Monday, August 26, 2013

1 Notation

The *Euclidean space* $\mathbb{R}^N$ is the space of all N-tuples $\mathbf{x} = (x_1, \ldots, x_N)$ of real numbers. The elements of $\mathbb{R}^N$ are called *vectors* or *points*. The *Euclidean inner product* between two vectors $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ is defined as

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_N y_N.$$

The *Euclidean norm* of a vector $\mathbf{x} = (x_1, \ldots, x_N)$ is defined as

$$\|\mathbf{x}\| := \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \cdots + x_N^2}.$$

The vectors

$$\mathbf{e}_1 := (1, 0, \cdots, 0)$$
$$\vdots$$
$$\mathbf{e}_N := (0, \cdots, 0, 1)$$

form the *standard orthonormal* or *Euclidean basis* of $\mathbb{R}^N$. A *direction* is a vector $\mathbf{v} \in \mathbb{R}^N$ of norm one.

Let $E \subseteq \mathbb{R}^N$, let $u : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. Given a direction $\mathbf{v} \in \mathbb{R}^N$, let L be the line through $\mathbf{x}_0$ in the direction $\mathbf{v}$, that is,

$$L := \left\{\mathbf{x} \in \mathbb{R}^N : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v},\, t \in \mathbb{R}\right\},$$

and assume that $\mathbf{x}_0$ is an accumulation point of the set $E \cap L$. The *directional derivative* of u at $\mathbf{x}_0$ in the direction $\mathbf{v}$ is defined as

$$\frac{\partial u}{\partial \mathbf{v}}(\mathbf{x}_0) := \lim_{t \to 0} \frac{u(\mathbf{x}_0 + t\mathbf{v}) - u(\mathbf{x}_0)}{t},$$

provided the limit exists in $\mathbb{R}$. In the special case in which $\mathbf{v} = \mathbf{e}_i$, the directional derivative $\frac{\partial u}{\partial \mathbf{e}_i}(\mathbf{x}_0)$, if it exists, is called the *partial derivative* of u with respect to x_i and is denoted $\frac{\partial u}{\partial x_i}(\mathbf{x}_0)$ or $u_{x_i}(\mathbf{x}_0)$ or $\partial_{x_i} u(\mathbf{x}_0)$. If all the partial derivatives of u at $\mathbf{x}_0$ exist, the vector

$$\left(\frac{\partial u}{\partial x_1}(\mathbf{x}_0), \ldots, \frac{\partial u}{\partial x_N}(\mathbf{x}_0)\right) \in \mathbb{R}^N$$

is called the *gradient* of u at $\mathbf{x}_0$ and is denoted by $\nabla u(\mathbf{x}_0)$.

Let $E \subseteq \mathbb{R}^N$, let $u : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. Let $i \in \{1, \ldots, N\}$ and assume that there exists the partial derivative $\frac{\partial u}{\partial x_i}(\mathbf{x})$ for all $\mathbf{x} \in E$. If $j \in \{1, \ldots, N\}$ and $\mathbf{x}_0$ is an accumulation point of $E \cap L$, where L is the line through $\mathbf{x}_0$ in the direction $\mathbf{e}_j$, then we can consider the partial derivative of the function $\frac{\partial u}{\partial x_i}$ with respect to x_j, that is,

$$\frac{\partial^2 u}{\partial x_j \partial x_i}(\mathbf{x}_0) := \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\right)(\mathbf{x}_0),$$

provided it exists.

The *Hessian matrix* of u at $\mathbf{x}_0$ is the $N \times N$ matrix

$$\nabla^2 u(\mathbf{x}_0) := \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2}(\mathbf{x}_0) & \cdots & \frac{\partial^2 u}{\partial x_N \partial x_1}(\mathbf{x}_0) \\ \vdots & & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_N}(\mathbf{x}_0) & \cdots & \frac{\partial^2 u}{\partial x_N^2}(\mathbf{x}_0) \end{pmatrix},$$

whenever it is defined. Note that in general $\nabla^2 u(\mathbf{x}_0)$ is not symmetric.

Exercise 1 *Let*

$$u(x,y) := \begin{cases} y^2 \arctan \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Prove that $\frac{\partial^2 u}{\partial x \partial y}(0,0) \neq \frac{\partial^2 u}{\partial y \partial x}(0,0)$.

However, we have the following.

Theorem 2 (Schwartz) *Let $E \subseteq \mathbb{R}^N$, let $u : E \to \mathbb{R}$, let $\mathbf{x}_0 \in E^\circ$, and let $i, j \in \{1, \ldots, N\}$. Assume that there exists $r > 0$ such that $B(\mathbf{x}_0, r) \subseteq E$ and the mixed partial derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial^2 u}{\partial x_j \partial x_i}$ exist for all $\mathbf{x} \in B(\mathbf{x}_0, r)$ and are continuous at $\mathbf{x}_0$. Then*

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\mathbf{x}_0) = \frac{\partial^2 u}{\partial x_j \partial x_i}(\mathbf{x}_0).$$

Note that Schwartz theorem (when it can be applied) implies that the $N \times N$ matrix $\nabla^2 u(\mathbf{x}_0)$ is symmetric.

Let $E \subseteq \mathbb{R}^N$, let $u : E \to \mathbb{R}$ and let $\mathbf{x}_0 \in E$. For every $k \in \mathbb{N}$, $\nabla^k u(\boldsymbol{x}_0)$ denotes the set of all derivatives

$$\frac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}}(\mathbf{x}_0)$$

of u of order k at $\mathbf{x}_0$, whenever it is defined. Here $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}$. Hence, $\nabla^k u(\boldsymbol{x}_0)$ can be identified with a tensor of order k. We will denote by $\mathbb{T}_N^k$ the space of tensors of order k in $\mathbb{R}^N$. Thus, $\mathbb{T}_N^1$ can be identified with $\mathbb{R}^N$, $\mathbb{T}_N^2$ can be identified with the space of all $N \times N$ matrices.

When Schwartz theorem can be applied, we can use the multi-index notation. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A *multi-index* α is a vector $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N}_0)^N$. The *length* of a multi-index is defined as

$$|\alpha| := \alpha_1 + \cdots + \alpha_N.$$

Given a multi-index α, the partial derivative $\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}$ is defined as

$$\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}},$$

where $\mathbf{x} = (x_1, \ldots, x_N)$. If $\alpha = \mathbf{0}$, we set $\frac{\partial^{\mathbf{0}} u}{\partial \mathbf{x}^{\mathbf{0}}} := u$.

Example 3 *If $N = 3$ and $\alpha = (2, 1, 0)$, then*

$$\frac{\partial^{(2,1,0)}}{\partial (x, y, z)^{(2,1,0)}} = \frac{\partial^3}{\partial x^2 \partial y}.$$

Given a multi-index α and $\mathbf{x} \in \mathbb{R}^N$, we set

$$\alpha! := \alpha_1! \cdots \alpha_N!, \quad \mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

If $\alpha = \mathbf{0}$, we set $\mathbf{x}^{\mathbf{0}} := 1$.

Given an open set $\Omega \subseteq \mathbb{R}^N$, for every nonnegative integer $m \in \mathbb{N}_0$, we denote by $C^m(\Omega)$ the space of all functions that are continuous together with their partial derivatives up to order m. We set $C^\infty(\Omega) := \bigcap_{m=0}^{\infty} C^m(\Omega)$. Note that for a function u in the space $C^m(\Omega)$, $m \geq 2$, we can apply Schwartz theorem, so that the order in which we take the derivatives is not important. Hence, for every $\boldsymbol{x} \in \Omega$ and $2 \leq k \leq m$, $\nabla^k u(\boldsymbol{x})$ will be a symmetric tensor of order k. Note that this tensor *does not* coincide with the set

$$\left\{ \frac{\partial^\alpha u}{\partial \mathbf{x}^\alpha}(\boldsymbol{x}) : \alpha \text{ multi-index}, \ |\alpha| = k \right\},$$

since the latter set has significantly less elements.

Theorem 4 (Taylor's Formula) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u \in C^m(\Omega)$, $m \in \mathbb{N}$, and let $\mathbf{x}_0 \in \Omega$. Then for every $\mathbf{x} \in \Omega$,*

$$u(\mathbf{x}) = \sum_{\alpha \text{ multi-index}, \ 0 \leq |\alpha| \leq m} \frac{1}{\alpha!} \frac{\partial^\alpha u}{\partial \mathbf{x}^\alpha}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)^\alpha + R_m(\mathbf{x}),$$

where

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{R_m(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^m} = 0.$$

2 Partial Differential Equations

Given a set $E \subseteq \mathbb{R}^M \times \mathbb{R} \times \mathbb{T}_M^1 \times \cdots \times \mathbb{T}_M^k$, $M, k \in \mathbb{N}$, and a function $F : E \to \mathbb{R}$, a *partial differential equation* (PDE) is an equation of the form

$$F\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^k u(\boldsymbol{y})\right) = 0 \qquad \text{(PDE)}$$

where $u = u(\boldsymbol{y})$ is the unknown. The *order* of a PDE is the order of the highest derivative, k-th order, in this case.[1]

A *(local) classical solution* of (PDE) is a function $u : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^M$, such that

(i) u has continuous partial derivatives of order k in A,

(ii) for every $\boldsymbol{y} \in A$, $\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^k u(\boldsymbol{y})\right) \in E$,

(iii) (PDE) holds for all $\boldsymbol{y} \in A$.

There are several notion of weak solutions, in the sense of distributions, in Sobolev spaces, viscosity solutions, etc.. To illustrate the idea of weak solutions, let's start from an ordinary differential equation. Consider the initial value problem

$$\begin{cases} \frac{du}{dt}(t) = f(t, u(t)), \\ u(t_0) = u_0. \end{cases} \tag{ODE}$$

To solve it, we transform it into the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds.$$

The advantage is that this integral equation is defined for continuous functions, which may not be differentiable. Under appropriate hypotheses on f we may solve the integral equation (e.g., using a fixed point theorem) in the space of continuous functions and then prove that the solution is actually differentiable and solves the initial problem.

For PDEs the approach is the same. First one proves existence of weak solutions and then uses regularity theory to prove that weak solutions are actually classical solutions.

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Note that in (ODE) t usually represents the time and $u(t_0) = u_0$ is the initial condition. For PDEs there are two important special cases.

1. $\boldsymbol{y} = (\boldsymbol{x}, t)$, where $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N$ represents space and $t \in [t_0, T]$ represents time. Usually Ω is an open set. In this case we have

   (a) *initial conditions*: what happens at time $t = t_0$, (for example, one could prescribe the initial datum $u(\boldsymbol{x}, t_0) = u_0(\boldsymbol{x})$, $\boldsymbol{x}, \in \Omega$),

   (b) *boundary conditions:* what happens on $\partial\Omega \times [t_0, T]$, (for example, one could prescribe the boundary datum $u(\boldsymbol{x}, t) = v_0(\boldsymbol{x}, t)$ for $(\boldsymbol{x}, t) \in \partial\Omega \times [t_0, T]$),

2. $\boldsymbol{y} = \boldsymbol{x}$, where $\boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N$ represents space. In this case we only have *boundary conditions:* what happens on $\partial\Omega$ (for example, one could prescribe the value of u or of some of its derivatives on $\partial\Omega$).

[1] Of course, we are assuming that there is a true dependence of F on the set of variables $\nabla^k u(\boldsymbol{y})$, namely that F does not remain constant as $\nabla^k u(\boldsymbol{y})$ varies.

Given a PDE with some initial and/or boundary conditions, we say that the problem is *well-posed* if

(i) the problem has a classical solution,

(ii) the solution is unique,

(iii) the solution depends continuously on the data.

We say that the problem is *ill-posed* if it is not well-posed.

The part below was not done in class: please read

3 Examples of PDEs

A PDE is *linear* if it has the form

$$\sum_{j=0}^{k} \boldsymbol{A}_j(\boldsymbol{y}) \cdot \nabla^j u(\mathbf{y}) = f(\mathbf{y}),$$

where $\boldsymbol{A}_j$ and f are given functions. When $f = 0$, it is *homogeneous.* If $u \in C^k(\Omega)$ for some open set, then we can rewrite it as

$$\sum_{\alpha \text{ multi-index, } |\alpha| \leq k} a_\alpha(\boldsymbol{y}) \frac{\partial^\alpha u}{\partial \mathbf{y}^\alpha}(\boldsymbol{y}) = f(\mathbf{y}).$$

Important examples of linear PDEs are

- *Laplace equation*

  $$\Delta u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,$$

  where

  $$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

- *Heat equation*

  $$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N, \, t \geq 0,$$

  where

  $$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

- *Wave equation*

  $$u_{tt}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N, \, t \geq 0,$$

  where

  $$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

A PDE is *semilinear* if it has the form

$$\boldsymbol{A}(\boldsymbol{y}) \cdot \nabla^k u(\mathbf{y}) + b\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^{k-1} u(\boldsymbol{y})\right) = 0,$$

where $\boldsymbol{A}$ and b are given functions. If $u \in C^k(\Omega)$ for some open set, then we can rewrite it as

$$\sum_{\alpha \text{ multi-index, } |\alpha|=k} a_\alpha(\boldsymbol{y}) \frac{\partial^\alpha u}{\partial \mathbf{y}^\alpha}(\boldsymbol{y}) + b\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^{k-1} u(\boldsymbol{y})\right) = 0.$$

Important examples of semilinear PDEs are

- *Korteweg–de Vries equation (KdV)*

$$u_t(\boldsymbol{x}, t) + u(\boldsymbol{x}, t) u_x(\boldsymbol{x}, t) + u_{xxx}(\boldsymbol{x}, t) = 0, \quad x \in \Omega \subseteq \mathbb{R},\, t \geq 0.$$

A PDE is *quasilinear* if it has the form

$$\boldsymbol{A}\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^{k-1} u(\boldsymbol{y})\right) \cdot \nabla^k u(\mathbf{y}) + b\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^{k-1} u(\boldsymbol{y})\right) = 0,$$

where $\boldsymbol{A}$ and b are given functions. If $u \in C^k(\Omega)$ for some open set, then we can rewrite it as

$$\sum_{\alpha \text{ multi-index, } |\alpha|=k} a_\alpha(\boldsymbol{y}) \frac{\partial^\alpha u}{\partial \mathbf{y}^\alpha}(\boldsymbol{y}) + b\left(\boldsymbol{y}, u(\boldsymbol{y}), \nabla u(\boldsymbol{y}), \ldots, \nabla^{k-1} u(\boldsymbol{y})\right) = 0.$$

Important examples of quasilinear PDEs are

- *Porous media equation*

$$u_t(\boldsymbol{x}, t) - \Delta\left(u^m(\boldsymbol{x}, t)\right) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,\, t \geq 0,$$

  where $u \geq 0$, $m > 1$. Note that

$$\begin{aligned} \Delta\left(u^m(\boldsymbol{x}, t)\right) &= \operatorname{div}\left(\nabla\left(u^m(\boldsymbol{x}, t)\right)\right) = \operatorname{div}\left(m u^{m-1}(\boldsymbol{x}, t) \nabla u(\boldsymbol{x}, t)\right) \\ &= m u^{m-1}(\boldsymbol{x}, t) \Delta u(\boldsymbol{x}, t) + m(m-1) u^{m-2}(\boldsymbol{x}, t) \|\nabla u(\boldsymbol{x}, t)\|^2. \end{aligned}$$

- *p-Laplacian equation*

$$\operatorname{div}\left(\|\nabla u(\boldsymbol{x})\|^{p-2} \nabla u(\boldsymbol{x})\right) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,$$

  where $p > 1$.

- *Minimal surface equation*

$$\operatorname{div}\left(\frac{\nabla u(\boldsymbol{x})}{\sqrt{1 + \|\nabla u(\boldsymbol{x})\|^2}}\right) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N.$$

A PDE is *fully nonlinear* if it is nonlinear in the highest order derivatives. Important examples of fully nonlinear PDEs are

- *Eikonal equation*

$$\|\nabla u(\boldsymbol{x})\| = 1, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,$$

- *p-Laplacian Monge–Ampère equation*

$$\det\left(\nabla^2 u(\boldsymbol{x})\right) = 0, \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N.$$

Given a set $E \subseteq \mathbb{R}^M \times \mathbb{R} \times \left(\mathbb{T}_M^1\right)^d \times \cdots \times \left(\mathbb{T}_M^k\right)^d$, $d, M, k \in \mathbb{N}$, and a function $\boldsymbol{F} : E \to \mathbb{R}^\ell$, a *system of partial differential equations* (SPDE) is a system of the form

$$\boldsymbol{F}\left(\boldsymbol{y}, \boldsymbol{u}(\boldsymbol{y}), \nabla \boldsymbol{u}(\boldsymbol{y}), \ldots, \nabla^k \boldsymbol{u}(\boldsymbol{y})\right) = \mathbf{0} \tag{SPDE}$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{y})$ is the unknown.

Important examples of systems of PDEs are

- *Navier–Stokes equations for incompressible viscous flow*

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \Delta \boldsymbol{u} = -\nabla p \\ \operatorname{div} \boldsymbol{u} = \mathbf{0} \end{cases} \qquad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^3,$$

  where $\boldsymbol{u}$ is the flow velocity and p is the pressure.

- *Equilibrium equations of linear elasticity*

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\operatorname{div} \mathbf{u}) = \mathbf{0}.$$

  where λ and μ are the Lamé moduli.

End of part not done in class

4 First Order PDEs: Characteristics

Given an open set $\Omega \subseteq \mathbb{R}^N$, $\Gamma \subseteq \partial\Omega$, and two functions $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$, consider the *boundary value problem*

$$\begin{cases} F(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases} \tag{BVP}$$

Here g is the boundary datum. We will try to find local solutions. Fix $\boldsymbol{a} \in \Gamma$. We want to calculate $u(\boldsymbol{x})$ in a neighborhood of $\boldsymbol{a}$. For any $\boldsymbol{x}_0 \in \Omega$ close to $\boldsymbol{a}$ we find a curve γ with range in Ω through $\boldsymbol{x}_1$ and calculate u along this curve. This is called the *method of characteristics*.

To illustrate this method, let's begin with a simple linear equation of the form

$$\begin{cases} \sum_{i=1}^N b_i(\boldsymbol{x}) \frac{\partial u}{\partial x_i}(\boldsymbol{x}) = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$

Given a curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in Ω, by the chain rule

$$\frac{d}{ds}\left(u\left(\boldsymbol{x}(s)\right)\right) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\left(\boldsymbol{x}(s)\right) \frac{\partial x_i}{\partial s}(s).$$

Comparing this expression with our PDE, it is natural to require that u is constant along the curve, so that $0 = \frac{d}{ds}\left(u\left(\boldsymbol{x}(s)\right)\right)$ and that

$$\begin{cases} \frac{\partial x_i}{\partial s}(s) = b_i\left(\boldsymbol{x}(s)\right), \\ x_i(0) = x_{0,i}, \end{cases} \tag{1}$$

where $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,N})$. Let's see an example

Example 5 *Consider the initial value problem*

$$\begin{cases} u_t(x,t) + cu_x(x,t) = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $c > 0$ and $g \in C^1(\mathbb{R})$. Here, $\boldsymbol{x} = (x,t)$ and so (1) takes the form

$$\begin{cases} \frac{dx}{ds}(s) = c, \\ \frac{dt}{ds}(s) = 1, \\ x(0) = x_0, \quad t(0) = 0, \end{cases} \Leftrightarrow \begin{cases} x(s) = cs + x_0, \\ t(s) = s. \end{cases}$$

Hence, we get $t = s$ and $x = ct + x_0$. Recalling that u is constant along the curve $x = ct + x_0$, we have that

$$u(ct + x_0, t) = c_1.$$

To find c_1, take $t = 0$ to find

$$g(x_0) = u(x_0, 0) = c_1,$$

and so $c_1 = g(x_0)$. Hence,

$$u(x,t) = g(x_0) = g(x - ct)$$

is the solution (Check). Note that if we fix a time t, the function $u(\cdot, t)$ is obtained by translating the graph of u at time 0 by the amount ct. So the graph of u is a wave propagating to the right with velocity c without changing is shape. This is called a travelling wave.

Next we consider the quasilinear equation

$$\begin{cases} \sum_{i=1}^{N} b_i\left(\boldsymbol{x}, u(\boldsymbol{x})\right) \frac{\partial u}{\partial x_i}(\boldsymbol{x}) + c\left(\boldsymbol{x}, u(\boldsymbol{x})\right) = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$

Given a curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in Ω, again by the chain rule

$$\frac{d}{ds}\left(u\left(\boldsymbol{x}(s)\right)\right) = \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\left(\boldsymbol{x}(s)\right) \frac{\partial x_i}{\partial s}(s).$$

Comparing this expression with our PDE, if we ask as before that $\frac{\partial x_i}{\partial s}(s) = b_i(\boldsymbol{x}(s), u(\boldsymbol{x}(s)))$, we get

$$\begin{aligned}\frac{d}{ds}(u(\boldsymbol{x}(s))) &= \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\boldsymbol{x}(s)) \frac{\partial x_i}{\partial s}(s) \\ &= \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(\boldsymbol{x}(s)) \, b_i(\boldsymbol{x}(s), u(\boldsymbol{x}(s))) = -c(\boldsymbol{x}(s), u(\boldsymbol{x}(s))).\end{aligned}$$

This gives a differential equation for $z(s) := u(\boldsymbol{x}(s))$, namely, $\frac{dz}{ds}(s) = -c(\boldsymbol{x}(s), z(s))$. Thus, we get the initial value problem

$$\begin{cases} \frac{\partial x_i}{\partial s}(s) = b_i(\boldsymbol{x}(s), z(s)), \\ \frac{dz}{ds}(s) = -c(\boldsymbol{x}(s), z(s)), \\ x_i(0) = x_{0,i}, \quad z(0) = g(\boldsymbol{x}_0). \end{cases} \tag{2}$$

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Example 6 *Consider the initial value problem*

$$\begin{cases} u_x(x,y) + u_y(x,y) = u^2(x,y), & x \in \mathbb{R},\ y > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where $g \in C^1(\mathbb{R})$. Here, $\boldsymbol{x} = (x,y)$, $b_1(x,y) = b_2(x,y) = 1$, and $c(x,y,z) = -z^2$. Hence, (2) takes the form

$$\begin{cases} \frac{dx}{ds}(s) = 1, \\ \frac{dy}{ds}(s) = 1, \\ \frac{dz}{ds}(s) = z^2(s), \\ x(0) = x_0, \quad y(0) = 0, \quad z(0) = g(x_0) \end{cases} \Leftrightarrow \begin{cases} x(s) = s + c_1, \\ y(s) = s + c_2, \\ z(s) = \frac{1}{c_3 - s}. \end{cases}$$

We have,

$$\begin{cases} \frac{dx}{ds}(s) = 1, \\ \frac{dy}{ds}(s) = 1, \\ \frac{dz}{ds}(s) = z^2(s), \end{cases} \Leftrightarrow \begin{cases} x(s) = s + c_1, \\ y(s) = s + c_2, \\ z(s) = \frac{1}{c_3 - s}. \end{cases}$$

Using the initial conditions, $x(0) = x_0$, $y(0) = 0$, and $z(0) = g(x_0)$ gives $c_1 = x_0$, $c_2 = 0$, and $c_3 = 1/g(x_0)$. Hence,

$$\begin{cases} x(s) = s + x_0, \\ y(s) = s, \\ z(s) = \frac{g(x_0)}{1 - g(x_0)s}. \end{cases}$$

It follows that $x - y = x_0$, and so, recalling that $z(s) := u(x(s), y(s))$, the solution is given by

$$u(x,y) = z(y) = \frac{g(x-y)}{1 - g(x-y)\,y}.$$

Check. Note that if $g = 1$, we get

$$u(x,y) = \frac{1}{1-y},$$

so the solution does not exist everywhere.

Exercise 7 *Solve the initial value problem*

$$\begin{cases} xu_y(x,y) - yu_x(x,y) = u(x,y), & x \in \mathbb{R},\ y > 0, \\ u(x,0) = g(x), & x > 0, \end{cases}$$

where $g \in C^1(\mathbb{R})$.

Finally, we consider the general case (BVP). In what follows we write

$$F = F(\boldsymbol{x}, z, \boldsymbol{p}).$$

Theorem 8 *Given an open set $W \subseteq \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and a function $F : W \to \mathbb{R}$ of class $C^1(V)$, let $u \in C^2(V)$ be a local solution of the PDE*

$$F(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) = 0, \quad \boldsymbol{x} \in V. \tag{3}$$

Let $I \subseteq \mathbb{R}$ be an interval and let $\boldsymbol{x} : I \to \mathbb{R}^N$ be a solution of the system of ODEs

$$\frac{d\boldsymbol{x}}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), u(\boldsymbol{x}(s)), \nabla u(\boldsymbol{x}(s))). \tag{4}$$

Then

$$z(s) := u(\boldsymbol{x}(s)), \quad \boldsymbol{p}(s) := \nabla u(\boldsymbol{x}(s))$$

solve the the system of ODEs

$$\begin{aligned} &\frac{dz}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) \cdot \boldsymbol{p}(s), \\ &\frac{d\boldsymbol{p}}{ds}(s) = -\nabla_{\boldsymbol{x}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) - \partial_z F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s))\, \boldsymbol{p}(s). \end{aligned}$$

Proof. By differentiating (3) with respect to x_i we get

$$\begin{aligned} &\frac{\partial F}{\partial x_i}(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) + \frac{\partial F}{\partial z}(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) \frac{\partial u}{\partial x_i}(\boldsymbol{x}) \\ &\qquad\qquad + \sum_{j=1}^{N} \frac{\partial F}{\partial p_j}(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) \frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) = 0 \end{aligned}$$

for all $\boldsymbol{x} \in V$. By evaluating the previous expression along the curve $\boldsymbol{x} = \boldsymbol{x}(s)$ and using (4) we get

$$\begin{aligned} &\frac{\partial F}{\partial x_i}(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) + \frac{\partial F}{\partial z}(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s))\, p_i(s) \\ &\qquad\qquad + \sum_{j=1}^{N} \frac{dx_j}{ds}(\boldsymbol{x}(s)) \frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) = 0 \end{aligned} \tag{5}$$

Since $p_i(s) := \frac{\partial u}{\partial x_i}(\boldsymbol{x}(s))$, by the chain rule and (5),

$$\begin{aligned}\frac{dp_i}{ds}(s) &= \frac{d}{ds}\left(\frac{\partial u}{\partial x_i}(\boldsymbol{x}(s))\right) = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}(s)) \frac{dx_j}{ds}(\boldsymbol{x}(s)) \\ &= -\frac{\partial F}{\partial x_i}(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) - \frac{\partial F}{\partial z}(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) p_i(s).\end{aligned}$$

On the other hand, since $z(s) := u(\boldsymbol{x}(s))$, by the chain rule and (4)

$$\begin{aligned}\frac{dz}{ds}(s) &= \frac{d}{ds}(u(\boldsymbol{x}(s))) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(\boldsymbol{x}(s)) \frac{dx_j}{ds}(\boldsymbol{x}(s)) \\ &= \sum_{j=1}^{N} p_j(s) \frac{\partial F}{\partial p_j}(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)).\end{aligned}$$

This concludes the proof. ■

The equations

$$\begin{cases} \frac{d\boldsymbol{x}}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)), \\ \frac{dz}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) \cdot \boldsymbol{p}(s), \\ \frac{d\boldsymbol{p}}{ds}(s) = -\nabla_{\boldsymbol{x}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) - \partial_z F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) \boldsymbol{p}(s), \\ F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) = 0 \end{cases} \tag{6}$$

are called the *characteristic equations* of the PDE, and the functions $(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s))$ are called *characteristics* of the PDE. The function $\boldsymbol{x}(s)$ is called the *projected characteristic* of the PDE.

Concerning the initial conditions for (6), note that the system of differential equations in (6) is autonomous, that is, s does not appear explicitly, so we can always replace s with $s + s_0$ without changing the system. Hence, we can take as initial time $s = 0$ and prescribe

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \Gamma, \quad z(0) = g(\boldsymbol{x}_0).$$

The initial condition $\boldsymbol{p}(0)$ is trickier. If we want regular solutions, we need to require that $\boldsymbol{p}(0)$ satisfies

$$F(\boldsymbol{x}_0, g(\boldsymbol{x}_0), \boldsymbol{p}(0)) = 0.$$

We will see that if $\partial\Omega$ is a manifold and Γ contains a relative neighborhood of $\boldsymbol{x}_0$, then the fact that $u = g$ on Γ implies that the tangential part of $\boldsymbol{p}(0)$ has to coincide with the tangential part of $\nabla g(\boldsymbol{x}_0)$. This will determine $N - 1$ components of $\boldsymbol{p}(0)$. The remaining component will be found using the equation $F(\boldsymbol{x}_0, g(\boldsymbol{x}_0), \boldsymbol{p}(0)) = 0$. We will make all this precise.

Monday, September 2, 2013

Labor day. No classes

Wednesday, September 4, 2013

Definition 9 *Given* $1 \leq k < N$*, a nonempty set* $M \subseteq \mathbb{R}^N$ *is called a* k*-*dimensional differential surface *or* manifold *if for every* $\mathbf{x}_0 \in M$ *there exist an open set* Ω *containing* $\mathbf{x}_0$ *and a differentiable function* $\varphi : W \to \mathbb{R}^N$*, where* $W \subseteq \mathbb{R}^k$ *is an open set such that*

(i) $\varphi : W \to M \cap \Omega$ *is a homeomorphism, that is, it is invertible and continuous together with its inverse* $\varphi^{-1} : M \cap \Omega \to W$*,*

(ii) $J_\varphi(\mathbf{y})$ *has maximum rank* k *for all* $\mathbf{y} \in W$*.*

The function φ *is called a* local chart *or a* system of local coordinates *or a* local parametrization *around* $\mathbf{x}_0$*. We say that* M *is of class* C^m*,* $m \in \mathbb{N}$*, (respectively,* C^∞*) if all local charts are of class* C^m *(respectively,* C^∞*).*

Definition 10 *Given a* k*-dimensional surface* M *and a point* $\boldsymbol{x}_0 \in M$*, a vector* $\boldsymbol{t}$ *is a* tangent vector *to* M *at* $\boldsymbol{x}_0$ *if there exists a function* $\boldsymbol{\psi} : (-r, r) \to M$ *such that* $\boldsymbol{\psi}(0) = \boldsymbol{x}_0$ *and* $\boldsymbol{\psi}$ *is differentiable at* $t = 0$ *with* $\boldsymbol{\psi}'(0) = \boldsymbol{t}$*. The set of all tangent vectors to* M *at* $\boldsymbol{x}_0$ *is called the* tangent space *to* M *at* $\boldsymbol{x}_0$ *and is denoted* $T_{\boldsymbol{x}_0}M$*.*

Remark 11 *If* M *is a* k*-dimensional surface of class* C^1*, then* $T_{\boldsymbol{x}_0}M$ *is a vector space of dimension* k*. Moreover, if* $\varphi : W \to \mathbb{R}^N$*, where* $W \subseteq \mathbb{R}^k$ *is an open set, is a local chart with* $\varphi(\boldsymbol{y}_0) = \boldsymbol{x}_0$ *for some* $\boldsymbol{y}_0 \in W$*, then the partial derivatives* $\frac{\partial \varphi}{\partial y_1}(\boldsymbol{y}_0), \ldots, \frac{\partial \varphi}{\partial y_k}(\boldsymbol{y}_0)$ *form a basis for* $T_{\boldsymbol{x}_0}M$*.*

Definition 12 *Given a* k*-dimensional surface* M *and a point* $\boldsymbol{x}_0 \in M$*, a vector* $\boldsymbol{n}$ *is a* normal vector *to* M *at* $\boldsymbol{x}_0$ *if* $\boldsymbol{n} \cdot \boldsymbol{t} = \mathbf{0}$ *for all* $\boldsymbol{t} \in T_{\boldsymbol{x}_0}M$*.*

Given a k-dimensional surface M of class C^1 and a function $u : M \to \mathbb{R}$, we say that u is *differentiable at* $\boldsymbol{x}_0 \in M$ if u can be extended to a function $v : B(\boldsymbol{x}_0, r) \to \mathbb{R}$ differentiable at $\boldsymbol{x}_0$, for some $r > 0$. The *differential* $d^M_{\boldsymbol{x}_0}u$ of u at $\boldsymbol{x}_0$ is the restriction of the differential $d_{\boldsymbol{x}_0}v : \mathbb{R}^N \to \mathbb{R}$ to the tangent space $T_{\boldsymbol{x}_0}M$. It can be shown that $d^M_{\boldsymbol{x}_0}u$ does not depend on the particular extension v. Since $d_{\boldsymbol{x}_0}v$ is given by the formula

$$(d_{\boldsymbol{x}_0}v)(\boldsymbol{h}) = \nabla v(\boldsymbol{x}_0) \cdot \boldsymbol{h}$$

for all $\boldsymbol{h} \in \mathbb{R}^N$, we have that

$$\left(d^M_{\boldsymbol{x}_0}u\right)(\boldsymbol{t}) = \nabla v(\boldsymbol{x}_0) \cdot \boldsymbol{t}$$

for all $\boldsymbol{t} \in T_{\boldsymbol{x}_0}M$. Hence, if $w : B(\boldsymbol{x}_0, r) \to \mathbb{R}$ is another extension of u differentiable at $\boldsymbol{x}_0$, then

$$\nabla w(\boldsymbol{x}_0) \cdot \boldsymbol{t} = \nabla v(\boldsymbol{x}_0) \cdot \boldsymbol{t} \tag{7}$$

for all $\boldsymbol{t} \in T_{\boldsymbol{x}_0}M$. Note that in view of Remark 11, condition (7) holds if and only if

$$\nabla w(\boldsymbol{x}_0) \cdot \frac{\partial \varphi}{\partial y_i}(\boldsymbol{y}_0) = \nabla v(\boldsymbol{x}_0) \cdot \frac{\partial \varphi}{\partial y_i}(\boldsymbol{y}_0) \tag{8}$$

for all $i = 1, \ldots, k$.

Next we define regular domains.

Given $i \in \{1, \ldots, N\}$ and $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ we denote by $\boldsymbol{x}_i$ the vector of $\mathbb{R}^{N-1}$ obtained by removing the i-th component from $\boldsymbol{x}$. With a slight abuse of notation, we write

$$\boldsymbol{x} = (\boldsymbol{x}_i, x_i) \in \mathbb{R}^{N-1} \times \mathbb{R}. \tag{9}$$

Definition 13 *Given an open set $\Omega \subseteq \mathbb{R}^N$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we say that $\partial\Omega$ is of class C^m if for each point $\boldsymbol{x}_0 \in \partial\Omega$ there exist $r > 0$, $i \in \{1, \ldots, N\}$ and a function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^m such that either*

$$\Omega \cap B(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, r) : x_i > f(\boldsymbol{x}_i)\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, r) : x_i < f(\boldsymbol{x}_i)\}.$$

In a similar way we can define Lipschitz, $C^{m,\alpha}$, C^∞, and analytic boundaries. Note that if $\partial\Omega$ is of class C^m with $m \geq 1$, then $\partial\Omega$ is a C^m surface of dimension $N-1$. Hence, it makes sense to talk about tangent and normal vectors to $\partial\Omega$.

Consider now the boundary value problem (BVP), where we assume that $\partial\Omega$ is of class C^1 and $g \in C^1(\mathbb{R}^N)$.

Definition 14 *A triple $(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \in \Gamma \times \mathbb{R} \times \mathbb{R}^N$ is called*

- admissible *if*

$$z_0 = g(\boldsymbol{x}_0), \quad F(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) = 0, \quad \boldsymbol{p}_0 \cdot \boldsymbol{t} = \nabla g(\boldsymbol{x}_0) \cdot \boldsymbol{t} \tag{10}$$

  for all $\boldsymbol{t} \in T_{\boldsymbol{x}_0}\partial\Omega$,

- non-characteristic *if*

$$\nabla_{\boldsymbol{p}} F(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \cdot \boldsymbol{n} \neq 0, \tag{11}$$

  where $\boldsymbol{n}$ is a unit normal to $\partial\Omega$ at $\boldsymbol{x}_0$.

Friday, September 6, 2013

Before we state the main theorem, let's discuss these two conditions with some examples.

Example 15 *Consider the initial value problem*

$$\begin{cases} u_x(x,y)\, u_y(x,y) = u(x,y), & x > 0,\ y \in \mathbb{R} \\ u(0,y) = y^2, & y \in \mathbb{R}. \end{cases}$$

Here, $\boldsymbol{x} = (x,y)$, $\boldsymbol{p} = (p_1, p_2)$, and $F(\boldsymbol{x}, z, \boldsymbol{p}) = p_1 p_2 - z$. Hence, (6) *takes the form*

$$\begin{cases} \frac{dx}{ds}(s) = p_2(s), \\ \frac{dy}{ds}(s) = p_1(s), \\ \frac{dz}{ds}(s) = 2p_1(s)\, p_2(s), \\ \frac{dp_1}{ds}(s) = p_1(s), \\ \frac{dp_2}{ds}(s) = p_2(s) \\ p_1(s)\, p_2(s) - z(s) = 0 \end{cases} \Leftrightarrow \begin{cases} p_1(s) = c_1 e^s, \\ p_2(s) = c_2 e^s, \\ z(s) = p_1(s)\, p_2(s) = c_1 c_2 e^{2s}, \\ x(s) = c_2 e^s + c_3, \\ y(s) = c_1 e^s + c_4. \end{cases}$$

As initial conditions, we have

$$x(0)=0, \quad y(0)=y_0, \quad z(0)=y_0^2$$

but we are missing initial conditions for p_1 *and* p_2*. Since* $u(0,y)=y^2$*, we have that* $u_y(0,y)=2y$*. Hence,* $p_2(0)=2y_0$ *and from the equation* $p_1(s)\,p_2(s)-z(s)=0$*, we get*

$$p_1(0)\,2y_0-y_0^2=0.$$

If $y_0\neq 0$*, this gives* $p_1(0)=\frac{1}{2}y_0$*. It follows from* $p_2(s)=c_2e^s$ *that* $c_2=2y_0$*, while from* $x(s)=c_2e^s+c_3$*, that* $c_3=-c_2=-2y_0$*. If* $y_0\neq 0$*, from* $p_1(s)=c_1e^s$*, we get* $c_1=\frac{1}{2}y_0$*, and in turn, from* $y(s)=c_1e^s+c_4$*,* $c_4=\frac{1}{2}y_0$*. In conclusion,*

$$\begin{cases} x(s)=2y_0\left(e^s-1\right), \\ y(s)=\frac{1}{2}y_0\left(e^s+1\right). \\ z(s)=y_0^2e^{2s}, \\ p_1(s)=\frac{1}{2}y_0e^s, \\ p_2(s)=2y_0e^s. \end{cases}$$

It follows that

$$\frac{x}{2y_0}+1=e^s \Leftrightarrow y=\frac{1}{2}y_0\left(\frac{x}{2y_0}+2\right)=\frac{x}{4}+y_0$$

so that

$$y_0=\frac{4y-x}{4} \quad \text{and} \quad e^s=\frac{x}{2\left(y-\frac{x}{4}\right)}+1=\frac{4y+x}{4y-x}$$

Hence,

$$\begin{aligned} u(x,y)=z(s)=z\left(\log\left(\frac{4y+x}{4y-x}\right)\right)&=\left(\frac{4y-x}{4}\right)^2e^{2\log\left(\frac{4y+x}{4y-x}\right)} \\ =\left(\frac{4y-x}{4}\right)^2\left(\frac{4y+x}{4y-x}\right)^2&=\frac{1}{16}\left(4y+x\right)^2. \end{aligned}$$

On the other hand, if $y_0=0$*, then*

$$\begin{cases} x(s)=0, \\ y(s)=-c_4e^s+c_4, \\ z(s)=0, \\ p_2(s)=0, \\ p_1(s)=-c_4e^s, \end{cases}$$

Example 16 *Consider the Eikonal equation*

$$\begin{cases} \|\nabla u(\boldsymbol{x})\|=1, & \boldsymbol{x}\in\mathbb{R}_+^N, \\ u(\boldsymbol{x}',0)=g(\boldsymbol{x}'), & \boldsymbol{x}'\in\mathbb{R}^{N-1}. \end{cases}$$

In this case we can take

$$F(\boldsymbol{x},z,\boldsymbol{p})=\|\boldsymbol{p}\|^2-1.$$

Let $\boldsymbol{x}_0' \in \mathbb{R}^{N-1}$. If $\|\nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0')\|_{N-1} > 1$, then there cannot be any local solution of class C^1 in a neighborhood of $(\boldsymbol{x}_0', 0)$. In this case the condition $(10)_2$ is violated. Assume next that $\|\nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0')\|_{N-1} \leq 1$ and let $c \in \mathbb{R}$ be such that

$$\|(\nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0'), c)\| = 1.$$

Since tangent vectors to the hyperplane $\left\{\boldsymbol{x} \in \mathbb{R}^N : x_N = 0\right\}$ are of the form $(\boldsymbol{t}', 0)$, the triple $((\boldsymbol{x}_0', 0), g(\boldsymbol{x}_0', 0), (\nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0'), c))$ is admissible. On the other hand, $\nabla_{\boldsymbol{p}} F(\boldsymbol{x}, z, \boldsymbol{p}) = \boldsymbol{p}$ and $\boldsymbol{n} = \boldsymbol{e}_N$, and so

$$\nabla_{\boldsymbol{p}} F(\boldsymbol{x}, z, \boldsymbol{p}) \cdot \boldsymbol{n} = p_N.$$

Hence, $((\boldsymbol{x}_0', 0), g(\boldsymbol{x}_0', 0), (\nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0'), c))$ is non-characteristic if $c \neq 0$. This shows that the notion of being non-characteristic depends strongly on $\boldsymbol{p}_0$. Note that this is not the case for linear or quasilinear equations.

Observe that in $(10)_2$ all the tangential components of ∇u near $\boldsymbol{p}_0$ are uniquely determined by $\nabla g(\boldsymbol{x}_0)$, but $(10)_2$ gives no information about the normal component $\nabla u \cdot \boldsymbol{n}$. On the other hand, we will see that the nontangential condition (11) will allow us to use the implicit function theorem to deduce $\nabla u \cdot \boldsymbol{n}$ from the equation in (BVP).

Example 17 *Consider the boundary value problem*

$$\begin{cases} \frac{\partial u}{\partial x_1}(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^N_+, \\ u(\boldsymbol{x}', 0) = g(\boldsymbol{x}'), & \boldsymbol{x}' \in \mathbb{R}^{N-1}. \end{cases}$$

If $\frac{\partial g}{\partial x_1}(\boldsymbol{x}') \neq 0$, then there cannot be any solutions whose partial derivatives are continuous up to the boundary.

