi_{\mathbf{x}_i}$, $r_i := r_{\mathbf{x}_i}$, $A_i := A_{\mathbf{x}_i}$, and $U_i := U_{\mathbf{x}_i}$. Note that

$$g := \sum_{i=1}^{n} g_i > 0 \quad \text{in } M,$$

since $A_1, \ldots, A_n$ that still cover M and $g_i > 0$ in A_i. Define

$$h_i(\mathbf{x}) := \begin{cases} \frac{g_i(\mathbf{x})}{g(\mathbf{x})} & \text{if } g_i(\mathbf{x}) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h_i \in C^1(\mathbb{R}^N)$, $h_i = 0$ outside U_i and $\sum_{i=1}^{n} h_i = 1$ in M. The family $\{h_i\}$ is called a *partition of unity subordinated to the family of open sets* $U_1, \ldots, U_n$.

Definition 248 *Given a* k*-dimensional surface* M *of class* C^m, $1 \le k < N$, $m \in \mathbb{N}$, *a set* $E \subseteq M$ *is* $\mathcal{H}^k$-measurable *if* $\varphi^{-1}(E)$ *is Peano–Jordan measurable in* $\mathbb{R}^k$ *for every local charts* $\varphi : V \to M$.

Given a k-dimensional surface M of class C^m, $1 \le k < N$, $m \in \mathbb{N}$, an $\mathcal{H}^k$-measurable set $E \subseteq M$ and a bounded function $f : E \to \mathbb{R}$, consider the partition of unity $\{h_i\}$ constructed above. For every $i = 1, \ldots, n$, the function $f h_i : E \to \mathbb{R}$ is zero at all points of E outside $U_i \cap M = \varphi_i(W_i)$, thus we can use (50) for the surface integral of fh_i over the set $E \cap \varphi_i(W_i)$. Hence, we can define the *k-dimensional surface integral* of f over E to be

$$\int_E f\, d\mathcal{H}^k := \sum_{i=1}^n \int_{E\cap\varphi_i(W_i)} fh_i\, d\mathcal{H}^k, \tag{57}$$

whenever all the surface integrals $\int_{E\cap\varphi_i(W_i)} fh_i\, d\mathcal{H}^k$ exist in the sense of (50). Note that this is always the case if f is bounded and continuous.

We need to show that this definition does not depend on the choice of the partition of unity. Let $\psi_j : V_j \to M \cap D_j$, $j = 1, \ldots, m$, be other local charts covering M, with $V_j \subseteq \mathbb{R}^k$ and $D_j \subseteq \mathbb{R}^N$ open and let $p_j \in C^1\left(\mathbb{R}^N\right)$, $j = 1, \ldots, m$, be a partition of unity subordinated to the family of open sets D_1, $\ldots$, D_m. Since $\sum_{i=1}^n h_i = 1$ in M, for every $j = 1, \ldots, m$ and every $\mathbf{x} \in M$,

$$f(\mathbf{x})\, p_j(\mathbf{x}) = \sum_{i=1}^n f(\mathbf{x})\, p_j(\mathbf{x})\, h_i(\mathbf{x}),$$

and so using the surface integral defined in (50) for the local chart ψ_j (note that fp_j is zero outside D_j), we have

$$\begin{aligned}\int_{E\cap\psi_j(V_j)} fp_j\, d\mathcal{H}^k &= \int_{E\cap\psi_j(V_j)} \sum_{i=1}^n fp_jh_i\, d\mathcal{H}^k = \sum_{i=1}^n \int_{E\cap\psi_j(V_j)} fp_jh_i\, d\mathcal{H}^k \\ &= \sum_{i=1}^n \int_{E\cap\psi_j(V_j)\cap\varphi_i(W_i)} fp_jh_i\, d\mathcal{H}^k.\end{aligned}$$

Hence, summing over j,

$$\sum_{j=1}^m \int_{E\cap\psi_j(V_j)} fp_j\, d\mathcal{H}^k = \sum_{j=1}^m \sum_{i=1}^n \int_{E\cap\psi_j(V_j)\cap\varphi_i(W_i)} fp_jh_i\, d\mathcal{H}^k. \tag{58}$$

Similarly, starting with fh_i in place of fp_j, we get that

$$\int_{E\cap\varphi_i(W_i)} fh_i\, d\mathcal{H}^k = \sum_{j=1}^m \int_{E\cap\psi_j(V_j)\cap\varphi_i(W_i)} fp_jh_i\, d\mathcal{H}^k,$$

and so, summing over i,

$$\sum_{i=1}^n \int_{E\cap\varphi_i(W_i)} fh_i\, d\mathcal{H}^k = \sum_{i=1}^n \sum_{j=1}^m \int_{E\cap\psi_j(V_j)\cap\varphi_i(W_i)} fp_jh_i\, d\mathcal{H}^k. \tag{59}$$

Since the right-hand sides of (58) and (59) are equal, it follows that

$$\sum_{j=1}^m \int_{E\cap\psi_j(V_j)} fp_j\, d\mathcal{H}^k = \sum_{i=1}^n \int_{E\cap\varphi_i(W_i)} fh_i\, d\mathcal{H}^k,$$

which shows that the definition (57) does not depend on the particular choice of the local charts and partition of unity.