Monday, September 9, 2013

4.1 Existence of Local Solutions

Under appropriate hypotheses, we will prove local existence of solutions of (BVP).

Theorem 18 (Local existence) *Assume that $\Omega \subseteq \mathbb{R}^N$ is an open set with $\partial\Omega$ of class C^2, that $F \in C^2\left(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N\right)$, $g \in C^3\left(\mathbb{R}^N\right)$, and that there exists a non-characteristic triple $(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \in \Gamma \times \mathbb{R} \times \mathbb{R}^N$. Then there exists a neighborhood V of $\boldsymbol{x}_0$ and a function $u \in C^2(V)$ such that*

$$\begin{cases} F(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) = 0 & \text{in } \Omega \cap V, \\ u = g & \text{on } \Gamma \cap V. \end{cases}$$

Proof. Step 1: We assume that $\partial\Omega$ is flat in a neighborhood of $\boldsymbol{x}_0$, more precisely, that there exists $R > 0$ such that

$$\Omega \cap B(\boldsymbol{x}_0, R) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R) : x_N > 0\}.$$

In this case, we can take the vector $\boldsymbol{e}_N$ as unit normal to $\partial\Omega$ at $\boldsymbol{x}_0$, so that $(10)_3$ and (11) become

$$\boldsymbol{p}_0' = \nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0), \quad \frac{\partial F}{\partial p_N}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \neq 0. \tag{12}$$

We claim that all points of Γ sufficiently close to $\boldsymbol{x}_0$ give rise to admissible triples. To see this, we apply the implicit function theorem to the function

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{p}) := (\boldsymbol{p}' - \nabla_{\boldsymbol{x}'} g(\boldsymbol{x}), F(\boldsymbol{x}, g(\boldsymbol{x}), \boldsymbol{p})).$$

Note that $\boldsymbol{G}(\boldsymbol{x}_0, \boldsymbol{p}_0) = (\boldsymbol{p}_0' - \nabla_{\boldsymbol{x}'} g(\boldsymbol{x}_0), F(\boldsymbol{x}_0, g(\boldsymbol{x}_0), \boldsymbol{p}_0)) = \boldsymbol{0}$ by (10) and (12), while

$$\nabla_{\boldsymbol{p}} \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{p}) = \begin{pmatrix} & 0 \\ I_{N-1} & \vdots \\ & 0 \\ \nabla_{\boldsymbol{p}'} F(\boldsymbol{x}, g(\boldsymbol{x}), \boldsymbol{p}) & \frac{\partial F}{\partial p_N}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \end{pmatrix}$$

and so

$$\det \nabla_{\boldsymbol{p}} \boldsymbol{G}(\boldsymbol{x}_0, \boldsymbol{p}_0) = \frac{\partial F}{\partial p_N}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \neq 0$$

by (12). By the implicit function theorem that there exist $0 < r < R$ and a function $\boldsymbol{q} \in C^2\left(B(\boldsymbol{x}_0, r); \mathbb{R}^N\right)$ such that $\boldsymbol{q}(\boldsymbol{x}_0) = \boldsymbol{p}_0$ and $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})) = \boldsymbol{0}$ for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$. Hence,

$$\boldsymbol{q}'(\boldsymbol{x}) = \nabla_{\boldsymbol{x}'} g(\boldsymbol{x}), \quad F(\boldsymbol{x}, g(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x})) = 0 \tag{13}$$

for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$. In particular, $(\boldsymbol{x}, g(\boldsymbol{x}), \boldsymbol{q}(\boldsymbol{x}))$ is admissible for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r) \cap \Gamma$.

Step 2: Given $\boldsymbol{y} \in B(\boldsymbol{x}_0, r) \cap \Gamma$, consider the Cauchy problem

$$\begin{cases} \frac{d\boldsymbol{x}}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)), \\ \frac{dz}{ds}(s) = \nabla_{\boldsymbol{p}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) \cdot \boldsymbol{p}(s), \\ \frac{d\boldsymbol{p}}{ds}(s) = -\nabla_{\boldsymbol{x}} F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) - \partial_z F(\boldsymbol{x}(s), z(s), \boldsymbol{p}(s)) \boldsymbol{p}(s), \\ \boldsymbol{x}(0) = \boldsymbol{y}, \quad z(0) = g(\boldsymbol{y}), \quad \boldsymbol{p}(0) = \boldsymbol{q}(\boldsymbol{y}). \end{cases} \tag{14}$$

Since F is of class C^2, $\nabla_{\boldsymbol{p}} F$, $\nabla_{\boldsymbol{x}} F$ and $\partial_z F$ are locally Lipschitz and so there exists a local solution. Moreover, we have continuous dependence on the initial data. Hence, by taking $r > 0$ smaller, if necessary, we can find $\delta > 0$ such that the Cauchy problem (14) admits a unique solution $\boldsymbol{x}(s, \boldsymbol{y})$, $z(s, \boldsymbol{y})$, $\boldsymbol{p}(s, \boldsymbol{y})$ for all $s \in (-\delta, \delta)$ and all $\boldsymbol{y} \in B(\boldsymbol{x}_0, r) \cap \Gamma$. Since F is of class C^2, the solution is of class C^2 in the s variable, of class C^1 in the $\boldsymbol{y}$ variables and the mixed second order derivatives exist and are continuous.

We claim that

$$F(\boldsymbol{x}(s, \boldsymbol{y}), z(s, \boldsymbol{y}), \boldsymbol{p}(s, \boldsymbol{y})) = 0 \tag{15}$$

for all $s \in (-\delta, \delta)$ and all $\boldsymbol{y} \in B(\boldsymbol{x}_0, r) \cap \Gamma$. To see this, fix $\boldsymbol{y} \in B(\boldsymbol{x}_0, r) \cap \Gamma$ and consider the function

$$f(s) := F(\boldsymbol{x}(s, \boldsymbol{y}), z(s, \boldsymbol{y}), \boldsymbol{p}(s, \boldsymbol{y})).$$

Then

$$f(0) = F(\boldsymbol{x}(0, \boldsymbol{y}), z(0, \boldsymbol{y}), \boldsymbol{p}(0, \boldsymbol{y})) = F(\boldsymbol{y}, g(\boldsymbol{y}), \boldsymbol{q}(\boldsymbol{y})) = 0$$

by Step 1, while by the chain rule and (14),

$$\begin{aligned} f'(s) &= \nabla_{\boldsymbol{x}} F \cdot \partial_s \boldsymbol{x} + \partial_z F \partial_s z + \nabla_{\boldsymbol{p}} F \cdot \partial_s \boldsymbol{p} \\ &= \nabla_{\boldsymbol{x}} F \cdot \nabla_{\boldsymbol{p}} F + \partial_z F \nabla_{\boldsymbol{p}} F \cdot \boldsymbol{p} + \nabla_{\boldsymbol{p}} F \cdot (-\nabla_{\boldsymbol{x}} F - \partial_z F \boldsymbol{p}) = 0, \end{aligned}$$

where $\nabla_{\boldsymbol{x}} F$, $\nabla_{\boldsymbol{p}} F$, and $\partial_z F$ are evaluated at $(\boldsymbol{x}(s, \boldsymbol{y}), z(s, \boldsymbol{y}), \boldsymbol{p}(s, \boldsymbol{y}))$ and $\partial_s \boldsymbol{x}$, $\partial_s z$, and $\partial_s \boldsymbol{p}$ at $(s, \boldsymbol{y})$. It follows that $f(s) = 0$ for all $s \in (-\delta, \delta)$, which proves the claim. ■

Wednesday, September 11, 2013

Proof. Step 3: We claim that the C^1 function $\boldsymbol{H} : (-\delta, \delta) \times B_{N-1}(\boldsymbol{x}_0', r) \to \mathbb{R}^N$, defined by

$$\boldsymbol{H}(s, \boldsymbol{y}') := \boldsymbol{x}(s, \boldsymbol{y}', 0),$$

is locally invertible, where $\boldsymbol{x}_0 = (\boldsymbol{x}_0', 0)$. Note that

$$\boldsymbol{H}(0, \boldsymbol{y}') = \boldsymbol{x}(0, \boldsymbol{y}', 0) = (\boldsymbol{y}', 0) \tag{16}$$

by (14). Hence,

$$\nabla_{\boldsymbol{y}'} \boldsymbol{H}(0, \boldsymbol{y}') = \begin{pmatrix} & I_{N-1} & \\ 0 & \cdots & 0 \end{pmatrix}$$

while by (14),

$$\partial_s \boldsymbol{H}(0, \boldsymbol{y}') = \partial_s \boldsymbol{x}(0, \boldsymbol{y}', 0) = \nabla_{\boldsymbol{p}} F(\boldsymbol{y}', 0, g(\boldsymbol{y}', 0), \boldsymbol{p}(\boldsymbol{y}', 0)).$$

In particular,

$$\det \nabla \boldsymbol{H}(0, \boldsymbol{x}_0') = \begin{pmatrix} 1 & \cdots & 0 & \frac{\partial F}{\partial p_1}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & \frac{\partial F}{\partial p_{N-1}}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \\ 0 & \cdots & 0 & \frac{\partial F}{\partial p_N}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \end{pmatrix} = \frac{\partial F}{\partial p_N}(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0) \neq 0$$

by (12). By the inverse function theorem that there exist $0 < \delta_1 < \delta$, $0 < r_1 < r$ and a neighborhood $W \subseteq B(\boldsymbol{x}_0, r)$ of $\boldsymbol{x}_0$ such that the function

$$\boldsymbol{H} : (-\delta_1, \delta_1) \times B_{N-1}(\boldsymbol{x}_0', r_1) \to W$$

is invertible and the inverse is of class C^1. Let $\boldsymbol{H}^{-1}(\boldsymbol{x}) = (s(\boldsymbol{x}), \boldsymbol{y}'(\boldsymbol{x}))$ be the inverse and let $\boldsymbol{y}(\boldsymbol{x}) := (\boldsymbol{y}'(\boldsymbol{x}), 0)$. For $\boldsymbol{x} \in W$ define

$$u(\boldsymbol{x}) := z(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})), \quad \boldsymbol{p}(\boldsymbol{x}) := \boldsymbol{p}(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})). \tag{17}$$

Taking $s = s(\boldsymbol{x})$ and $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ in (15) and using the fact that

$$\boldsymbol{x}(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})) = \boldsymbol{x}\left(\boldsymbol{H}^{-1}(\boldsymbol{x}), 0\right) = \boldsymbol{H}\left(\boldsymbol{H}^{-1}(\boldsymbol{x})\right) = \boldsymbol{x} \tag{18}$$

gives

$$F(\boldsymbol{x}, u(\boldsymbol{x}), \boldsymbol{p}(\boldsymbol{x})) = 0$$

for all $\boldsymbol{x} \in W$. Note that for $\boldsymbol{x} \in W \cap \Gamma$, $\boldsymbol{x} = (x', 0)$, by (16),

$$(0, \boldsymbol{x}') = \boldsymbol{H}^{-1}(\boldsymbol{H}(0, \boldsymbol{x}')) = \boldsymbol{H}^{-1}(\boldsymbol{x}', 0) = \boldsymbol{H}^{-1}(\boldsymbol{x}),$$

so that $s(\boldsymbol{x}) = 0$ and $\boldsymbol{y}'(\boldsymbol{x}) = \boldsymbol{x}'$. In turn, by (14),

$$u(\boldsymbol{x}) = z(0, \boldsymbol{x}) = g(\boldsymbol{x}).$$

To conclude the proof in the flat case, it remains to show that

$$\nabla u(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x}) \tag{19}$$

for all $\boldsymbol{x} \in W$.

For fixed $\boldsymbol{y} \in \Gamma$ and $i = 1, \ldots, N-1$, define

$$r_i(s) := \frac{\partial z}{\partial y_i}(s, \boldsymbol{y}) - \sum_{j=1}^{N} p_j(s, \boldsymbol{y}) \frac{\partial x_j}{\partial y_i}(s, \boldsymbol{y}).$$

We claim that $r_i \equiv 0$. By (14), $\boldsymbol{x}(0) = \boldsymbol{y}$, $z(0) = g(\boldsymbol{y})$, $\boldsymbol{p}(0) = \boldsymbol{q}(\boldsymbol{y})$, and so

$$\begin{aligned} r_i(0) &= \frac{\partial z}{\partial y_i}(0, \boldsymbol{y}) - \sum_{j=1}^{N} p_j(0, \boldsymbol{y}) \frac{\partial x_j}{\partial y_i}(0, \boldsymbol{y}) \\ &= \frac{\partial g}{\partial y_i}(\boldsymbol{y}) - \sum_{j=1}^{N} q_j(\boldsymbol{y}) \frac{\partial y_j}{\partial y_i} = \frac{\partial g}{\partial y_i}(\boldsymbol{y}) - q_i(\boldsymbol{y}) = 0, \end{aligned}$$

where we have used (13). On the other hand,

$$\frac{dr_i}{ds}(s) = \frac{\partial^2 z}{\partial s \partial y_i}(s, \boldsymbol{y}) - \sum_{j=1}^{N} \left(\frac{\partial p_j}{\partial s}(s, \boldsymbol{y}) \frac{\partial x_j}{\partial y_i}(s, \boldsymbol{y}) - p_j(s, \boldsymbol{y}) \frac{\partial^2 x_j}{\partial s \partial y_i}(s, \boldsymbol{y}) \right). \tag{20}$$

From the first two differential equations in (14) we have that

$$\frac{\partial z}{\partial s}(s, \boldsymbol{y}) = \sum_{j=1}^{N} p_j(s, \boldsymbol{y}) \frac{\partial x_j}{\partial s}(s, \boldsymbol{y}). \tag{21}$$

By differentiating this expression with respect to y_i, and using the fact that the mixed second derivatives are continuous, by Schwartz theorem, we get

$$\frac{\partial^2 z}{\partial s \partial y_i}(s, \boldsymbol{y}) = \frac{\partial^2 z}{\partial y_i \partial s}(s, \boldsymbol{y}) = \sum_{j=1}^{N} \left(\frac{\partial p_j}{\partial y_i}(s, \boldsymbol{y}) \frac{\partial x_j}{\partial s}(s, \boldsymbol{y}) + p_j(s, \boldsymbol{y}) \frac{\partial^2 x_j}{\partial s \partial y_i}(s, \boldsymbol{y}) \right).$$

Substituting this expression in (20) and using again (14), we get

$$\begin{aligned} \frac{dr_i}{ds} &= \sum_{j=1}^{N} \left(\frac{\partial p_j}{\partial y_i} \frac{\partial x_j}{\partial s} - \frac{\partial p_j}{\partial s} \frac{\partial x_j}{\partial y_i} \right) \\ &= \sum_{j=1}^{N} \left(\frac{\partial p_j}{\partial y_i} \frac{\partial F}{\partial p_j} - \left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z} p_j \right) \frac{\partial x_j}{\partial y_i} \right). \end{aligned} \tag{22}$$

On the other hand, differentiating (15) with respect to y_i gives

$$\sum_{j=1}^{N} \frac{\partial x_j}{\partial y_i} \frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y_i} + \sum_{j=1}^{N} \frac{\partial p_j}{\partial y_i} \frac{\partial F}{\partial p_j} = 0.$$

Substituting this expression in (22) yields

$$\frac{dr_i}{ds} = \frac{\partial F}{\partial z} \left[-\frac{\partial z}{\partial y_i} + \sum_{j=1}^{N} p_j \frac{\partial x_j}{\partial y_i} \right] = \frac{\partial F}{\partial z} r_i.$$

In conclusion, we have shown that r_i satisfies the Cauchy problem

$$\begin{cases} \frac{dr_i}{ds}(s) = \frac{\partial F}{\partial z}(\boldsymbol{x}(s, \boldsymbol{y}), z(s, \boldsymbol{y}), \boldsymbol{p}(s, \boldsymbol{y}))\, r_i(s) \\ r_i(0) = 0. \end{cases}$$

It follows that $r_i \equiv 0$, so that,

$$\frac{\partial z}{\partial y_i}(s, \boldsymbol{y}) = \sum_{j=1}^{N} p_j(s, \boldsymbol{y}) \frac{\partial x_j}{\partial y_i}(s, \boldsymbol{y}). \tag{23}$$

We are finally ready to prove (19). By (14), (17), (21), and the fact that $\boldsymbol{y}(\boldsymbol{x}) := (\boldsymbol{y}'(\boldsymbol{x}), 0)$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x_\ell}(\boldsymbol{x}) &= \frac{\partial}{\partial x_\ell}\left(z\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\right) \\
&= \frac{\partial z}{\partial s}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial s}{\partial x_\ell}(\boldsymbol{x}) + \sum_{i=1}^{N-1}\frac{\partial z}{\partial y_i}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial y_i}{\partial x_\ell}(\boldsymbol{x}) \\
&= \frac{\partial s}{\partial x_\ell}(\boldsymbol{x})\sum_{j=1}^{N} p_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial x_j}{\partial s}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right) \\
&\quad + \sum_{i=1}^{N-1}\frac{\partial y_i}{\partial x_\ell}(\boldsymbol{x})\sum_{j=1}^{N} p_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial x_j}{\partial y_i}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right) \\
&= \sum_{j=1}^{N} p_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\left[\frac{\partial x_j}{\partial s}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial s}{\partial x_\ell}(\boldsymbol{x}) + \sum_{i=1}^{N-1}\frac{\partial x_j}{\partial y_i}\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial y_i}{\partial x_\ell}(\boldsymbol{x})\right] \\
&= \sum_{j=1}^{N} p_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial}{\partial x_\ell}\left(x_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\right) \\
&= \sum_{j=1}^{N} p_j\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right)\frac{\partial x_j}{\partial x_\ell} = p_\ell\left(s(\boldsymbol{x}), \boldsymbol{y}(\boldsymbol{x})\right),
\end{aligned}$$

where in the second last equality we have used (18). This completes the proof of (19). ■

Friday, September 13, 2013

Proof. Step 4: We will remove the additional hypothesis that $\partial\Omega$ is flat in a neighborhood of $\boldsymbol{x}_0$. Since $\partial\Omega$ is of class C^2, there exist $r > 0$, $i \in \{1, \dots, N\}$ and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^2 such that

$$\Omega \cap B(\boldsymbol{x}_0, r) = \left\{\boldsymbol{x} \in B(\boldsymbol{x}_0, r) : x_i > h(\boldsymbol{x}_i)\right\}.$$

Without loss of generality, we take $i = N$ and write $\boldsymbol{x}' := \boldsymbol{x}_N = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}$. Consider the transformation $\boldsymbol{\Psi} : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\boldsymbol{\Psi}(\boldsymbol{x}) := (\boldsymbol{x}', x_N - h(\boldsymbol{x}')) := \boldsymbol{y}. \tag{24}$$

Note that Ψ sends points of the form $(\boldsymbol{x}', h(\boldsymbol{x}'))$ into $(\boldsymbol{x}', 0)$, so it flattens the boundary. Moreover,

$$\boldsymbol{\Psi}^{-1}(\boldsymbol{y}) = (\boldsymbol{y}', y_N + h(\boldsymbol{y}')) = \boldsymbol{x}$$

and

$$\nabla_{\boldsymbol{x}}\boldsymbol{\Psi}(\boldsymbol{x}) = \begin{pmatrix} & 0 \\ I_{N-1} & \vdots \\ & 0 \\ -\nabla_{\boldsymbol{x}'}h(\boldsymbol{x}') & 1 \end{pmatrix},$$

and $\nabla_{\boldsymbol{x}}^T \boldsymbol{\Psi}(\boldsymbol{x})\, \boldsymbol{e}_N = (-\nabla_{\boldsymbol{x}'} h(\boldsymbol{x}'), 1)$, which is normal to $\partial\Omega$ at $\boldsymbol{x}$.

Given a function $u = u(\boldsymbol{x})$ of class C^1, consider the function

$$v(\boldsymbol{y}) := u\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right).$$

Then $u(\boldsymbol{x}) = v(\boldsymbol{\Psi}(\boldsymbol{x}))$ and so

$$\nabla_{\boldsymbol{x}} u(\boldsymbol{x}) = \nabla_{\boldsymbol{y}} v(\boldsymbol{\Psi}(\boldsymbol{x}))\, \nabla_{\boldsymbol{x}} \boldsymbol{\Psi}(\boldsymbol{x}),$$

while

$$F(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x})) = F\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y}), v(\boldsymbol{y}), \nabla_{\boldsymbol{y}} v(\boldsymbol{y})\, \nabla_{\boldsymbol{x}} \boldsymbol{\Psi}\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right)\right).$$

Define

$$F_1(\boldsymbol{y}, z, \boldsymbol{q}) := F\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y}), z, \boldsymbol{q} \nabla_{\boldsymbol{x}} \boldsymbol{\Psi}\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right)\right), \quad g_1(\boldsymbol{y}) := g\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right),$$

and consider the boundary value problem

$$\begin{cases} F_1(\boldsymbol{y}, v(\boldsymbol{y}), \nabla_{\boldsymbol{y}} v(\boldsymbol{y})) = 0 & \text{in } \boldsymbol{\Psi}(\Omega), \\ v(\boldsymbol{y}) = g_1(\boldsymbol{y}) & \text{on } \boldsymbol{\Psi}(\Gamma). \end{cases} \tag{25}$$

Set $R := \nabla_{\boldsymbol{x}} \boldsymbol{\Psi}(\boldsymbol{x}_0)$, $\boldsymbol{y}_0 := (\boldsymbol{x}_0', 0)$, $\boldsymbol{n}_0 := (-\nabla_{\boldsymbol{x}'} h(\boldsymbol{x}_0'), 1)$. To apply the previous steps, we need to prove that $\left(\boldsymbol{y}_0, z_0, \boldsymbol{p}_0 R^{-1}\right)$ is a non-characteristic triple for (25).

Note that the function

$$\varphi(\boldsymbol{x}') := (\boldsymbol{x}', h(\boldsymbol{x}')), \quad \boldsymbol{x}' \in \mathbb{R}^{N-1},$$

is a local chart for $\partial\Omega$ at $\boldsymbol{x}_0$. Hence, a basis for $T_{\boldsymbol{x}_0}\partial\Omega$ is given by $\frac{\partial \varphi}{\partial x_i}(\boldsymbol{x}_0') = \left(\boldsymbol{e}_i', \frac{\partial h}{\partial x_i}(\boldsymbol{x}_0')\right)$, where $\boldsymbol{e}_i'$ is the standard basis in $\mathbb{R}^{N-1}$. It follows that $(10)_2$ is equivalent to

$$p_{0,i} + p_{0,N} \frac{\partial h}{\partial x_i}(\boldsymbol{x}_0') = \frac{\partial g}{\partial x_i}(\boldsymbol{x}_0) + \frac{\partial g}{\partial x_N}(\boldsymbol{x}_0) \frac{\partial h}{\partial x_i}(\boldsymbol{x}_0')$$

for all $i = 1, \dots, N-1$, where $\boldsymbol{p}_0 = (p_{0,1}, \dots, p_{0,N})$.

On the other hand, the tangent space to $\{y_N = 0\}$ at any point is $\mathbb{R}^{N-1} \times \{0\}$. Hence, $(10)_2$ for (25) is equivalent to

$$\left(\boldsymbol{p}_0 R^{-1}\right)_i = \frac{\partial g_1}{\partial y_i}(\boldsymbol{x}_0', 0)$$

for all $i = 1, \dots, N-1$. Since $g_1(\boldsymbol{y}) = g(\boldsymbol{y}', y_N + h(\boldsymbol{y}'))$, we have that

$$\frac{\partial g_1}{\partial y_i}(\boldsymbol{y}) = \frac{\partial g}{\partial x_i}(\boldsymbol{y}', y_N + h(\boldsymbol{y}')) + \frac{\partial g}{\partial x_N}(\boldsymbol{y}', y_N + h(\boldsymbol{y}')) \frac{\partial h}{\partial x_i}(\boldsymbol{y}'),$$

while

$$\boldsymbol{p}_0 R^{-1} = \boldsymbol{p}_0 \left(\nabla_{\boldsymbol{x}} \boldsymbol{\Psi}\left(\boldsymbol{x}_0\right)\right)^{-1} = \boldsymbol{p}_0 \begin{pmatrix} & 0 \\ I_{N-1} & \vdots \\ & 0 \\ \nabla_{\boldsymbol{x}'} h\left(\boldsymbol{x}'\right) & 1 \end{pmatrix} \tag{26}$$

$$= \left(p_{0,1} + p_{0,N} \frac{\partial h}{\partial x_1}\left(\boldsymbol{x}_0'\right), \ldots, p_{0,N-1} + p_{0,N} \frac{\partial h}{\partial x_{N-1}}\left(\boldsymbol{x}_0'\right), p_N\right).$$

It follows that $\left(\boldsymbol{y}_0, z_0, \boldsymbol{p}_0 R^{-1}\right)$ is admissible for (25). To prove that it is a non-characteristic triple, note that

$$\begin{aligned} \nabla_{\boldsymbol{q}} F_1\left(\boldsymbol{y}_0, z_0, \boldsymbol{p}_0 R^{-1}\right) \cdot \boldsymbol{e}_N &= \nabla_{\boldsymbol{p}} F\left(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0\right) R \cdot \boldsymbol{e}_N \\ &= \nabla_{\boldsymbol{p}} F\left(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0\right) \cdot R^T \boldsymbol{e}_N \\ &= \nabla_{\boldsymbol{p}} F\left(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0\right) \cdot \boldsymbol{n}_0 \neq 0. \end{aligned}$$

Hence, we are in a position to apply the previous steps to (25) to find a local solution Then there exists a neighborhood W of $\boldsymbol{y}_0$ and a function $v \in C^2(W)$ such that

$$\begin{cases} F_1\left(\boldsymbol{y}, v\left(\boldsymbol{y}\right), \nabla_{\boldsymbol{y}} v\left(\boldsymbol{y}\right)\right) = 0 & \text{in } \boldsymbol{\Psi}\left(\Omega\right) \cap W, \\ v\left(\boldsymbol{y}\right) = g_1\left(\boldsymbol{y}\right) & \text{on } \boldsymbol{\Psi}\left(\Gamma\right) \cap W. \end{cases}$$

The function $u(\boldsymbol{x}) := v(\boldsymbol{\Psi}(\boldsymbol{x}))$ is a local solution of (BVP). ■

For a first order linear PDE

$$\boldsymbol{b}\left(\boldsymbol{x}\right) \cdot \nabla u\left(\boldsymbol{x}\right) + c\left(\boldsymbol{x}\right) u\left(\boldsymbol{x}\right) = d\left(\boldsymbol{x}\right),$$

we have

$$F\left(\boldsymbol{x}, z, \boldsymbol{p}\right) = \boldsymbol{b}\left(\boldsymbol{x}\right) \cdot \boldsymbol{p} + c\left(\boldsymbol{x}\right) z - d\left(\boldsymbol{x}\right).$$

Hence, an admissible triple $(\boldsymbol{x}_0, z_0, \boldsymbol{p}_0)$ is non-characteristic if

$$\boldsymbol{b}\left(\boldsymbol{x}_0\right) \cdot \boldsymbol{n} \neq 0.$$

The next exercise shows that local existence may fail if a triple is not admissible or characteristics.

Exercise 19 *Consider the initial value problem*

$$\begin{cases} x u_y\left(x, y\right) - y u_x\left(x, y\right) = u\left(x, y\right), & x \in \mathbb{R},\ y > 0, \\ u\left(x, 0\right) = g\left(x\right), & x \in \mathbb{R}, \end{cases}$$

where $g \in C^1(\mathbb{R})$, *with* $g(0) = 0$ *and* $g'(0) \neq 0$. *Prove that there is no regular solution near* $(0, 0)$.

Monday, September 16, 2013

4.2 Conservation Laws

A *scalar conservation law* in one space dimension is a first order partial equation of the form

$$u_t + (f \circ u)_x = 0,$$

where $f \in C^1(\mathbb{R})$. Here u is the *conserved quantity* and f is the *flux*. These types of equation describes transport phenomena. Assume that u is a classical solution. Integrating this equation with respect to x between a and b , and we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\int_a^b u(x,t)\,dx\right) &= \int_a^b u_t(x,t)\,dx = -\int_a^b (f \circ u)_x(x,t)\,dx \\
&= -f(u(b,t)) + f(u(a,t)) \\
&= -\text{out flow at } b + \text{ in flow at } a.
\end{aligned}$$

This says that the total amount of u in the interval $[a,b]$ can change only due to the flow of u across the boundary points a and b. Hence, u is conserved. Examples are the density of cars in an highway.

Let $f \in C^2(\mathbb{R})$ and consider the initial value problem

$$\begin{cases} u_t + f'(u)\,u_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases} \tag{27}$$

where $g \in C^1(\mathbb{R})$. This is a quasilinear PDE. Here, $\boldsymbol{x} = (x,t)$, $\boldsymbol{b}(x,t,z) = (f'(z),1)$, and $c(x,t,z) = 0$. Hence, (2) takes the form

$$\begin{cases} \frac{dx}{ds}(s) = f'(z), \\ \frac{dt}{ds}(s) = 1, \\ \frac{dz}{ds}(s) = 0, \end{cases} \quad \Leftrightarrow \begin{cases} x(s) = f'(c_1)\,s + c_3, \\ t(s) = s + c_2, \\ z(s) = c_1. \end{cases}$$

The initial data are

$$x(0) = x_0, \quad t(0) = 0, \quad z(0) = g(x_0).$$

Hence, $c_2 = 0$, $c_3 = x_0$, $c_1 = g(x_0)$, so that

$$\begin{cases} x(s) = f'(g(x_0))\,s + x_0, \\ t(s) = s, \\ z(s) = g(x_0). \end{cases}$$

The projected characteristic is $x = f'(g(x_0))\,t + x_0$. Hence, $x_0 = x - g(x_0)\,t$, and so the solution is given implicitly by

$$u(x,t) = z(t) = g(x_0) = g(x - f'(u(x,t))\,t),$$

and it is impossible to determine where it exists using this expression.

Now let's look at an admissible triple $((x_0,0), g(x_0), p_{01}, p_{02})$. Here, $\Gamma = \{(x,0) : x \in \mathbb{R}\}$, so that $\boldsymbol{t} = (1,0)$, $F(x,t,z,p_1,p_2) = p_2 + f'(z)\,p_1$, so that (10) becomes

$$\begin{cases} p_{02} + f'(g(x_0))\,p_{01} = 0, \\ p_{01} = g'(x_0), \end{cases} \Leftrightarrow \begin{cases} p_{01} = g'(x_0), \\ p_{02} = -f'(g(x_0))\,g'(x_0), \end{cases}$$

while (11) reduces to

$$\frac{\partial F}{\partial p_2}(x,t,z,p_1,p_2) = 1 \neq 0.$$

Hence, the triple $((x_0,0), g(x_0), g'(x_0), -f'(g(x_0))\,g'(x_0))$ is non-characteristics. By Theorem 18, there exists a local solution in a neighborhood of $(x_0,0)$.

The projected characteristic through $(x_0,0)$ is $x = f'(g(x_0))\,t + x_0$. If we consider another point $(x_1,0)$, with $x_1 > x_0$, the projected characteristic through $(x_1,0)$ is $x = f'(g(x_1))\,t + x_1$. These two lines intersect at

$$\begin{cases} x = f'(g(x_0))\,t + x_0, \\ x = f'(g(x_1))\,t + x_1 \end{cases} \Leftrightarrow \begin{cases} x = f'(g(x_1))\,t + x_1, \\ f'(g(x_0))\,t + x_0 = f'(g(x_1))\,t + x_1, \end{cases}$$
$$\Leftrightarrow \begin{cases} x = g(x_1)\,t + x_1, \\ (f'(g(x_0)) - f'(g(x_1)))\,t = x_1 - x_0 > 0, \end{cases}$$

hence, if $f'(g(x_0)) - f'(g(x_1)) > 0$, then the two lines will meet at time

$$t_* = \frac{x_1 - x_0}{f'(g(x_0)) - f'(g(x_1))},$$

with

$$u(x_0,t_*) = g(x_0) \neq g(x_1) = u(x_1,t_*).$$

It follows that if f' is strictly increasing and g is not decreasing, the solution cannot have a global C^1 solution. However, it is possible to define a weak solution. To see this, assume that u is a classical solution of (27). Multiply the equation by a function $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$, integrate over $\mathbb{R} \times \mathbb{R}_+$. By integrating by parts we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_0^\infty \varphi u_t \; dtdx + \int_0^\infty \int_{\mathbb{R}} \varphi\,(f \circ u)_x \; dxdt \\ &= -\int_{\mathbb{R}} \int_0^\infty \varphi_t u \; dtdx - \int_{\mathbb{R}} \varphi(x,0)\,g(x) \; dx - \int_0^\infty \int_{\mathbb{R}} \varphi_x f \circ u \; dxdt. \end{aligned}$$

Definition 20 *Given a locally integrable function $g : \mathbb{R} \to \mathbb{R}$, we say that a locally integrable function $u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ is a* weak solution *of (27) if*

$$0 = \int_{\mathbb{R}} \int_0^\infty (\varphi_t u + \varphi_x f \circ u) \; dtdx + \int_{\mathbb{R}} \varphi(x,0)\,g(x) \; dx \tag{28}$$

for all $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$. Next we will study the discontinuities of weak solutions.

Consider an open set $U \subseteq \mathbb{R} \times \mathbb{R}_+$ and assume that a curve γ parametrized by $x = \xi(t)$, $t \in (a,b)$, of class C^1 divides U into two open regions U_- and U_+. Assume that a weak solution u has limit as we approach the curves from the left and from the right, denoted u^- and u^+, respectively. Set $u^- := u$ in U^+ and $u^+ := u$ in U^+ and assume that $u^\pm \in C^1\left(\overline{U^\pm}\right)$ and that $u^\pm$ is a classical solution of (27) in $U^\pm$. For every $\varphi \in C_c^1(U)$, we can rewrite (28) as

$$\begin{aligned}
0 &= \int_{\mathbb{R}} \int_0^\infty (\varphi_t u + \varphi_x f \circ u)\ dtdx + \int_{\mathbb{R}} \varphi(x,0)\, g(x)\ dx \\
&= \iint_U (\varphi_t u + \varphi_x f \circ u)\ dtdx \\
&= \iint_{U+} (\varphi_t u + \varphi_x f \circ u)\ dtdx + \iint_{U-} (\varphi_t u + \varphi_x f \circ u)\ dtdx \\
&= -\iint_{U+} \varphi\left(u_t + (f \circ u)_x\right)\ dtdx + \int_{\gamma^+} \varphi u^+\ dx + \int_{\gamma^+} \varphi f \circ u^+\ dt \\
&\quad - \iint_{U-} \varphi\left(u_t + (f \circ u)_x\right)\ dtdx + \int_{\gamma^-} \varphi u^-\ dx + \int_{\gamma^-} \varphi f \circ u^-\ dt \\
&= \int_a^b \varphi(x(t),t) \left[\left(u^+(x(t),t) - u^-(x(t),t)\right) \frac{dx(t)}{dt} - \left(f\left(u^+(x(t),t)\right) - f\left(u^-(x(t),t)\right)\right)\right] dt,
\end{aligned}$$

where we have used Gauss–Green formulas.[2] Since this is true for every $\varphi \in C_c^1(U)$, it follows that x satisfies the ordinary differential equation

$$\underbrace{\left(u^+(x(t),t) - u^-(x(t),t)\right)}_{\text{jump in the state}} \underbrace{\frac{dx(t)}{dt}}_{\text{speed of the shock}} = \underbrace{f\left(u^+(x(t),t)\right) - f\left(u^-(x(t),t)\right)}_{\text{jump in the flux}}, \tag{31}$$

which is called the *shock* or *Rankine–Hugoniot condition.*

Wednesday, September 18, 2013

In the case $f(z) = \frac{z^2}{2}$ we get the Burgers' equation

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{32}$$

The shock condition for the Burger's equation is

$$\frac{dx(t)}{dt} = \frac{1}{2}u^-(x(t),t) + \frac{1}{2}u^+(x(t),t). \tag{33}$$

[2] *Gauss–Green formulas*

$$\int_U \frac{\partial f}{\partial x}(x,y)\ dxdy = \int_{\partial U} f\, dy. \tag{29}$$

$$\int_U \frac{\partial g}{\partial y}(x,y)\ dxdy = -\int_{\partial U} g\, dx. \tag{30}$$

Example 21 *Consider* (32), *where*

$$g(x) := \begin{cases} 1 & \text{if } x \le 0, \\ 1-x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x \ge 1. \end{cases}$$

The projected characteristic through $(x_0, 0)$ *is* $x = g(x_0)t + x_0$. *If* $x_0 \le 0$, *then* $g(x_0) = 1$, *and so the projected characteristic through* $(x_0, 0)$ *is* $x = t + x_0 \le t$. *Along this line, we have*

$$u(x,y) = g(x_0) = 1.$$

If $x_0 \ge 1$, *then* $g(x_0) = 0$, *and so the projected characteristic through* $(x_0, 0)$ *is* $x = x_0 \ge 1$. *Along this line, we have*

$$u(x,y) = g(x_0) = 0.$$

If $0 < x_0 < 1$, *then* $g(x_0) = 1 - x_0$, *and so the projected characteristic through* $(x_0, 0)$ *is* $x = (1 - x_0)t + x_0$. *In turn,* $x - 1 = (1 - x_0)(t - 1)$, *and so along this line, we have*

$$u(x,y) = g(x_0) = 1 - x_0 = \frac{x-1}{t-1}.$$

Moreover, $t = \frac{x - x_0}{1 - x_0} \le x$, *since* $x - x_0 \le x - x x_0$. *It follows that the solution is given by*

$$u(x,t) := \begin{cases} 1 & \text{if } x \le t, \quad 0 \le t < 1, \\ \dfrac{x-1}{t-1} & \text{if } t \le x \le 1, \quad 0 \le t < 1 \\ 0 & \text{if } x \ge 1, \quad 0 \le t < 1. \end{cases}$$

Now for $t \ge 1$, *the projected characteristics cross, and so we cannot use the method of characteristics.*

Consider the curve $x(t) = \frac{t+1}{2}$, $t \ge 1$, *and let*

$$u(x,t) := \begin{cases} 1 & \text{if } x < \frac{t+1}{2}, \quad t \ge 1, \\ 0 & \text{if } x \ge \frac{t+1}{2}, \quad t \ge 1. \end{cases}$$

Then (33) *is satisfied. This function is a weak solution (Exercise).*

The next example shows that weak solutions may not be unique.

Example 22 *Consider* (32), *where*

$$g(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \ge 0. \end{cases}$$

The projected characteristic through $(x_0, 0)$ *is* $x = g(x_0)t + x_0$. *If* $x_0 < 0$, *then* $g(x_0) = 0$, *and so the projected characteristic through* $(x_0, 0)$ *is* $x = x_0$. *Along this line, we have*

$$u(x,y) = g(x_0) = 0.$$

If $x_0 \geq 0$, then $g(x_0) = 1$, and so the projected characteristic through $(x_0, 0)$ is $x = t + x_0 \geq t$. Along this line, we have

$$u(x, y) = g(x_0) = 1.$$

In the region $\{0 < x < t\}$ the method of characteristics gives no information. It can be shown that

$$u_1(x,t) := \begin{cases} 0 & \text{if } x < \frac{t}{2}, \\ 1 & \text{if } x \geq \frac{t}{2}, \end{cases}$$

and

$$u_2(x,t) := \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{x}{t} & \text{if } 0 < x < t, \\ 1 & \text{if } x > t \end{cases}$$

are both weak solutions (Exercise). Note that (33) *is satisfied.*

In several applications from physics, there is only one physically preferred solution. Hence, different selection criteria have been proposed to recover uniqueness of solutions. We only list a few of them:

- **Lax's shock condition** selects solutions that jump down across a shock, precisely, for the equation (27) in the situation when there is a shock, we ask that
$$f\left(u^+(x(t), t)\right) < \frac{dx(t)}{dt} < f\left(u^-(x(t), t)\right).$$
- **Entropy/flux pair:** A function $R \in C^1(\mathbb{R})$ is called an *entropy* for (27) with *entropy flux* $S \in C^1(\mathbb{R})$ if
$$S'(z) = f'(z) R'(z)$$
for all $z \in \mathbb{R}$ (for example for the Burger's equation we can take $S(z) = \frac{3}{4}z^4$ and $R(z) = z^3$) and assume that u is a classical solution of (27). Multiplying the PDE by $R'(u)$ we get
$$\begin{aligned} R'(u)(u_t + f'(u) u_x) = 0 &\Leftrightarrow R'(u) u_t + S'(u) u_x = 0 \\ &\Leftrightarrow (R \circ u)_t + (S \circ u)_x = 0. \end{aligned} \tag{34}$$
The latter equation is in *divergence form* and it has the same structure of (27). It is an additional conservation law for (27). Multiply the equation by a function $\varphi \in C_c^1(\mathbb{R} \times [0, \infty))$, integrate over $\mathbb{R} \times \mathbb{R}_+$. By integrating by parts we get
$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_0^\infty \varphi (R \circ u)_t \, dt dx + \int_0^\infty \int_{\mathbb{R}} \varphi (S \circ u)_x \, dx dt \\ &= -\int_{\mathbb{R}} \int_0^\infty \varphi_t R \circ u \, dt dx - \int_{\mathbb{R}} \varphi(x, 0) g(x) \, dx - \int_0^\infty \int_{\mathbb{R}} \varphi_x S \circ u \, dx dt. \end{aligned}$$

A weak solution u of (27) is *entropy admissible* if it is locally bounded and

$$\int_{\mathbb{R}}\int_0^{\infty} \varphi_t R\circ u\ dtdx + \int_0^{\infty}\int_{\mathbb{R}} \varphi_x S\circ u\ dxdt \geq 0$$

for all nonnegative $\varphi \in C_c^1(\mathbb{R}\times\mathbb{R}_+)$ and for every pair (R, S), where R is a convex entropy and S is the corresponding entropy flux. Note that if we have a shock, reasoning exactly as before, we get

$$\left(R\left(u^+\left(x\left(t\right),t\right)\right) - R\left(u^-\left(x\left(t\right),t\right)\right)\right)\frac{dx\left(t\right)}{dt} \geq S\left(u^+\left(x\left(t\right),t\right)\right) - S\left(u^-\left(x\left(t\right),t\right)\right).$$

- **Vanishing viscosity solutions**: A weak solution u of (27) is *admissible in the vanishing viscosity sense* if there exist solutions u_ε of the second order partial differential equation

  $$u_t^\varepsilon + f'\left(u^\varepsilon\right)u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, \quad x\in\mathbb{R},\, t>0,$$

  where $\varepsilon > 0$ such that $u_\varepsilon \to u$ in L^1_{loc}. Solutions of this equation (which is parabolic, we will study it next semester) exist and are regular.

Remark 23 *Given a convex function $R\in C^1(\mathbb{R})$, one can always find a corresponding entropy flux by taking*

$$S\left(z\right) := \int_{z_0}^{z} f'\left(s\right)R'\left(s\right)\ ds.$$

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Remark 24 *By Rademacher's theorem, a Lipschitz (or locally Lipschitz) function is differentiable a.e. Hence, in (28) one could consider a Lipschitz function f. Note that the definition of weak solution does not involve any derivative of f and thus it makes sense in this case. Similarly, it is also possible to consider convex entropies R that are locally Lipschitz. Indeed, R is differentiable for all but countably many points. Thus, it is enough to require that the pair (R, S) satisfies $S'(z) = f'(z)R'(z)$ for $\mathcal{L}^1$ a.e. $z\in\mathbb{R}$.*

Remark 25 *In what follows we are going to use the following property: If $E\subset\mathbb{R}$ has Lebesgue measure zero and $u\in C^1(\Omega)$, where $\Omega\subseteq\mathbb{R}^N$ is an open set, then*

$$\frac{\partial u}{\partial x_i}\left(\boldsymbol{x}\right) = 0 \text{ for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x}\in u^{-1}\left(E\right)$$

and all $i = 1,\ldots,N$. In particular, since any singleton $\{c\}$ has Lebesgue measure zero in $\mathbb{R}$, we have that

$$\frac{\partial u}{\partial x_i}\left(\boldsymbol{x}\right) = 0 \text{ for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x}\in\Omega_c,$$

where $\Omega_c := \{\boldsymbol{x}\in\Omega:\ u(\boldsymbol{x}) = c\}$.

Exercise 26 *Prove that a bounded solution which is admissible in the vanishing viscosity sense is entropy admissible.*

The following theorem was proved by A.I. Volpert (1967).

Theorem 27 *Let $f \in C^2(\mathbb{R})$ and let $g \in L^\infty(\mathbb{R})$. Then there exists a weak solution u of (27), defined for all $x \in \mathbb{R}$ and all $t \geq 0$, which is admissible in the vanishing viscosity sense.*

On the other hand S. Kruzhkov (1970) proved the following theorem.

Theorem 28 *Let $f \in C^2(\mathbb{R})$ and let $g \in L^\infty(\mathbb{R})$. Then there exists a unique weak solution of (27), which is admissible for the entropy/entropy flux pairs*

$$R_k(z) := |z-k|, \quad S_k(z) = (f(z) - f(k)) \operatorname{sign}(z-k),$$

where $k \in \mathbb{Z}$.

In what follows we will sketch the proof of Volpert's theorem,
Proof of Theorem 27. Step 1: Let $f \in C_c^\infty(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. It can be shown that solutions of the equation

$$\begin{cases} u_t^\varepsilon + f'(u^\varepsilon) u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, & x \in \mathbb{R},\, t > 0, \\ u^\varepsilon(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

exist for all time, are C^∞, bounded together with all their derivatives, and decay fast at infinity together with all their derivatives. Differentiate the equation with respect to x to get

$$\begin{cases} (u_x^\varepsilon)_t + f'(u^\varepsilon)(u_x^\varepsilon)_x + f''(u^\varepsilon)(u_x^\varepsilon)^2 = \varepsilon (u_x^\varepsilon)_{xx}, & x \in \mathbb{R},\, t > 0, \\ u_x^\varepsilon(x,0) = g'(x), & x \in \mathbb{R}. \end{cases} \tag{35}$$

Let $v = u_x^\varepsilon$. Then v solves the equation

$$\begin{cases} v_t + (f'(u^\varepsilon) v)_x = \varepsilon v_{xx}, & x \in \mathbb{R},\, t > 0, \\ v(x,0) = g'(x). & x \in \mathbb{R}. \end{cases}$$

Fix $\delta > 0$ and consider an increasing function $\varphi_\delta \in C^1(\mathbb{R})$ such that $\varphi_\delta(z) = -1$ if $z \leq -\delta$, $\varphi_\delta(z) = 1$ if $z \geq \delta$ and $0 \leq \varphi_\delta' \leq C/\delta$. Multiply the equation by $\varphi_\delta \circ v$ and integrate to get

$$\int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v) v_t \, dtdx + \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v)(f'(u^\varepsilon) v)_x \, dxdt = \varepsilon \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v) v_{xx} \, dxdt.$$

Let's study all these integrals separately. Since v_t is integrable and $|\varphi_\delta \circ v| \leq 1$ we can apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} \lim_{\delta \to 0^+} \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v) v_t \, dtdx &= \int_{\mathbb{R}} \int_0^T (\operatorname{sign} v) v_t \, dtdx \\ &= \int_{\mathbb{R}} \int_0^T \frac{\partial |v|}{\partial t} \, dtdx = \int_{\mathbb{R}} |v(x,T)| \, dx - \int_{\mathbb{R}} |g'(x)| \, dx, \end{aligned}$$

where we used the fact that $v_t = 0\ \mathcal{L}^2$ a.e. in the set $\{(x,t) \in \mathbb{R} \times (0,T) : \ v(x,t) = 0\}$ (see Remark 25).

Integrating by parts in x we have

$$\begin{aligned}\left| \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v) \left(f'(u) v\right)_x \ dxdt \right| &= \left| - \int_{\mathbb{R}} \int_0^T (\varphi'_\delta \circ v) v_x f'(u) v \ dxdt \right| \\ &= \left| \iint_{\{|v| \leq \delta\}} (\varphi'_\delta \circ v) v_x f'(u) v \ dxdt \right| \\ &\leq \frac{C}{\delta} \iint_{\{|v| \leq \delta\}} |v_x| |v| \ dxdt \leq C \iint_{\{|v| \leq \delta\}} |v_x| \ dxdt \to 0\end{aligned}$$

as $\delta \to 0^+$ again by the Lebesgue dominated convergence and the fact that $v_x = 0\ \mathcal{L}^2$ a.e. in the set $\{(x,t) \in \mathbb{R} \times (0,T) : \ v(x,t) = 0\}$ (see Remark 25). Note that we have used the fact that f' is bounded.

Finally, integrating by parts in x we have

$$\varepsilon \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ v) v_{xx} \ dxdt = -\varepsilon \int_{\mathbb{R}} \int_0^T (\varphi'_\delta \circ v) v_x^2 \ dxdt \leq 0$$

since $\varphi'_\delta \geq 0$. In conclusion we have shown that

$$\int_{\mathbb{R}} |v(x,T)| \ dx \leq \int_{\mathbb{R}} |g'(x)| \ dx.$$

It follows that for all $t > 0$ and $\varepsilon > 0$,

$$\int_{\mathbb{R}} |u_x^\varepsilon(x,t)| \ dx \leq \int_{\mathbb{R}} |g'(x)| \ dx. \tag{36}$$

Next, differentiate the equation with respect to t to get

$$\begin{cases} (u_t^\varepsilon)_t + f'(u^\varepsilon)(u_t^\varepsilon)_x + f''(u^\varepsilon) u_x^\varepsilon u_t^\varepsilon = \varepsilon (u_t^\varepsilon)_{xx}, & x \in \mathbb{R},\ t > 0, \\ u_t^\varepsilon(x,0) = f'(g(x)) g'(x) - \varepsilon g''(x), & x \in \mathbb{R}. \end{cases}$$

Let $w = u_t^\varepsilon$. Then w solves the equation

$$\begin{cases} w_t + \left(f'(u^\varepsilon) w\right)_x = \varepsilon w_{xx}, & x \in \mathbb{R},\ t > 0, \\ w(x,0) = f'(g(x)) g'(x) - \varepsilon g''(x). & x \in \mathbb{R}. \end{cases}$$

Hence, repeating the same argument, we obtain that for all $t > 0$ and $\varepsilon > 0$,

$$\int_{\mathbb{R}} |u_t^\varepsilon(x,t)| \ dx \leq \int_{\mathbb{R}} |f'(g(x)) g'(x)| \ dx + \varepsilon \int_{\mathbb{R}} |g''(x)| \ dx. \tag{37}$$

■

Monday, September 23, 2013

To continue the proof of Volpert's theorem, we need to introduce Sobolev spaces and BV spaces.

4.3 Sobolev and BV Spaces

Definition 29 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. The* Sobolev space $W^{1,p}(\Omega)$ *is the space of all functions $u \in L^p(\Omega)$ whose distributional first order partial derivatives belong to $L^p(\Omega)$, that is, for all $i = 1, \ldots, N$ there exists a function $g_i \in L^p(\Omega)$ such that*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} = -\int_\Omega g_i \varphi\, d\boldsymbol{x}$$

for all $\varphi \in C_c^\infty(\Omega)$. The function g_i is called the weak *or* distributional partial derivative *of u with respect to x_i and is denoted $\frac{\partial u}{\partial x_i}$.*

Exercise 30 *Show that the weak derivatives are unique.*

The space $W^{1,p}(\Omega)$ is a normed space with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}.$$

Example 31 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. If $u \in C^1(\Omega) \cap L^p(\Omega)$ and its (classical) partial derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega)$, then it follows by integration by parts that*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi\, d\boldsymbol{x}$$

for all $\varphi \in C_c^\infty(\Omega)$. Hence, the (classical) partial derivatives $\frac{\partial u}{\partial x_i}$ are also the weak derivatives.

Definition 32 *Let $\Omega \subset \mathbb{R}^N$ be an open set. We define the* space of functions of bounded variation $BV(\Omega)$ *as the space of all functions $u \in L^1(\Omega)$ whose distributional first-order partial derivatives are finite signed measures; that is, for all $i = 1, \ldots, N$ there exists a signed measure $\lambda_i : \mathcal{B}(\Omega) \to \mathbb{R}$ such that*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} = -\int_\Omega \varphi\, d\lambda_i \tag{38}$$

for all $\varphi \in C_c^\infty(\Omega)$. The measure λ_i is called the weak, *or* distributional, partial derivative *of u with respect to x_i and is denoted $D_i u$.*

For $u \in BV(\Omega)$ we set

$$Du := (D_1 u, \ldots, D_N u).$$

Given a signed measure $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}$, the *total variation measure* of λ, defined by

$$|\lambda|(E) := \sup \left\{ \sum_{n=1}^\infty |\lambda(E_n)| \right\}, \quad E \in \mathcal{B}(\Omega), \tag{39}$$

where the supremum is taken over all partitions $\{E_n\} \subset \mathcal{B}(\Omega)$ of E, is a finite measure. Since the space of all signed measures may be identified with the dual of $C_0(\Omega)$, it can be shown that

$$\begin{aligned} |\lambda|(\Omega) &= \|\lambda\|_{(C_0(\Omega))'} \\ &= \sup\left\{\sum_{i=1}^{N}\int_{\Omega}\varphi\, d\lambda : \varphi \in C_0(\Omega), \|\varphi\|_{C_0(\Omega)} \leq 1\right\}. \end{aligned}$$

The space $BV(\Omega)$ is a normed space with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} |D_i u|(\Omega),$$

where $|D_i u|(\Omega)$ is defined by (39) for $E = \Omega$.

Remark 33 *If $u \in W^{1,1}(\Omega)$, then u belongs to $BV(\Omega)$ with weak derivatives $D_i u : \mathcal{B}(\Omega) \to \mathbb{R}$ defined by*

$$D_i u(E) := \int_E \frac{\partial u}{\partial x_i}\, d\boldsymbol{x}, \quad E \in \mathcal{B}(\Omega).$$

Note that the measure $D_i u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ (restricted to Ω). Conversely, if $u \in BV(\Omega)$ and all its weak derivatives $D_i u$ are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$, then by the Radon-Nikodym theorem, we can find functions $g_i \in L^1(\Omega)$ such that

$$D_i u(E) = \int_E g_i\, d\boldsymbol{x}, \quad E \in \mathcal{B}(\Omega).$$

It follows that u belongs to $W^{1,1}(\Omega)$ and the weak derivatives $\frac{\partial u}{\partial x_i}$ are given by g_i.

The next example shows that the inclusion is strict. We put ourselves in the same setting of a shock condition for scalar conservations laws.

Example 34 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume that an $(N-1)$ dimensional surface M of class C^1 subdivides the domain Ω into two open sets Ω^- and Ω^+. Let $u^- \in C^1\left(\overline{\Omega^-}\right)$ and $u^+ \in C^1\left(\overline{\Omega^+}\right)$ and define the function*

$$u(\boldsymbol{x}) := \begin{cases} u^-(\boldsymbol{x}) & \boldsymbol{x} \in \Omega^-, \\ u^+(\boldsymbol{x}) & \boldsymbol{x} \in \Omega^+. \end{cases}$$

Note that u is not defined on M, which has Lebesgue measure zero. We claim that the function u belongs to $BV(\Omega)$ but not to $W^{1,p}(\Omega)$ for any p. To see this, let $\varphi \in C_c^\infty(\Omega)$. Then

$$\int_\Omega u\frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} = \int_{\Omega^-} u\frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x} + \int_{\Omega^+} u\frac{\partial \varphi}{\partial x_i}\, d\boldsymbol{x}.$$

Since M is of class C^1 and $\varphi \in C_c^\infty(\Omega)$, so $\varphi = 0$ near $\partial\Omega^\pm \setminus M$, we can integrate by parts in Ω^- and Ω^+ to get

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} = \int_{\Omega^-} u^- \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x} + \int_{\Omega^+} u^+ \frac{\partial \varphi}{\partial x_i} \, d\boldsymbol{x}$$
$$= -\int_{\Omega^-} \varphi \frac{\partial u^-}{\partial x_i} \, d\boldsymbol{x} - \int_{\Omega^+} \varphi \frac{\partial u^+}{\partial x_i} \, d\boldsymbol{x} + \int_M \varphi \left(u^+ - u^-\right) n_i \, dS,$$

where $n_i(\boldsymbol{x})$ is the i-th component of the outward unit normal to Ω^+ at $\boldsymbol{x} \in M \cap \partial\Omega^+$. This implies that u belongs to $BV(\Omega)$, since we can take as weak derivatives $D_i u$ the signed measures $D_i u : \mathcal{B}(\Omega) \to \mathbb{R}$ defined by

$$D_i u(E) := \int_{E\cap\Omega^-} \frac{\partial u^-}{\partial x_i} \, d\boldsymbol{x} + \int_{E\cap\Omega^+} \frac{\partial u^+}{\partial x_i} \, d\boldsymbol{x} + \int_{E\cap M} \left(u^+ - u^-\right) n_i \, dS, \quad E \in \mathcal{B}(\Omega).$$

Observe that measure is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^N$ (restricted to Ω) if and only if $u^-(\boldsymbol{x}) n_i(\boldsymbol{x}) = u^+(\boldsymbol{x}) n_i(\boldsymbol{x})$ for all $\boldsymbol{x} \in M \cap \Omega$. Thus, if this condition is violated, it follows that u cannot be in $W^{1,p}(\Omega)$ for any p.

Example 35 *Consider the function $u : (-1,1) \to \mathbb{R}$ defined by*

$$u(x) := \begin{cases} 0 & x \in (-1,0), \\ 1 & x \in (0,1). \end{cases}$$

By adapting the previous example, we can show that $u \in BV(-1,1)$ with $Du = \delta_0$, a Dirac delta centered at 0. For $\varepsilon > 0$ consider the sequence of functions

$$u_\varepsilon(x) := \begin{cases} 0 & x \in (-1,-\varepsilon), \\ 1 + \frac{1}{2\varepsilon}(x-\varepsilon) & x \in [-\varepsilon,\varepsilon], \\ 1 & x \in (\varepsilon,1). \end{cases}$$

Then $u_\varepsilon \in W^{1,1}(-1,1)$ with

$$u_\varepsilon'(x) := \begin{cases} 0 & x \in (-1,-\varepsilon), \\ \frac{1}{2\varepsilon} & x \in (-\varepsilon,\varepsilon), \\ 0 & x \in (\varepsilon,1). \end{cases}$$

Note that

$$\int_{-1}^1 |u_\varepsilon - u| \; dx = \int_{-\varepsilon}^\varepsilon |u_\varepsilon - u| \; dx \le 4\varepsilon \to 0$$

as $\varepsilon \to 0^+$, while

$$\int_{-1}^1 |u_\varepsilon'| \; dx = \frac{1}{2\varepsilon} 2\varepsilon = 1.$$

Hence, the sequence $\{u_\varepsilon\}$ is bounded in $W^{1,1}(-1,1)$ and converges to $u \in BV(-1,1)$ in $L^1(-1,1)$. Note that for $p > 1$,

$$\int_{-1}^1 |u_\varepsilon'|^p \; dx = \frac{1}{(2\varepsilon)^p} 2\varepsilon = \frac{1}{(2\varepsilon)^{p-1}} \to \infty,$$

so the sequence is not bounded in $W^{1,p}(-1,1)$ for any $p > 1$.

Wednesday, September 25, 2013

4.4 Compactness in Sobolev and BV Spaces

Given a normed space X, the dual of X is the space X' of all functions $L : X \to \mathbb{R}$ linear and continuous. The space X' is a normed space with the norm

$$\|L\|_{X'} := \sup_{x \neq 0} \frac{|L(x)|}{\|x\|}.$$

We say that X is reflexive if it can be identified with its bidual $X'' := (X')'$.

Definition 36 *Given a normed space, we say that a sequence* $\{x_n\} \subseteq X$ converges weakly *to some* $x \in X$ *if* $L(x_n) \to L(x)$ *for every* $L \in X'$. *We write* $x_n \rightharpoonup x$ *in* X.

Remark 37 *It can be shown that if* $x_n \rightharpoonup x$, *then*

$$\|x\| \leq \liminf_{n \to \infty} \|x_n\|.$$

Theorem 38 *Let* X *be a reflexive space and let* $\{x_n\} \subseteq X$ *be a bounded sequence. Then there exist a subsequence* $\{x_{n_k}\}$ *and* $x \in X$ *such that* $x_{n_k} \rightharpoonup x$.

Definition 39 *Given a normed space, we say that a sequence* $\{L_n\} \subseteq X'$ converges weakly star *to some* $L \in X'$ *if* $L_n(x) \to L(x)$ *for every* $x \in X$. *We write* $L_n \overset{*}{\rightharpoonup} L$ *in* X'.

Remark 40 *It can be shown that if* $L_n \overset{*}{\rightharpoonup} L$ *in* X', *then*

$$\|L\|_{X'} \leq \liminf_{n \to \infty} \|L_n\|_{X'}.$$

Theorem 41 *Let* X *be a separable space and let* $\{L_n\} \subseteq X'$ *be a bounded sequence. Then there exist a subsequence* $\{L_{n_k}\}$ *and* $L \in X'$ *such that* $L_{n_k} \overset{*}{\rightharpoonup} L$ *in* X'..

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\{v_n\} \subset L^p(\Omega)$ be a bounded sequence, $1 \leq p \leq \infty$. For $1 < p < \infty$, the space $L^p(\Omega)$ is reflexive. Indeed, it can be shown that for every $L \in (L^p(\Omega))'$ there exists a unique function $g \in L^{p'}(\Omega)$, where $p' := \frac{p}{p-1}$, such that

$$L(f) = \int_\Omega fg \, d\boldsymbol{x}$$

for all $f \in L^p(\Omega)$ and $\|L\|_{(L^p(\Omega))'} = \|g\|_{L^{p'}(\Omega)}$. Conversely, given $g \in L^{p'}(\Omega)$, it follows by Holder's inequality that L defined as above is linear and continuous. Hence, by identifying L with g, we can identify $(L^p(\Omega))'$ with $L^{p'}(\Omega)$. In turn,

$$(L^p(\Omega))'' = \left((L^p(\Omega))'\right)' \simeq \left(L^{p'}(\Omega)\right)' \simeq L^p(\Omega),$$

since $(p')' = \frac{p'}{p'-1} = p$. It follows by Theorem 38 that there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a function $v \in L^p(\Omega)$ such that $v_{n_k} \rightharpoonup v$ in $L^p(\Omega)$, that is,

$$\lim_{k\to\infty} \int_\Omega v_{n_k} g \, d\boldsymbol{x} = \int_\Omega vg \, d\boldsymbol{x}$$

for all $g \in L^{p'}(\Omega)$, where $p' := \frac{p}{p-1}$.

For $p = \infty$, reasoning as above, the space $L^\infty(\Omega)$ can be identified with the dual of $L^1(\Omega)$, which is a separable space. Hence, there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a function $v \in L^\infty(\Omega)$ such that $v_{n_k} \overset{*}{\rightharpoonup} v$ in $L^\infty(\Omega)$, that is,

$$\lim_{k\to\infty} \int_\Omega v_{n_k} g \, d\boldsymbol{x} = \int_\Omega vg \, d\boldsymbol{x}$$

for all $g \in L^1(\Omega)$.

For $p = 1$ we have no compactness theorem. However, we can define the signed measures $\lambda_n : \mathcal{B}(\Omega) \to \mathbb{R}$ defined by

$$\lambda_n(E) := \int_E v_n \, d\boldsymbol{x}, \quad E \in \mathcal{B}(\Omega).$$

Then it can be shown that

$$\|\lambda_n\|_{(C_0(\Omega))'} = |\lambda_n|(\Omega) = \int_\Omega |v_n| \, d\boldsymbol{x} < C$$

and since $(C_0(\Omega))'$ is the dual of the space $C_0(\Omega)$, which is separable, by Theorem 41 there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a signed measure $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}$ such that $\lambda_{n_k} \overset{*}{\rightharpoonup} \lambda$ in $(C_0(\Omega))'$, that is,

$$\lim_{k\to\infty} \int_\Omega g v_{n_k} \, d\boldsymbol{x} = \lim_{k\to\infty} \int_\Omega g \, d\lambda_{n_k} = \int_\Omega g \, d\lambda$$

for all $g \in C_0(\Omega)$.

Theorem 42 *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let $\{u_n\} \subset W^{1,p}(\Omega)$ be a bounded sequence. If $1 < p < \infty$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in W^{1,p}(\Omega)$ such that $u_{n_k} \to u$ in $L^p(\Omega)$, $\frac{\partial u_{n_k}}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i}$ in $L^p(\Omega)$ for all i, and*

$$\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega \left| \frac{\partial u_{n_k}}{\partial x_i} \right|^p \, d\boldsymbol{x}.$$

If $p = \infty$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in W^{1,\infty}(\Omega)$ such that $u_{n_k} \to u$ in $L^\infty(\Omega)$, $\frac{\partial u_{n_k}}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i}$ in $L^\infty(\Omega)$ for all i, and

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\Omega)} \leq \liminf_{k\to\infty} \left\| \frac{\partial u_{n_k}}{\partial x_i} \right\|_{L^\infty(\Omega)}.$$

If $p = 1$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in BV(\Omega)$ such that $u_{n_k} \to u$ in $L^1(\Omega)$, $D_i u_{n_k} \overset{*}{\rightharpoonup} D_i u$ in $(C_0(\Omega))'$ and

$$|D_i u|(\Omega) \leq \liminf_{k\to\infty} \int_\Omega \left|\frac{\partial u_{n_k}}{\partial x_i}\right|^p \, d\boldsymbol{x}.$$

Corollary 43 *Let $\Omega \subseteq \mathbb{R}^N$ and let $\{u_n\} \subset W^{1,p}(\Omega)$ be a bounded sequence. If $1 < p < \infty$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in W^{1,p}(\Omega)$ such that $u_{n_k} \to u$ in $L^p_{\text{loc}}(\Omega)$ and*

$$\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p \, d\boldsymbol{x} \leq \liminf_{k\to\infty} \int_\Omega \left|\frac{\partial u_{n_k}}{\partial x_i}\right|^p \, d\boldsymbol{x}.$$

If $p = \infty$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in W^{1,\infty}(\Omega)$ such that $u_{n_k} \to u$ in $L^\infty_{\text{loc}}(\Omega)$, $\frac{\partial u_{n_k}}{\partial x_i} \overset{*}{\rightharpoonup} \frac{\partial u}{\partial x_i}$ in $L^\infty(\Omega)$ for all i, and

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{L^\infty(\Omega)} \leq \liminf_{k\to\infty} \left\|\frac{\partial u_{n_k}}{\partial x_i}\right\|_{L^\infty(\Omega)}.$$

If $p = 1$ there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in BV(\Omega)$ such that $u_{n_k} \to u$ in $L^1_{\text{loc}}(\Omega)$, $D_i u_{n_k} \overset{*}{\rightharpoonup} D_i u$ in $(C_0(\Omega))'$ and

$$|D_i u|(\Omega) \leq \liminf_{k\to\infty} \int_\Omega \left|\frac{\partial u_{n_k}}{\partial x_i}\right|^p \, d\boldsymbol{x}.$$

A similar theorem holds in BV.

Theorem 44 *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let $\{u_n\} \subset BV(\Omega)$ be a bounded sequence. Thenexist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in BV(\Omega)$ such that $u_{n_k} \to u$ in $L^1(\Omega)$, $D_i u_{n_k} \overset{*}{\rightharpoonup} D_i u$ in $(C_0(\Omega))'$ and*

$$|D_i u|(\Omega) \leq \liminf_{k\to\infty} |D_i u_{n_k}|(\Omega).$$

Corollary 45 *Let $\Omega \subseteq \mathbb{R}^N$ and let $\{u_n\} \subset BV(\Omega)$ be a bounded sequence. Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in BV(\Omega)$ such that $u_{n_k} \to u$ in $L^1_{\text{loc}}(\Omega)$, $D_i u_{n_k} \overset{*}{\rightharpoonup} D_i u$ in $(C_0(\Omega))'$ and*

$$|D_i u|(\Omega) \leq \liminf_{k\to\infty} |D_i u_{n_k}|(\Omega).$$

Friday, September 27, 2013

4.5 Conservation Laws, Continued

We are now ready to conclude the sketch of the proof of Volpert's Theorem.

Proof of Theorem 27, continued. Integrating (36) and (37) in time in $[0,T]$ we get

$$\int_0^T\int_{\mathbb{R}}|u_x^\varepsilon(x,t)|\ dxdt\le T\int_{\mathbb{R}}|g'(x)|\ dx,$$
$$\int_0^T\int_{\mathbb{R}}|u_t^\varepsilon(x,t)|\ dxdt\le T\int_{\mathbb{R}}|f'(g(x))\,g'(x)|\ dx+\varepsilon T\int_{\mathbb{R}}|g''(x)|\ dx.$$

[3]Moreover, by the fundamental theorem of calculus for $0\le t\le T$ and $x\in\mathbb{R}$,

$$|u^\varepsilon(x,t)-g(x)|\le\int_0^t|u_t^\varepsilon(x,s)|\ ds.$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}}|u^\varepsilon(x,t)-g(x)|\ dx&\le\int_0^T\int_{\mathbb{R}}|u_t^\varepsilon(x,t)|\ dxdt\\&\le T\int_{\mathbb{R}}|f'(g(x))\,g'(x)|\ dx+\varepsilon T\int_{\mathbb{R}}|g''(x)|\ dx.\end{aligned}$$

In turn,

$$\int_0^T\int_{\mathbb{R}}|u^\varepsilon(x,t)-g(x)|\ dxdt\le T^2\int_{\mathbb{R}}|f'(g(x))\,g'(x)|\ dx+\varepsilon T^2\int_{\mathbb{R}}|g''(x)|\ dx.$$

This shows that the sequence $\{u^\varepsilon\}$ is bounded in $W^{1,1}(\mathbb{R}\times(0,T))$ for every $T>0$. Hence, there exists a subsequence $\varepsilon_{n,1}\to0$ and a function $u_1\in BV(\mathbb{R}\times(0,1))$ such that $u^{\varepsilon_{n,1}}\to u_1$ in $L^1_{\text{loc}}(\mathbb{R}\times(0,1))$. Since the sequence $\{u^{\varepsilon_{n,1}}\}$ is bounded in $W^{1,1}(\mathbb{R}\times(0,2))$ we can find a subsequence $\{\varepsilon_{n,2}\}$ of $\{\varepsilon_{n,1}\}$ and a function $u_2\in BV(\mathbb{R}\times(0,2))$ such that $u^{\varepsilon_{n,2}}\to u_2$ in $L^1_{\text{loc}}(\mathbb{R}\times(0,2))$. But since $\{\varepsilon_{n,2}\}\subset\{\varepsilon_{n,2}\}$ and $u^{\varepsilon_{n,1}}\to u_1$ in $L^1(\mathbb{R}\times(0,1))$, it follows that $u^{\varepsilon_{n,2}}\to u_1$ in $L^1_{\text{loc}}(\mathbb{R}\times(0,1))$. By the uniqueness of limits, we have that $u_1=u_2$ for $\mathcal{L}^2$ a.e. $(x,t)\in\mathbb{R}\times(0,1)$. By induction, for every k we can find a subsequence $\{\varepsilon_{n,k}\}$ of $\{\varepsilon_{n,k-1}\}$ and a function $u_k\in BV(\mathbb{R}\times(0,k))$ such that $u^{\varepsilon_{n,k}}\to u_k$ in $L^1_{\text{loc}}(\mathbb{R}\times(0,k))$. As before, we have that $u_k=u_{k-1}$ for $\mathcal{L}^2$ a.e. $(x,t)\in\mathbb{R}\times(0,k-1)$. Hence, we can uniquely define a function $u\in L^1_{\text{loc}}(\mathbb{R}\times(0,\infty))$ such that the diagonal sequence $u^{\varepsilon_{k,k}}\to u$ in $L^1_{\text{loc}}(\mathbb{R}\times(0,\infty))$. Moreover, $u\in BV(\mathbb{R}\times(0,k))$ for every k. Let $T>0$. Then

$$|D_xu|(\mathbb{R}\times(0,T))\le\liminf_{k\to\infty}\int_0^T\int_{\mathbb{R}}\left|u_x^{\varepsilon_{k,k}}\right|\ dxdt\le T\int_{\mathbb{R}}|g'|\ dx,\tag{40}$$
$$|D_tu|(\mathbb{R}\times(0,T))\le\liminf_{k\to\infty}\int_0^T\int_{\mathbb{R}}\left|u_t^{\varepsilon_{k,k}}\right|\ dxdt\le T\sup|f'|\int_{\mathbb{R}}|g'|\ dx+0,\tag{41}$$

$$\int_0^T\int_{\mathbb{R}}|u(x,t)-g(x)|\ dxdt\le T^2\sup|f'|\int_{\mathbb{R}}|g'(x)|\ dx+0.\tag{42}$$

[3]Note that without loss of generality we can take $f(0)=0$ and so taking $g_2=0$, we get an estimate for u^ε from Step 2.

It remains to show that u is a weak solution. Multiply the equation in (35) by a function $\varphi \in C_c^1(\mathbb{R} \times [0,\infty))$, integrate over $\mathbb{R} \times \mathbb{R}_+$. By integrating by parts we get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty \varphi u_t^{\varepsilon_{k,k}} \; dtdx + \int_0^\infty \int_{\mathbb{R}} \varphi \left(f \circ u^{\varepsilon_{k,k}}\right)_x \; dxdt - \varepsilon_{k,k} \int_0^\infty \int_{\mathbb{R}} \varphi u_{xx}^{\varepsilon_{k,k}} \; dxdt \\
&= - \int_{\mathbb{R}} \int_0^\infty \varphi_t u^{\varepsilon_{k,k}} \; dtdx - \int_{\mathbb{R}} \varphi(x,0) g(x) \; dx - \int_0^\infty \int_{\mathbb{R}} \varphi_x f \circ u^{\varepsilon_{k,k}} \; dxdt \\
&\quad - \varepsilon_{k,k} \int_0^\infty \int_{\mathbb{R}} \varphi_{xx} u^{\varepsilon_{k,k}} \; dxdt.
\end{aligned}$$

Then,

$$\begin{aligned}
\left| \int_{\mathbb{R}} \int_0^\infty \varphi_t \left(u^{\varepsilon_{k,k}} - u\right) \; dtdx \right| &\leq \sup |\varphi_t| \iint_{\operatorname{supp} \varphi_t} \left|u^{\varepsilon_{k,k}} - u\right| \; dtdx \to 0, \\
\left| \int_0^\infty \int_{\mathbb{R}} \varphi_x \left(f \circ u^{\varepsilon_{k,k}} - f \circ u\right) \; dxdt \right| &\leq \sup |\varphi_x| \iint_{\operatorname{supp} \varphi_x} \left|f \circ u^{\varepsilon_{k,k}} - f \circ u\right| \; dtdx \\
&\leq \sup |\varphi_x| \sup |f'| \iint_{\operatorname{supp} \varphi_s} \left|u^{\varepsilon_{k,k}} - u\right| \; dtdx \to 0,
\end{aligned}$$

while

$$\begin{aligned}
\left| \varepsilon_{k,k} \int_0^\infty \int_{\mathbb{R}} \varphi_{xx} u^{\varepsilon_{k,k}} \; dxdt \right| &\leq \varepsilon_{k,k} \sup |\varphi_{xx}| \iint_{\operatorname{supp} \varphi_{xx}} \left|u^{\varepsilon_{k,k}}\right| \; dtdx \\
&\to 0 \times \iint_{\operatorname{supp} \varphi_{xx}} |u| \; dtdx = 0.
\end{aligned}$$

This shows that u is a weak solution.
Step 2: Given g_1, $g_2 \in C_c^\infty\left(\mathbb{R}^N\right)$, let u^ε and v^ε be solutions of

$$\begin{cases} u_t^\varepsilon + f'(u^\varepsilon) u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, & x \in \mathbb{R},\, t > 0, \\ u^\varepsilon(x,0) = g_1(x), & x \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} v_t^\varepsilon + f'(v^\varepsilon) v_x^\varepsilon = \varepsilon v_{xx}^\varepsilon, & x \in \mathbb{R},\, t > 0, \\ v^\varepsilon(x,0) = g_2(x), & x \in \mathbb{R}. \end{cases}$$

Let $z := u^\varepsilon - v^\varepsilon$. Then

$$\begin{cases} z_t + f'(u^\varepsilon) u_x^\varepsilon - f'(v^\varepsilon) v_x^\varepsilon = \varepsilon z_{xx}, & x \in \mathbb{R},\, t > 0, \\ z(x,0) = g_1(x) - g_2(x), & x \in \mathbb{R}. \end{cases}$$

Multiply the equation by $\varphi_\delta \circ z$ and integrate to get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ z) z_t \; dtdx + \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ z) \left[\left(f(u^\varepsilon) - f(v^\varepsilon)\right)_x\right] \; dxdt \\
&= \varepsilon \int_{\mathbb{R}} \int_0^T (\varphi_\delta \circ z) z_{xx} \; dxdt.
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_0^T (\varphi_\delta \circ z)\left[(f(u^\varepsilon) - f(v^\varepsilon))_x\right]\ dxdt\right| \\
&= \left|\int_{\mathbb{R}}\int_0^T (\varphi_\delta' \circ z)\, z_x \left(\frac{f(u^\varepsilon) - f(v^\varepsilon)}{u^\varepsilon - v^\varepsilon}\right) z\ dxdt\right| \\
&\le \frac{C\,\mathrm{Lip}\, f}{\delta}\iint_{\{|z|\le\delta\}} |z_x|\,|z|\ dxdt \le C\,\mathrm{Lip}\, f \iint_{\{|z|\le\delta\}} |z_x|\ dxdt \to 0.
\end{aligned}$$

Hence, reasoning as in Step 1, we get

$$\int_{\mathbb{R}} |u^\varepsilon(x,t) - v^\varepsilon(x,t)|\ dx \le \int_{\mathbb{R}} |g_1(x) - g_2(x)|\ dx$$

for all $t > 0$. In turn,

$$\int_0^T \int_{\mathbb{R}} |u^\varepsilon(x,t) - v^\varepsilon(x,t)|\ dxdt \le T\int_{\mathbb{R}} |g_1(x) - g_2(x)|\ dx$$

for all $T > 0$. Letting $\varepsilon \to 0$ and considering the corresponding vanishing viscosity solutions u and v, we get

$$\int_0^T \int_{\mathbb{R}} |u(x,t) - v(x,t)|\ dxdt \le T\int_{\mathbb{R}} |g_1(x) - g_2(x)|\ dx \tag{43}$$

for all $T > 0$.
Step 3: Given $g \in L^\infty(\mathbb{R})$, using mollifiers we can construct a sequence $\{g_n\} \subset C_c^\infty(\mathbb{R})$ such that $g_n \to g$ in $L^1(\mathbb{R})$ (we will see this later). Let u_n be the vanishing viscosity solution corresponding to g_n. Then by (43),

$$\int_0^T \int_{\mathbb{R}} |u_n(x,t) - u_m(x,t)|\ dxdt \le T\int_{\mathbb{R}} |g_n(x) - g_m(x)|\ dx \to 0$$

as $n, m \to \infty$. Hence, $\{u_n\}$ is a Cauchy sequence in $L^1_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$ and thus converge in $L^1_{\mathrm{loc}}(\mathbb{R}\times(0,\infty))$ to a function u, which will be a weak solution of (27).
Step 4: Given $f \in C^2(\mathbb{R})$, we should approximate f with a sequence $\{f_n\} \subset C_c^\infty(\mathbb{R})$. I will skip this part. ■

Remark 46 *If $g \in BV(\mathbb{R})$ then we will see later we can we can construct a sequence $\{g_n\} \subset C_c^\infty(\mathbb{R})$ such that $g_n \to g$ in $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} |g_n'(x)|\ dx \to |Dg|(\mathbb{R}).$$

In particular,

$$\int_{\mathbb{R}} |g_n'(x)|\ dx \le C$$

for all n. Hence, in Step 3, using (40)–(42), we get

$$|D_x u_n|(\mathbb{R}\times(0,T)) \le T\int_{\mathbb{R}} |g_n'(x)|\ dx \le CT,$$
$$|D_t u_n|(\mathbb{R}\times(0,T)) \le T\sup|f'|\int_{\mathbb{R}} |g_n'(x)|\ dx \le T\sup|f'|\, C.$$

Hence, u will belong to $BV(\mathbb{R}\times(0,T))$ for every $T>0$. Moreover, by the semicontinuity of the norms,

$$\begin{aligned}|D_x u|(\mathbb{R}\times(0,T)) &\le \liminf_{n\to\infty} |D_x u_n|(\mathbb{R}\times(0,T))\\ &\le T\liminf_{n\to\infty}\int_{\mathbb{R}} |g_n'(x)|\ dx = T\,|Dg|(\mathbb{R}),\\ |D_t u|(\mathbb{R}\times(0,T)) &\le \liminf_{n\to\infty} |D_t u_n|(\mathbb{R}\times(0,T))\\ &\le T\sup|f'|\int_{\mathbb{R}} |g_n'(x)|\ dx \le T(\sup|f'|)\,|Dg|(\mathbb{R}).\end{aligned}$$

Monday, September 30, 2013

PIRE workshop, no class

Wednesday, October 2, 2013

PIRE workshop, no class

Friday, October 4, 2013

PIRE workshop, no class

Monday, October 7, 2013

5 First Order Systems

The notion of non-characteristic initial data can be extended to systems of partial differential equations. Consider the boundary value problem

$$\begin{cases} \boldsymbol{F}(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x}),\nabla\boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{0} & \text{in } \Omega,\\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma,\end{cases} \tag{44}$$

where $\Omega\subseteq\mathbb{R}^N$ is an open set of class C^1, $\Gamma\subseteq\partial\Omega$, $\boldsymbol{u}:\Omega\to\mathbb{R}^d$, $\boldsymbol{F}:\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}\to\mathbb{R}^d$ is C^1 and $\boldsymbol{g}:\mathbb{R}^N\to\mathbb{R}^d$ is of class C^1. Here $\nabla\boldsymbol{u}(\boldsymbol{x})$ is the $d\times N$ matrix given by

$$\nabla\boldsymbol{u}(\boldsymbol{x}) = \begin{pmatrix}\nabla u_1(\boldsymbol{x})\\ \vdots\\ \nabla u_d(\boldsymbol{x})\end{pmatrix}.$$

Given a $d\times N$ matrix $\boldsymbol{P}$, we write

$$\boldsymbol{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,N}\\ \vdots & \vdots & \vdots\\ p_{d,1} & \cdots & p_{d,N}\end{pmatrix} = \begin{pmatrix}\boldsymbol{p}^{(1)}\\ \vdots\\ \boldsymbol{p}^{(d)}\end{pmatrix} = \begin{pmatrix}\boldsymbol{p}_{(1)} & \cdots & \boldsymbol{p}_{(N)}\end{pmatrix}.$$

Definition 47 *A triple* $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{P}) \in \Gamma \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ *is called*

- admissible *if*

$$\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{P}) = \boldsymbol{0}, \quad \boldsymbol{p}^{(k)} \cdot \boldsymbol{t} = \nabla g_k(\boldsymbol{x}) \cdot \boldsymbol{t}$$

  for all $\boldsymbol{t} \in T_{\boldsymbol{x}}\partial\Omega$ *and all* $i = k, \ldots, d$*,*

- non-characteristic *if*

$$\det\left(\left(\nabla_{\boldsymbol{p}^{(k)}} F_j(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{P}) \cdot \boldsymbol{n}\right)_{k,j}\right) \neq 0, \tag{45}$$

  where $\boldsymbol{n}$ *is a normal unit vector to* $\partial\Omega$ *at* $\boldsymbol{x}$*.*

In the case of a quasilinear first order partial differential system

$$\begin{cases} \sum_{i=1}^{N} \boldsymbol{B}_i(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) \frac{\partial \boldsymbol{u}}{\partial x_i}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{0} & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \end{cases} \tag{46}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set, $\Gamma \subseteq \partial\Omega$, $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, $\boldsymbol{B}_i : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $\boldsymbol{C} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, we have that

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{P}) = \sum_{i=1}^{N} \boldsymbol{B}_i(\boldsymbol{x}, \boldsymbol{z})\, \boldsymbol{p}_{(i)} + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{z}),$$

and so (45) reduces to

$$\det\left(\sum_{i=1}^{N} \boldsymbol{B}_i(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x}))\, n_i(\boldsymbol{x})\right) \neq 0. \tag{47}$$

Note that this condition does not depend on $\boldsymbol{P}$. Hence, we say that $\boldsymbol{x} \in \Gamma$ is *non-characteristic* for (46) if (47) holds. We say that Γ is *non-characteristic* for (46) if every point of Γ is non-characteristic.

The first striking difference with the case $d = 1$ is that in general (46) may not have any local solutions, even if Γ is non-characteristic.

Definition 48 *Given an open set* $\Omega \subseteq \mathbb{R}^N$ *and a function* $f : \Omega \to \mathbb{R}$ *of class* C^∞*, we say that* f *is (real)* analytic *at* $\boldsymbol{x}_0 \in \Omega$ *if there exists* $r > 0$ *such that*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{a}} \frac{1}{\boldsymbol{a}!} \frac{\partial^{\boldsymbol{a}} f}{\partial \boldsymbol{x}^{\boldsymbol{a}}}(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0)^{\boldsymbol{a}}$$

for all $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$*. We say that* f *is* analytic *if it is analytic at every* $\boldsymbol{x} \in \Omega$*.*

Example 49 *The function*

$$f(\boldsymbol{x}) := \begin{cases} \exp\left(\frac{1}{\|\boldsymbol{x}\|^2 - 1}\right) & \text{if } \|\boldsymbol{x}\| < 1, \\ 0 & \text{if } \|\boldsymbol{x}\| \geq 1, \end{cases}$$

is of class C^∞ *but not analytic at points* $\|\boldsymbol{x}\| = 1$*. Indeed, at those points one can show that* $\frac{\partial^{\boldsymbol{a}} f}{\partial \boldsymbol{x}^{\boldsymbol{a}}}(\boldsymbol{x}) = 0$ *for all* $\boldsymbol{a}$ *but* f *is not zero in a neighborhood of* $\boldsymbol{x}$*.*

Given a open set $\Omega \subseteq \mathbb{R}^N$, we say that its boundary $\partial\Omega$ is *analytic at* $\boldsymbol{x}_0 \in \partial\Omega$ if the function f given in Definition 13 is analytic at $\boldsymbol{x}_0$. We say that $\partial\Omega$ is *analytic* if it is analytic at every $\boldsymbol{x} \in \partial\Omega$.

Example 50 *Consider the system of partial differential equations*

$$\begin{cases} u_x(x,y) - v_y(x,y) = 0, \\ u_y(x,y) + v_x(x,y) = 0. \end{cases} \tag{48}$$

Let's prove that every curve in $\mathbb{R}^2$ is non-characteristic. Rewrite (48) as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then for $\boldsymbol{n} = (n_1, n_2) \neq \boldsymbol{0}$, (47) becomes

$$\begin{aligned} \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} n_1 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n_2 \right) &= \det \begin{pmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{pmatrix} \\ &= n_1^2 + n_2^2 \neq 1. \end{aligned}$$

Let now $\Omega = \mathbb{R}^2_+ := \left\{(x,y) \in \mathbb{R}^2 : y > 0\right\}$ and consider the boundary value problem

$$\begin{cases} u_x(x,y) - v_y(x,y) = 0, \\ u_y(x,y) + v_x(x,y) = 0, \\ u(x,0) = g(x), \quad v(x,0) = 0. \end{cases} \tag{49}$$

Define the complex function $\boldsymbol{w}(x,y) := u(x,y) + \boldsymbol{i}v(x,y)$. Then in view of (48), $\boldsymbol{w}$ satisfies the Cauchy-Riemann equation, and so it is holomorphic in $\mathbb{R}^2_+$. Since $\boldsymbol{w}(x,0) = g(x)$, which is real-valued, it follows by the reflection principle that the function

$$\boldsymbol{h}(x,y) := \begin{cases} \boldsymbol{w}(x,y) & \text{if } x \in \mathbb{R} \text{ and } y \geq 0, \\ \boldsymbol{w}(x,-y) & \text{if } x \in \mathbb{R} \text{ and } y < 0 \end{cases}$$

is holomorphic in $\mathbb{R}^2$. In particular, $\boldsymbol{w}(x,0) = g(x)$ is analytic. Hence, if g is a C^∞ function, which is not analytic, the boundary value problem (49) has no solution.

A more striking result is the following.

Theorem 51 (H. Lewy, 1957) *There exists a function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{C}$ such that the PDE*

$$-\boldsymbol{u}_x(x,y,t) - \boldsymbol{i}\boldsymbol{u}_y(x,y,t) + 2\boldsymbol{i}(x + y\boldsymbol{i})\boldsymbol{u}_t(x,y,t) = \boldsymbol{f}(x,y,t)$$

has no solution $\boldsymbol{u}$ of class C^2 on any *open set $\Omega \subseteq \mathbb{R}^3$.*

Remark 52 *Writing* $\boldsymbol{u}(x,y,t) = u_1(x,y,t) + u_2(x,y,t)\,\boldsymbol{i}$*, where* u_1 *and* u_2 *are real-valued, we have that*

$$\begin{aligned}&\boldsymbol{u}_x(x,y,t) + \boldsymbol{i}\boldsymbol{u}_y(x,y,t) + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{u}_t(x,y,t)\\&= \frac{\partial u_1}{\partial x}(x,y,t) - \frac{\partial u_2}{\partial y}(x,y,t) - 2x\frac{\partial u_2}{\partial t}(x,y,t) - 2y\frac{\partial u_1}{\partial t}(x,y,t)\\&\quad + \left(\frac{\partial u_2}{\partial x}(x,y,t) + \frac{\partial u_1}{\partial y}(x,y,t) + 2x\frac{\partial u_1}{\partial t}(x,y,t) + 2y\frac{\partial u_2}{\partial t}(x,y,t)\right)\boldsymbol{i}.\end{aligned}$$

Hence, the PDE is equivalent to the system of PDEs

$$\begin{cases}\frac{\partial u_1}{\partial x}(x,y,t) - \frac{\partial u_2}{\partial y}(x,y,t) - 2x\frac{\partial u_2}{\partial t}(x,y,t) - 2y\frac{\partial u_1}{\partial t}(x,y,t) = f_1(x,y,t),\\ \frac{\partial u_2}{\partial x}(x,y,t) + \frac{\partial u_1}{\partial y}(x,y,t) + 2x\frac{\partial u_1}{\partial t}(x,y,t) + 2y\frac{\partial u_2}{\partial t}(x,y,t) = f_2(x,y,t).\end{cases}$$

The problem here is that $\boldsymbol{f}$ is C^∞. Next we show that if $\boldsymbol{f}$ is analytic, then for non-characteristic Γ, there exists a local solution. More generally, we have the following result.

Theorem 53 (Cauchy–Kowalevski) *Consider the boundary value problem* (44)*, where* $\Omega \subseteq \mathbb{R}^N$ *is an open set of class* C^∞*,* $\Gamma \subseteq \partial\Omega$*,* $\boldsymbol{F} : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to \mathbb{R}^d$ *is* C^∞ *and* $\boldsymbol{g} : \mathbb{R}^N \to \mathbb{R}^d$ *is of class* C^∞*. Let* $\boldsymbol{x}_0 \in \Gamma$ *and assume that the triple* $(\boldsymbol{x}_0, \boldsymbol{g}(\boldsymbol{x}_0), \boldsymbol{P}_0)$ *is non-characteristic, with* $\partial\Omega$ *and* $\boldsymbol{g}$ *analytic at* $\boldsymbol{x}_0$*, and* $\boldsymbol{F}$ *is analytic at* $(\boldsymbol{x}_0, \boldsymbol{g}(\boldsymbol{x}_0), \boldsymbol{P}_0)$*. Then there exists a neighborhood* V *of* $\boldsymbol{x}_0$ *and a function* $\boldsymbol{u} \in C^\infty(V;\mathbb{R}^d)$*, analytic at* $\boldsymbol{x}_0$*, such that*

$$\begin{cases}\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla\boldsymbol{u}(\boldsymbol{x})) = \boldsymbol{0} & \text{in } \Omega\cap V,\\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma\cap V.\end{cases} \tag{50}$$

Moreover, $\boldsymbol{u}$ *is the unique solution of* (50) *which is analytic at* $\boldsymbol{x}_0$*.*

The proof may be found in Evans' book.

The part below was not done in class: please read

Proof of Lewy's Theorem. Step 1: Let $t_0 \in \mathbb{R}$, $\delta > 0$, let

$$\Omega := \left\{(x,y,t) \in \mathbb{R}^3 : x^2 + y^2 < \delta,\ |t - t_0| < \delta\right\}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^∞. We will prove that if $\boldsymbol{u} : \Omega \to \mathbb{C}$ is a solution of class C^1 of the PDE

$$-\boldsymbol{u}_x(x,y,t) - \boldsymbol{i}\boldsymbol{u}_y(x,y,t) + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{u}_t(x,y,t) = g'(t), \tag{51}$$

then g is analytic at t_0. Hence, if we take g to be a C^∞ function, which is not analytic at t_0, then the PDE cannot have a classical solution in Ω.

To prove the claim, write $\boldsymbol{z} := x + y\boldsymbol{i} = re^{\boldsymbol{i}\theta}$ and consider the function

$$\mathbf{v}(r,t) := \int_{\|\mathbf{z}\|=r} \boldsymbol{u}(x,y,t)\,d\boldsymbol{z} = \int_0^{2\pi} \boldsymbol{u}(r\cos\theta, r\sin\theta, t)\,r\boldsymbol{i}e^{\boldsymbol{i}\theta}d\theta,$$

where $0 < r < \delta$ and $|t - t_0| < \delta$. By the Gauss–Green theorem[4]

$$\begin{aligned}\mathbf{v}(r,t) &= \int_{\|\mathbf{z}\|=r} \boldsymbol{u}(x,y,t)\, d\boldsymbol{z} = \boldsymbol{i} \iint_{\|\mathbf{z}\|<r} \left(\boldsymbol{u}_x(x,y,t) + \boldsymbol{i}\boldsymbol{u}_y(x,y,t)\right) dxdy \\ &= \boldsymbol{i}\int_0^{2\pi}\int_0^r \left(\boldsymbol{u}_x(\rho\cos\theta,\rho\sin\theta,t) + \boldsymbol{i}\boldsymbol{u}_y(\rho\cos\theta,\rho\sin\theta,t)\right)\rho d\rho d\theta.\end{aligned}$$

Hence, using the PDE,

$$\begin{aligned}\mathbf{v}_r(r,t) &= \boldsymbol{i}\int_0^{2\pi} \left(\boldsymbol{u}_x(r\cos\theta,r\sin\theta,t) + \boldsymbol{i}\boldsymbol{u}_y(r\cos\theta,r\sin\theta,t)\right) r d\theta \\ &= \int_{\|\mathbf{z}\|=r} \left(\boldsymbol{u}_x(x,y,t) + \boldsymbol{i}\boldsymbol{u}_y(x,y,t)\right) r \frac{d\boldsymbol{z}}{\boldsymbol{z}} \\ &= \int_{\|\mathbf{z}\|=r} \left(2\boldsymbol{i}\boldsymbol{z}\boldsymbol{u}_t(x,y,t) - g'(t)\right) r \frac{d\boldsymbol{z}}{\boldsymbol{z}} \\ &= 2r\boldsymbol{i}\int_{\|\mathbf{z}\|=r} \boldsymbol{u}_t(x,y,t)\, d\boldsymbol{z} - g'(t)\, r \int_{\|\mathbf{z}\|=r} \frac{d\boldsymbol{z}}{\boldsymbol{z}} \\ &= 2r\boldsymbol{i}\mathbf{v}_t(r,t) - g'(t)\, r 2\pi \boldsymbol{i}\end{aligned}$$

Let $s = r^2$ and define $\boldsymbol{w}(s,t) := \mathbf{v}(\sqrt{s},t)$. Then by the chain rule

$$\begin{aligned}\boldsymbol{w}_s(s,t) &= \mathbf{v}_r\left(\sqrt{s},t\right)\frac{1}{2\sqrt{s}} = \left(2\sqrt{s}\boldsymbol{i}\mathbf{v}_t\left(\sqrt{s},t\right) - g'(t)\sqrt{s}2\pi\boldsymbol{i}\right)\frac{1}{2\sqrt{s}} \\ &= \boldsymbol{i}\mathbf{w}_t(s,t) - \pi g'(t)\,\boldsymbol{i} = \boldsymbol{i}\left(\mathbf{w}(s,t) - \pi g(t)\right)_t.\end{aligned}$$

Hence, if we consider the function $\boldsymbol{h}(s,t) := \boldsymbol{w}(s,t) - \pi g(t)$, we have that

$$\boldsymbol{h}_s(s,t) = \boldsymbol{i}\boldsymbol{h}_s(s,t),$$

which shows that $\boldsymbol{h}$ satisfies the Cauchy–Riemann equations. Thus, $\boldsymbol{h}$ is holomorphic as a function of $t + s\boldsymbol{i}$ in the region $0 < s < \delta^2$ and $|t - t_0| < \delta$. Moreover, $\boldsymbol{h}$ is continuous at $s = 0$. Since $\mathbf{w}(0,t) = 0$, we have that $\boldsymbol{h}(0,t) = -\pi g(t)$, which is real-valued. It follows by the reflection principle that the function

$$\boldsymbol{h}(s,t) := \begin{cases} \boldsymbol{h}(s,t) & \text{if } 0 \le s < \delta^2,\ |t-t_0| < \delta, \\ \boldsymbol{h}(-s,t) & \text{if } -\delta^2 < s \le 0,\ |t-t_0| < \delta \end{cases}$$

is holomorphic near $(0, t_0)$. In turn, $\boldsymbol{h}(0,t) = -\pi g(t)$ is analytic near near t_0. This proves the claim.

[4] *Gauss-Green formulas*

$$\int_U \frac{\partial f}{\partial x}(x,y)\, dxdy = \int_{\partial U} f\, dy. \tag{52}$$

$$\int_U \frac{\partial g}{\partial y}(x,y)\, dxdy = -\int_{\partial U} g\, dx. \tag{53}$$

Step 2: Let $(x_0, y_0, t_0) \in \mathbb{R}^3$ and let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^∞. We will prove that if $\boldsymbol{u} : \Omega \to \mathbb{C}$ is a solution of class C^1 of the PDE

$$-\boldsymbol{u}_x(x,y,t) - \boldsymbol{i}\boldsymbol{u}_y(x,y,t) + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{u}_t(x,y,t) = g'(t - 2y_0x + 2x_0y)$$

in a neighborhood of (x_0, y_0, t_0), then g is analytic at t_0. Let

$$\boldsymbol{w}(x,y,t) := \boldsymbol{u}(x + x_0, y + y_0, t + 2y_0x - 2x_0y).$$

Then $\boldsymbol{w}$ is of class C^1 in a neighborhood of $(0, 0, t_0)$ and satisfies

$$\begin{aligned}
-\boldsymbol{w}_x - \boldsymbol{i}\boldsymbol{w}_y + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{w}_t &= -\boldsymbol{u}_x - 2y_0\boldsymbol{u}_t - \boldsymbol{i}\boldsymbol{u}_y + 2x_0\boldsymbol{i}\boldsymbol{u}_t + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{u}_t \\
&= -\boldsymbol{u}_x - \boldsymbol{i}\boldsymbol{u}_y + 2\boldsymbol{i}(x + x_0 + (y+y_0)\,\boldsymbol{i})\boldsymbol{u}_t \\
&= g'(t).
\end{aligned}$$

Hence, by Step 1, g is analytic at t_0.

Step 3: Consider the Banach space ℓ_∞ of all real-valued bounded sequences $\boldsymbol{s} := \{s_n\}$ with the norm

$$\|\boldsymbol{s}\|_\infty := \sup_n |s_n|.$$

Let $\{(x_n, y_n, t_n)\}$ be dense in $\mathbb{R}^3$, let $g : \mathbb{R} \to \mathbb{R}$ be a periodic function of class C^∞, and for every $\boldsymbol{s} \in \ell_\infty$ define the function

$$f_{\boldsymbol{s}}(x,y,z) = \sum_{n=1}^{\infty} s_n \varepsilon_n g'(t - 2y_nx + 2x_ny),$$

where

$$\varepsilon_n := \frac{1}{2^n} e^{-|x_n|-|y_n|}.$$

We claim that $f_{\boldsymbol{s}}$ is of class $C^\infty(\mathbb{R}^3)$. Since g is periodic, for every $k \in \mathbb{N}$,

$$M_k := \max_{t\in\mathbb{R}} \left| g^{(k)}(t) \right| < \infty,$$

and given a multi-index $\boldsymbol{\alpha} \in (\mathbb{N}_0)^3$,

$$\left|\partial^{\boldsymbol{\alpha}}(s_n\varepsilon_n g'(t - 2y_nx + 2x_ny))\right| \le \frac{1}{2^n} e^{-|x_n|-|y_n|} (|x_n| + |y_n|)^{|\boldsymbol{\alpha}|} 2^{|\boldsymbol{\alpha}|} M_{|\boldsymbol{\alpha}|+1} \|\boldsymbol{s}\|_\infty,$$

and so

$$\sum_{n=1}^{\infty} \left|\partial^{\boldsymbol{\alpha}}(s_n\varepsilon_n g'(t - 2y_nx + 2x_ny))\right| \le 2^{|\boldsymbol{\alpha}|} M_{|\boldsymbol{\alpha}|+1} \|\boldsymbol{s}\|_\infty \sum_{n=1}^{\infty} \frac{1}{2^n} e^{-|x_n|-|y_n|} (|x_n| + |y_n|)^{|\boldsymbol{\alpha}|}.$$

It follows that the series is uniformly convergent. In turn, $f_{\boldsymbol{s}}$ is of class $C^\infty(\mathbb{R}^3)$.

Step 4: Let $g : \mathbb{R} \to \mathbb{R}$ be a periodic function of class C^∞, which is nowhere analytic. We now use Baire's category theorem. Let $C_{n,k}$ be the set of all $\boldsymbol{s} \in \ell_\infty$ for which there exists a solution $\boldsymbol{u}^{\boldsymbol{s}}$ of class C^1 of the PDE

$$-\boldsymbol{u}^{\boldsymbol{s}}_x(x,y,t) - \boldsymbol{i}\boldsymbol{u}^{\boldsymbol{s}}_y(x,y,t) + 2\boldsymbol{i}(x+y\boldsymbol{i})\boldsymbol{u}^{\boldsymbol{s}}_t(x,y,t) = f_{\boldsymbol{s}}(x,y,t) \tag{54}$$

in the ball $B\left(\left(x_n, y_n, t_n\right), 1/k\right)$ and with the properties that

$$\boldsymbol{u}^{\boldsymbol{s}}\left(x_n, y_n, t_n\right)=0, \quad\left|\partial^{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{s}}(x, y, t)\right| \leq k \tag{55}$$

for all $\boldsymbol{\alpha} \in\left(\mathbb{N}_0\right)^3$ with $|\boldsymbol{\alpha}| \leq 1$ and all $(x, y, t) \in B\left(\left(x_n, y_n, t_n\right), 1/k\right)$, and

$$\left|\partial^{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{s}}(x, y, t)-\partial^{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{s}}\left(x^{\prime}, y^{\prime}, t^{\prime}\right)\right| \leq k\left\|(x, y, t)-\left(x^{\prime}, y^{\prime}, t^{\prime}\right)\right\|^{1/k} \tag{56}$$

for all $\boldsymbol{\alpha} \in\left(\mathbb{N}_0\right)^3$ with $|\boldsymbol{\alpha}|=1$ and all $(x, y, t),\left(x^{\prime}, y^{\prime}, t^{\prime}\right) \in B\left(\left(x_n, y_n, t_n\right), 1/k\right)$.

The sets $C_{n,k}$ are closed. Indeed, let $\boldsymbol{s}_{\ell} \in C_{n,k}$ be such that $\boldsymbol{s}_{\ell} \rightarrow \boldsymbol{s}$. Then

$$\left|f_{\boldsymbol{s}_{\ell}}(x, y, z)-f_{\boldsymbol{s}}(x, y, z)\right| \leq\left\|\boldsymbol{s}_{\ell}-\boldsymbol{s}\right\|_{\infty} M_1 \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow 0$$

as $\ell \rightarrow \infty$. By (55) and (56), the sequences $\{\boldsymbol{u}^{\boldsymbol{s}_{\ell}}\}$ and $\{\partial^{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{s}_{\ell}}\}$ are equi-bounded and equi-continuous. Hence, by the Ascoli–Arzelà we have that $\{\boldsymbol{u}^{\boldsymbol{s}_{\ell}}\}$ and $\{\partial^{\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{s}_{\ell}}\}$ converge uniformly to continuous functions $\{\boldsymbol{u}\}$ and $\{\boldsymbol{v}_{\boldsymbol{\alpha}}\}$. It follows that $\boldsymbol{u}$ is of class C^1 with $\partial^{\boldsymbol{\alpha}} \boldsymbol{u}=\boldsymbol{v}_{\boldsymbol{\alpha}}$. Moreover, taking $\boldsymbol{s}=\boldsymbol{s}_{\ell}$ in (54), (55), and (56) and letting $\ell \rightarrow \infty$ shows that $\boldsymbol{u}$ satisfies (54), (55), and (56), which shows that $\boldsymbol{s} \in C_{n,k}$. Hence, $C_{n,k}$ is closed.

Next we prove that $C_{n,k}$ is nowhere dense, that is, it has empty interior. Assume by contradiction that $C_{n,k}$ has an interior point $\boldsymbol{s}$ and consider the sequence $\boldsymbol{s}_n:=\{s_{\ell}^n\}$ where $s_{\ell}^n:=0$ if $n \neq \ell$ and $s_n^n:=1/\varepsilon_n$. Then

$$f_{\boldsymbol{s}_n}(x, y, z)=g^{\prime}\left(t-2 y_n x+2 x_n y\right).$$

Let $h>0$ be so small that $\boldsymbol{t}:=\boldsymbol{s}+h \boldsymbol{s}_n \in C_{n,k}$. Let $\boldsymbol{u}^{\boldsymbol{s}}$ and $\boldsymbol{u}^{\boldsymbol{t}}$ be the solutions corresponding to $\boldsymbol{s}$ and $\boldsymbol{t}$, respectively and consider the function

$$\boldsymbol{v}:=\frac{\boldsymbol{u}^{\boldsymbol{t}}-\boldsymbol{u}^{\boldsymbol{s}}}{h}.$$

Then $\boldsymbol{v}$ is a solution of the PDE

$$-\boldsymbol{v}_x(x, y, t)-\boldsymbol{i} \boldsymbol{v}_y(x, y, t)+2 \boldsymbol{i}(x+y \boldsymbol{i}) \boldsymbol{v}_t(x, y, t)=g^{\prime}\left(t-2 y_n x+2 x_n y\right)$$

in $B\left(\left(x_n, y_n, t_n\right), 1/k\right)$. It follows by Step 2 that g is analytic at t_n, which is a contradiction. Hence, $C_{n,k}$ is nowhere dense.

Step 4: We are now ready to prove the theorem. We claim that there exists $\boldsymbol{s} \in \ell_{\infty}$ for which the PDE

$$-\boldsymbol{u}_x(x, y, t)-\boldsymbol{i} \boldsymbol{u}_y(x, y, t)+2 \boldsymbol{i}(x+y \boldsymbol{i}) \boldsymbol{u}_t(x, y, t)=f_{\boldsymbol{s}}(x, y, t) \tag{57}$$

has no solution class C^2 in any open set. Assume by contradiction that this is not the case and let $\boldsymbol{s} \in \ell_{\infty}$. Then there exist an open set $\Omega \subseteq \mathbb{R}^3$ and solution $\boldsymbol{u}$ of class C^2 of (57) in Ω. Since $\{(x_n, y_n, t_n)\}$ is dense in $\mathbb{R}^3$, there exist $n, k \in \mathbb{N}$ such that $\overline{B\left(\left(x_n, y_n, t_n\right), 1/k\right)} \subseteq \Omega$. Moreover, by taking k larger, if necessary, and using the fact that $\boldsymbol{u}$ is of class C^2 in Ω, we have that $(54)_2$

and (55). Then the function $\boldsymbol{u}^{\boldsymbol{s}} := \boldsymbol{u} - \boldsymbol{u}(x_n, y_n, t_n)$ satisfies (54), (55), and (56), and so $\boldsymbol{s} \in C_{n,k}$. This shows that

$$\ell_\infty = \bigcup_{n,k} C_{n,k},$$

which contradicts Baire's category theorem and completes the proof. ■

Wednesday, October 9, 2013

6 Second Order Elliptic PDEs

Consider a second order partial differential equation of the form

$$F\left(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x}), \nabla^2 u(\boldsymbol{x})\right) = 0, \quad \boldsymbol{x} \in \Omega, \tag{58}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set. Observe that if we are looking for a classical solution u of class $C^2(\Omega)$, then by the Schwartz theorem,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 u}{\partial x_j \partial x_i}(\mathbf{x}).$$

Hence, the $N \times N$ matrix $\nabla^2 u(\boldsymbol{x})$ is symmetric. Write $F(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A})$. Note that by replacing F with

$$G(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A}) := F\left(\boldsymbol{x}, z, \boldsymbol{p}, \frac{\boldsymbol{A} + \boldsymbol{A}^T}{2}\right),$$

we are not changing the PDE (in the case of classical solutions). Hence, we can assume that the $N \times N$ matrix

$$\nabla_{\boldsymbol{A}} F(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A})$$

is symmetric. This implies that its eigenvalues are real. We say the the PDE (58) is *elliptic* at a point $(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A})$ if the matrix $\nabla_{\boldsymbol{A}} F(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A})$ is positive definite, namely, if all eigenvalues are positive. We say the the PDE (58) is *elliptic* if it is elliptic at all points $(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A})$.

Exercise 54 *Prove that an elliptic equation has no characteristic surfaces.*

The typical example of an elliptic equation is *Poisson's equation*

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

where Δ is the *Laplace* operator or *Laplacian*

$$\Delta u(\boldsymbol{x}) := \operatorname{div}(\nabla u(\boldsymbol{x})) = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}(\boldsymbol{x}).$$

In this case

$$F(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A}) = \sum_{i=1}^{N} a_{i,i} + f(\boldsymbol{x}),$$

and so

$$\nabla_{\boldsymbol{A}} F(\boldsymbol{x}, z, \boldsymbol{p}, \boldsymbol{A}) = I_N,$$

which has all eigenvalues one. Note that the Laplacian $\Delta u(\boldsymbol{x})$ is the trace of the Hessian matrix $\nabla^2 u(\boldsymbol{x})$.

Since elliptic equations have no characteristic surfaces, if $\Omega \subseteq \mathbb{R}^N$ and $f, g, h : \mathbb{R}^N \to \mathbb{R}$ are analytic, by the Cauchy–Kowalevski theorem the boundary value problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}), \quad \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = h(\boldsymbol{x}) & \text{on } \Gamma \end{cases} \tag{59}$$

admits a local solution in a neighborhood of every $\boldsymbol{x}_0 \in \Gamma \subseteq \partial\Omega$.

However, in general there are no global solution. To see this, we consider Green's formula. Consider an open bounded set $\Omega \subseteq \mathbb{R}^N$ of class C^1 and let $u, v \in C^2(\overline{\Omega})$. Then

$$\int_\Omega v\Delta u \; d\boldsymbol{x} + \int_\Omega \nabla v \cdot \nabla u \; d\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial v}{\partial \boldsymbol{n}} \; dS. \tag{60}$$

To see this, note that

$$\operatorname{div}(v\nabla u) = v\Delta u + \nabla v \cdot \nabla u$$

and so by the divergence theorem applied to the vector function $v\nabla u$ we get

$$\begin{aligned} \int_\Omega v\Delta u \; d\boldsymbol{x} + \int_\Omega \nabla v \cdot \nabla u \; d\boldsymbol{x} &= \int_\Omega \operatorname{div}(v\nabla u) \; d\boldsymbol{x} \\ &= \int_{\partial\Omega} (v\nabla u) \cdot \boldsymbol{n} \; dS = \int_{\partial\Omega} v \frac{\partial u}{\partial \boldsymbol{n}} \; dS. \end{aligned}$$

Taking $u = v$ gives

$$\int_\Omega u\Delta u \; d\boldsymbol{x} + \int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x} = \int_{\partial\Omega} u \frac{\partial u}{\partial \boldsymbol{n}} \; dS.$$

Hence, if we take $\Gamma = \partial\Omega$ and $f = g = 0$, we obtain

$$\int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x} = 0,$$

which implies that $u = 0$ in Ω. It follows that if $h \neq 0$, then (59) has no global solution.

We will see later on that for elliptic equations, one should assign only one boundary condition. Hence, instead of (59), one considers the *Dirichlet problem*

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ u(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega, \end{cases} \tag{61}$$

or the *Neumann problem*

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = h(\boldsymbol{x}) & \text{on } \partial\Omega. \end{cases} \tag{62}$$

7 Harmonic Functions

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$. The function u is called

- *harmonic* in Ω if $\Delta u(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \Omega$,
- *subharmonic* in Ω if $\Delta u(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \Omega$,
- *superharmonic* in Ω if $\Delta u(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in \Omega$.

Theorem 55 (Mean Value Formulas) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$ be an harmonic function. Then for every ball $B(\boldsymbol{x}, R) \Subset \Omega$,*

$$u(\boldsymbol{x}) = \frac{1}{N\alpha_N R^{N-1}} \int_{\partial B(\boldsymbol{x},R)} u(\boldsymbol{y})\ dS(\boldsymbol{y}), \quad u(\boldsymbol{x}) = \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} u(\boldsymbol{y})\ d\boldsymbol{y},$$

where α_N is the measure of the unit ball.

Proof. For $0 < r \leq R$ define the function

$$f(r) = \frac{1}{N\alpha_N r^{N-1}} \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y})\ dS(\boldsymbol{y}). \tag{63}$$

Using spherical coordinates and writing $\boldsymbol{y} = \boldsymbol{x} + r\boldsymbol{z}$, we get

$$f(r) = \frac{1}{N\alpha_N} \int_{\partial B(\boldsymbol{0},1)} u(\boldsymbol{x} + r\boldsymbol{z})\ dS(\boldsymbol{z}).$$

By the theorem on derivation under the integral sign (here we are using the fact that u is $C^1\left(\overline{B(\boldsymbol{x}, R)}\right)$), we have

$$\begin{aligned} f'(r) &= \frac{1}{N\alpha_N} \int_{\partial B(\boldsymbol{0},1)} \nabla u(\boldsymbol{x} + r\boldsymbol{z}) \cdot \boldsymbol{z}\ dS(\boldsymbol{z}) \\ &= \frac{1}{N\alpha_N r^{N-1}} \int_{\partial B(\boldsymbol{x},r)} \nabla u(\boldsymbol{x} + r\boldsymbol{z}) \cdot \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{r}\right)\ dS(\boldsymbol{y}) \\ &= \frac{1}{N\alpha_N r^{N-1}} \int_{\partial B(\boldsymbol{x},r)} \frac{\partial u}{\partial \boldsymbol{n}}\ dS(\boldsymbol{y}) = \frac{1}{N\alpha_N r^{N-1}} \int_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})\ d\boldsymbol{y} = 0. \end{aligned} \tag{64}$$

This implies that f is constant in $(0, R]$. On the other hand, by the continuity of u, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|u(\boldsymbol{y}) - u(\boldsymbol{x})| \leq \varepsilon$$

for all $\boldsymbol{y} \in B(\boldsymbol{x}, \delta)$. Hence, for $0 < r < \delta$,

$$|f(r) - u(\boldsymbol{x})| = \frac{1}{N\alpha_N r^{N-1}} \left| \int_{\partial B(\boldsymbol{x},r)} (u(\boldsymbol{y}) - u(\boldsymbol{x})) \ dS(\boldsymbol{y}) \right| \leq \varepsilon.$$

Since $f(R) = f(r)$, it follows that $|f(R) - u(\boldsymbol{x})| \leq \varepsilon$ for all $\varepsilon > 0$, which shows that $f(R) = u(\boldsymbol{x})$.

Using spherical coordinates we have

$$\begin{aligned}\int_{B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ d\boldsymbol{y} &= \int_0^R \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y}) \ dS(\boldsymbol{y}) \ dr \\ &= u(\boldsymbol{x}) N\alpha_N \int_0^R r^{N-1} \ dr = u(\boldsymbol{x}) N\alpha_N R^N.\end{aligned}$$

This concludes the proof. ■

Remark 56 *If in the previous theorem, we assume that u is subharmonic instead of harmonic, then the same proof gives*

$$u(\boldsymbol{x}) \leq \frac{1}{N\alpha_N R^{N-1}} \int_{\partial B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ dS(\boldsymbol{y}), \quad u(\boldsymbol{x}) \leq \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ d\boldsymbol{y}.$$

Similarly, for superharmonic functions, we get

$$u(\boldsymbol{x}) \geq \frac{1}{N\alpha_N R^{N-1}} \int_{\partial B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ dS(\boldsymbol{y}), \quad u(\boldsymbol{x}) \geq \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ d\boldsymbol{y}.$$

Friday, October 11, 2013

The opposite implication is also true, namely, we can show that a function satisfying the mean value property is smooth and harmonic. To prove this result, we need to talk about mollifiers. Let

$$\varphi(\boldsymbol{x}) := \begin{cases} c \exp\left(\frac{1}{\|\boldsymbol{x}\|^2 - 1}\right) & \text{if } \|\boldsymbol{x}\| < 1, \\ 0 & \text{if } \|\boldsymbol{x}\| \geq 1, \end{cases} \tag{65}$$

where $c > 0$ is chosen so that

$$\int_{\mathbb{R}^N} \varphi(\boldsymbol{x}) \ d\boldsymbol{x} = 1.$$

Note that $\varphi \in C^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \varphi \subset \overline{B(\boldsymbol{0}, 1)}$. For every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(\boldsymbol{x}) := \frac{1}{\varepsilon^N} \varphi\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

The functions φ_ε are called *mollifiers*. Note that $\operatorname{supp} \varphi_\varepsilon \subset \overline{B(\boldsymbol{0}, \varepsilon)}$ and

$$\int_{\mathbb{R}^N} \varphi_\varepsilon(\boldsymbol{x}) \ d\boldsymbol{x} = 1. \tag{66}$$

Given an open set $\Omega \subset \mathbb{R}^N$ and a function $u \in C(\Omega)$, we may define

$$u_\varepsilon(\boldsymbol{x}) := \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \tag{67}$$

for $\boldsymbol{x} \in \Omega_\varepsilon$, where the open set Ω_ε is given by

$$\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

Note that if $\boldsymbol{x} \in \Omega_\varepsilon$, then $\overline{B(\boldsymbol{x}, \varepsilon)} \subset \Omega$ and $u_\varepsilon(\boldsymbol{x}) = \int_{\overline{B(\boldsymbol{x},\varepsilon)}} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y}$. Since u is bounded in $\overline{B(\boldsymbol{x}, \varepsilon)}$, the integral is well-defined. The function $u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ is called a *mollification* of u.

Remark 57 *Note that if $\Omega = \mathbb{R}^N$, then $\Omega_\varepsilon = \mathbb{R}^N$, thus u_ε is defined in the entire space $\mathbb{R}^N$.*

The first main result of this subsection is the following theorem.

Theorem 58 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in C(\Omega)$. Then $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ uniformly on compact subsets of Ω. Moreover, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $0 < \varepsilon < 1$ and for every multi-index α,*

$$\frac{\partial^\alpha u_\varepsilon}{\partial x^\alpha}(\boldsymbol{x}) = \int_\Omega \frac{\partial^\alpha \varphi_\varepsilon}{\partial x^\alpha}(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \tag{68}$$

for all $\boldsymbol{x} \in \Omega_\varepsilon$.

Proof. Let $K \subset \mathbb{R}^N$ be a compact set. For any fixed

$$0 < \eta < \operatorname{dist}(K, \partial\Omega)$$

let

$$K_\eta := \left\{\boldsymbol{x} \in \mathbb{R}^N : \operatorname{dist}(\boldsymbol{x}, K) \le \eta\right\}.$$

so that $K_\eta \subset \Omega$. Note that for $\varepsilon > 0$ sufficiently small we have that $K_\eta \subset \Omega_\varepsilon$. Since K_η is compact and u is uniformly continuous on K_η, for every $\rho > 0$ there exists $\delta = \delta(\eta, K, \rho) > 0$ such that

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \le \rho \tag{69}$$

for all $\boldsymbol{x}, y \in K_\eta$, with $\|\boldsymbol{x} - \boldsymbol{y}\| \le \delta$. Let $0 < \varepsilon < \min\{\delta, \eta\}$. Then for all $\boldsymbol{x} \in K$,

$$\begin{aligned} |u_\varepsilon(\boldsymbol{x}) - u(\boldsymbol{x})| &= \left|\int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} - u(\boldsymbol{x})\right| \qquad (70) \\ &= \left|\int_{B(\boldsymbol{x},\varepsilon)} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, [u(\boldsymbol{y}) - u(\boldsymbol{x})]\, d\boldsymbol{y}\right| \le \rho \end{aligned}$$

where we have used the facts that $B(\boldsymbol{x}, \varepsilon) \subset \Omega$ and that $\int_{B(\boldsymbol{x},\varepsilon)} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{y} = 1$. It follows by (69) that

$$|u_\varepsilon(\boldsymbol{x}) - u(\boldsymbol{x})| \le \rho$$

for all $\boldsymbol{x} \in K$, and so $\|u_\varepsilon - u\|_{C(K)} \leq \rho$.

Fix $\boldsymbol{x} \in \Omega_\varepsilon$ and $0 < \eta < \operatorname{dist}(\boldsymbol{x}, \partial\Omega) - \varepsilon$. Let $\boldsymbol{e}_i$, $i = 1, \ldots, N$, be an element of the canonical basis of $\mathbb{R}^N$ and for every $h \in \mathbb{R}$, with $0 < |h| \leq \eta$, consider

$$\begin{aligned}
&\left| \frac{u_\varepsilon(\boldsymbol{x} + h\boldsymbol{e}_i) - u_\varepsilon(\boldsymbol{x})}{h} - \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \right| \\
&= \left| \int_\Omega \left[\frac{\varphi_\varepsilon(\boldsymbol{x} + h\boldsymbol{e}_i - \boldsymbol{y}) - \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})}{h} - \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) \right] u(\boldsymbol{y})\, d\boldsymbol{y} \right| \\
&= \left| \int_\Omega \left(\frac{1}{h} \int_0^h \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t e_i)\, dt - \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) \right) u(\boldsymbol{y})\, d\boldsymbol{y} \right| \\
&\leq \left| \frac{1}{h} \int_0^h \left(\int_{B(\boldsymbol{x}, \varepsilon + \eta)} \left| \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t e_i) - \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) \right| |u(\boldsymbol{y})|\, d\boldsymbol{y} \right) dt \right|,
\end{aligned}$$

where we have used Fubini's theorem and the fact that $\operatorname{supp} \varphi_\varepsilon \subset \overline{B(0, \varepsilon)}$.

Since $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ its partial derivatives are uniformly continuous. Hence for every $\rho > 0$ there exists $\delta = \delta(\eta, \boldsymbol{x}, \rho, \varepsilon) > 0$ such that

$$\left| \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{z}) - \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{w}) \right| \leq \frac{\rho}{1 + \|u\|_{L^1(B(\boldsymbol{x}, \varepsilon + \eta))}}$$

for all $\boldsymbol{z}, \boldsymbol{y} \in B(\boldsymbol{x}, \varepsilon + \eta)$, with $\|\boldsymbol{z} - \boldsymbol{w}\| \leq \delta$. Then for $0 < |h| \leq \min\{\eta, \delta\}$ we have

$$\left| \frac{u_\varepsilon(\boldsymbol{x} + h e_i) - u_\varepsilon(\boldsymbol{x})}{h} - \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \right| \leq \rho,$$

which shows that

$$\frac{\partial u_\varepsilon}{\partial x_i}(\boldsymbol{x}) = \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y}.$$

Note that the only properties that we have used on the function φ_ε are that $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \varphi_\varepsilon \subset \overline{B(0, \varepsilon)}$. Hence the same proof carries over if we replace φ_ε with $\psi_\varepsilon := \frac{\partial \varphi_\varepsilon}{\partial x_i}$. Thus by induction we may prove that for every multi-index α there holds

$$\frac{\partial^\alpha u_\varepsilon}{\partial x^\alpha}(\boldsymbol{x}) = \int_\Omega \frac{\partial^\alpha \varphi_\varepsilon}{\partial x^\alpha}(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y}.$$

This completes the proof. ■

Theorem 59 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C(\Omega)$ be such that for every $\boldsymbol{x} \in \Omega$ and every ball $B(\boldsymbol{x}, R) \Subset \Omega$,*

$$u(\boldsymbol{x}) = \frac{1}{N \alpha_N R^{N-1}} \int_{\partial B(\boldsymbol{x}, R)} u(\boldsymbol{y})\, dS(\boldsymbol{y}). \tag{71}$$

Then $u \in C^\infty(\Omega)$ and is harmonic in Ω.

Proof. For $\varepsilon > 0$, let u_ε be a mollification of u. Note that the function φ in (65) is radial, and so we can write $\varphi_\varepsilon(\boldsymbol{x}) = g_\varepsilon(\|\boldsymbol{x}\|)$ for every $\boldsymbol{x} \in \Omega$. Fix $\boldsymbol{x} \in \Omega$ and let $0 < \varepsilon < \operatorname{dist}(\boldsymbol{x}, \partial\Omega)$. Then by the change of variables $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, the fact that $\operatorname{supp}\varphi_\varepsilon \subset \overline{B(\boldsymbol{0}, \varepsilon)}$,

$$\begin{aligned} u_\varepsilon(\boldsymbol{x}) &= \int_\Omega \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{B(\boldsymbol{0},\varepsilon)} \varphi_\varepsilon(\boldsymbol{z})\, u(\boldsymbol{x} - \boldsymbol{z})\, d\boldsymbol{z} \\ &= \int_0^\varepsilon r^{N-1} \int_{\partial B(\boldsymbol{0},1)} \varphi_\varepsilon(r\boldsymbol{w})\, u(\boldsymbol{x} - r\boldsymbol{w})\, dS(\boldsymbol{w})\, dr \\ &= \int_0^\varepsilon r^{N-1} g_\varepsilon(r) \int_{\partial B(\boldsymbol{0},1)} u(\boldsymbol{x} - r\boldsymbol{w})\, dS(\boldsymbol{w})\, dr \\ &= \int_0^\varepsilon r^{N-1} g_\varepsilon(r) \frac{1}{r^{N-1}} \int_{\partial B(\boldsymbol{x},r)} u(\boldsymbol{y})\, dS(\boldsymbol{y})\, dr \\ &= u(\boldsymbol{x})\, N\alpha_N \int_0^\varepsilon r^{N-1} g_\varepsilon(r)\, dr = u(\boldsymbol{x}), \end{aligned}$$

where we have used (66) and (71). In view of the previous theorem, we have that $u \in C^\infty(\Omega)$.

Define the function f as in (63). Since f is constant by (71), it follows that $f' = 0$. Hence, by (64),

$$0 = f'(r) = \frac{1}{N\alpha_N r^{N-1}} \int_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})\, d\boldsymbol{y},$$

and so $\int_{B(\boldsymbol{x},r)} \Delta u(\boldsymbol{y})\, d\boldsymbol{y} = 0$ for every $r > 0$. Since Δu is continuous, it follows that $\Delta u(\boldsymbol{x}) = 0$. This shows that u is harmonic. ■

Remark 60 *With a similar proof, one can show that if $u : \Omega \to \mathbb{R}$ is measurable, integrable on compact sets and satisfies*

$$\int_\Omega u(\boldsymbol{y})\, \Delta\varphi(\boldsymbol{y})\, d\boldsymbol{y} = 0$$

for all $\varphi \in C_c^\infty(\Omega)$, then u is harmonic.

Monday, October 14, 2013

7.1 Interior Estimates

Using the mean value formulas we can obtain interior estimates of the derivatives of an harmonic function. Since an harmonic function u in Ω is of class $C^\infty(\Omega)$, for every multi-index $\boldsymbol{\alpha}$, we have that

$$0 = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}}(\Delta u) = \Delta\left(\frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}}\right)$$

and thus $\frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}}$ is also harmonic in Ω. This simple observation has important applications.

Theorem 61 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$ be an harmonic function. Then for every ball $B(\boldsymbol{x}, R) \Subset \Omega$,*

$$\|\nabla u(\boldsymbol{x})\| \leq \frac{N}{R} \max_{\partial B(\boldsymbol{x},R)} |u| .$$

Proof. Since ∇u is harmonic, by the mean value formula applied to ∇u,

$$\begin{aligned}\nabla u(\boldsymbol{x}) &= \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} \nabla u(\boldsymbol{y}) \; d\boldsymbol{y} \\ &= \frac{1}{\alpha_N R^N} \int_{\partial B(\boldsymbol{x},R)} u(\boldsymbol{y})\, \boldsymbol{n}((\boldsymbol{y})) \; dS(\boldsymbol{y}),\end{aligned}$$

where we have used the divergence theorem. Hence,

$$\|\nabla u(\boldsymbol{x})\| \leq \frac{1}{\alpha_N R^N} \max_{\partial B(\boldsymbol{x},R)} |u| \int_{\partial B(\boldsymbol{x},R)} 1 \; dS(\boldsymbol{y}) = \frac{N}{R} \max_{\partial B(\boldsymbol{x},R)} |u| .$$

■

Remark 62 *Note that the previous theorem uses only the fact that ∇u is harmonic, and thus continues to apply if $u \in C^3(\Omega)$ satisfies $\Delta u(\boldsymbol{x}) = c$ for all $\boldsymbol{x} \in \Omega$ and for some constant $c \in \mathbb{R}$.*

Corollary 63 (Liouville Theorem) *Let $u \in C^2(\mathbb{R}^N)$ be a bounded harmonic function. Then u is constant.*

Proof. Let $M > 0$ be such that $|u| \leq M$ in $\mathbb{R}^N$. Then by the previous theorem

$$\|\nabla u(\boldsymbol{x})\| \leq \frac{N}{R} M$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. Letting $R \to \infty$ shows that $\nabla u(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \mathbb{R}^N$, which implies that u is constant. ■

Remark 64 *Liouville's theorem continues to work if we assume that $u \in C^2(\mathbb{R}^N)$ is an bounded harmonic function bounded from below or from above. More generally, one can show that if $u \in C^2(\mathbb{R}^N)$ is an harmonic function such that*

$$\liminf_{\|\boldsymbol{x}\| \to \infty} \frac{u(\boldsymbol{x})}{\|\boldsymbol{x}\|} \geq 0,$$

then u is constant.

Theorem 65 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$ be an harmonic function. Then for every multi-index $\boldsymbol{\alpha}$ and for every ball $B(\boldsymbol{x}, R) \Subset \Omega$,*

$$\left| \frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}}(\boldsymbol{x}) \right| \leq \left(\frac{N}{R} \right)^{|\boldsymbol{\alpha}|} e^{|\boldsymbol{\alpha}|-1} |\boldsymbol{\alpha}|! \max_{B(\boldsymbol{x},R)} |u| .$$

Proof. The proof is by induction on the number $n := |\boldsymbol{\alpha}|$. For $n = 1$, the estimate follows from Theorem 61. Assume the estimate is true for all multi-indeces $\boldsymbol{\beta}$ of length $n-1$, and let's prove it for a multi-index $\boldsymbol{\alpha}$ of length n. Then we can write $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{e}_i$ for some multi-index $\boldsymbol{\beta}$ of length $n-1$ and some $i \in \{1, \dots, N\}$. Let $0 < r < R$. Since $\frac{\partial^{\boldsymbol{\beta}} u}{\partial \boldsymbol{x}_{\boldsymbol{\beta}}}$ is harmonic, by Theorem 61, we have

$$\left| \frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}} (\boldsymbol{x}) \right| \leq \left\| \nabla \left(\frac{\partial^{\boldsymbol{\beta}} u}{\partial \boldsymbol{x}_{\boldsymbol{\beta}}} \right) (\boldsymbol{x}) \right\| \leq \frac{N}{r} \max_{\partial B(\boldsymbol{x}, r)} \left| \frac{\partial^{\boldsymbol{\beta}} u}{\partial \boldsymbol{x}_{\boldsymbol{\beta}}} \right| .$$

If $\boldsymbol{y} \in B(\boldsymbol{x}, r)$, then $B(\boldsymbol{y}, R - r) \subset B(\boldsymbol{x}, R) \Subset \Omega$, and so by the induction hypothesis

$$\begin{aligned} \left| \frac{\partial^{\boldsymbol{\beta}} u}{\partial \boldsymbol{x}_{\boldsymbol{\beta}}} (\boldsymbol{y}) \right| &\leq \left(\frac{N}{R - r} \right)^{n-1} e^{n-2} (n-1)! \max_{B(\boldsymbol{y}, R-r)} |u| \\ &\leq \left(\frac{N}{R - r} \right)^{n-1} e^{n-2} (n-1)! \max_{B(\boldsymbol{x}, R)} |u| . \end{aligned}$$

Since u is of class C^∞, this estimate holds for all $\boldsymbol{y} \in \overline{B(\boldsymbol{x}, r)}$. By combining these two inequalities, we obtain

$$\left| \frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}} (\boldsymbol{x}) \right| \leq \frac{N}{r} \left(\frac{N}{R - r} \right)^{n-1} e^{n-2} (n-1)! \max_{B(\boldsymbol{x}, R)} |u| .$$

We now choose $r = \frac{1}{n} R$. Then

$$\frac{1}{r} \frac{1}{(R - r)^{n-1}} = \frac{n}{R^n} \left(\frac{n}{n-1} \right)^{n-1} = \frac{n}{R^n} \left(1 + \frac{1}{n-1} \right)^{n-1} < \frac{ne}{R^N} .$$

This concludes the proof. ■

Remark 66 *Using these estimates, one can show that harmonic functions are analytic.*

Remark 67 *The previous theorem continues to hold if $u \in C^3(\Omega)$ satisfies $\Delta u(\boldsymbol{x}) = c$ for all $\boldsymbol{x} \in \Omega$ and for some constant $c \in \mathbb{R}$.*

Corollary 68 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$ be an harmonic function. Then for every multi-index $\boldsymbol{\alpha}$ and for all $U \Subset V \Subset \Omega$,*

$$\left| \frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}} (\boldsymbol{x}) \right| \leq \left(\frac{N}{\operatorname{dist}(U, \partial V)} \right)^{|\boldsymbol{\alpha}|} e^{|\boldsymbol{\alpha}| - 1} |\boldsymbol{\alpha}|! \max_{\overline{V}} |u|$$

for all $\boldsymbol{x} \in U$.

Proof. Let $R < \operatorname{dist}(U, \partial V)$. Then for $\boldsymbol{x} \in U$, $B(\boldsymbol{x}, R) \subset V \Subset \Omega$, and so we may apply the previous theorem to conclude that

$$\left| \frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}} (\boldsymbol{x}) \right| \leq \left(\frac{N}{R} \right)^{|\boldsymbol{\alpha}|} e^{|\boldsymbol{\alpha}| - 1} |\boldsymbol{\alpha}|! \max_{B(\boldsymbol{x}, R)} |u| \leq \left(\frac{N}{R} \right)^{|\boldsymbol{\alpha}|} e^{|\boldsymbol{\alpha}| - 1} |\boldsymbol{\alpha}|! \max_{\overline{V}} |u| .$$

Letting $R \to \operatorname{dist}(U, \partial V)$ gives the desired result. ■

Remark 69 *The reason why we need to introduce the set V is that an harmonic function can behaves really badly near the boundary. For example, the function*

$$u(x,y) = \frac{x}{x^2+y^2}$$

is harmonic but is unbounded near $(0,0)$. More generally, we can construct harmonic functions that are unbounded in a neighborhood of every point of the boundary.

7.2 Maximum Principle

Theorem 70 (Strong Maximum Principle) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in C^2(\Omega)$ be subharmonic function. Assume that there exists $\boldsymbol{x}_0 \in \Omega$ such that*

$$u(\boldsymbol{x}_0) \geq u(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \Omega$. Then u is constant.

In particular, an harmonic function cannot assume an interior maximum or minimum value unless it is constant.

Proof. Let $M := u(\boldsymbol{x}_0)$ and define

$$E := \{\boldsymbol{x} \in \Omega : \ u(\boldsymbol{x}) = M\}.$$

The set E is nonempty, since $\boldsymbol{x}_0 \in E$, and relatively closed, since u is continuous and $E = u^{-1}(\{M\})$.

Fix $\boldsymbol{x} \in E$ and let $B(\boldsymbol{x}, R) \Subset \Omega$. By the mean values formulas (see Remark 56), and that fact that $u \leq M$,

$$M = u(\boldsymbol{x}) \leq \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} u(\boldsymbol{y}) \ d\boldsymbol{y} \leq \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} M \ d\boldsymbol{y} = M.$$

It follows that

$$\frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},R)} (M - u(\boldsymbol{y})) \ d\boldsymbol{y} = 0,$$

but since u is continuous and $M - u \geq 0$, necessarily, $M - u(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in B(\boldsymbol{x}, R)$. This shows that E is open. But since E is also relatively closed, it follows that E must be Ω.

To prove the second part of the theorem, it suffices to observe that if u is harmonic, then u and $-u$ are subharmonic. ■

Wednesday, October 16, 2013

Theorem 71 (Weak Maximum Principle) *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a subharmonic function in Ω. Then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

In particular, if u is harmonic, then

$$\min_{\partial\Omega} u \leq u(\boldsymbol{x}) \leq \max_{\partial\Omega} u$$

for all $\boldsymbol{x} \in \Omega$*.*

First proof. Let $\boldsymbol{x}_0 \in \overline{\Omega}$ be such that $u(\boldsymbol{x}_0) = \max_{\overline{\Omega}} u$. If $\boldsymbol{x}_0 \in \partial\Omega$, then we are done. Thus, assume that $\boldsymbol{x}_0 \in \Omega$. Then by the strong maximum principle, u is constant in Ω. But since u is continuous in $\overline{\Omega}$, it is constant in $\overline{\Omega}$ and the proof is complete. ■

Second proof. This second proof does not rely on the mean value formulas. For $\varepsilon > 0$, let

$$u_\varepsilon(\boldsymbol{x}) := u(\boldsymbol{x}) + \varepsilon \|\boldsymbol{x}\|^2 .$$

Then $\Delta u_\varepsilon(\boldsymbol{x}) = \Delta u(\boldsymbol{x}) + 2N\varepsilon > 0$. Hence, u_ε is subharmonic. Let $\boldsymbol{x}_\varepsilon \in \overline{\Omega}$ be such that $u_\varepsilon(\boldsymbol{x}_\varepsilon) = \max_{\overline{\Omega}} u_\varepsilon$. If $\boldsymbol{x}_\varepsilon \in \Omega$, then $\nabla^2 u_\varepsilon(\boldsymbol{x}_0)$ is semipositive definite, which means that

$$\left(\nabla^2 u_\varepsilon(\boldsymbol{x}_\varepsilon)\, \boldsymbol{y}^T\right) \cdot \boldsymbol{y} \leq 0$$

for all $\boldsymbol{y} \in \mathbb{R}^N$. Taking $\boldsymbol{y} = \boldsymbol{e}_i$, gives $\frac{\partial^2 u_\varepsilon}{\partial x_i^2}(\mathbf{x}_\varepsilon) \leq 0$. In turn, $\Delta u_\varepsilon(\boldsymbol{x}_\varepsilon) \leq 0$, which is a contradiction. Hence, $\max_{\overline{\Omega}} u_\varepsilon = \max_{\partial\Omega} u_\varepsilon$ and so

$$u_\varepsilon(\boldsymbol{x}) = u(\boldsymbol{x}) + \varepsilon \|\boldsymbol{x}\|^2 \leq u_\varepsilon(\boldsymbol{x}_\varepsilon) = u(\boldsymbol{x}_\varepsilon) + \varepsilon \|\boldsymbol{x}_\varepsilon\|^2$$

for all $\boldsymbol{x} \in \Omega$. Since $\partial\Omega$ is compact, there exists $\boldsymbol{x}_0 \in \partial\Omega$ such that, up to a subsequence, $\boldsymbol{x}_\varepsilon \to \boldsymbol{x}_0$. Letting $\varepsilon \to 0^+$ in the previous inequality shows that $u(\boldsymbol{x}) \leq u(\boldsymbol{x}_0)$ for all $\boldsymbol{x} \in \Omega$. ■

Corollary 72 (Uniqueness of the Dirichlet problem) *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded open set and let* $u, v \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ *be such that u is harmonic functions in* Ω *and v subharmonic in* Ω *with* $u = v$ *on* $\partial\Omega$*. Then* $u \geq v$*.*

In particular, if v is also harmonic, then $u = v$*.*

Proof. Let $w := v - u$. Then w is subharmonic in Ω and $w = 0$ on $\partial\Omega$. Hence, by the weak maximum principle,

$$w(\boldsymbol{x}) \leq \max_{\partial\Omega} w = 0$$

for all $\boldsymbol{x} \in \Omega$. If v is harmonic, the by the weak maximum principle,

$$\min_{\partial\Omega} w = 0 \leq w(\boldsymbol{x}) \leq \max_{\partial\Omega} w = 0$$

for all $\boldsymbol{x} \in \Omega$. ■

Remark 73 *The previous corollary motivates the term "subharmonic". This corollary is very useful in applications. If you want to construct a bound for an harmonic function, it is sufficient to construct a subharmonic function which equals to u on the boundary.*

Example 74 *The weak maximum principle does not hold on unbounded domains. To see this, let*

$$\Omega := \left\{ (x,y) \in \mathbb{R}^2 : \ 0 < y < \pi \right\}$$

and $u(x,y) = e^x \sin y$. *Then* u *is harmonic and* $u(x,0) = u(x,\pi) = 0$ *but* u *is not identically zero.*

Theorem 75 (Harnack's Inequality) *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and let* $u \in C^2(\Omega)$ *be a nonnegative harmonic function in* Ω. *Then for every connected open set* $U \Subset \Omega$ *there exists a constant* $C = C(\Omega, U) > 0$ *such that*

$$\sup_U u \le C \inf_U u.$$

Proof. Step 1: Let $\boldsymbol{x} \in \Omega$ and let $R > 0$ be such that $B(\boldsymbol{x}, 4R) \subseteq \Omega$. Take $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\boldsymbol{x}, R)$. By the mean value property and the fact that $u \ge 0$,

$$\begin{aligned} u(\boldsymbol{x}_1) &= \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x}_1,R)} u(\boldsymbol{y}) \ d\boldsymbol{y} \le \frac{1}{\alpha_N R^N} \int_{B(\boldsymbol{x},2R)} u(\boldsymbol{y}) \ d\boldsymbol{y}, \\ u(\boldsymbol{x}_2) &= \frac{1}{\alpha_N (3R)^N} \int_{B(\boldsymbol{x}_2,3R)} u(\boldsymbol{y}) \ d\boldsymbol{y} \ge \frac{1}{\alpha_N (3R)^N} \int_{B(\boldsymbol{x},2R)} u(\boldsymbol{y}) \ d\boldsymbol{y}. \end{aligned}$$

It follows that

$$u(\boldsymbol{x}_1) \le 3^N u(\boldsymbol{x}_2).$$

Since this is true for all $\boldsymbol{x}_1 \in B(\boldsymbol{x}, R)$, we get

$$\sup_{B(\boldsymbol{x},R)} u \le 3^N u(\boldsymbol{x}_2),$$

and since this is true for all $\boldsymbol{x}_2 \in B(\boldsymbol{x}, R)$, it follows that

$$\sup_{B(\boldsymbol{x},R)} u \le 3^N \inf_{B(\boldsymbol{x},R)} u.$$

Step 2: Let $0 < 4R < \operatorname{dist}(U, \partial\Omega)$. The family of balls $\left\{ B(\boldsymbol{x}, R) : \boldsymbol{x} \in \overline{U} \right\}$ covers the compact set $\overline{U}$ and so there exists a number $\ell = \ell(U, \Omega)$ such that

$$\overline{U} \subset \bigcup_{i=1}^{\ell} B(\boldsymbol{x}_i, R).$$

Given $\boldsymbol{x}_1, \boldsymbol{x}_2 \in U$, since U is open and connected, it is pathwise connected, and so there is continuous curve $\boldsymbol{\gamma} : [0,1] \to U$ joining $\boldsymbol{x}_1$ with $\boldsymbol{x}_2$. The range $\boldsymbol{\gamma}([0,1])$ of the curve γ will be covered by some of the balls, say,

$$\boldsymbol{\gamma}([0,1]) \subset \bigcup_{i=1}^{m} B(\boldsymbol{y}_i, R).$$

Without loss of generality, we may assume that $\boldsymbol{x}_1 \in B(\boldsymbol{y}_1, R)$, $\boldsymbol{x}_2 \in B(\boldsymbol{y}_m, R)$, and $B(\boldsymbol{y}_i, R) \cap B(\boldsymbol{y}_{i+1}, R) \neq \emptyset$. Note however, that the balls could be repeated, so the number m could be much larger than ℓ. To avoid this problem, we replace the curve $\boldsymbol{\gamma}$ with a polygonal path $\boldsymbol{\gamma}_1$ obtained by joining $\boldsymbol{x}_1$ with $\boldsymbol{y}_1$, $\boldsymbol{y}_i$ with $\boldsymbol{y}_{i+1}$, for $i = 1, \ldots, m-1$, and $\boldsymbol{y}_m$ with $\boldsymbol{x}_2$. If $\boldsymbol{y}_i = \boldsymbol{y}_j$ for some $i < j$, we remove the balls $B(\boldsymbol{y}_i, R)$, ..., $B(\boldsymbol{y}_{j-1}, R)$. This elimination process guarantees that $m \leq \ell$. Choose $\boldsymbol{z}_i \in B(\boldsymbol{y}_i, R) \cap B(\boldsymbol{y}_{i+1}, R)$,

By applying Step 1 we have that

$$\begin{aligned} u(\boldsymbol{x}_1) &\leq \sup_{B(\boldsymbol{y}_1,R)} u \leq 3^N \inf_{B(\boldsymbol{y}_1,R)} u \leq 3^N u(\boldsymbol{z}_1) \leq 3^N \sup_{B(\boldsymbol{y}_2,R)} u \leq 3^{2N} \inf_{B(\boldsymbol{y}_2,R)} u \\ &\leq 3^{2N} u(\boldsymbol{z}_2) \leq \cdots \leq 3^{mN} \inf_{B(\boldsymbol{y}_m,R)} u \leq 3^{mN} u(\boldsymbol{x}_2). \end{aligned}$$

Hence, $u(\boldsymbol{x}_1) \leq 3^{\ell N} u(\boldsymbol{x}_2)$ for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in U$, where ℓ depends only on U and Ω. It follows that $\sup_U u \leq 3^{\ell N} \inf_U u$. ■

Exercise 76 *Prove that if u is a radial function, that is, $u(\boldsymbol{x}) = g(\|\boldsymbol{x}\|)$, with g of class C^2, then in $\mathbb{R}^N \setminus \{\mathbf{0}\}$,*

$$\Delta u(\boldsymbol{x}) = g''(\|\boldsymbol{x}\|) + \frac{N-1}{\|\boldsymbol{x}\|} g'(\|\boldsymbol{x}\|).$$

Example 77 *Harnack's inequality fails for superharmonic and subharmonic functions. To see this, let $N = 2$, fix $\varepsilon > 0$, and consider the radial function u_ε defined by $u_\varepsilon(\boldsymbol{x}) = g_\varepsilon(\|\boldsymbol{x}\|)$, where*

$$g_\varepsilon(r) := \begin{cases} \log \frac{R}{r} & \text{if } r \geq \varepsilon, \\ \log \frac{R}{\varepsilon} + \frac{3}{4} - \frac{r^2}{\varepsilon^2} + \frac{r^4}{4\varepsilon^4} & \text{if } 0 \leq r < \varepsilon, \end{cases}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq R$. Then

$$\frac{g_\varepsilon(0)}{g_\varepsilon\left(\frac{R}{2}\right)} = \frac{\log \frac{R}{\varepsilon} + \frac{3}{4}}{\log \frac{1}{2}} \to \infty$$

as $\varepsilon \to 0^+$ (so Harnack's inequality fails) but

$$g_\varepsilon'(r) = \begin{cases} -\frac{1}{r} & \text{if } r > \varepsilon, \\ -\frac{2r}{\varepsilon^2} + \frac{r^3}{\varepsilon^4} & \text{if } 0 \leq r < \varepsilon, \end{cases} \qquad g_\varepsilon''(r) = \begin{cases} \frac{1}{r^2} & \text{if } r > \varepsilon, \\ -\frac{2}{\varepsilon^2} + \frac{3r^2}{\varepsilon^4} & \text{if } 0 \leq r < \varepsilon, \end{cases}$$

and so

$$\begin{aligned} g_\varepsilon''(r) + \frac{1}{r} g_\varepsilon(r) &= \begin{cases} \frac{1}{r^2} - \frac{1}{r^2} & \text{if } r > \varepsilon, \\ -\frac{2}{\varepsilon^2} + \frac{3r^2}{\varepsilon^4} - \frac{2}{\varepsilon^2} + \frac{r^2}{\varepsilon^4} & \text{if } 0 \leq r < \varepsilon, \end{cases} \\ &= \begin{cases} 0 & \text{if } r > \varepsilon, \\ \frac{4}{\varepsilon^4}\left(-\varepsilon^2 + r^2\right) & \text{if } 0 \leq r < \varepsilon, \end{cases} \end{aligned}$$

which shows that g_ε is superharmonic. A similar result can be shown for subharmonic. It suffices to consider $-g_\varepsilon(r) + C_\varepsilon$.

Exercise 78 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\{u_n\} \subseteq C^2(\Omega)$ be a decreasing sequence of harmonic functions in Ω. Assume that there exists $\boldsymbol{x}_0 \in \Omega$ such that the limit*

$$\lim_{n\to\infty} u_n(\boldsymbol{x}_0) \in \mathbb{R}$$

and prove that $\{u_n\}$ converges uniformly on compact sets of Ω to an harmonic function u.

Friday, October 18, 2013

Midsemester break. Make-up classes. Two hours

8 Schauder Estimates for the Neumann Problem

8.1 Interior Schauder Estimates

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $f \in C(\Omega)$, and consider the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

We are interested in then the regularity of u. We would expect that $u \in C^2(\Omega)$. However, this is not the case. See Homework #5.

Theorem 79 (Weak Maximum Principle) *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set, let $f \in C(\overline{\Omega})$ and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Poisson's equation*

$$-\Delta u = f \quad \textit{in } \Omega.$$

Then

$$\sup_{\Omega} |u| \leq \max_{\partial\Omega} |u| + \frac{(\operatorname{diam}\Omega)^2}{4N} \max_{\overline{\Omega}} |f|.$$

Proof. [5]Let $R := \frac{1}{2}\operatorname{diam}\Omega$ and, without loss of generality, assume that $\Omega \subseteq B(\mathbf{0}, R)$. Define,

$$L := \max_{\overline{\Omega}} |f|, \quad M := \max_{\partial\Omega} |u|$$

and set

$$w(\boldsymbol{x}) := M + \frac{L}{2N}\left(R^2 - \|\boldsymbol{x}\|^2\right).$$

Then $\Delta w = -L$ in Ω and $w \geq M$ on $\partial\Omega$. Hence,

$$\Delta(w \pm u) = -L \mp f \leq 0 \quad \text{in } \Omega$$

and $w \pm u \geq M \pm u \geq 0$ on $\partial\Omega$. Since $w \pm u$ is superharmonic, by the weak maximum principle,

$$\min_{\overline{\Omega}} (w \pm u) = \min_{\partial\Omega} (w \pm u),$$

[5]I did not prove this in class. Please read it.

that is,

$$(w \pm u)(\boldsymbol{x}) \geq \min_{\partial\Omega}(w \pm u) \geq 0 \quad \text{for all } \boldsymbol{x} \in \Omega.$$

This implies that

$$-w(\boldsymbol{x}) \leq u(\boldsymbol{x}) \leq w(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega$$

and concludes the proof. ■

Definition 80 *Let $E \subseteq \mathbb{R}^N$ and let $f : E \to \mathbb{R}$. The* modulus of continuity *of f is the increasing function $\omega_f : [0, \infty) \to [0, \infty)$ defined by*

$$\omega_f(t) := \sup\{|f(\boldsymbol{x}) - f(\boldsymbol{y})| : \ \boldsymbol{x}, \boldsymbol{y} \in E, \ \|\boldsymbol{x} - \boldsymbol{y}\| \leq t\}.$$

The function f is Dini's continuous *if*

$$\int_0^1 \frac{\omega_f(t)}{t}\, dt < \infty.$$

Definition 81 *Let $E \subseteq \mathbb{R}^N$ and let $f : E \to \mathbb{R}$. The function f is* Hölder continuous *with exponent $\alpha \in (0,1)$ if*

$$|f|_{C^{0,\alpha}(E)} := \sup\left\{\frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha} : \ \boldsymbol{x}, \boldsymbol{y} \in E, \ \boldsymbol{x} \neq \boldsymbol{y}\right\} < \infty.$$

Remark 82 *A Lipschitz function or a Hölder continuous function is Dini's continuous since $\omega_f(t) \leq Lt^\alpha$ for $0 < \alpha \leq 1$.*

Theorem 83 (Schauder Estimate in a Ball) *Let $f : B(\mathbf{0}, 1) \to \mathbb{R}$ be Dini's continuous and let $u \in C^2(B(\mathbf{0}, 1)) \cap C\left(\overline{B(\mathbf{0}, 1)}\right)$ be a solution of*

$$-\Delta u = f \quad \text{in } B(\mathbf{0}, 1).$$

Then for all $\boldsymbol{x}, \boldsymbol{y} \in B(\mathbf{0}, 1/4)$,

$$\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\| \leq C(N)\left\{\|\boldsymbol{x} - \boldsymbol{y}\| \sup_{B(\mathbf{0},1)}|u| + \int_0^{8\|\boldsymbol{x}-\boldsymbol{y}\|}\frac{\omega_f(t)}{t}\, dt + \|\boldsymbol{x} - \boldsymbol{y}\|\int_{\|\boldsymbol{x}-\boldsymbol{y}\|}^1 \frac{\omega_f(t)}{t^2}\, dt\right\}.$$

In particular, if f is Hölder continuous with exponent $\alpha \in (0,1)$, then

$$\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\mathbf{0},1/2))} \leq C(N)\left\{\sup_{B(\mathbf{0},1)}|u| + \frac{1}{\alpha(1-\alpha)}|f|_{C^{0,\alpha}(B(\mathbf{0},1))}\right\},$$

while if f is Lipschitz continuous, then for all $\boldsymbol{x}, \boldsymbol{y} \in B(\mathbf{0}, 1/4)$,

$$\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\| \leq C(N)\|\boldsymbol{x} - \boldsymbol{y}\|\left\{\sup_{B(\mathbf{0},1)}|u| + \operatorname{Lip} f\,|\log\|\boldsymbol{x} - \boldsymbol{y}\||\right\}.$$

Proof. Step 1: Fix $\boldsymbol{y} \in B(\mathbf{0}, 1/4)$. For $n \in \mathbb{N}$ let u_n be the solution of

$$\begin{cases} -\Delta u_n(\boldsymbol{x}) = f(\boldsymbol{y}) & \text{in } B(\boldsymbol{y}, 1/2^n), \\ u_n(\boldsymbol{x}) = u(\boldsymbol{x}) & \text{on } \partial B(\boldsymbol{y}, 1/2^n). \end{cases}$$

We will prove that $\nabla^2 u_n(\boldsymbol{y}) \to \nabla^2 u(\boldsymbol{y})$.

We have $-\Delta(u - u_n)(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\boldsymbol{y})$ in $B(\boldsymbol{y}, 1/2^n)$ with $u - u_n = 0$ on $\partial B(\boldsymbol{y}, 1/2^n)$ and so, by the weak maximum principle,

$$\begin{aligned} \sup_{B(\boldsymbol{y},1/2^n)} |u - u_n| &\le \frac{1}{2^{2n+1}N} \max_{\boldsymbol{x} \in \overline{B(\boldsymbol{y},1/2^n)}} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\ &\le \frac{1}{2^{2n+1}N} \omega_f\left(\frac{1}{2^n}\right). \end{aligned} \tag{72}$$

In turn, since ω_f is increasing

$$\begin{aligned} \sup_{B(\boldsymbol{y},1/2^{n+1})} |u_{n+1} - u_n| &\le \sup_{B(\boldsymbol{y},1/2^{n+1})} |u_{n+1} - u| + \sup_{B(\boldsymbol{y},1/2^n)} |u - u_n| \\ &\le \frac{1}{2^{2n+2}N} \omega_f\left(\frac{1}{2^{n+1}}\right) + \frac{1}{2^{2n+1}N} \omega_f\left(\frac{1}{2^n}\right) \\ &\le \frac{1}{2^{2n}N} \omega_f\left(\frac{1}{2^n}\right). \end{aligned} \tag{73}$$

On the other hand, since $u_{n+1} - u_n$ is harmonic in $B\left(\boldsymbol{y}, 1/2^{n+1}\right)$, by Theorem 65,

$$\left| \frac{\partial^{\boldsymbol{\alpha}} (u_{n+1} - u_n)}{\partial \boldsymbol{x}_{\boldsymbol{\alpha}}} (\boldsymbol{x}) \right| \le \left(\frac{N}{R}\right)^{|\boldsymbol{\alpha}|} e^{|\boldsymbol{\alpha}|-1} |\boldsymbol{\alpha}|! \max_{\overline{B(\boldsymbol{x},R)}} |u_{n+1} - u_n|$$

for every $\overline{B(\boldsymbol{x}, R)} \subset B\left(\boldsymbol{y}, 1/2^{n+1}\right)$. In particular,

$$\begin{aligned} \sup_{B(\boldsymbol{y},1/2^{n+2})} \left\| \nabla^2 (u_{n+1} - u_n) \right\| &\le C(N) \left(2^{n+2}\right)^2 \max_{\overline{B(\boldsymbol{y},1/2^{n+1})}} |u_{n+1} - u_n| \\ &\le C(N) \frac{2^{2n+4}}{2^{2n+1}} \omega_f\left(\frac{1}{2^n}\right) = C(N) 2^3 \omega_f\left(\frac{1}{2^n}\right). \end{aligned} \tag{74}$$

Let

$$p(\boldsymbol{x}) := u(\boldsymbol{y}) + \nabla u(\boldsymbol{y})(\boldsymbol{x} - \boldsymbol{y}) + \frac{1}{2} (\boldsymbol{x} - \boldsymbol{y})^T \nabla^2 u(\boldsymbol{y}) (\boldsymbol{x} - \boldsymbol{y}).$$

Since $u \in C^2$, by Taylor's formula, for all n sufficiently large we have

$$\sup_{B(\boldsymbol{y},1/2^n)} |u - p| = o\left(\frac{1}{2^{2n}}\right). \tag{75}$$

Hence, by (72),

$$\begin{aligned} \sup_{B(\boldsymbol{y},1/2^n)} |u_n - p| &\le \sup_{B(\boldsymbol{y},1/2^n)} |u_n - u| + \sup_{B(\boldsymbol{y},1/2^n)} |u - p| \\ &\le \frac{1}{2^{2n+1}N} \omega_f\left(\frac{1}{2^n}\right) + o\left(\frac{1}{2^{2n}}\right) = o\left(\frac{1}{2^{2n}}\right). \end{aligned}$$

Since, $\Delta p = \Delta u(\boldsymbol{y}) = -f(\boldsymbol{y})$, the function $u_n - p$ is harmonic in $B(\boldsymbol{y}, 1/2^n)$, and so, by Theorem 65 applied to $u_n - p$, and by the previous inequality,

$$\left\|\nabla^2 u_n(\boldsymbol{y}) - \nabla^2 u(\boldsymbol{y})\right\| = \left\|\nabla^2 u_n(\boldsymbol{y}) - \nabla^2 p(\boldsymbol{y})\right\| \le C(N)\, 2^{2n} \sup_{B(\boldsymbol{y},1/2^n)} |u_n - p| = o(1).$$

It follows that $\nabla^2 u_n(\boldsymbol{y}) \to \nabla^2 u(\boldsymbol{y})$.

Step 2: Given $\boldsymbol{z} \in B(\boldsymbol{0}, 1/4) \setminus \{\boldsymbol{y}\}$, write

$$\begin{aligned}\left\|\nabla^2 u(\boldsymbol{z}) - \nabla^2 u(\boldsymbol{y})\right\| &\le \left\|\nabla^2 u_\ell(\boldsymbol{y}) - \nabla^2 u(\boldsymbol{y})\right\| + \left\|\nabla^2 u_\ell(\boldsymbol{z}) - \nabla^2 u(\boldsymbol{z})\right\| \\ &\quad + \left\|\nabla^2 u_\ell(\boldsymbol{z}) - \nabla^2 u_\ell(\boldsymbol{y})\right\| =: I_1 + I_2 + I_3,\end{aligned} \tag{76}$$

where ℓ is so large that $\frac{1}{2^{\ell+2}} \le \|\boldsymbol{z} - \boldsymbol{y}\| < \frac{1}{2^{\ell+1}}$. Then by (74) and the fact that $\nabla^2 u_n(\boldsymbol{y}) \to \nabla^2 u(\boldsymbol{y})$, using telescopic series we have

$$\begin{aligned} I_1 &= \left\|\sum_{n=1}^{\ell-1} \left(\nabla^2 u_{n+1} - \nabla^2 u_n\right)(\boldsymbol{y}) - \sum_{n=1}^{\infty} \left(\nabla^2 u_{n+1} - \nabla^2 u_n\right)(\boldsymbol{y})\right\| \\ &\le \sum_{n=\ell}^{\infty} \left\|\left(\nabla^2 u_{n+1} - \nabla^2 u_n\right)(\boldsymbol{y})\right\| \le C(N) \sum_{n=\ell}^{\infty} \omega_f\left(\frac{1}{2^n}\right) \\ &= C(N) \sum_{n=\ell}^{\infty} 2 \frac{2^{n-1}}{2^n} \omega_f\left(\frac{1}{2^n}\right) \le C(N) \sum_{n=\ell}^{\infty} \int_{1/2^n}^{1/2^{n-1}} \frac{\omega_f(t)}{t}\, dt \\ &\le C(N) \int_0^{8\|\boldsymbol{z}-\boldsymbol{y}\|} \frac{\omega_f(t)}{t}\, dt. \end{aligned} \tag{77}$$

To estimate I_2, let v_n be the solution of

$$\begin{cases} -\Delta v_n(\boldsymbol{x}) = f(\boldsymbol{z}) & \text{in } B(\boldsymbol{z}, 1/2^n), \\ v_n(\boldsymbol{x}) = u(\boldsymbol{x}) & \text{on } \partial B(\boldsymbol{z}, 1/2^n) \end{cases}$$

for $n \ge \ell$. Then

$$I_2 \le \left\|\nabla^2 v_\ell(\boldsymbol{z}) - \nabla^2 u(\boldsymbol{z})\right\| + \left\|\nabla^2 v_\ell(\boldsymbol{z}) - \nabla^2 u_\ell(\boldsymbol{z})\right\| =: II_1 + II_2. \tag{78}$$

Reasoning as in the first part of the proof, with $\boldsymbol{y}$ replaced by $\boldsymbol{z}$, we have that

$$II_1 = \left\|\nabla^2 v_\ell(\boldsymbol{z}) - \nabla^2 u(\boldsymbol{z})\right\| \le C(N) \int_0^{8\|\boldsymbol{z}-\boldsymbol{y}\|} \frac{\omega_f(t)}{t}\, dt. \tag{79}$$

To estimate II_2, let $\Omega_\ell := B\left(\boldsymbol{z}, 1/2^\ell\right) \cap B\left(\boldsymbol{y}, 1/2^\ell\right)$. Then $u_\ell - v_\ell$ satisfies the equation

$$-\Delta(v_\ell - u_\ell)(\boldsymbol{x}) = f(\boldsymbol{z}) - f(\boldsymbol{y}) \quad \text{in } \Omega_\ell,$$

and thus by the weak maximum principle

$$\sup_{\Omega_\ell} |v_\ell - u_\ell| \le \max_{\partial\Omega_\ell} |v_\ell - u_\ell| + \frac{(\operatorname{diam}\Omega_\ell)^2}{4N} |f(\boldsymbol{z}) - f(\boldsymbol{y})|.$$

Since $u_\ell(\boldsymbol{x}) = u(\boldsymbol{x})$ on $\partial B(\boldsymbol{y}, 1/2^\ell)$ and $v_\ell(\boldsymbol{x}) = u(\boldsymbol{x})$ on $\partial B(\boldsymbol{z}, 1/2^\ell)$, we have that

$$\begin{aligned}\max_{\partial\Omega_\ell} |v_\ell - u_\ell| &= \max\left\{\max_{B(\boldsymbol{z},1/2^\ell)\cap\partial B(\boldsymbol{y},1/2^\ell)} |v_\ell - u|, \max_{B(\boldsymbol{y},1/2^\ell)\cap\partial B(\boldsymbol{z},1/2^\ell)} |u - v_\ell|\right\} \\ &\le \frac{1}{2^{2\ell+1}N}\omega_f\left(\frac{1}{2^\ell}\right)\end{aligned}$$

by (72) for u_ℓ and by the analogous estimate for v_ℓ. In turn,

$$\begin{aligned}\sup_{\Omega_\ell} |v_\ell - u_\ell| &\le \frac{1}{2^{2\ell+1}N}\omega_f\left(\frac{1}{2^\ell}\right) + \frac{1}{2^{2\ell+2}N}|f(\boldsymbol{z}) - f(\boldsymbol{y})| \\ &\le \frac{1}{2^{2\ell-1}N}\omega_f\left(\frac{1}{2^\ell}\right).\end{aligned}$$

By Theorem 65, Remark 67, with $R = \frac{1}{2^{\ell+1}}$, and the previous estimate

$$II_2 \le C(N)\, 2^{2\ell} \max_{B\left(\boldsymbol{z},\frac{1}{2^{\ell+1}}\right)} |v_\ell - u_\ell| \le C(N)\,\omega_f\left(\frac{1}{2^\ell}\right) \le C(N)\int_{1/2^\ell}^{1/2^{\ell-1}} \frac{\omega_f(t)}{t}\,dt, \tag{80}$$

where we have used the fact that $\overline{B\left(\boldsymbol{z}, \frac{1}{2^{\ell+1}}\right)} \subset \Omega_\ell$, since $\|\boldsymbol{z} - \boldsymbol{y}\| < \frac{1}{2^{\ell+1}}$. Combining (78), (79), and (80) gives

$$I_2 \le C(N)\int_0^{8\|\boldsymbol{z}-\boldsymbol{y}\|} \frac{\omega_f(t)}{t}\,dt. \tag{81}$$

Finally, it remains to estimate I_3. Let $h_n := u_n - u_{n-1}$, for $n = 2, \dots, \ell$. Then by the mean value theorem

$$\left|\frac{\partial^2 h_n}{\partial x_i \partial x_j}(\boldsymbol{z}) - \frac{\partial^2 h_n}{\partial x_i \partial x_j}(\boldsymbol{y})\right| \le \left\|\nabla^3 h_n(\boldsymbol{w})\right\| \|\boldsymbol{z} - \boldsymbol{y}\|$$

for some $\boldsymbol{w}$ between $\boldsymbol{z}$ and $\boldsymbol{y}$. In turn, by Theorem 65 with $R = \frac{1}{2^{n+1}}$, and (73),

$$\left\|\nabla^3 h_n(\boldsymbol{w})\right\| \le C(N)\, 2^{3n} \max_{B\left(\boldsymbol{w},\frac{1}{2^{n+1}}\right)} |h_n| \le C(N)\, 2^n \omega_f\left(\frac{1}{2^n}\right).$$

Hence,

$$\left\|\nabla^2 h_n(\boldsymbol{z}) - \nabla^2 h_n(\boldsymbol{y})\right\| \le C(N)\, 2^n \omega_f\left(\frac{1}{2^n}\right)\|\boldsymbol{z} - \boldsymbol{y}\|.$$

Similarly, by the mean value theorem, Theorem 65, Remark 67, and (72),

$$\begin{aligned}\left|\frac{\partial^2 u_1}{\partial x_i \partial x_j}(\boldsymbol{z})-\frac{\partial^2 u_1}{\partial x_i \partial x_j}(\boldsymbol{y})\right| &\leq \left\|\nabla^3 u_1(\boldsymbol{\zeta})\right\|\|\boldsymbol{z}-\boldsymbol{y}\| \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\frac{N}{1/2}\right)^3 e^{3-1} 3! \max_{B(\boldsymbol{y},1/2)}|u_1| \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\max_{B(\boldsymbol{y},1/2)}|u_1-u|+\max_{B(\boldsymbol{y},1/2)}|u|\right) \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\frac{1}{2N}\omega_f\left(\frac{1}{2}\right)+\max_{B(\boldsymbol{y},1/2)}|u|\right),\end{aligned}$$

and so

$$\left\|\nabla^2 u_1(\boldsymbol{z})-\nabla^2 u_1(\boldsymbol{y})\right\| \leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\frac{1}{2N}\omega_f\left(\frac{1}{2}\right)+\max_{B(\boldsymbol{y},1/2)}|u|\right).$$

Using telescopic sums

$$\begin{aligned}I_3 &\leq \left\|\nabla^2 u_1(\boldsymbol{z})-\nabla^2 u_1(\boldsymbol{y})\right\|+\sum_{n=2}^{\ell}\left\|\nabla^2 h_n(\boldsymbol{z})-\nabla^2 h_n(\boldsymbol{y})\right\| \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\max_{B(\boldsymbol{y},1/2)}|u|+\omega_f(1/2)\right)+C(N)\sum_{n=2}^{\ell} 2^n \omega_f\left(\frac{1}{2^n}\right)\|\boldsymbol{z}-\boldsymbol{y}\| \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\max_{B(\boldsymbol{y},1/2)}|u|+\sum_{n=1}^{\ell}\int_{1/2^n}^{1/2^{n-1}}\frac{\omega_f(t)}{t^2}\,d\right) \\ &= C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\max_{B(\boldsymbol{y},1/2)}|u|+\int_{1/2^\ell}^{1}\frac{\omega_f(t)}{t^2}\,d\right) \\ &\leq C(N)\|\boldsymbol{z}-\boldsymbol{y}\|\left(\max_{B(\boldsymbol{y},1/2)}|u|+\int_{\|\boldsymbol{x}-\boldsymbol{y}\|}^{1}\frac{\omega_f(t)}{t^2}\,dt\right).\end{aligned}$$

■

Monday, October 21, 2013

Remark 84 *Let* $f: B(\boldsymbol{x}_0, R) \to \mathbb{R}$ *be Dini's continuous and let* $u \in C^2(B(\boldsymbol{x}_0, R)) \cap C\left(\overline{B(\boldsymbol{x}_0, R)}\right)$ *be a solution of*

$$-\Delta u = f \quad \text{in } B(\boldsymbol{x}_0, R).$$

Consider the function $v(\boldsymbol{y}) := u(\boldsymbol{x}_0 + R\boldsymbol{y})$, $\boldsymbol{y} \in B(\boldsymbol{0}, 1)$. *Then*

$$\frac{\partial^2 v}{\partial y_i \partial y_j}(\boldsymbol{y}) = R^2 \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}_0 + R\boldsymbol{y}),$$

and so $-\Delta_{\boldsymbol{y}} v(\boldsymbol{y}) = R^2 f(\boldsymbol{x}_0 + R\boldsymbol{y}) := f_R(\boldsymbol{y})$*. Note that*

$$\begin{aligned}
\omega_{f_R}(t) &= \sup\{|f_R(\boldsymbol{y}_1) - f(\boldsymbol{y}_2)| : \ \boldsymbol{y}_1, \boldsymbol{y}_2 \in B(\boldsymbol{0}, 1), \ \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \le t\} \\
&= R^2 \sup\{|f(\boldsymbol{x}_0 + R\boldsymbol{y}_1) - f(\boldsymbol{x}_0 + R\boldsymbol{y}_1)| : \ \boldsymbol{y}_1, \boldsymbol{y}_2 \in B(\boldsymbol{0}, 1), \ \|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \le t\} \\
&= R^2 \sup\{|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| : \ \boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\boldsymbol{x}_0, R), \ \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \le Rt\} = R^2 \omega_f(Rt).
\end{aligned}$$

and so for $\boldsymbol{y}_1, \boldsymbol{y}_2 \in B(\boldsymbol{0}, 1/4)$*, letting* $\boldsymbol{x}_1 := \boldsymbol{x}_0 + R\boldsymbol{y}_1$ *and* $\boldsymbol{x}_2 := \boldsymbol{x}_0 + R\boldsymbol{y}_2$*, using the change of variables* $s = Rt$*, we get*

$$\begin{aligned}
\int_0^{8\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|} \frac{\omega_{f_R}(t)}{t}\, dt &= R^2 \int_0^{8\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|} \frac{\omega_f(Rt)}{t}\, dt \\
&= R^2 \int_0^{8\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|} \frac{\omega_f(s)}{s}\, ds,
\end{aligned}$$

and similarly

$$\|\boldsymbol{y}_1 - \boldsymbol{y}_2\| \int_{\|\boldsymbol{y}_1 - \boldsymbol{y}_2\|}^1 \frac{\omega_{f_R}(t)}{t^2}\, dt = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| R^2 \int_{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|}^R \frac{\omega_f(s)}{s^2}\, ds.$$

Hence, by the previous theorem,

$$\begin{aligned}
\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{x}_2)\right\| &= \frac{1}{R^2} \left\|\nabla^2_{\boldsymbol{y}} v(\boldsymbol{y}_1) - \nabla^2_{\boldsymbol{y}} v(\boldsymbol{y}_2)\right\| \\
&\le \frac{C(N)}{R^2} \left\{ \frac{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|}{R} \sup_{B(\boldsymbol{x}_0, R)} |u| + R^2 \int_0^{8\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|} \frac{\omega_f(s)}{s}\, ds \right. \\
&\qquad \left. + \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| R^2 \int_{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|}^R \frac{\omega_f(s)}{s^2}\, ds \right\},
\end{aligned}$$

that is,

$$\begin{aligned}
\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{x}_2)\right\| \le C(N) &\left\{ \frac{1}{R^3} \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \sup_{B(\boldsymbol{x}_0, R)} |u| + \int_0^{8\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|} \frac{\omega_f(t)}{t}\, dt \right. \\
&\qquad \left. + \|\boldsymbol{x}_1 - \boldsymbol{x}_2\| \int_{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|}^R \frac{\omega_f(t)}{t^2}\, dt \right\}.
\end{aligned}$$

In particular, if f *is Hölder continuous with exponent* $\alpha \in (0, 1)$*, then*

$$\begin{aligned}
\frac{\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{x}_2)\right\|}{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^\alpha} &\le C \left\{ \frac{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^{1-\alpha}}{R^3} \sup_{B(\boldsymbol{x}_0, R)} |u| + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0, R))} \right\} \\
&\le C \left\{ \frac{1}{R^{2+\alpha}} \sup_{B(\boldsymbol{x}_0, R)} |u| + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0, R))} \right\},
\end{aligned}$$

where $C = C(N, \alpha)$*.*

Next we extend this result to linear elliptic equations of the form

$$-\sum_{i,j=1}^{N} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } B(\boldsymbol{x}_0, R),$$

where the matrix $\boldsymbol{A} := (a_{i,j})_{i,j}$ is symmetric and positive definite. For simplicity we consider only the Hölder continuous case.

Theorem 85 *Let $f \in C^{0,\alpha}(B(\boldsymbol{x}_0, R))$, $\alpha \in (0,1)$, $\boldsymbol{x}_0 \in \mathbb{R}^N$, $R > 0$, and let $u \in C^2(B(\boldsymbol{x}_0, R)) \cap C\left(\overline{B(\boldsymbol{x}_0, R)}\right)$ be a solution of*

$$-\sum_{i,j=1}^{N} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } B(\boldsymbol{x}_0, R),$$

where the matrix $\boldsymbol{A} := (a_{i,j})_{i,j}$ is symmetric and positive definite. Then, for all $0 < r \le \sqrt{\frac{\lambda_1}{\lambda_N}} \frac{R}{4}$,

$$\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0, r))} \le C \left\{ \frac{1}{r^{2+\alpha}} \sup_{B(\boldsymbol{x}_0, R)} |u| + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0, R))} \right\},$$

where $\lambda_1 < \cdots < \lambda_N$ are the eigenvalues of the matrix $\boldsymbol{A}$ and $C = C(N, \alpha, \lambda_1, \lambda_N) > 0$.

Proof. Without loss of generality, take $\boldsymbol{x}_0 = \boldsymbol{0}$. If $\boldsymbol{B}$ is an invertible $N \times N$ matrix, with the change of variables $\boldsymbol{y} := \boldsymbol{x}\boldsymbol{B}$ and $v(\boldsymbol{y}) := u\left(\boldsymbol{y}\boldsymbol{B}^{-1}\right)$, we have that

$$\sum_{i,j=1}^{N} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = \sum_{i,j=1}^{N} c_{i,j} \frac{\partial^2 v}{\partial y_i \partial y_j}(\boldsymbol{y}),$$

where $\boldsymbol{C} := \boldsymbol{B}^T \boldsymbol{A} \boldsymbol{B}$.

Find an orthogonal matrix $\boldsymbol{O}$ (that is $\boldsymbol{O}^T \boldsymbol{O} = \boldsymbol{I}_N$) such that

$$\boldsymbol{O}^T \boldsymbol{A} \boldsymbol{O} = \operatorname{diag}(\lambda_1, \ldots, \lambda_N).$$

Since an orthogonal matrix preserves length, that is,

$$\boldsymbol{x}_1 \boldsymbol{O} \cdot \boldsymbol{x}_2 \boldsymbol{O} = \boldsymbol{x}_1 \cdot \boldsymbol{x}_2,$$

the change of variables $\boldsymbol{x} \mapsto \boldsymbol{x}\boldsymbol{O}$ maps a ball into a ball of same radius. Let $\boldsymbol{D} = \operatorname{diag}\left(\lambda_1^{-1/2}, \ldots, \lambda_N^{-1/2}\right)$. Then

$$\boldsymbol{D}^T \left(\boldsymbol{O}^T \boldsymbol{A} \boldsymbol{O}\right) \boldsymbol{D} = \boldsymbol{D}^T (\operatorname{diag}(\lambda_1, \ldots, \lambda_N)) \boldsymbol{D} = I_N \tag{82}$$

and the change variables $\boldsymbol{z} \mapsto \boldsymbol{z}\boldsymbol{D}$ is such that $\lambda_N^{-1/2} \|\boldsymbol{z}\| \le \|\boldsymbol{z}\boldsymbol{D}\| \le \lambda_1^{-1/2} \|\boldsymbol{z}\|$.

Finally, let $\boldsymbol{B} := \boldsymbol{ODR}$, where $\boldsymbol{R}$ is a rotation such that the half space $\{x_N > 0\}$ is mapped into the half space $\{y_N > 0\}$, and consider $\boldsymbol{y} := \boldsymbol{xB}$ and $v(\boldsymbol{y}) := u\left(\boldsymbol{yB}^{-1}\right)$. Then $u(\boldsymbol{x}) = v(\boldsymbol{xB})$ and so

$$\frac{\partial u}{\partial x_j}(\boldsymbol{x}) = \sum_{k=1}^{N} b_{j,k} \frac{\partial v}{\partial y_k}(\boldsymbol{xB}), \tag{83}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = \sum_{k,l=1}^{N} b_{i,l} b_{j,k} \frac{\partial^2 v}{\partial y_l \partial y_k}(\boldsymbol{xB}), \tag{84}$$

which implies that

$$\begin{aligned}
\sum_{i,j=1}^{N} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) &= \sum_{i,j,k,l=1}^{N} a_{i,j} b_{i,l} b_{j,k} \frac{\partial^2 v}{\partial y_l \partial y_k}(\boldsymbol{xB}) \\
&= \sum_{k,l=1}^{N} \frac{\partial^2 v}{\partial y_l \partial y_k}(\boldsymbol{xB}) \sum_{i,j=1}^{N} a_{i,j} b_{i,l} b_{j,k} \\
&= \sum_{k,l=1}^{N} \frac{\partial^2 v}{\partial y_l \partial y_k}(\boldsymbol{xB}) \delta_{l,k} = \sum_{k=1}^{N} \frac{\partial^2 v}{\partial y_k^2}(\boldsymbol{xB}),
\end{aligned}$$

where we used (82). This shows that

$$\sum_{i,j=1}^{N} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = \Delta_{\boldsymbol{y}} v(\boldsymbol{y}).$$

Moreover,

$$\sqrt{\lambda_1} \|\boldsymbol{y}\| \le \|\boldsymbol{x}\| \le \sqrt{\lambda_N} \|\boldsymbol{y}\|. \tag{85}$$

Hence, if $\boldsymbol{\Psi}(\boldsymbol{x}) := \boldsymbol{xB}$, then for all $t > 0$ sufficeintly small

$$B(\boldsymbol{0}, t) \subseteq \boldsymbol{\Psi}\left(B\left(\boldsymbol{0}, 4\sqrt{\lambda_N} t\right)\right) \subseteq B\left(\boldsymbol{0}, 4\sqrt{\frac{\lambda_N}{\lambda_1}} t\right). \tag{86}$$

Consider the equation

$$-\Delta_{\boldsymbol{y}} v(\boldsymbol{y}) = f_1(\boldsymbol{y}) \quad \text{in } B(\boldsymbol{0}, 4s),$$

where $f_1(\boldsymbol{y}) := f\left(\boldsymbol{yB}^{-1}\right)$. Observe that if $\boldsymbol{y}_1, \boldsymbol{y}_2 \in B(\boldsymbol{0}, 4s)$, then

$$\begin{aligned}
|f_1(\boldsymbol{y}_1) - f_1(\boldsymbol{y}_2)| &= \left|f\left(\boldsymbol{y}_1 \boldsymbol{B}^{-1}\right) - f\left(\boldsymbol{y}_2 \boldsymbol{B}^{-1}\right)\right| \\
&\le |f|_{C^{0,\alpha}\left(B\left(\boldsymbol{0}, 4\sqrt{\lambda_N} s\right)\right)} \left\|(\boldsymbol{y}_1 - \boldsymbol{y}_2) \boldsymbol{B}^{-1}\right\|^{\alpha} \\
&\le |f|_{C^{0,\alpha}\left(B\left(\boldsymbol{0}, 4\sqrt{\lambda_N} s\right)\right)} \lambda_N^{\alpha/2} \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|^{\alpha}
\end{aligned}$$

and so

$$|f_1|_{C^{0,\alpha}(B(\boldsymbol{0}, 4s))} \le |f|_{C^{0,\alpha}\left(B\left(\boldsymbol{0}, 4\sqrt{\lambda_N} s\right)\right)} \lambda_N^{\alpha/2}. \tag{87}$$

By the previous theorem and Remark 84,

$$\left|\nabla^2 v\right|_{C^{0,\alpha}(B(\mathbf{0},s))} \leq C\left\{\frac{1}{s^{2+\alpha}}\sup_{B(\mathbf{0},4s)}|v| + |f_1|_{C^{0,\alpha}(B(\mathbf{0},4s))}\right\}.$$

Let $0 < r \leq \sqrt{\frac{\lambda_1}{\lambda_N}}\frac{R}{4}$ and take $s := \frac{r}{\sqrt{\lambda_1}}$. By (84), (85), and (86), for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B(\mathbf{0}, r)$,

$$\begin{aligned}
&\left\|\nabla^2_{\boldsymbol{x}} u(\boldsymbol{x}_1) - \nabla^2_{\boldsymbol{x}} u(\boldsymbol{x}_2)\right\| \leq \frac{C}{\lambda_1}\left\|\nabla^2_{\boldsymbol{y}} v(\boldsymbol{x}_1\boldsymbol{B}) - \nabla^2_{\boldsymbol{y}} v(\boldsymbol{x}_2\boldsymbol{B})\right\| \\
&\leq \frac{C}{\lambda_1}\left\|(\boldsymbol{x}_1 - \boldsymbol{x}_2)\boldsymbol{B}\right\|^\alpha \left\{\frac{\lambda_1^{(2+\alpha)/2}}{r^{2+\alpha}}\sup_{B\left(\mathbf{0},4r/\sqrt{\lambda_1}\right)}|v| + |f_1|_{C^{0,\alpha}\left(B\left(\mathbf{0},4r/\sqrt{\lambda_1}\right)\right)}\right\} \qquad (88)\\
&\leq \frac{C}{\lambda_1^{1+\alpha/2}}\left\|\boldsymbol{x}_1 - \boldsymbol{x}_2\right\|^\alpha \left\{\frac{\lambda_1^{(2+\alpha)/2}}{r^{2+\alpha}}\sup_{B(\mathbf{0},R)}|u| + \lambda_N^{\alpha/2}|f|_{C^{0,\alpha}(B(\mathbf{0},R))}\right\}.
\end{aligned}$$

■

Next we extend this result to linear elliptic equations of the form

$$-\sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x})\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } B(\boldsymbol{x}_0, r),$$

where the matrix $\boldsymbol{A}(\boldsymbol{x}) := (a_{i,j}(\boldsymbol{x}))_{i,j}$ is symmetric and positive definite. We will also consider elliptic equations in divergence form, that is,

$$-\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}) \quad \text{in } B(\boldsymbol{x}_0, r).$$

Definition 86 *Given an open set $\Omega \subseteq \mathbb{R}^N$, a nonnegative integer m and $\alpha \in (0,1)$, the space $C^{m,\alpha}(\Omega)$ is the space of all functions $u \in C^m(\Omega)$ such that*

$$|\nabla^m u|_{C^{0,\alpha}(\Omega)} < \infty.$$

It is endowed with the norm

$$\|u\|_{C^{m,\alpha}(\Omega)} := \|u\|_{C^m(\Omega)} + |\nabla^m u|_{C^{0,\alpha}(\Omega)}.$$

Exercise 87 *Let $u \in C^{2,\alpha}\left(\overline{B(\mathbf{0},r)}\right)$, $\alpha \in (0,1)$. Prove that there exists $C(N) > 0$ such that*

$$\sup_{B(\mathbf{0},r)}\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| \leq \varepsilon\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{C^{0,\alpha}(B(\mathbf{0},r))} + \varepsilon\left|\frac{\partial^2 u}{\partial x_i^2}\right|_{C^{0,\alpha}(B(\mathbf{0},r))} + \frac{C}{\varepsilon^{2/\alpha}}\sup_{B(\mathbf{0},r)}|u|$$

for all $0 < \varepsilon << r$.

Theorem 88 *Let* $a_{i,j} \in C^{0,\alpha}(B(\boldsymbol{x}_0, R))$, $i, j = 1, \ldots, N$, $\alpha \in (0,1)$, $\boldsymbol{x}_0 \in \mathbb{R}^N$, $R > 0$, *let* $f \in C^{0,\alpha}(B(\boldsymbol{x}_0, R))$, *and let* $u \in C^2(B(\boldsymbol{x}_0, R)) \cap C\left(\overline{B(\boldsymbol{x}_0, R)}\right)$ *be a solution of*

$$-\sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } B(\boldsymbol{x}_0, R),$$

where the matrix $\mathbf{A}(\boldsymbol{x}) := (a_{i,j}(\boldsymbol{x}))_{i,j}$ *is symmetric, positive definite, and*

$$\lambda \|\boldsymbol{\xi}\|^2 \le \sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}) \xi_i \xi_j \le \Lambda \|\boldsymbol{\xi}\|^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$, $\boldsymbol{x} \in B(\boldsymbol{x}_0, R)$ *and for some* $0 < \lambda < \Lambda$. *Then, for all* $0 < r \le \sqrt{\frac{\lambda}{\Lambda}} \frac{R}{4}$,

$$\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0, r))} \le C\left(\|u\|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))} + \|\nabla u\|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))} + \|f\|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}\right),$$

where $C = C(\lambda, \Lambda, \alpha, L, R) > 0$ *and*

$$L := \sum_{i,j=1}^{N} |a_{i,j}|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}.$$

Wednesday, October 23, 2013

Proof. Step 1: Assume that $u \in C_c^2(B(\boldsymbol{x}_0, R))$. We will use the method of freezing the coefficients. Let $\boldsymbol{x}_1, \boldsymbol{y}_1 \in B(\boldsymbol{x}_0, R)$. We rewrite our PDE as follows

$$-\sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}_1) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = \sum_{i,j=1}^{N} (a_{i,j}(\boldsymbol{x}) - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) + f(\boldsymbol{x}) =: f_1(\boldsymbol{x}).$$

Fix $s > 0$. If $\boldsymbol{y}_1 \in B(\boldsymbol{x}_1, s)$, then by the previous theorem,

$$\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\| \le C \|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha \left\{\frac{1}{s^{2+\alpha}} \sup_{B(\boldsymbol{x}_1, S)} |u| + |f_1|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))}\right\},$$

where $C = C(N, \alpha, \lambda, \Lambda)$ and $S := 4\sqrt{\frac{\Lambda}{\lambda}} s$. On the other hand,

$$|f_1|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \le |f|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} + \sum_{i,j=1}^{N} \left|(a_{i,j} - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))}.$$

Since $|hg|_{C^{0,\alpha}} \leq \|h\|_\infty |g|_{C^{0,\alpha}} + \|g\|_\infty |h|_{C^{0,\alpha}}$. We have

$$\begin{aligned}
&\left|(a_{i,j} - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \\
&\leq \|a_{i,j} - a_{i,j}(\boldsymbol{x}_1)\|_{L^\infty(B(\boldsymbol{x}_1,S))} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \\
&\quad + \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^\infty(B(\boldsymbol{x}_1,S))} |a_{i,j} - a_{i,j}(\boldsymbol{x}_1)|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \\
&\leq S^\alpha |a_{i,j}|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} + \left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^\infty(B(\boldsymbol{x}_1,S))} |a_{i,j}|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))}.
\end{aligned} \tag{89}$$

By the previous exercise, with $\varepsilon = S^\alpha$,

$$\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^\infty(B(\boldsymbol{x}_1,S))} \leq S^\alpha \left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} + \frac{C}{S^2} \|u\|_{L^\infty(B(\boldsymbol{x}_1,S))}.$$

Hence,

$$\begin{aligned}
|f_1|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} &\leq |f|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \\
&\quad + L\left[2S^\alpha \left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} + \frac{C}{S^2} \|u\|_{L^\infty(B(\boldsymbol{x}_1,S))}\right].
\end{aligned}$$

In turn,

$$\begin{aligned}
\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\| &\leq C \|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha \left\{\left(\frac{1}{s^{2+\alpha}} + \frac{L}{s^2}\right) \sup_{B(\boldsymbol{x}_1,S)} |u| \right. \\
&\qquad \left. + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} + s^\alpha \left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))}\right\} \\
&\leq C \|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha \left\{\left(\frac{1}{s^{2+\alpha}} + \frac{1}{s^2}\right) \sup_{B(\boldsymbol{x}_1,S)} |u| \right. \\
&\qquad \left. + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))}\right\} + \frac{1}{2} \left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))},
\end{aligned}$$

provided

$$Cs^\alpha = \frac{1}{2}.$$

Since u has compact support in $B(\boldsymbol{x}_0, R)$ (and so does f), we have that

$$\begin{aligned}
\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\| &\leq C \|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha \left\{\sup_{B(\boldsymbol{x}_0,R)} |u| \right. \\
&\qquad \left. + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}\right\} + \frac{1}{2} \left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}.
\end{aligned}$$

On the other hand, if $\|\boldsymbol{x}_1 - \boldsymbol{y}_1\| \geq s$,

$$\begin{aligned}\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\| &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha}\left\{\left\|\nabla^2 u(\boldsymbol{x}_1)\right\| + \left\|\nabla^2 u(\boldsymbol{y}_1)\right\|\right\} \\ &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha} 2\left\|\nabla^2 u\right\|_{L^\infty(B(\boldsymbol{x}_0,R))} \\ &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha}\left(\frac{s^\alpha}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))} + \frac{C}{s^2}\|u\|_{L^\infty(B(\boldsymbol{x}_0,R))}\right),\end{aligned}$$

where in the last line we have used the previous exercise with $\varepsilon = \frac{1}{4}s^\alpha$. By combining these two inequalities we get

$$\begin{aligned}\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))} \leq C&\left\{\left(\frac{1}{s^{2+\alpha}} + \frac{1}{s^2}\right)\sup_{B(\boldsymbol{x}_0,R)}|u|\right. \\ &\left.+ |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}\right\} + \frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))},\end{aligned}$$

and so

$$\frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))} \leq C\left\{\left(\frac{1}{s^{2+\alpha}} + \frac{1}{S^2}\right)\sup_{B(\boldsymbol{x}_0,R)}|u| + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_0,R))}\right\}.$$

Step 2: Exercise. ■

Friday, October 25, 2013

8.2 Boundary Schauder Estimates

Next we consider Schauder estimates for the Neumann problem. In what follows, for $s > 0$,

$$\Gamma_s := \partial B^+(\mathbf{0}, s) \cap \left\{x \in \mathbb{R}^N : x_N = 0\right\}.$$

Theorem 89 *Let* $f \in C^{0,\alpha}(B^+(\mathbf{0},1))$, $g \in C^{1,\alpha}(\Gamma_1)$, $\alpha \in (0,1)$, *and let* $u \in C^2\left(\overline{B^+(\mathbf{0},1)}\right)$ *be a solution of*

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } B^+(\mathbf{0},1), \\ \frac{\partial u}{\partial x_N}(\boldsymbol{x}', 0) = g(\boldsymbol{x}', 0) & \text{on } \Gamma_1. \end{cases}$$

Then for all $\boldsymbol{x}, \boldsymbol{y} \in B^+\left(\mathbf{0}, \frac{1}{4}\right)$,

$$\begin{aligned}\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\| \leq C\|\boldsymbol{x} - \boldsymbol{y}\|&\left(\|u\|_{C(B^+(\mathbf{0},1))} + \|g\|_{C(\Gamma)}\right) \\ &+ \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha\left(|f|_{C^{0,\alpha}(B^+(\mathbf{0},1))} + \|g\|_{C^{1,\alpha}(\Gamma_1)}\right),\end{aligned}$$

where $C = C(N, \alpha) > 0$.

Proof. Step 1: Assume that $g = 0$ and $R = 1$. We extend u and f by reflection in $B(\mathbf{0}, 1)$, precisely,

$$\hat{u}(\boldsymbol{x}', x_N) := \begin{cases} u(\boldsymbol{x}) & \text{if } x_N \geq 0, \\ u(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \end{cases}$$

$$\hat{f}(\boldsymbol{x}', x_N) := \begin{cases} f(\boldsymbol{x}) & \text{if } x_N \geq 0, \\ f(\boldsymbol{x}', -x_N) & \text{if } x_N < 0. \end{cases}$$

Note that

$$\frac{\partial \hat{u}}{\partial x_N}(\boldsymbol{x}', x_n) = \begin{cases} \frac{\partial u}{\partial x_N}(\boldsymbol{x}) & \text{if } x_N \geq 0, \\ -\frac{\partial u}{\partial x_N}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \end{cases}$$

which is continuous since $\frac{\partial u}{\partial x_N}(\boldsymbol{x}', 0) = 0$, while

$$\frac{\partial^2 \hat{u}}{\partial x_N^2}(\boldsymbol{x}', x_n) = \begin{cases} \frac{\partial^2 u}{\partial x_N^2}(\boldsymbol{x}) & \text{if } x_N \geq 0, \\ \frac{\partial^2 u}{\partial x_N^2}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0. \end{cases}$$

By differentiating $\frac{\partial u}{\partial x_N}(\boldsymbol{x}', 0) = 0$ with respect to x_i, we get $\frac{\partial^2 u}{\partial x_i \partial x_N}(\boldsymbol{x}', 0) = 0$, and so

$$\frac{\partial^2 \hat{u}}{\partial x_i \partial x_N}(\boldsymbol{x}', x_n) = \begin{cases} \frac{\partial^2 u}{\partial x_i \partial x_N}(\boldsymbol{x}) & \text{if } x_N \geq 0, \\ -\frac{\partial^2 u}{\partial x_i \partial x_N}(\boldsymbol{x}', -x_N) & \text{if } x_N < 0, \end{cases}$$

is continuous. This shows that $\hat{u}$ is of class $C^2(B(\mathbf{0}, 1))$. Moreover,

$$-\Delta \hat{u}(\boldsymbol{x}) = \hat{f}(\boldsymbol{x}) \quad \text{in } B(\mathbf{0}, 1).$$

Hence, by Theorem 83,

$$\left\|\nabla^2 \hat{u}(\boldsymbol{x}) - \nabla^2 \hat{u}(\boldsymbol{y})\right\| \leq C \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha \left\{ \|\boldsymbol{x} - \boldsymbol{y}\|^{1-\alpha} \|\hat{u}\|_{C(B(\mathbf{0},1))} + \left|\hat{f}\right|_{C^{0,\alpha}(B(\mathbf{0},1))} \right\}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in B\left(\mathbf{0}, \frac{1}{4}\right)$. Now $\left|\hat{f}\right|_{C^{0,\alpha}(B(\mathbf{0},1))} \leq |f|_{C^{0,\alpha}(B^+(\mathbf{0},1))}$. Indeed, observe that if $\boldsymbol{x}$ and $\boldsymbol{y} \in B(\mathbf{0}, 1)$ with $x_N < 0 < y_N$, then

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{y}\| &= \sqrt{\|\boldsymbol{x}' - \boldsymbol{y}'\|_{N-1}^2 + (y_N - x_N)^2} \\ &\geq \sqrt{\|\boldsymbol{x}' - \boldsymbol{y}'\|_{N-1}^2 + (y_N + x_N)^2} = \|(\boldsymbol{x}', -x_N) - \boldsymbol{y}\|, \end{aligned}$$

and so

$$\frac{\left|\hat{f}(\boldsymbol{x}) - \hat{f}(\boldsymbol{y})\right|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha} \leq \begin{cases} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|}{\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha} & \text{if } x_N, y_N \geq 0, \\ \frac{\left|f(\boldsymbol{x}', -x_N) - f(\boldsymbol{y}', -y_N)\right|}{\|(\boldsymbol{x}', -x_N) - (\boldsymbol{y}', -y_N)\|^\alpha} & \text{if } x_N, y_N \leq 0, \\ \frac{\left|f(\boldsymbol{x}', -x_N) - f(\boldsymbol{y})\right|}{\|(\boldsymbol{x}', -x_N) - \boldsymbol{y}\|^\alpha} & \text{if } x_N < 0 < y_N. \end{cases}$$

Since $\left|\nabla^2 \hat{u}\right|_{C^{0,\alpha}\left(B^+\left(\mathbf{0}, \frac{1}{4}\right) \cup \Gamma_{\frac{1}{4}}\right)} \leq \left|\nabla^2 \hat{u}\right|_{C^{0,\alpha}\left(B\left(\mathbf{0}, \frac{1}{4}\right)\right)}$, this concludes the proof.

Step 2: Extend $g(\cdot, 0)$ to a function $h \in C_c^{1,\alpha}\left(\mathbb{R}^{N-1}\right)$ with

$$\|h\|_{C(\mathbb{R}^{N-1})} \le C(N)\|g\|_{C(\Gamma_1)}, \quad \|\nabla_{\boldsymbol{x}'} h\|_{C(\mathbb{R}^{N-1})} \le C\|g\|_{C^1(\Gamma_1)}$$

and

$$|\nabla_{\boldsymbol{x}'} h|_{C^{0,\alpha}(\mathbb{R}^{N-1})} \le C\|g\|_{C^{1,\alpha}(\Gamma_1)},$$

where $C = C(N)$. Let

$$\varphi(\boldsymbol{x}') := \begin{cases} c\exp\left(\frac{1}{\|\boldsymbol{x}'\|_{N-1}^2 - 1}\right) & \text{if } \|\boldsymbol{x}'\|_{N-1} < 1, \\ 0 & \text{if } \|\boldsymbol{x}'\|_{N-1} \ge 1, \end{cases}$$

where $c > 0$ is chosen so that

$$\int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{x}')\, d\boldsymbol{x}' = 1$$

and define

$$\psi(\boldsymbol{x}) := x_N \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}')\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}'.$$

Then $\psi \in C^{2,\alpha}(B(\mathbf{0}, R))$, $\frac{\partial \psi}{\partial x_N}(\boldsymbol{x}', 0) = h(\boldsymbol{x}')$, and by integration by parts

$$\begin{aligned} \frac{\partial \psi}{\partial x_i}(\boldsymbol{x}) &= x_N \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}') \frac{\partial h}{\partial x_i}(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}' \\ &= -\int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}') \frac{\partial}{\partial y_i}(h(\boldsymbol{x}' - x_N \boldsymbol{y}'))\, d\boldsymbol{y}' \\ &= \int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial y_i}(\boldsymbol{y}')\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}', \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x_j \partial x_i}(\boldsymbol{x}) &= \int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial y_i}(\boldsymbol{y}') \frac{\partial h}{\partial x_j}(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}', \\ \frac{\partial^2 \psi}{\partial x_N \partial x_i}(\boldsymbol{x}) &= -\sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \frac{\partial \varphi}{\partial y_i}(\boldsymbol{y}') \frac{\partial h}{\partial x_i}(\boldsymbol{x}' - x_N \boldsymbol{y}')\, y_i\, d\boldsymbol{y}', \end{aligned}$$

while

$$\begin{aligned} \frac{\partial \psi}{\partial x_N}(\boldsymbol{x}) &= \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}')\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}' - \sum_{i=1}^{N-1} x_N \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}') \frac{\partial h}{\partial x_i}(\boldsymbol{x}' - x_N \boldsymbol{y}')\, y_i\, d\boldsymbol{y}' \\ &= \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}')\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}' + \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}')\, y_i \frac{\partial}{\partial y_i}(h(\boldsymbol{x}' - x_N \boldsymbol{y}'))\, d\boldsymbol{y}' \\ &= \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}')\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}' - \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial y_i}(y_i \varphi(\boldsymbol{y}'))\, h(\boldsymbol{x}' - x_N \boldsymbol{y}')\, d\boldsymbol{y}' \end{aligned}$$

and

$$\frac{\partial^2 \psi}{\partial x_N^2}(\boldsymbol{x}) = -\sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} \varphi(\boldsymbol{y}') \frac{\partial h}{\partial x_i}(\boldsymbol{x}' - x_N \boldsymbol{y}') y_i \, d\boldsymbol{y}'$$
$$+ \sum_{i,j=1}^{N-1} \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial y_i}(y_i \varphi(\boldsymbol{y}')) \frac{\partial h}{\partial x_j}(\boldsymbol{x}' - x_N \boldsymbol{y}') y_i \, d\boldsymbol{y}'.$$

It follows that

$$\|\psi\|_{C(B(\mathbf{0},1))} \le C(N) \|h\|_{C(\mathbb{R}^{N-1})} \le C(N) \|g\|_{C(\Gamma_1)},$$

while

$$\left|\nabla^2 \psi\right|_{C^{0,\alpha}(B^+(\mathbf{0},1))} \le C(N) |\nabla_{\boldsymbol{x}'} h|_{C^{0,\alpha}(\mathbb{R}^{N-1})} \le C(N) \|g\|_{C^{1,\alpha}(\Gamma_1)}$$

The function $v := u - \psi$ solves the Neumann problem

$$\begin{cases} -\Delta v(\boldsymbol{x}) = f(\boldsymbol{x}) - \Delta\psi(\boldsymbol{x}) & \text{in } B^+(\mathbf{0},1), \\ \frac{\partial v}{\partial x_N}(\boldsymbol{x}', 0) = 0 & \text{on } \Gamma_1. \end{cases}$$

Hence, by the previous step,

$$\left\|\nabla^2 v(\boldsymbol{x}) - \nabla^2 v(\boldsymbol{y})\right\| \le C \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha \left\{ \|\boldsymbol{x} - \boldsymbol{y}\|^{1-\alpha} \|v\|_{C(B(\mathbf{0},1))} + |f - \Delta\psi|_{C^{0,\alpha}(B(\mathbf{0},1))} \right\}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in B\left(\mathbf{0}, \frac{1}{4}\right)$. In turn,

$$\begin{aligned} &\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\| \le \left\|\nabla^2 \psi(\boldsymbol{x}) - \nabla^2 \psi(\boldsymbol{y})\right\| \\ &\quad + C \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha \left\{ \|\boldsymbol{x} - \boldsymbol{y}\|^{1-\alpha} \|u - \psi\|_{C(B^+(\mathbf{0},1))} + |f - \Delta\psi|_{C^{0,\alpha}(B^+(\mathbf{0},1))} \right\} \\ &\le C \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha \left\{ \|\boldsymbol{x} - \boldsymbol{y}\|^{1-\alpha} \left(\|u\|_{C(B^+(\mathbf{0},1))} + \|g\|_{C(\Gamma_1)} \right) + |f|_{C^{0,\alpha}(B^+(\mathbf{0},1))} + \|g\|_{C^{1,\alpha}(\Gamma_1)} \right\}. \end{aligned}$$

This completes the proof. ■

Remark 90 *As is Remark 84, if $B^+(\mathbf{0},1)$ is replaced by $B^+(\mathbf{0},R)$, consider $v(\boldsymbol{y}) := u(R\boldsymbol{y})$ for $\boldsymbol{y} \in B^+(\mathbf{0},1)$. Then v satisfies the problem*

$$\begin{cases} -\Delta_{\boldsymbol{y}} v(\boldsymbol{y}) = R^2 f(R\boldsymbol{y}) =: f_R(\boldsymbol{y}) & \text{in } B^+(\mathbf{0},1), \\ \frac{\partial v}{\partial x_N}(\boldsymbol{y}', 0) = Rg(R\boldsymbol{y}', 0) =: g_R(\boldsymbol{y}', 0) & \text{on } \Gamma_1. \end{cases}$$

Note that

$$\|g_R\|_{C(\Gamma_1)} = R\|g\|_{C(\Gamma_R)}, \quad \|\nabla_{\boldsymbol{y}'} g_R\|_{C(\Gamma_1;\mathbb{R}^{N-1})} = R^2 \|\nabla_{\boldsymbol{x}'} g\|_{C(\Gamma_R;\mathbb{R}^{N-1})},$$
$$|\nabla_{\boldsymbol{y}'} g_R|_{C^{0,\alpha}(\Gamma_1)} = R^{2+\alpha} |\nabla_{\boldsymbol{x}'} g|_{C^{0,\alpha}(\Gamma_R)}, \quad |f_R|_{C^{0,\alpha}(B^+(\mathbf{0},1))} = R^{2+\alpha} |f|_{C^{0,\alpha}(B^+(\mathbf{0},R))}$$

Hence, for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B^+\left(\mathbf{0}, \frac{R}{4}\right)$,

$$\begin{aligned}
&\left\|\nabla^2 u\left(\boldsymbol{x}_1\right)-\nabla^2 u\left(\boldsymbol{x}_2\right)\right\|=\frac{1}{R^2}\left\|\nabla_{\boldsymbol{y}}^2 v\left(\frac{\boldsymbol{x}_1}{R}\right)-\nabla_{\boldsymbol{y}}^2 v\left(\frac{\boldsymbol{x}_2}{R}\right)\right\| \\
&\leq C\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|\left(\frac{1}{R^3}\|u\|_{C\left(B^+(\mathbf{0}, R)\right)}+\frac{1}{R^2}\|g\|_{C\left(\Gamma_R\right)}\right) \\
&\quad+\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|^\alpha\left(|f|_{C^{0, \alpha}\left(B^+(\mathbf{0}, R)\right)}+\frac{1}{R^{1+\alpha}}\|g\|_{C\left(\Gamma_R\right)}+\frac{1}{R^\alpha}\left\|\nabla_{\boldsymbol{x}^{\prime}} g\right\|_{C\left(\Gamma_R ; \mathbb{R}^{N-1}\right)}+\left|\nabla_{\boldsymbol{x}^{\prime}} g\right|_{C^{0, \alpha}\left(\Gamma_R\right)}\right) \\
&\leq C\left\|\boldsymbol{x}_1-\boldsymbol{x}_2\right\|^\alpha\left(\frac{1}{R^{2+\alpha}}\|u\|_{C\left(B^+(\mathbf{0}, R)\right)}+|f|_{C^{0, \alpha}\left(B^+(\mathbf{0}, R)\right)}\right. \\
&\quad\left.+\frac{1}{R^{1+\alpha}}\|g\|_{C\left(\Gamma_R\right)}+\frac{1}{R^\alpha}\left\|\nabla_{\boldsymbol{x}^{\prime}} g\right\|_{C\left(\Gamma_R ; \mathbb{R}^{N-1}\right)}+\left|\nabla_{\boldsymbol{x}^{\prime}} g\right|_{C^{0, \alpha}\left(\Gamma_R\right)}\right) .
\end{aligned}$$

Remark 91 *Note that the reflection argument in Step 1 would not work for a different elliptic equation with constant coefficients, since* $\frac{\partial^2 \hat{u}}{\partial x_i \partial x_N}$ *changes sign, unlike* $\frac{\partial^2 \hat{u}}{\partial x_i^2}$.

Theorem 92 *Let* $f \in C^{0,\alpha}\left(B^+(\mathbf{0}, R)\right)$, $g \in C^{1,\alpha}\left(\partial B^+(\mathbf{0}, R) \cap \{x_N = 0\}\right)$, $\alpha \in (0,1)$, $R > 0$, *and let* $u \in C^2\left(\overline{B^+(\mathbf{0}, R)}\right)$ *be a solution of*

$$\begin{cases} -\sum_{i,j=1}^N a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } B^+(\mathbf{0}, R), \\ \left(\boldsymbol{A} \nabla u\left(\boldsymbol{x}', 0\right)\right) \cdot e_N = g\left(\boldsymbol{x}', 0\right) & \text{on } \Gamma_R, \end{cases}$$

where the matrix $\boldsymbol{A} := (a_{i,j})_{i,j}$ *is symmetric and positive definite. Then, for all* $0 < r \leq \sqrt{\frac{\lambda_1}{\lambda_N}} \frac{R}{4}$,

$$\begin{aligned}
\left|\nabla^2 u\right|_{C^{0, \alpha}\left(B^+(\mathbf{0}, r) \cup \Gamma_r\right)} \leq C &\left\{\frac{1}{r^{2+\alpha}} \sup_{B^+(\mathbf{0}, r)}|u|+|f|_{C^{0, \alpha}\left(B^+(\mathbf{0}, r)\right)}\right. \\
&\left.+\frac{1}{r^{1+\alpha}}\|g\|_{C\left(\Gamma_R\right)}+\frac{1}{r^\alpha}\left\|\nabla_{\boldsymbol{x}^{\prime}} g\right\|_{C^0\left(\Gamma_R ; \mathbb{R}^{N-1}\right)}+\left|\nabla_{\boldsymbol{x}^{\prime}} g\right|_{C^{0, \alpha}\left(\Gamma_R\right)}\right\},
\end{aligned}$$

where $\lambda_1 < \cdots < \lambda_N$ *are the eigenvalues of the matrix* $\boldsymbol{A}$ *and* $C = C(N, \alpha, \lambda_1, \lambda_N) > 0$.

Proof. [6]We proceed as in the proof of Theorem 85, and let $\boldsymbol{\Psi}(\boldsymbol{x}) = \boldsymbol{y} := \boldsymbol{x}\boldsymbol{B}$ and $v(\boldsymbol{y}) := u\left(\boldsymbol{y}\boldsymbol{B}^{-1}\right)$. Recalling that the rotation $\boldsymbol{R}$ was chosen so that the half space $\{x_N > 0\}$ is mapped into the half space $\{y_N > 0\}$, we get that $\boldsymbol{B}\boldsymbol{e}_N = \boldsymbol{e}_N$. Hence,

$$\begin{aligned}
\left(\boldsymbol{A} \nabla_{\boldsymbol{x}} u\left(\boldsymbol{x}', 0\right)\right) \cdot e_N &= \left(\boldsymbol{A}\boldsymbol{B} \nabla_{\boldsymbol{y}} v\left(\boldsymbol{y}', 0\right)\right) \cdot \boldsymbol{B} e_N = \left(\boldsymbol{B}^T \boldsymbol{A}\boldsymbol{B} \nabla_{\boldsymbol{y}} v\left(\boldsymbol{y}', 0\right)\right) \cdot e_N \\
&= \left(I_N \nabla_{\boldsymbol{y}} v\left(\boldsymbol{y}', 0\right)\right) \cdot e_N = \frac{\partial v}{\partial y_N}\left(\boldsymbol{y}', 0\right).
\end{aligned}$$

[6]We only sketched this proof in class. Please read it.

Hence, v satisfies the Neumann problem

$$\begin{cases} -\Delta_{\boldsymbol{y}} v(\boldsymbol{y}) = f\left(\boldsymbol{y}\boldsymbol{B}^{-1}\right) =: f_1(\boldsymbol{y}) & \text{in } \boldsymbol{\Psi}\left(B^+(\mathbf{0},R)\right), \\ \frac{\partial v}{\partial y_N}(\boldsymbol{y}',0) = g\left((\boldsymbol{y}',0)\boldsymbol{B}^{-1}\right) =: g_1(\boldsymbol{y}',0) & \text{on } \partial\boldsymbol{\Psi}\left(B^+(\mathbf{0},R)\right)\cap\{y_N=0\}. \end{cases}$$

By (85), $B^+\left(\mathbf{0},\frac{R}{\sqrt{\lambda_N}}\right) \subseteq \boldsymbol{\Psi}\left(B^+(\mathbf{0},R)\right)$ and so by the previous theorem,

$$\begin{aligned} \left|\nabla^2_{\boldsymbol{y}} v\right|_{C^{0,\alpha}(B^+(\mathbf{0},s)\cup\Gamma_s)} \leq C \Bigg\{ & \frac{1}{s^{2+\alpha}} \sup_{B^+(\mathbf{0},4s)} |v| + |f_1|_{C^{0,\alpha}(B^+(\mathbf{0},4s))} \\ & + \frac{1}{s^{1+\alpha}} \|g_1\|_{C(\Gamma_{4s})} + \frac{1}{s^{\alpha}} \|\nabla_{\boldsymbol{y}'} g_1\|_{C^0(\Gamma_{4s};\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{y}'} g|_{C^{0,\alpha}(\Gamma_{4s})} \Bigg\} \end{aligned}$$

for all $0 < s \leq \frac{R}{4\sqrt{\lambda_N}}$. By (83), (85),

$$\|g_1\|_{C(\Gamma_{4s})} \leq \|g\|_{C(\Gamma_R)}, \quad \|\nabla_{\boldsymbol{y}'} g_1\|_{C^0(\Gamma_{4s};\mathbb{R}^{N-1})} \leq C\lambda_N^{1/2} \|\nabla_{\boldsymbol{x}'} g\|_{C^{0,\alpha}(\Gamma_R;\mathbb{R}^{N-1})}$$

$$|\nabla_{\boldsymbol{y}'} g_1|_{C^{0,\alpha}(\Gamma_{4s})} \leq C\lambda_N^{1+\alpha/2} |\nabla_{\boldsymbol{x}'} g|_{C^{0,\alpha}(\Gamma_R)}.$$

Let $0 < r \leq \sqrt{\frac{\lambda_1}{\lambda_N}}\frac{R}{4}$ and take $s := \frac{r}{\sqrt{\lambda_1}}$. Then, also by (84), (85), (86), and (87), for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in B^+(\mathbf{0},r)\cup\Gamma_r$,

$$\begin{aligned} &\left\|\nabla^2_{\boldsymbol{x}} u(\boldsymbol{x}_1) - \nabla^2_{\boldsymbol{x}} u(\boldsymbol{x}_2)\right\| \leq \frac{C}{\lambda_1} \left\|\nabla^2_{\boldsymbol{y}} v(\boldsymbol{x}_1\boldsymbol{B}) - \nabla^2_{\boldsymbol{y}} v(\boldsymbol{x}_2\boldsymbol{B})\right\| \\ &\leq \frac{C}{\lambda_1} \|(\boldsymbol{x}_1-\boldsymbol{x}_2)\boldsymbol{B}\|^{\alpha} \Bigg\{ \frac{\lambda_1^{(2+\alpha)/2}}{r^{2+\alpha}} \sup_{B^+\left(\mathbf{0},4r/\sqrt{\lambda_1}\right)} |v| + |f_1|_{C^{0,\alpha}\left(B^+\left(\mathbf{0},4r/\sqrt{\lambda_1}\right)\right)} \\ &\quad + \frac{\lambda_1^{(1+\alpha)/2}}{r^{1+\alpha}} \|g_1\|_{C\left(\Gamma_{4r/\sqrt{\lambda_1}}\right)} + \frac{\lambda_1^{\alpha/2}}{r^{\alpha}} \|\nabla_{\boldsymbol{y}'} g_1\|_{C^0\left(\Gamma_{4r/\sqrt{\lambda_1}};\mathbb{R}^{N-1}\right)} + |\nabla_{\boldsymbol{y}'} g|_{C^{0,\alpha}\left(\Gamma_{4r/\sqrt{\lambda_1}}\right)} \Bigg\} \\ &\leq \frac{C}{\lambda_1^{1+\alpha/2}} \|\boldsymbol{x}_1-\boldsymbol{x}_2\|^{\alpha} \Bigg\{ \frac{\lambda_1^{(2+\alpha)/2}}{r^{2+\alpha}} \sup_{B^+(\mathbf{0},R)} |u| + \lambda_N^{\alpha/2} |f|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \\ &\quad + \frac{\lambda_1^{(1+\alpha)/2}}{r^{1+\alpha}} \|g\|_{C(\Gamma_R)} + \frac{\lambda_1^{\alpha/2}\lambda_N^{1/2}}{r^{\alpha}} \|\nabla_{\boldsymbol{x}'} g\|_{C^0(\Gamma_R;\mathbb{R}^{N-1})} + \lambda_N^{1+\alpha/2} |\nabla_{\boldsymbol{x}'} g|_{C^{0,\alpha}(\Gamma_R)} \Bigg\}. \end{aligned}$$

■

Monday, October 28, 2013

Finally, we consider the case

$$\begin{aligned} -\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}) \quad & \text{in } B^+(\mathbf{0},R), \\ (\boldsymbol{A}(\boldsymbol{x}',0)\nabla u(\boldsymbol{x}',0))\cdot \boldsymbol{e}_N = g(\boldsymbol{x}',0) \quad & \text{on } \Gamma_R, \end{aligned}$$

Theorem 93 *Let $a_{i,j} \in C^{1,\alpha}(B^+(\mathbf{0},R))$, $i,j = 1,\ldots,N$, $\alpha \in (0,1)$, $R > 0$, let $f \in C^{0,\alpha}(B^+(\mathbf{0},R))$, and let $u \in C^2\left(\overline{B^+(\mathbf{0},R)}\right)$ be a solution of*

$$\begin{aligned} -\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}) \quad & \text{in } B^+(\mathbf{0},R), \\ (\boldsymbol{A}(\boldsymbol{x}',0)\nabla u(\boldsymbol{x}',0))\cdot \boldsymbol{e}_N = g(\boldsymbol{x}',0) \quad & \text{on } \Gamma_R, \end{aligned}$$

where the matrix $\boldsymbol{A}(\boldsymbol{x}) := (a_{i,j}(\boldsymbol{x}))_{i,j}$ *is symmetric, positive definite, and*

$$\lambda \|\boldsymbol{\xi}\|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}) \xi_i \xi_j \leq \Lambda \|\boldsymbol{\xi}\|^2$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$, $\boldsymbol{x} \in B^+(\mathbf{0}, R)$ *and for some* $0 < \lambda \leq 1 \leq \Lambda$. *Then, for all* $0 < r \leq \sqrt{\frac{\lambda}{\Lambda}} \frac{R}{4}$,

$$\begin{aligned}|\nabla^2 u|_{C^{0,\alpha}(B^+(\mathbf{0},r)\cup\Gamma_r)} \leq C \Big(&\|u\|_{C(B^+(\mathbf{0},R))} + \|\nabla u\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \\ &+ \|f\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} + \|g\|_{C^{1,\alpha}(\Gamma_R)} \Big),\end{aligned}$$

where $C = C(\lambda, \Lambda, \alpha, L, M, r)$ *and*

$$L := \sum_{i,j=1}^{N} \|a_{i,j}\|_{C^{0,\alpha}(B^+(\mathbf{0},R))}, \quad M := \sum_{i,j=1}^{N} \|a_{i,j}\|_{C^{1,\alpha}(B^+(\mathbf{0},R))}.$$

Proof. [7]**Step 1:** Assume that $u \in C_c^2(B^+(\mathbf{0}, R) \cup \Gamma_R)$. We will use the method of freezing the coefficients. Let $\boldsymbol{x}_1, \boldsymbol{y}_1 \in B^+(\mathbf{0}, R) \cup \Gamma_R$. Since,

$$\begin{aligned}\operatorname{div}(\boldsymbol{A}(\boldsymbol{x}) \nabla u(\boldsymbol{x})) &= \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{N} a_{i,j}(\boldsymbol{x}) \frac{\partial u}{\partial x_j}(\boldsymbol{x}) \right) \\ &= \sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) + \sum_{i,j=1}^{N} \frac{\partial a_{i,j}}{\partial x_i}(\boldsymbol{x}) \frac{\partial u}{\partial x_j}(\boldsymbol{x}).\end{aligned}$$

We rewrite our PDE as follows

$$\begin{aligned}-\sum_{i,j=1}^{N} a_{i,j}(\boldsymbol{x}_1) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) = &\sum_{i,j=1}^{N} (a_{i,j}(\boldsymbol{x}) - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}) \\ &+ \sum_{i,j=1}^{N} \frac{\partial a_{i,j}}{\partial x_i}(\boldsymbol{x}) \frac{\partial u}{\partial x_j}(\boldsymbol{x}) + f(\boldsymbol{x}) =: f_1(\boldsymbol{x}).\end{aligned}$$

Similarly, we write

$$\begin{aligned}(\boldsymbol{A}(\boldsymbol{x}_1) \nabla u(\boldsymbol{x}', 0)) \cdot \boldsymbol{e}_N &= g(\boldsymbol{x}', 0) + ((\boldsymbol{A}(\boldsymbol{x}_1) - \boldsymbol{A}(\boldsymbol{x}', 0)) \nabla u(\boldsymbol{x}', 0)) \cdot \boldsymbol{e}_N \\ &=: g_1(\boldsymbol{x}', 0).\end{aligned}$$

Then u satisfies the Neumann problem

$$\begin{cases} -\operatorname{div}(\boldsymbol{A}(\boldsymbol{x}_1) \nabla u(\boldsymbol{x})) = f_1(\boldsymbol{x}) & \text{in } B^+(\mathbf{0}, R), \\ (\boldsymbol{A}(\boldsymbol{x}_1) \nabla u(\boldsymbol{x}', 0)) \cdot \boldsymbol{e}_N = g_1(\boldsymbol{x}', 0) & \text{on } \Gamma_R, \end{cases}$$

[7] We only sketched this proof in class. Please read it.

which is of the type studied in Theorem 92.

Fix $0 < s \leq 1$. If $B(\boldsymbol{x}_1, s)$ intersects Γ_R, let $\mathbf{z}_1 := (\boldsymbol{x}_1', 0)$ be the projection of $\boldsymbol{x}_1$ on the hyperplane $x_N = 0$. Then $\|\mathbf{z}_1 - \mathbf{x}_1\| < s$ and so $B(\boldsymbol{x}_1, s) \subset B^+(\boldsymbol{z}_1, 2s)$. It follows by Theorem 92 applied to the ball $B^+(\boldsymbol{z}_1, 2s)$ that if $\boldsymbol{y}_1 \in B(\boldsymbol{x}_1, s)$,

$$\frac{\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\|}{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^{\alpha}} \leq C \frac{\Lambda^{1/2}}{\lambda^{1+\alpha/2}} \left\{ \frac{1}{s^{2+\alpha}} \|u\|_{C^0(B^+(\boldsymbol{z}_1, S))} + |f_1|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \right. \tag{90}$$

$$\left. + \frac{1}{s^{1+\alpha}} \|g_1\|_{C(\Gamma_S)} + \frac{1}{s^{\alpha}} \|\nabla_{\boldsymbol{x}'} g_1\|_{C^0(\Gamma_S; \mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \right\},$$

where $S := 8\sqrt{\frac{\Lambda}{\lambda}} s$. Reasoning as in the proof of Theorem 88, we have that

$$|f_1|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \leq |f|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} + \sum_{i,j=1}^{N} \left| (a_{i,j} - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))}$$

$$+ \sum_{i,j=1}^{N} \left| \frac{\partial a_{i,j}}{\partial x_i} \frac{\partial u}{\partial x_j} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))}$$

with

$$\begin{aligned} &\left| (a_{i,j} - a_{i,j}(\boldsymbol{x}_1)) \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \\ &\leq |a_{i,j}|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \left(2S^{\alpha} \left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} + \frac{C}{S^2} \|u\|_{C(B^+(\boldsymbol{z}_1, S))} \right) \\ &\leq L \left(16 \left(\frac{\Lambda}{\lambda} \right)^{\alpha/2} s^{\alpha} \left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} + \frac{C}{s^2} \|u\|_{C(B^+(\boldsymbol{z}_1, S))} \right), \end{aligned}$$

while

$$\begin{aligned} \left| \frac{\partial a_{i,j}}{\partial x_i} \frac{\partial u}{\partial x_j} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} &\leq \left\| \frac{\partial a_{i,j}}{\partial x_i} \right\|_{C(B^+(\boldsymbol{z}_1, S))} \left| \frac{\partial u}{\partial x_j} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \\ &\quad + \left\| \frac{\partial u}{\partial x_j} \right\|_{C(B^+(\boldsymbol{z}_1, S))} \left| \frac{\partial a_{i,j}}{\partial x_i} \right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \\ &\leq 2M \|\nabla u\|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))}. \end{aligned}$$

In turn,

$$|f_1|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \leq |f|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} + 16L \left(\frac{\Lambda}{\lambda} \right)^{\alpha/2} s^{\alpha} \left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} \tag{91}$$

$$+ CM \|\nabla u\|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1, S))} + \frac{CL}{s^2} \|u\|_{C(B^+(\boldsymbol{z}_1, S))}.$$

On the other hand, by the mean value theorem

$$\|(\boldsymbol{A}(\boldsymbol{x}_1) - \boldsymbol{A}(\boldsymbol{x}', 0)) \nabla u(\boldsymbol{x}', 0)\| \le S \|\nabla u(\boldsymbol{x}', 0)\| \sum_{i,j=1}^{N} \left\| \frac{\partial a_{i,j}}{\partial x_i} \right\|_{C(B^+(\boldsymbol{z}_1, S) \cup \Gamma_S)},$$

and so

$$\|g_1\|_{C(\Gamma_S)} \le \|g\|_{C(\Gamma_S)} + C\sqrt{\frac{\Lambda}{\lambda}} sM \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})},$$

while by the mean value theorem and Exercise 87,

$$\begin{aligned}
& C \sum_{i,j=1}^{N} |a_{i,j}(\boldsymbol{x}_1) - a_{i,j}(\boldsymbol{x}', 0)| \left\|\nabla^2_{\boldsymbol{x}'} u\right\|_{C(\Gamma_S;\mathbb{R}^{N-1})} \\
& \le CS \sum_{i,j=1}^{N} \left\| \frac{\partial a_{i,j}}{\partial x_i} \right\|_{C(B^+(\boldsymbol{z}_1, S) \cup \Gamma_S)} \left\|\nabla^2_{\boldsymbol{x}'} u\right\|_{C(\Gamma_S;\mathbb{R}^{N-1})} \\
& \le CM\sqrt{\frac{\Lambda}{\lambda}} s \left(s^\alpha \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)} + \frac{1}{s^2} \|u\|_{C(\Gamma_S)} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\nabla_{\boldsymbol{x}'} g_1\|_{C(\Gamma_S;\mathbb{R}^{N-1})} \le{}& \|\nabla_{\boldsymbol{x}'} g\|_{C(\Gamma_S;\mathbb{R}^{N-1})} + CM \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} \\
& + CM\sqrt{\frac{\Lambda}{\lambda}} \left(s^{1+\alpha} \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)} + \frac{1}{s} \|u\|_{C(\Gamma_S)} \right).
\end{aligned}$$

Similarly, reasoning as in (89) and using the the mean value theorem and Exercise 87,

$$\begin{aligned}
& \left| (a_{i,j}(\boldsymbol{x}_1) - a_{i,j}(\cdot, 0)) \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{C^{0,\alpha}(\Gamma_S)} \\
& \le \|a_{i,j}(\boldsymbol{x}_1) - a_{i,j}(\cdot, 0)\|_{C(\Gamma_S)} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{C^{0,\alpha}(\Gamma_S)} \\
& \quad + \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C(\Gamma_S)} |a_{i,j}(\cdot, 0)|_{C^{0,\alpha}(\Gamma_S)} \\
& \le 8\sqrt{\frac{\Lambda}{\lambda}} sM \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)} + L \left(s \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)} + \frac{1}{s^{2/\alpha}} \|u\|_{C(\Gamma_S)} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
|\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \le{}& |\nabla_{\boldsymbol{x}'} g|_{C^{0,\alpha}(\Gamma_S)} + CM \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} \\
& + 8\sqrt{\frac{\Lambda}{\lambda}} s (M + L) \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)} + \frac{L}{s^{2/\alpha}} \|u\|_{C(\Gamma_S)}.
\end{aligned}$$

In turn, using the fact that $\lambda \leq 1 \leq \Lambda$ and $s \leq 1$,

$$
\begin{aligned}
&\frac{1}{s^{1+\alpha}} \|g_1\|_{C(\Gamma_S)} + \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} g_1\|_{C^0(\Gamma_S;\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \\
&\leq \frac{1}{s^{1+\alpha}} \|g\|_{C(\Gamma_S)} + \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} g\|_{C^0(\Gamma_S;\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g|_{C^{0,\alpha}(\Gamma_S)} \qquad (92) \\
&\quad + \frac{1}{s^{2/\alpha}} \|u\|_{C(\Gamma_S)} + \frac{CM}{s^\alpha} \sqrt{\frac{\Lambda}{\lambda}} \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} + C(M+L) \sqrt{\frac{\Lambda}{\lambda}} s \left|\nabla^2_{\boldsymbol{x}'} u\right|_{C^{0,\alpha}(\Gamma_S)}
\end{aligned}
$$

Combining (90), (91), and (92), and using the facts that $\lambda \leq 1 \leq \Lambda$ and $s \leq 1$, it follows that

$$
\begin{aligned}
\frac{\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\|}{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha} \leq C\frac{\Lambda}{\lambda^2}(L+M) \Bigg\{ &\left(\frac{1}{s^{2+\alpha}} + \frac{1}{s^{2/\alpha}}\right) \|u\|_{C(B^+(\boldsymbol{z}_1,S))} + |f|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1,S))} \\
&+ \frac{1}{s^{1+\alpha}} \|g\|_{C(\Gamma_S)} + \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} g_1\|_{C^0(\Gamma_S;\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \\
&+ \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} + \|\nabla u\|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1,S))} + s\left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{z}_1,S))} \Bigg\}.
\end{aligned}
$$

Taking

$$
C\frac{\Lambda}{\lambda^2}(L+M)s \leq \frac{1}{2}
$$

and using the fact that u has compact support gives

$$
\begin{aligned}
\frac{\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\|}{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha} \leq C\frac{\Lambda}{\lambda^2}(L+M) \Bigg\{ &\left(\frac{1}{s^{2+\alpha}} + \frac{1}{s^{2/\alpha}}\right) \|u\|_{C(B^+(\boldsymbol{0},R))} + |f|_{C^{0,\alpha}(B^+(\boldsymbol{0},R))} \\
&+ \frac{1}{s^{1+\alpha}} \|g\|_{C(\Gamma_S)} + \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} g_1\|_{C^0(\Gamma_S;\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \\
&+ \frac{1}{s^\alpha} \|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} + \|\nabla u\|_{C^{0,\alpha}(B^+(\boldsymbol{0},R))} \Bigg\} + \frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{0},R))}.
\end{aligned}
$$

On the other hand, if $B(\boldsymbol{x}_1, s)$ does not intersect Γ_R, then we can Theorem 83 to get

$$
\begin{aligned}
\frac{\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\|}{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha} &\leq C\frac{\Lambda^{\alpha/2}}{\lambda^{1+\alpha/2}} \left\{ \frac{1}{s^{2+\alpha}} \sup_{B(\boldsymbol{x}_1,S)} |u| + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \right\} \\
&\quad + \frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B(\boldsymbol{x}_1,S))} \\
&\leq C\frac{\Lambda^{\alpha/2}}{\lambda^{1+\alpha/2}} \left\{ \frac{1}{s^{2+\alpha}} \sup_{B^+(\boldsymbol{0},R)} |u| + |f|_{C^{0,\alpha}(B^+(\boldsymbol{0},R))} \right\} \\
&\quad + \frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\boldsymbol{0},R))}
\end{aligned}
$$

provided

$$
C\frac{\Lambda}{\lambda^2} L s^\alpha \leq \frac{1}{2}.
$$

On the other hand, if $\|\boldsymbol{x}_1 - \boldsymbol{y}_1\| \geq s$,

$$\begin{aligned}\left\|\nabla^2 u(\boldsymbol{x}_1) - \nabla^2 u(\boldsymbol{y}_1)\right\| &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha}\left\{\left\|\nabla^2 u(\boldsymbol{x}_1)\right\| + \left\|\nabla^2 u(\boldsymbol{y}_1)\right\|\right\} \\ &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha} 2\left\|\nabla^2 u\right\|_{C(B^+(\mathbf{0},R))} \\ &\leq \frac{\|\boldsymbol{x}_1 - \boldsymbol{y}_1\|^\alpha}{s^\alpha}\left(\frac{s^\alpha}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\mathbf{0},R))} + \frac{C}{s^2}\|u\|_{C(B^+(\mathbf{0},R))}\right),\end{aligned}$$

where in the last line we have used Exercise 87 with $\varepsilon = \frac{1}{4}s^\alpha$. By combining these tree inequalities we get

$$\begin{aligned}\frac{1}{2}\left|\nabla^2 u\right|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \leq C\frac{\Lambda}{\lambda^2}(L+M)\Bigg\{&\left(\frac{1}{s^{2+\alpha}} + \frac{1}{s^{2/\alpha}}\right)\|u\|_{C(B^+(\mathbf{0},R))} + |f|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \\ &+ \frac{1}{s^{1+\alpha}}\|g\|_{C(\Gamma_S)} + \frac{1}{s^\alpha}\|\nabla_{\boldsymbol{x}'} g_1\|_{C^0(\Gamma_S;\mathbb{R}^{N-1})} + |\nabla_{\boldsymbol{x}'} g_1|_{C^{0,\alpha}(\Gamma_S)} \\ &+ \frac{1}{s^\alpha}\|\nabla_{\boldsymbol{x}'} u\|_{C(\Gamma_S;\mathbb{R}^{N-1})} + \|\nabla u\|_{C^{0,\alpha}(B^+(\mathbf{0},R))}\Bigg\},\end{aligned}$$

where

$$s = \min\left\{\frac{1}{2}\frac{\lambda^2}{C\Lambda(L+M)}, \left(\frac{1}{2}\frac{\lambda^2}{C\Lambda L}\right)^{1/\alpha}\right\}.$$

Step 2: Exercise. ■

Now we consider the case in which the boundary is not flat. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $C^{2,\alpha}$ boundary, $\alpha \in (0,1)$ and let $\boldsymbol{x}_0 \in \partial\Omega$. Then there exist $R > 0$, $k \in \{1, \ldots, N\}$ and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class $C^{2,\alpha}$ such that either

$$\Omega \cap B(\boldsymbol{x}_0, R) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R) : x_k > h(\boldsymbol{x}_k)\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, R) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R) : x_k < h(\boldsymbol{x}_k)\}.$$

Without loss of generality, we take $k = N$ and assume that the first case holds. Also, by a translation, we may assume that $\boldsymbol{x}_0 = \mathbf{0}$.

Theorem 94 *Let $\Omega \subset \mathbb{R}^N$ be an open set with $\mathbf{0} \in \partial\Omega$. Assume that there exist $R > 0$ and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class $C^{2,\alpha}$, $\alpha \in (0,1)$ such that*

$$\Omega \cap B(\mathbf{0}, R) = \{\boldsymbol{x} \in B(\mathbf{0}, R) : x_N > h(\boldsymbol{x}')\}.$$

Let $f \in C^{0,\alpha}(\Omega \cap B(\mathbf{0}, R))$, let $g \in C^{1,\alpha}\left(\overline{\Omega \cap B(\mathbf{0}, R)}\right)$ and let $u \in C^2\left(\overline{\Omega \cap B(\mathbf{0}, R)}\right)$ be a solution of the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega \cap B(\mathbf{0}, R), \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega \cap B(\mathbf{0}, R). \end{cases}$$

Then, for all

$$0 < r \le \frac{R}{16\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C(B_{N-1}(\mathbf{0},R);\mathbb{R}^{N-1})}\right)}$$

we have

$$\begin{aligned}\left|\nabla^2 u\right|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},r))} \le C\Bigg(&\frac{1}{r^{2/\alpha}}\|u\|_{C(\Omega\cap B(\mathbf{0},R))} + \|u\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))} \\ &+ |f|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} + \|g\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))}\Bigg)\end{aligned}$$

where $C = C\left(\alpha, \|h\|_{C^{1,\alpha}(B_{N-1}(\mathbf{0},R))}, \|h\|_{C^{2,\alpha}(B_{N-1}(\mathbf{0},R))}\right)$.

Proof. [8]Consider the transformation $\mathbf{\Psi} : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\mathbf{\Psi}(\boldsymbol{x}) := (\boldsymbol{x}', x_N - h(\boldsymbol{x}')) := \boldsymbol{y}.$$

Note that Ψ sends points of the form $(\boldsymbol{x}', h(\boldsymbol{x}'))$ into $(\boldsymbol{x}', 0)$, so it flattens the boundary. Moreover,

$$\mathbf{\Psi}^{-1}(\boldsymbol{y}) = (\boldsymbol{y}', y_N + h(\boldsymbol{y}')) = \boldsymbol{x}$$

and

$$\nabla_{\boldsymbol{x}}\mathbf{\Psi}(\boldsymbol{x}) = \begin{pmatrix} & 0 \\ I_{N-1} & \vdots \\ & 0 \\ -\nabla_{\boldsymbol{x}'} h(\boldsymbol{x}') & 1 \end{pmatrix}.$$

Consider the function

$$v(\boldsymbol{y}) := u\left(\mathbf{\Psi}^{-1}(\boldsymbol{y})\right) = u(\boldsymbol{y}', y_N + h(\boldsymbol{y}')).$$

Then $u(\boldsymbol{x}) = v(\mathbf{\Psi}(\boldsymbol{x})) = v(\boldsymbol{x}', x_N - h(\boldsymbol{x}'))$ and so for $i, j = 1, \dots, N-1$ and $\boldsymbol{y} = \mathbf{\Psi}(\boldsymbol{x})$,

$$\begin{aligned}\frac{\partial u}{\partial x_i}(\boldsymbol{x}) &= \frac{\partial v}{\partial y_i}(\boldsymbol{y}) - \frac{\partial v}{\partial y_N}(\boldsymbol{y})\frac{\partial h}{\partial x_i}(\boldsymbol{y}'), \\ \frac{\partial u}{\partial x_N}(\boldsymbol{x}) &= \frac{\partial v}{\partial y_N}(\boldsymbol{y}),\end{aligned} \tag{93}$$

[8]We only sketched this proof in class. Please read it.

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) &= \frac{\partial^2 v}{\partial y_j \partial y_i}(\boldsymbol{y}) - \frac{\partial^2 v}{\partial y_N \partial y_i}(\boldsymbol{y}) \frac{\partial h}{\partial x_j}(\boldsymbol{y}') \\ &\quad - \frac{\partial v}{\partial y_N}(\boldsymbol{y}) \frac{\partial^2 h}{\partial x_j \partial x_i}(\boldsymbol{y}') - \frac{\partial^2 v}{\partial y_j \partial y_N}(\boldsymbol{y}) \frac{\partial h}{\partial x_i}(\boldsymbol{y}') + \frac{\partial^2 v}{\partial y_N^2}(\boldsymbol{y}) \frac{\partial h}{\partial x_j}(\boldsymbol{y}') \frac{\partial h}{\partial x_i}(\boldsymbol{y}'), \end{aligned} \tag{94}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x_N \partial x_i}(\boldsymbol{x}) &= \frac{\partial^2 v}{\partial y_N \partial y_i}(\boldsymbol{y}) - \frac{\partial^2 v}{\partial y_N^2}(\boldsymbol{y}) \frac{\partial h}{\partial x_i}(\boldsymbol{y}'), \\ \frac{\partial^2 u}{\partial x_N^2}(\boldsymbol{x}) &= \frac{\partial^2 v}{\partial y_N^2}(\boldsymbol{y}).\end{aligned}$$

It follows that

$$\Delta_{\boldsymbol{x}} u(\boldsymbol{x}) = \operatorname{div}_{\boldsymbol{y}}\left(A_h(\boldsymbol{y}') \nabla_{\boldsymbol{y}} v(\boldsymbol{y})\right),$$

where

$$A_h(\boldsymbol{y}') := \begin{pmatrix} & & & -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') \\ & I_{N-1} & & \vdots \\ & & & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') \\ -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') & \cdots & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') & 1 + \sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}(\boldsymbol{y}')\right)^2 \end{pmatrix}.$$

On the other hand, for $\boldsymbol{x} \in \partial\Omega \cap B(\boldsymbol{0}, R)$,

$$\begin{aligned} g(\boldsymbol{x}) = \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) &= \nabla_{\boldsymbol{x}} u(\boldsymbol{x}) \cdot \frac{(\nabla_{\boldsymbol{x}'} h(\boldsymbol{x}'), -1)}{\sqrt{1 + \|\nabla_{\boldsymbol{x}'} h(\boldsymbol{x}')\|_{N-1}^2}} \\ &= \frac{(A_h(\boldsymbol{y}') \nabla_{\boldsymbol{y}} v_n(\boldsymbol{y})) \cdot (-\boldsymbol{e}_N)}{\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|_{N-1}^2}}. \end{aligned}$$

Hence, (124) becomes

$$\begin{cases} -\operatorname{div}_{\boldsymbol{y}}\left(A_h(\boldsymbol{y}') \nabla_{\boldsymbol{y}} v_n(\boldsymbol{y})\right) = f_1(\boldsymbol{y}) & \text{in } B^+(\boldsymbol{0}, R), \\ (A_h(\boldsymbol{y}') \nabla_{\boldsymbol{y}} v(\boldsymbol{y}', 0)) \cdot \boldsymbol{e}_N = g_1(\boldsymbol{y}') & \text{on } \partial B^+(\boldsymbol{0}, R) \cap \{y_N = 0\}, \end{cases}$$

where $f_1(\boldsymbol{y}) := f\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right)$ and $g_1(\boldsymbol{y}') := -g\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y}', 0)\right)\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|_{N-1}^2}$.
■

Wednesday, October 30, 2013

Proof. Let's check the ellipticity. For $\boldsymbol{\xi} \in \mathbb{R}^N$ we have

$$\begin{aligned}
\boldsymbol{\xi} A_h(\boldsymbol{y}')\boldsymbol{\xi}^T &= \begin{pmatrix} & & & -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') \\ & I_{N-1} & & \vdots \\ & & & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') \\ -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') & \cdots & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') & 1+\sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}(\boldsymbol{y}')\right)^2 \end{pmatrix} \\
&= \boldsymbol{\xi}\cdot\left(\boldsymbol{\xi}' - \xi_N \nabla_{\boldsymbol{y}'} h, -\boldsymbol{\xi}'\cdot\nabla_{\boldsymbol{y}'} h + \xi_N + \xi_N \sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}\right)^2\right) \\
&= \xi_1^2 + \cdots + \xi_N^2 - 2\boldsymbol{\xi}'\cdot\nabla_{\boldsymbol{y}'} h \xi_N + \xi_N^2 \sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}\right)^2 \\
&= \|\boldsymbol{\eta}\|^2,
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\eta} = \boldsymbol{\xi}\nabla_{\boldsymbol{y}}\boldsymbol{\Psi}(\boldsymbol{y}) &= \boldsymbol{\xi}\begin{pmatrix} & 0 \\ I_{N-1} & \vdots \\ & 0 \\ -\nabla_{\boldsymbol{x}'} h(\boldsymbol{y}') & 1 \end{pmatrix} \\
&= \left(\boldsymbol{\xi}' - \xi_N \nabla_{\boldsymbol{y}'} h, \xi_N\right).
\end{aligned}$$

Since $\nabla_{\boldsymbol{y}}\boldsymbol{\Psi}(\boldsymbol{y})\nabla_{\boldsymbol{y}}\boldsymbol{\Psi}^{-1}(\boldsymbol{\Psi}(\boldsymbol{y})) = I_N$, we have that $\boldsymbol{\xi} = \boldsymbol{\eta}\nabla_{\boldsymbol{y}}\boldsymbol{\Psi}^{-1}(\boldsymbol{\Psi}(\boldsymbol{y}))$, and so

$$\|\boldsymbol{\xi}\|^2 \le 2\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C(B_{N-1}(\mathbf{0},R);\mathbb{R}^{N-1})}\right)\|\boldsymbol{\eta}\|^2.$$

In turn,

$$\boldsymbol{\xi} A_h(\boldsymbol{y}')\boldsymbol{\xi}^T = \|\boldsymbol{\eta}\|^2 \ge \frac{1}{2\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C(B_{N-1}(\mathbf{0},R);\mathbb{R}^{N-1})}\right)}\|\boldsymbol{\xi}\|^2.$$

Thus, we must have

$$\begin{aligned}
\lambda &\le \frac{1}{2\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C(B_{N-1}(\mathbf{0},R);\mathbb{R}^{N-1})}\right)}, \\
2&\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C(B_{N-1}(\mathbf{0},R);\mathbb{R}^{N-1})}\right) \le \Lambda.
\end{aligned}$$

By the previous theorem, for all $0 < r \le \sqrt{\frac{\lambda}{\Lambda}}\frac{R}{4}$,

$$\begin{aligned}
\left|\nabla^2_{\boldsymbol{y}} v\right|_{C^{0,\alpha}(B^+(\mathbf{0},r)\cup\Gamma_r)} \le C\Big(&\|v\|_{C(B^+(\mathbf{0},R))} + \|\nabla_{\boldsymbol{y}} v\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \\
&+ \|f_1\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} + \|g_1\|_{C^{1,\alpha}(\Gamma_R)}\Big),
\end{aligned}$$

where $C = C(\lambda, \Lambda, \alpha, L, M, r)$ and

$$L := \sum_{i,j=1}^{N} \|a_{i,j}\|_{C^{0,\alpha}(B^+(\mathbf{0},R))}, \quad M := \sum_{i,j=1}^{N} \|a_{i,j}\|_{C^{1,\alpha}(B^+(\mathbf{0},R))}.$$

Now, by the mean value theorem,

$$\begin{aligned}
|f_1(\boldsymbol{y}_1) - f_1(\boldsymbol{y}_2)| &= \left|f\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y}_1)\right) - f\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y}_2)\right)\right| \\
&\le \|f\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} \left\|\boldsymbol{\Psi}^{-1}(\boldsymbol{y}_1) - \boldsymbol{\Psi}^{-1}(\boldsymbol{y}_2)\right\|^{\alpha} \qquad (95) \\
&\le C\|f\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} \left(1 + \|\nabla_{\boldsymbol{y}'} h\|^{\alpha}_{C(B_{N-1}(\mathbf{0},R))}\right) \|\boldsymbol{y}_1 - \boldsymbol{y}_2\|^{\alpha},
\end{aligned}$$

so that

$$\|f\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \le CL^{\alpha} \|f\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))}.$$

Since $g_1(\boldsymbol{y}') = -g(\boldsymbol{y}', h(\boldsymbol{y}'))\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|^2_{N-1}}$, we have that

$$\begin{aligned}
\frac{\partial g_1}{\partial y_i}(\boldsymbol{y}') = &-\frac{\partial g}{\partial x_i}(\boldsymbol{y}', h(\boldsymbol{y}'))\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|^2_{N-1}} \\
&- \frac{\partial g}{\partial x_N}(\boldsymbol{y}', h(\boldsymbol{y}'))\frac{\partial h}{\partial y_i}(\boldsymbol{y}')\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|^2_{N-1}} \\
&- g(\boldsymbol{y}', h(\boldsymbol{y}'))\frac{\sum_{j=1}^{N-1} \frac{\partial h}{\partial y_j}(\boldsymbol{y}') \frac{\partial^2 h}{\partial y_i \partial y_j}(\boldsymbol{y}')}{\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h(\boldsymbol{y}')\|^2_{N-1}}},
\end{aligned}$$

and so, reasoning as in (95),

$$\begin{aligned}
&\left|\frac{\partial g}{\partial x_i}(\cdot, h(\cdot))\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right|_{C^{0,\alpha}(\Gamma_R)} \\
&\le \left\|\frac{\partial g}{\partial x_i}(\cdot, h(\cdot))\right\|_{C(\Gamma_S)} \left|\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right|_{C^{0,\alpha}(\Gamma_R)} \\
&\quad + \left|\frac{\partial g}{\partial x_i}(\cdot, h(\cdot))\right|_{C^{0,\alpha}(\Gamma_R)} \left\|\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right\|_{C(\Gamma_S)} \\
&\le CL^{1+\alpha}\|g\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left|\frac{\partial g}{\partial x_N}(\cdot, h(\cdot))\frac{\partial h}{\partial y_i}\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right|_{C^{0,\alpha}(\Gamma_R)} \\
&\le \left\|\frac{\partial g}{\partial x_N}(\cdot, h(\cdot))\right\|_{C(\Gamma_S)} \left|\frac{\partial h}{\partial y_i}\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right|_{C^{0,\alpha}(\Gamma_R)} \\
&\quad + \left|\frac{\partial g}{\partial x_N}(\cdot, h(\cdot))\right|_{C^{0,\alpha}(\Gamma_R)} \left\|\frac{\partial h}{\partial y_i}\sqrt{1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{N-1}}\right\|_{C(\Gamma_S)} \\
&\le CL^{2+\alpha}\|g\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))},
\end{aligned}$$

while

$$\begin{aligned}
&\left| g\left(\cdot, h\left(\boldsymbol{\cdot}\right)\right) \frac{\sum_{j=1}^{N-1} \frac{\partial h}{\partial y_j} \frac{\partial^2 h}{\partial y_i \partial y_j}}{\sqrt{1+\left\|\nabla_{\boldsymbol{y}'} h\right\|_{N-1}^2}} \right|_{C^{0,\alpha}(\Gamma_R)} \\
&\le \left\| g\left(\cdot, h\left(\boldsymbol{\cdot}\right)\right)\right\|_{C(\Gamma_S)} \left| \frac{\sum_{j=1}^{N-1} \frac{\partial h}{\partial y_j} \frac{\partial^2 h}{\partial y_i \partial y_j}}{\sqrt{1+\left\|\nabla_{\boldsymbol{y}'} h\right\|_{N-1}^2}} \right|_{C^{0,\alpha}(\Gamma_R)} \\
&\quad + \left| g\left(\cdot, h\left(\boldsymbol{\cdot}\right)\right)\right|_{C^{0,\alpha}(\Gamma_R)} \left\| \frac{\sum_{j=1}^{N-1} \frac{\partial h}{\partial y_j} \frac{\partial^2 h}{\partial y_i \partial y_j}}{\sqrt{1+\left\|\nabla_{\boldsymbol{y}'} h\right\|_{N-1}^2}} \right\|_{C(\Gamma_S)} \\
&\le CL^{2+\alpha} M \left\| g\right\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} .
\end{aligned}$$

Hence,

$$\left| \frac{\partial g_1}{\partial y_i} \right|_{C^{0,\alpha}(\Gamma_R)} \le CL^{2+\alpha} M \left\| g\right\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))} .$$

Similarly, using $v\left(\boldsymbol{y}\right) = u\left(\boldsymbol{y}', y_N + h\left(\boldsymbol{y}'\right)\right)$, we get

$$\left\| \frac{\partial v}{\partial y_i} \right\|_{C^{0,\alpha}(B^+(\mathbf{0},R))} \le CL^2 \left\| u\right\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))} .$$

Hence,

$$\begin{aligned}
\left|\nabla^2_{\boldsymbol{y}} v\right|_{C^{0,\alpha}(B^+(\mathbf{0},r)\cup\Gamma_r)} \le CL^{2+\alpha} M \Big(&\left\| u\right\|_{C(\Omega\cap B(\mathbf{0},R))} + \left\| u\right\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))} \\
&+ \left\| f\right\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} + \left\| g\right\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))} \Big) ,
\end{aligned}$$

Finally, using (94) and setting $D_r := B^+\left(\mathbf{0}, r\right) \cup \Gamma_r$,

$$\begin{aligned}
&\left| \frac{\partial^2 u}{\partial x_j \partial x_i} \right|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},r))} \le L^\alpha \left| \frac{\partial^2 v}{\partial y_j \partial y_i} \right|_{C^{0,\alpha}(D_r)} + L \left\| \frac{\partial^2 v}{\partial y_N \partial y_i} \right\|_{C(D_r)} \\
&+ L^{1+\alpha} \left| \frac{\partial^2 v}{\partial y_N \partial y_i} \right|_{C^{0,\alpha}(D_r)} + M \left\| \frac{\partial v}{\partial y_N} \right\|_{C(D_r)} + ML^\alpha \left| \frac{\partial v}{\partial y_N} \right|_{C^{0,\alpha}(D_r)} \\
&+ L \left\| \frac{\partial^2 v}{\partial y_N \partial y_j} \right\|_{C(D_r)} + L^{1+\alpha} \left| \frac{\partial^2 v}{\partial y_N \partial y_j} \right|_{C^{0,\alpha}(D_r)} + L^2 \left\| \frac{\partial^2 v}{\partial y_N^2} \right\|_{C(D_r)} \\
&+ L^{2+\alpha} \left| \frac{\partial^2 v}{\partial y_N^2} \right|_{C^{0,\alpha}(D_r)} .
\end{aligned}$$

By applying Exercise 87 with $\varepsilon = r$, we get

$$\begin{aligned}\left|\frac{\partial^2 u}{\partial x_j \partial x_i}\right|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},r))} &\le CL^{2+\alpha}\left|\nabla_{\boldsymbol{y}}^2 v\right|_{C^{0,\alpha}(D_r)} + CML^{\alpha}\left\|\frac{\partial v}{\partial y_N}\right\|_{C^{0,\alpha}(D_r)} + \frac{C}{r^{2/\alpha}}\|v\|_{C(D_r)} \\ &\le CL^{2+\alpha}\left|\nabla_{\boldsymbol{y}}^2 v\right|_{C^{0,\alpha}(D_r)} + CML^{2\alpha}\left\|\frac{\partial u}{\partial x_N}\right\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},r))} + \frac{C}{r^{2/\alpha}}\|u\|_{C(\Omega\cap B(\mathbf{0},r))} \\ &\le CL^{4+2\alpha}M^2\left(\frac{1}{r^{2/\alpha}}\|u\|_{C(\Omega\cap B(\mathbf{0},R))} + \|u\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))}\right. \\ &\quad \left. + \|f\|_{C^{0,\alpha}(\Omega\cap B(\mathbf{0},R))} + \|g\|_{C^{1,\alpha}(\Omega\cap B(\mathbf{0},R))}\right).\end{aligned}$$

Similar, simpler estimates can be obtained for $\frac{\partial^2 u}{\partial x_N \partial x_i}$ and $\frac{\partial^2 u}{\partial x_N^2}$. ■

We are now ready to combine all these theorems to prove a global Schauder estimate. For simplicity we will consider the Laplacian, but the result holds also for equation in divergence form.

Theorem 95 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with $C^{2,\alpha}$ boundary, $\alpha \in (0,1)$, let $f \in C^{0,\alpha}(\Omega)$, let $g \in C^{1,\alpha}\left(\overline{\Omega}\right)$ and let $u \in C^2\left(\overline{\Omega}\right)$ be a solution of the Neumann problem*

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\left\|u - \frac{1}{\mathcal{L}^N(\Omega)}\int_\Omega u\,d\boldsymbol{x}\right\|_{C^{2,\alpha}(\Omega)} \le C(N,\alpha,\Omega)\left(|f|_{C^{0,\alpha}(\Omega)} + \|g\|_{C^{1,\alpha}(\overline{\Omega})}\right).$$

Proof. Step 1: We first prove that

$$\begin{aligned}\left|\nabla^2 u\right|_{C^{0,\alpha}(\Omega)} \le C(N,\alpha,\Omega)\Big(&\|u\|_{C(\Omega)} + \|u\|_{C^{1,\alpha}(\Omega)} \\ &+ |f|_{C^{0,\alpha}(\Omega)} + \|g\|_{C^{1,\alpha}(\overline{\Omega})}\Big).\end{aligned}$$

For all $\boldsymbol{x}_0 \in \Omega$, there exists $B(\boldsymbol{x}_0, 4r_{\boldsymbol{x}_0}) \subseteq \Omega$, while for all $\boldsymbol{x}_0 \in \partial\Omega$ there exist $R_{\boldsymbol{x}_0} > 0$, $k \in \{1,\ldots,N\}$ and a function $h_{\boldsymbol{x}_0} : \mathbb{R}^{N-1} \to \mathbb{R}$ of class $C^{2,\alpha}$ such that either

$$\Omega \cap B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) : x_k > h_{\boldsymbol{x}_0}(\boldsymbol{x}_k)\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) : x_k < h_{\boldsymbol{x}_0}(\boldsymbol{x}_k)\}.$$

Let

$$r_{\boldsymbol{x}_0} := \frac{R_{\boldsymbol{x}_0}}{16\left(1 + \|\nabla_{\boldsymbol{y}'} h\|^2_{C\left(B_{N-1}\left(\mathbf{0},R_{\boldsymbol{x}_0}\right);\mathbb{R}^{N-1}\right)}\right)}.$$

Then

$$\overline{\Omega} \subset \bigcup_{\boldsymbol{x}_0 \in \overline{\Omega}} B(\boldsymbol{x}_0, r_{\boldsymbol{x}_0})$$

and by compactness there exists ℓ such that

$$\overline{\Omega} \subset \bigcup_{n=1}^{\ell} B\left(\boldsymbol{x}_n, \frac{1}{2} r_{\boldsymbol{x}_n}\right).$$

Let $r = \min_n r_{\boldsymbol{x}_n}$. Given $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, if $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_n, r_{\boldsymbol{x}_n}) \subset \Omega$ for some n, then by Remark 84,

$$\begin{aligned}\frac{\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\|}{\|\boldsymbol{x}-\boldsymbol{y}\|^{\alpha}} &\le C(N,\alpha)\left\{\frac{1}{R_{\boldsymbol{x}_n}^{2+\alpha}}\|u\|_{C(B(\boldsymbol{x}_n, R_{\boldsymbol{x}_n}))} + |f|_{C^{0,\alpha}(B(\boldsymbol{x}_n, R_{\boldsymbol{x}_n}))}\right\}\\ &\le C(N,\alpha)\left\{\frac{1}{r^{2+\alpha}}\|u\|_{C(\Omega)} + |f|_{C^{0,\alpha}(\Omega)}\right\},\end{aligned}$$

while if $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_n, r_{\boldsymbol{x}_n})$ with $\boldsymbol{x}_n \in \partial\Omega$, then by the previous theorem

$$\begin{aligned}\frac{\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\|}{\|\boldsymbol{x}-\boldsymbol{y}\|^{\alpha}} \le C\Bigg(&\frac{1}{r^{2/\alpha}}\|u\|_{C(\Omega)} + \|u\|_{C^{1,\alpha}(\Omega)}\\ &+ |f|_{C^{0,\alpha}(\Omega)} + \|g\|_{C^{1,\alpha}(\Omega)}\Bigg)\end{aligned}$$

where $C = C\left(\alpha, \|h_{\boldsymbol{x}_n}\|_{C^{1,\alpha}(B_{N-1}(\boldsymbol{x}_n, R_{\boldsymbol{x}_n}))}, \|h\|_{C^{2,\alpha}(B_{N-1}(\boldsymbol{x}_n, R_{\boldsymbol{x}_n}))}\right)$. On the other hand, if there is no n such that $\boldsymbol{x}, \boldsymbol{y} \in B(\boldsymbol{x}_n, r_{\boldsymbol{x}_n})$, then let n be such that $\boldsymbol{x} \in B\left(\boldsymbol{x}_n, \frac{1}{2} r_{\boldsymbol{x}_n}\right)$, then $\|\boldsymbol{x}-\boldsymbol{y}\| \ge \frac{1}{2} r_{\boldsymbol{x}_n}$, since otherwise

$$\|\boldsymbol{x}_n - \boldsymbol{y}\| \le \|\boldsymbol{x} - \boldsymbol{x}_n\| + \|\boldsymbol{x}_n - \boldsymbol{y}\| < r_{\boldsymbol{x}_n}.$$

Hence,

$$\begin{aligned}\frac{\left\|\nabla^2 u(\boldsymbol{x}) - \nabla^2 u(\boldsymbol{y})\right\|}{\|\boldsymbol{x}-\boldsymbol{y}\|^{\alpha}} &\le \frac{2^{\alpha+1}}{r^{\alpha}}\left\{\left\|\nabla^2 u(\boldsymbol{x})\right\| + \left\|\nabla^2 u(\boldsymbol{y})\right\|\right\}\\ &\le \frac{C}{r^{\alpha}}\left(r^{\alpha}\left|\nabla^2 u\right|_{C^{0,\alpha}(\Omega)} + \frac{1}{r^2}\|u\|_{C(\Omega)}\right),\end{aligned}$$

where in the last line we have used Exercise 87 with $\varepsilon = r^{\alpha}$.

This completes the proof of the first step. Since $u - \frac{1}{\mathcal{L}^N(\Omega)}\int_\Omega u\,d\boldsymbol{x}$ satisfies the same Neuman problem, without loss of generality, we may assume that

$$\frac{1}{\mathcal{L}^N(\Omega)}\int_\Omega u\,d\boldsymbol{x} = 0.$$

■

Friday, November 01, 2013

Proof. Step 2: We claim that there exists a constant $C_1(N,\alpha,\Omega)$ such that

$$\|u\|_{C^{2,\alpha}(\Omega)} \le C_1(N,\alpha,\Omega)\left(|f|_{C^{0,\alpha}(\Omega)} + \|g\|_{C^{1,\alpha}(\overline{\Omega})}\right).$$

Assume by contradiction that this is not the case. Then for every $n \in \mathbb{N}$ there exist $u_n \in C^2\left(\overline{\Omega}\right)$, $f_n \in C^{0,\alpha}(\Omega)$ and $g_n \in C^{2,\alpha}(\Omega)$ such that $\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u_n \, d\boldsymbol{x} = 0$,

$$\begin{cases} -\Delta u_n(\boldsymbol{x}) = f_n(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g_n(\boldsymbol{x}) & \text{on } \partial\Omega, \end{cases}$$

but

$$\|u_n\|_{C^{2,\alpha}(\Omega)} > n \left(|f_n|_{C^{0,\alpha}(\Omega)} + \|g_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \right).$$

Consider

$$v_n := \frac{u_n}{\|u_n\|_{C^{2,\alpha}(\Omega)}}, \quad F_n := \frac{f_n}{\|u_n\|_{C^{2,\alpha}(\Omega)}}, \quad G_n := \frac{g_n}{\|u_n\|_{C^{2,\alpha}(\Omega)}}.$$

Then $\|v_n\|_{C^{2,\alpha}(\Omega)} = 1$, $\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega v_n \, d\boldsymbol{x} = 0$,

$$\begin{cases} -\Delta v_n(\boldsymbol{x}) = F_n(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial v_n}{\partial \boldsymbol{n}}(\boldsymbol{x}) = G_n(\boldsymbol{x}) & \text{on } \partial\Omega, \end{cases}$$

and

$$1 = \|v_n\|_{C^{2,\alpha}(\Omega)} > n \left(|F_n|_{C^{0,\alpha}(\Omega)} + \|G_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \right),$$

which implies that $|F_n|_{C^{0,\alpha}(\Omega)} \to 0$ and $\|G_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \to 0$. Then $\{v_n\}$ is bounded in $C^2(\Omega)$ and for every multi-index $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = 2$, we have that

$$\left\| \frac{\partial^{\boldsymbol{\beta}} v_n}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) - \frac{\partial^{\boldsymbol{\beta}} v_n}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{y}) \right\| \le \|\boldsymbol{x} - \boldsymbol{y}\|^\alpha .$$

By the Ascoli-Arzelà theorem, this implies that there exist $v \in C^{2,\alpha}\left(\overline{\Omega}\right)$ and a subsequence of $\{v_n\}$, not relabeled, . In turn,

$$\begin{aligned} -\Delta v(\boldsymbol{x}) &= -\lim_{n\to\infty} \Delta v_n(\boldsymbol{x}) = \lim_{n\to\infty} F_n(\boldsymbol{x}) = 0, \\ \frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{x}) &= \lim_{n\to\infty} \frac{\partial v_n}{\partial \boldsymbol{n}}(\boldsymbol{x}) = \lim_{n\to\infty} G_n(\boldsymbol{x}) = 0 \end{aligned}$$

and $\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega v \, d\boldsymbol{x} = 0$. By the uniqueness of the Neumann problem (up to constants), this implies that $v = 0$. On the other hand,

$$\begin{aligned} 1 = \|v_n\|_{C^{2,\alpha}(\Omega)} \le C(N, \alpha, \Omega) \Big(& \|v_n\|_{C^{1,\alpha}(\Omega)} \\ & + |F_n|_{C^{0,\alpha}(\Omega)} + \|G_n\|_{C^{1,\alpha}\left(\overline{\Omega}\right)} \Big) \to 0, \end{aligned}$$

and we have reached a contradiction. ■

9 Existence of Weak Solutions for the Neumann Problem

Consider the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega. \end{cases} \tag{96}$$

Assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set with C^1 boundary, $u \in C^2\left(\overline{\Omega}\right)$, that $f \in C\left(\overline{\Omega}\right)$ and that $g \in C^1\left(\overline{\Omega}\right)$. By (60),

$$\int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} + \int_\Omega v \Delta u \, d\boldsymbol{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial \boldsymbol{n}} \, dS$$

for all $v \in C^2\left(\overline{\Omega}\right)$. Hence,

$$\int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} + \int_\Omega v f \, d\boldsymbol{x} = \int_{\partial\Omega} v g \, dS. \tag{97}$$

We will use (97) to define weak solutions.

Before we do that, we need the notion of trace of a function in $H^1(\Omega)$.

Definition 96 *The boundary $\partial\Omega$ of an open set $\Omega \subset \mathbb{R}^N$ is* uniformly Lipschitz *if there exist $\varepsilon, L > 0$, $M \in \mathbb{N}$, and a locally finite countable open cover $\{\Omega_n\}$ of $\partial\Omega$ such that*

(i) if $x \in \partial\Omega$, then $B(x, \varepsilon) \subset \Omega_n$ for some $n \in \mathbb{N}$,

(ii) no point of $\mathbb{R}^N$ is contained in more than M of the Ω_n's,

(iii) for each n there exist local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and a Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ (both depending on n), with $\operatorname{Lip} f \leq L$, such that

$$\Omega_n \cap \Omega = \Omega_n \cap \left\{(y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_N > f(y')\right\}.$$

Theorem 97 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz and let $1 \leq p < \infty$. Then there exists a continuous linear operator*

$$\operatorname{Tr} : W^{1,p}(\Omega) \to L^p\left(\partial\Omega, \mathcal{H}^{N-1}\right)$$

such that

(i) $\operatorname{Tr}(u) = u$ on $\partial\Omega$ for all $u \in W^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right)$,

(ii) for all $\psi \in C_c^1\left(\mathbb{R}^N\right)$, $u \in W^{1,p}(\Omega)$, and $i = 1, \ldots, N$,

$$\int_\Omega u \frac{\partial \psi}{\partial x_i} \, dx = -\int_\Omega \psi \frac{\partial u}{\partial x_i} \, dx + \int_{\partial\Omega} \psi \operatorname{Tr}(u) \nu_i \, d\mathcal{H}^{N-1},$$

where ν is the outward unit normal to $\partial\Omega$.

Theorem 98 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz. There exists a continuous linear operator*

$$\mathrm{Tr} : BV(\Omega) \to L^1\left(\partial\Omega, \mathcal{H}^{N-1}\right)$$

such that

(i) $\mathrm{Tr}(u) = u$ *on $\partial\Omega$ for all $u \in BV(\Omega) \cap C\left(\overline{\Omega}\right)$,*

(ii) for all $\psi \in C_c^1\left(\mathbb{R}^N\right)$, $u \in BV(\Omega)$, and $i = 1, \ldots, N$,

$$\int_\Omega u \frac{\partial \psi}{\partial x_i}\, dx = -\int_\Omega \psi D_i u\, dx + \int_{\partial\Omega} \psi \,\mathrm{Tr}(u)\, \nu_i \, d\mathcal{H}^{N-1},$$

where ν is the outward unit normal to $\partial\Omega$.

Definition 99 *Let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary. Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. A function $u \in H^1(\Omega)$ is a* weak solution *of* (96) *if it satisfies*

$$\int_\Omega \nabla v \cdot \nabla u \; d\boldsymbol{x} + \int_\Omega v f \; d\boldsymbol{x} = \int_{\partial\Omega} g \,\mathrm{tr}\, v \; dS$$

for all $v \in C_c^1\left(\mathbb{R}^N\right)$.

When Ω has is bounded, taking $v = 1$ in Ω gives the compatibility condition

$$\int_\Omega f \; d\boldsymbol{x} = \int_{\partial\Omega} g \; dS. \tag{98}$$

Monday, November 04, 2013

To prove the existence of a weak solution of the Neumann problem we will need the following results.

Theorem 100 (Poincaré's inequality) *Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a connected bounded domain whose boundary is of class C. Then there exists a constant $C = C(p, \Omega) > 0$ such that for all $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega |u(x) - u_\Omega|^p \; dx \leq C \int_\Omega |\nabla u(x)|^p \; dx.$$

Theorem 101 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded set whose boundary $\partial\Omega$ is Lipschitz and let $1 < p < \infty$. If $u_n, u \subset W^{1,p}(\Omega)$, $n \in \mathbb{N}$, are such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, then* $\mathrm{Tr}(u_n) \to \mathrm{Tr}(u)$ *in $L^p(\partial\Omega)$.*

Theorem 102 (Existence of a weak solution) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz boundary. Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be such that* (98). *Then there exists a weak solution u of the Neumann problem* (96). *This function is unique up to a constant. Moreover,*

$$\|u - u_\Omega\|_{H^1(\Omega)} \leq C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right). \tag{99}$$

Proof. Step 1: Let's prove uniqueness. Let u_1 and u_2 be two weak solutions of (96). Then

$$\int_\Omega \nabla v \cdot \nabla u_1 \; d\boldsymbol{x} + \int_\Omega v f \; d\boldsymbol{x} = \int_{\partial\Omega} g \operatorname{tr} v \; dS,$$
$$\int_\Omega \nabla v \cdot \nabla u_2 \; d\boldsymbol{x} + \int_\Omega v f \; d\boldsymbol{x} = \int_{\partial\Omega} g \operatorname{tr} v \; dS,$$

for all $v \in H^1(\Omega)$. By subtracting the two equations, we get

$$\int_\Omega \nabla v \cdot \nabla (u_1 - u_2) \; d\boldsymbol{x} = 0,$$

and taking $v = u_1 - u_2$ gives $\nabla (u_1 - u_2)(\boldsymbol{x}) = \mathbf{0}$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. Since Ω is connected, it follows that $u_1 - u_2$ is a constant.
Step 2: Next we will prove that a weak solution u satisfies (99). Taking $v = u - u_\Omega$ in (96) and using Hölder's inequality give

$$\begin{aligned}\int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x} &= -\int_\Omega (u - u_\Omega) f \; d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr}(u - u_\Omega) \; dS \\ &\le \|f\|_{L^2(\Omega)} \|u - u_\Omega\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \|\operatorname{tr}(u - u_\Omega)\|_{L^2(\partial\Omega)}.\end{aligned}$$

By Poincaré's inequality

$$\int_\Omega |u - u_\Omega|^2 \; d\boldsymbol{x} \le C(\Omega) \int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x}, \tag{100}$$

while by the trace theorem and (100),

$$\begin{aligned}\|\operatorname{tr}(u - u_\Omega)\|_{L^2(\partial\Omega)} &\le C(\Omega) \left(\|u - u_\Omega\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} \right) \\ &\le C(\Omega) \|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}.\end{aligned} \tag{101}$$

Hence,

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)} \le C(\Omega) \|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right),$$

which gives

$$\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} \le C(\Omega) \left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right).$$

Again by (100), we get (99).
Step 3: We will prove the existence of a weak solution of (96). Consider the functional

$$J(u) := \frac{1}{2} \int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x} - \int_\Omega u f \; d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} u \; dS$$

defined on the space $X := \left\{u \in H^1(\Omega) : \; u_\Omega = 0\right\}$. We begin by showing that J is bounded from below. Indeed, by Hölder's inequality, (100) and (101),

$$J(u) \geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^N)} - C(\Omega)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} \quad (102)$$

for all $u \in X$. It suffices to observe that the function

$$t \in \mathbb{R} \mapsto \frac{1}{2}t^2 - C(\Omega)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)t$$

is bounded from below.

Next let

$$m := \inf_{u \in X} J(u)$$

and, using the definition of infimum consider a sequence $\{u_n\} \subset X$ such that

$$m \leq J(u_n) \leq m + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} J(u_n) = m.$$

It follows from (102) and the fact that $J(u_n) \leq m + 1$ for all n, that $\{\nabla u_n\}$ is bounded in $L^2\left(\Omega; \mathbb{R}^N\right)$. In turn, by Poincaré's inequality, the sequence $\{u_n\}$ is bounded in $H^1(\Omega)$, and so, up to a subsequence, not relabelled, there exists $u \in H^1(\Omega)$ such that $\{u_n\}$ converges weakly to u in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. Since $\int_\Omega u_n \, d\boldsymbol{x} = 0$ for all n, letting $n \to \infty$ shows that $\int_\Omega u \, d\boldsymbol{x} = 0$, so that $u \in X$. We claim that $J(u) = m$. To see this, observe that

$$\begin{aligned}
J(u_n) &= \frac{1}{2}\int_\Omega \|\nabla u_n\|^2 \, d\boldsymbol{x} - \int_\Omega u_n f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} u_n \, dS \\
&= \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u + \nabla u\|^2 \, d\boldsymbol{x} - \int_\Omega (u_n - u + u) f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr}(u_n - u + u) \, dS \\
&= J(u) + \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u\|^2 \, d\boldsymbol{x} \\
&\quad + \int_\Omega (\nabla u_n - \nabla u)\cdot\nabla u \, d\boldsymbol{x} - \int_\Omega (u_n - u) f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr}(u_n - u) \, dS.
\end{aligned}$$

Hence,

$$\begin{aligned}
m &\leq J(u) \leq J(u) + \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u\|^2 \, d\boldsymbol{x} \\
&= J(u_n) - \int_\Omega (\nabla u_n - \nabla u)\cdot\nabla u \, d\boldsymbol{x} + \int_\Omega (u_n - u) f \, d\boldsymbol{x} - \int_{\partial\Omega} g \operatorname{tr}(u_n - u) \, dS.
\end{aligned}$$

Letting $n \to \infty$ and using the facts that $u_n \rightharpoonup u$ in $H^1(\Omega)$, Theorem 101, and $J(u_n) \to m$, shows that

$$m = J(u)$$

and that $\nabla u_n \to \nabla u$ in $L^2\left(\Omega;\mathbb{R}^N\right)$. ■

Wednesday, November 06, 2013

Proof. Next we will show that u is a weak solution of (96). To see this, let $v \in H^1(\Omega)$. Then $w := v - v_\Omega \in X$ and so for every $t \in \mathbb{R}$,

$$J(u) \le J(u + tw).$$

This shows that the real value function $\omega(t) := J(u + tw)$ has a minimum at $t = 0$. Hence, if ω is differentiable at $t = 0$, then $\omega'(0) = 0$. We have

$$\begin{aligned} J(u+tw) &= \frac{1}{2}\int_\Omega \|\nabla u + t\nabla w\|^2 \, d\boldsymbol{x} - \int_\Omega (u+tw) f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr}(u+tw) \, dS \\ &= J(u) + \frac{t^2}{2}\int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} \\ &\quad + t\int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - t\int_\Omega wf \, d\boldsymbol{x} + t\int_{\partial\Omega} g \operatorname{tr} w \, dS, \end{aligned}$$

and so

$$\begin{aligned} \frac{\omega(t) - \omega(0)}{t} &= \frac{J(u+tw) - J(u)}{t} \\ &= \frac{t}{2}\int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} + \int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - \int_\Omega wf \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} w \, dS. \end{aligned}$$

Letting $t \to 0^+$ it follows that ω is differentiable, which implies that $\omega'(0) = 0$, so that

$$\begin{aligned} 0 &= \int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - \int_\Omega wf \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} w \, dS. \\ &= \int_\Omega (\nabla v - 0) \cdot \nabla u \, d\boldsymbol{x} - \int_\Omega (v - v_\Omega) f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr}(v - v_\Omega) \, dS. \\ &= \int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} - \int_\Omega vf \, d\boldsymbol{x} + v_\Omega \left(\int_\Omega f \, d\boldsymbol{x} - \int_{\partial\Omega} g \, dS\right) \\ &= \int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} - \int_\Omega vf \, d\boldsymbol{x} - 0, \end{aligned}$$

by (98). ■

Exercise 103 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz boundary. Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and $\boldsymbol{F} \in H^1\left(\Omega;;\mathbb{R}^N\right)$ be such that*

$$\int_\Omega (f - \operatorname{div} \boldsymbol{F}) \, d\boldsymbol{x} = \int_{\partial\Omega} g \, dS.$$

Consider the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) - \operatorname{div} \boldsymbol{F}(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega. \end{cases}$$

Using the functional

$$J(u) := \frac{1}{2}\int_\Omega \|\nabla u\|^2 \; d\boldsymbol{x} - \int_\Omega uf \; d\boldsymbol{x} - \int_\Omega \nabla u \cdot \boldsymbol{F} \; d\boldsymbol{x} - \int_{\partial\Omega} g \operatorname{tr} u \; dS$$

prove that there exists a weak solution u. This function is unique up to a constant. Moreover,

$$\|u - u_\Omega\|_{H^1(\Omega)} \le C(\Omega)\left(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)} + \|\boldsymbol{F}\|_{L^2(\Omega;\mathbb{R}^N)}\right).$$

Next we show that the previous theorem can be extended to treat existence of weak solutions of a general class of equations.

Theorem 104 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz boundary, let $1 < p < \infty$. Let $f \in L^{p'}(\Omega)$ and $g \in L^{p'}(\partial\Omega)$ be such that (98). Assume that $F : \mathbb{R}^N \to \mathbb{R}$ is a convex function satisfying the conditions*

$$\frac{1}{C}\left(\|\boldsymbol{\xi}\|^p - 1\right) \le F(\boldsymbol{\xi}) \le C\left(1 + \|\boldsymbol{\xi}\|^p\right) \tag{103}$$

for some constant $C > 0$ and for all $\boldsymbol{\xi} \in \mathbb{R}^N$. Then the functional

$$J(u) := \int_\Omega F(\nabla u(\boldsymbol{x})) \; d\boldsymbol{x} - \int_\Omega uf \; d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} u \; dS$$

defined on the space $X := \left\{u \in W^{1,p}(\Omega) : \; u_\Omega = 0\right\}$ admits a minimum.

Proof. We begin by showing that J is bounded from below. Indeed, by (103), Hölder's inequality, (100) and (101),

$$\begin{aligned} J(u) \ge{}& \frac{1}{C}\left(\|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)} - \mathcal{L}^N(\Omega)\right) \\ &- C(\Omega)\left(\|f\|_{L^{p'}(\Omega)} + \|g\|_{L^{p'}(\partial\Omega)}\right)\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \end{aligned}$$

for all $u \in X$. Since $p > 1$, it suffices to observe that the function

$$t \in \mathbb{R} \mapsto \frac{1}{C}\left(t^p - \mathcal{L}^N(\Omega)\right) - C(\Omega)\left(\|f\|_{L^{p'}(\Omega)} + \|g\|_{L^{p'}(\partial\Omega)}\right) t$$

is bounded from below. Next let

$$m := \inf_{u \in X} J(u)$$

and, using the definition of infimum consider a sequence $\{u_n\} \subset X$ such that

$$m \le J(u_n) \le m + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} J(u_n) = m.$$

It follows from the fact that $J(u) \to \infty$ as $\|\nabla u\|_{L^p(\Omega;\mathbb{R}^N)} \to \infty$ and the fact that $J(u_n) \le m+1$ for all n, that $\{\nabla u_n\}$ is bounded in $L^p\left(\Omega;\mathbb{R}^N\right)$. In turn, by Poincaré's inequality, the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$, and so, up to a subsequence, not relabelled, there exists $u \in W^{1,p}(\Omega)$ such that $\{u_n\}$ converges weakly to u in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Since $\int_\Omega u_n \, d\boldsymbol{x} = 0$ for all n, letting $n \to \infty$ shows that $\int_\Omega u \, d\boldsymbol{x} = 0$, so that $u \in X$. We claim that $J(u) = m$. To see this, observe that by Theorem101, Theorem 105 below, and the fact that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$,

$$\begin{aligned} m &= \lim_{n\to\infty}\left(m + \frac{1}{n}\right) \ge \liminf_{n\to\infty} J(u_n) \\ &= \liminf_{n\to\infty} \int_\Omega F(\nabla u_n) \, d\boldsymbol{x} + \lim_{n\to\infty}\left(-\int_\Omega u_n f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} u_n \, dS\right) \\ &\ge \int_\Omega F(\nabla u) \, d\boldsymbol{x} - \int_\Omega u f \, d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} u \, dS = J(u) \ge m. \end{aligned}$$

This shows that $J(u) = m$. ■

Friday, November 08, 2013

The previous theorem used the lower semicontinuity of the functional $u \mapsto \int_\Omega F(\nabla u) \, d\boldsymbol{x}$ with respect to weak convergence in $W^{1,p}(\Omega)$. We will prove this fact below.

Theorem 105 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let $1 \le p < \infty$, and let $F : \mathbb{R}^N \to \mathbb{R}$ be a borel function bounded from below. Then the functional*

$$H(u) := \int_\Omega F(\nabla u(\boldsymbol{x})) \, d\boldsymbol{x}, \quad u \in W^{1,p}(\Omega),$$

is sequentially lower semicontinuous with respect to weak convergence in $W^{1,p}(\Omega)$ if and only if F is convex.

Proof of sufficiency. By replacing F with F+const, without loss of generality, we can assume that F is nonnegative. Assume that F is convex. Then F is continuous on $\mathbb{R}^N$. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be weakly convergent in $W^{1,p}(\Omega)$ to some $u \in W^{1,p}(\Omega)$. For $\varepsilon > 0$ let

$$\Omega_\varepsilon := \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}$$

and for $\boldsymbol{x} \in \Omega_\varepsilon$ define

$$u_{n,\varepsilon}(\boldsymbol{x}) := (\varphi_\varepsilon * u_n)(\boldsymbol{x}), \quad u_\varepsilon(\boldsymbol{x}) := (\varphi_\varepsilon * u)(\boldsymbol{x}),$$

where φ_ε is a standard mollifier. Fix open sets $U \Subset V \Subset \Omega$. Then for $\varepsilon > 0$ sufficiently small we have $V \Subset \Omega_\varepsilon$. Since

$$\nabla u_\varepsilon(\boldsymbol{x}) = \int_\Omega \nabla\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) \, u(\boldsymbol{y}) \, d\boldsymbol{y}$$

for all $\boldsymbol{x} \in U$, we have

$$\left|\nabla u_{n,\varepsilon}(\boldsymbol{x}) - \nabla u_{\varepsilon}(\boldsymbol{x})\right| \leq C(\varepsilon) \int_V \left|u_n(\boldsymbol{y}) - u(\boldsymbol{y})\right| \, d\boldsymbol{y}.$$

It follows from the fact that $u_n \to u$ in $L^1(V)$, that $\nabla u_{n,\varepsilon}(\boldsymbol{x}) \to \nabla u_{\varepsilon}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \overline{U}$. By Jensen's inequality for all $\boldsymbol{x} \in U$,

$$\begin{aligned} F(\nabla u_{\varepsilon}(\boldsymbol{x})) &\leq F(\nabla u_{n,\varepsilon}(\boldsymbol{x})) + \delta(n,\varepsilon) \\ &= F\left(\int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) \nabla u_n(\boldsymbol{y}) \, d\boldsymbol{y}\right) + \delta(n,\varepsilon) \\ &\leq \int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y} + \delta(n,\varepsilon) \\ &= \int_{\Omega_{\varepsilon}} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y} + \delta(n,\varepsilon), \end{aligned}$$

where

$$\begin{aligned} \delta(n,\varepsilon) := \max\Big\{ &|F(\boldsymbol{\xi}) - F(\boldsymbol{\xi}_1)| : \|\boldsymbol{\xi}\| \leq \|\nabla u_{\varepsilon}\|_{L^{\infty}(\overline{V};\mathbb{R}^N)}, \\ &\|\boldsymbol{\xi} - \boldsymbol{\xi}_1\| \leq C(\varepsilon) \|u_n - u\|_{L^1(V)} \Big\} \to 0 \end{aligned}$$

as $n \to \infty$. Since $F \geq 0$, by Fubini's theorem, and using the fact that

$$\int_{\Omega} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) \, d\boldsymbol{x} = \int_{\Omega_{\varepsilon}} \varphi_{\varepsilon}(\boldsymbol{x}-y) \, d\boldsymbol{x} = 1,$$

we have

$$\begin{aligned} \int_U F(\nabla u_{\varepsilon}(\boldsymbol{x})) \, d\boldsymbol{x} &\leq \int_U \left(\int_{\Omega_{\varepsilon}} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y}\right) d\boldsymbol{x} + \delta(n,\varepsilon) \mathcal{L}^N(U) \\ &\leq \int_{\Omega_{\varepsilon}} F(\nabla u_n(\boldsymbol{y})) \left(\int_{\Omega_{\varepsilon}} \varphi_{\varepsilon}(\boldsymbol{x}-\boldsymbol{y}) \, d\boldsymbol{x}\right) d\boldsymbol{y} + \delta(n,\varepsilon) \mathcal{L}^N(U) \\ &= \int_{\Omega} F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y} + \delta(n,\varepsilon) \mathcal{L}^N(U). \end{aligned}$$

Letting $n \to \infty$ yields

$$\int_U F(\nabla u_{\varepsilon}(\boldsymbol{x})) \, d\boldsymbol{x} \leq \liminf_{n\to\infty} \int_{\Omega} F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y}.$$

Since $\nabla u_{\varepsilon}(\boldsymbol{x}) = (\varphi_{\varepsilon} * \nabla u)(\boldsymbol{x}) \to \nabla u(\boldsymbol{x})$ at every Lebesgue point $\boldsymbol{x} \in U$, it follows by Fatou's lemma that

$$\begin{aligned} \int_U F(\nabla u(\boldsymbol{x})) \, d\boldsymbol{x} &= \int_U \liminf_{\varepsilon\to 0^+} F(\nabla u_{\varepsilon}(\boldsymbol{x})) \, d\boldsymbol{x} \leq \liminf_{\varepsilon\to 0^+} \int_U F(\nabla u_{\varepsilon}(\boldsymbol{x})) \, d\boldsymbol{x} \\ &\leq \liminf_{n\to\infty} \int_{\Omega} F(\nabla u_n(\boldsymbol{y})) \, d\boldsymbol{y} \end{aligned}$$

It now suffices to let $U \nearrow \Omega$ and use the Lebesgue monotone convergence theorem. ∎

Proof of necessity. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^N$ with $\boldsymbol{\xi}_1 \neq \boldsymbol{\xi}_2$ and $\theta \in (0,1)$. By rotating the axes, we can assume that $\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2$ is parallel to $\boldsymbol{e}_1$. Define

$$u_n(\boldsymbol{x}) := \boldsymbol{\xi}_2 \cdot \boldsymbol{x} + \frac{\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|}{n} \int_0^{nx_1} \chi(s)\, ds, \tag{104}$$

where χ is the characteristic function of the interval $[0,\theta]$ in $[0,1]$ extended periodically to $\mathbb{R}$ with period one. Then $u_n \in W^{1,p}(\Omega)$ with

$$\begin{aligned}\nabla u_n(\boldsymbol{x}) &= \boldsymbol{\xi}_2 + \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \chi(nx_1)\, \boldsymbol{e}_1 \\ &= \boldsymbol{\xi}_2 + \chi(nx_1)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2).\end{aligned}$$

Since the $\chi(nx_1) \rightharpoonup \theta$ (we will see this later), we have that

$$\nabla u_n(\boldsymbol{x}) \rightharpoonup \boldsymbol{\xi}_2 + \theta(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)$$

in $L^p(\Omega; \mathbb{R}^N)$, and so, up to a subsequence, we have

$$u_n \rightharpoonup (\theta\boldsymbol{\xi}_1 + (1-\theta)\boldsymbol{\xi}_2)\boldsymbol{x} + c \text{ in } W^{1,p}(\Omega)$$

for some $c \in \mathbb{R}$. Thus,

$$\begin{aligned}&F(\theta\boldsymbol{\xi}_1 + (1-\theta)\boldsymbol{\xi}_2)\mathcal{L}^N(\Omega) = \int_\Omega F(\boldsymbol{\xi}_1 + \theta(\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1))\, d\boldsymbol{x} \\ &\leq \liminf_{n\to\infty} \int_\Omega F(\boldsymbol{\xi}_2 + \chi(nx_1)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2))\, d\boldsymbol{x} \\ &= \liminf_{n\to\infty} \int_\Omega \chi(nx_1) F(\boldsymbol{\xi}_1) + (1 - \chi(nx_1)) F(\boldsymbol{\xi}_2)\, d\boldsymbol{x} \\ &= [\theta F(\boldsymbol{\xi}_1) + (1-\theta) F(\boldsymbol{\xi}_1)]\mathcal{L}^N(\Omega),\end{aligned}$$

therefore F is convex. This completes the proof. ∎

Monday, November 11, 2013

Proof. Let $\theta \in (0,1)$ and let χ be the characteristic function of the interval $[0,\theta]$ in $[0,1]$ extended periodically to $\mathbb{R}$ with period one. We claim that the sequence of functions $v_n(\boldsymbol{x}) := \chi(nx_N)$ converges weakly to θ in $L^p(\Omega)$. To see this, let $h \in C_c(\Omega)$. Let $Q_m = (-m,m)^N$ be a cube with $m \in \mathbb{N}$ such that $\Omega \subseteq Q_m$. Extend h to be zero outside Ω. Then by Fubini's theorem

$$\begin{aligned}\int_\Omega (\chi(nx_N) - \theta) h(\boldsymbol{x})\, d\boldsymbol{x} &= \int_{Q_m} (\chi(nx_N) - \theta) h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_{-m}^m (\chi(nx_N) - \theta)\left(\int_{Q'_m} h(\boldsymbol{x}', x_N)\, d\boldsymbol{x}'\right) dx_N \\ &= \int_{-m}^m (\chi(nx_N) - \theta)\psi(x_N)\, dx_N,\end{aligned}$$

where $\psi(x_N) := \int_{Q'_m} h(\boldsymbol{x}', x_N)\ d\boldsymbol{x}' \in C_c((-m,m))$. Since ψ is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\psi(s) - \psi(t)| \le \varepsilon$ for all $s, t \in (-m,m)$ with $|s-t| \le \delta$. Since $n\frac{i-1}{n} = i-1$, by periodicity of χ, we have that

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} (\chi(nx_N) - \theta)\ dx_N = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\chi\left(n\left(x_N - \frac{i-1}{n}\right)\right) - \theta\right)\ dx_N$$
$$= \frac{1}{n}\int_0^1 (\chi(z) - \theta)\ dz = 0,$$

where we have made the change of variables $n\left(x_N - \frac{i-1}{n}\right) = t$. Hence,

$$\begin{aligned}\left|\int_{-m}^{m} (\chi(nx_N) - \theta)\psi(x_N)\ dx_N\right| &= \left|\sum_i \int_{\frac{i-1}{n}}^{\frac{i}{n}} (\chi(nx_N) - \theta)\psi(x_N)\ dx_N\right| \\ &= \left|\sum_i \int_{\frac{i-1}{n}}^{\frac{i}{n}} (\chi(nx_N) - \theta)\left(\psi(x_N) - \psi\left(\frac{i-1}{n}\right)\right)\ dx_N\right| \\ &\le (1+\theta)\sum_i \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|\psi(x_N) - \psi\left(\frac{i-1}{n}\right)\right|\ dx_N \\ &\le (1+\theta)\,\varepsilon \sum_i \int_{\frac{i-1}{n}}^{\frac{i}{n}} 1\ dx_N = (1+\theta)\,\varepsilon m,\end{aligned}$$

provided n is so large that $n \ge 1/\delta$. This shows that

$$\lim_{n\to\infty} \int_\Omega (\chi(nx_N) - \theta)\, h(\boldsymbol{x})\ d\boldsymbol{x} = 0.$$

By density, the same is true for all $h \in L^{p'}(\Omega)$. Note that here we have used the fact that θ is the average of χ over $(0,1)$ and that χ is periodic with period 1 and bounded. ■

Theorem 106 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz boundary, let $1 < p < \infty$. Let $f \in L^{p'}(\Omega)$ and $g \in L^{p'}(\partial\Omega)$ be such that* (98). *Assume that $F : \mathbb{R}^N \to \mathbb{R}$ is a C^1 convex function satisfying the conditions*

$$\frac{1}{C}(\|\boldsymbol{\xi}\|^p - 1) \le F(\boldsymbol{\xi}) \le C(1 + \|\boldsymbol{\xi}\|^p)$$

for some constant $C > 0$ and for all $\boldsymbol{\xi} \in \mathbb{R}^N$. Then the Neumann problem

$$\begin{cases} -\operatorname{div}(\nabla_{\boldsymbol{\xi}} F(\nabla u)) = f & \text{in } \Omega, \\ (\nabla_{\boldsymbol{\xi}} F(\nabla u)) \cdot \boldsymbol{n} = g & \text{on } \partial\Omega. \end{cases}$$

admit a weak solution $u \in W^{1,p}(\Omega)$, that is,

$$0 = \int_\Omega \nabla_{\boldsymbol{\xi}} F(\nabla u) \cdot \nabla v\ d\boldsymbol{x} - \int_\Omega v f\ d\boldsymbol{x} + \int_{\partial\Omega} g \operatorname{tr} v\ dS$$

for all $v \in W^{1,p}(\Omega)$.

Proof. Step 1: We first prove that

$$\left|\frac{\partial F}{\partial \xi_i}(\boldsymbol{\xi})\right| \leq C\left(1+\|\boldsymbol{\xi}\|^{p-1}\right) \tag{105}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^N$. Fix $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ be fixed and consider

$$h(t) := F(\xi_1, \ldots, \xi_{i-1}, t, \xi_{i+1}, \ldots, \xi_m), \quad t \in \mathbb{R}.$$

Since h is convex and C^1, for all s and $t \in \mathbb{R}$ we have

$$h(t+s) \geq h(t) + h'(t)s.$$

Thus, with $s := 1 + \|\boldsymbol{\xi}\|$, $t := \xi_i$,

$$h'(t) = \frac{\partial F}{\partial \xi_i}(\boldsymbol{\xi}) \leq \frac{h(t+s) - h(t)}{s} \leq C\frac{(1+\|\boldsymbol{\xi}\|^p)}{1+\|\boldsymbol{\xi}\|} \leq C\left(1+\|\boldsymbol{\xi}\|^{p-1}\right).$$

Also, $h(t-s) \geq h(t) - h'(t)s$, and so

$$-h'(t) \leq \frac{h(t-s) - h(t)}{s} \leq C\frac{(1+\|\boldsymbol{\xi}\|^p)}{1+\|\boldsymbol{\xi}\|} \leq C\left(1+\|\boldsymbol{\xi}\|^{p-1}\right).$$

Hence (105) holds.
Step 2: Let u be the minimum of the functional J defined in Theorem 104. Let $v \in H^1(\Omega)$. Then $w := v - v_\Omega \in X$ and so for every $t \in \mathbb{R}$,

$$J(u) \leq J(u + tw).$$

This shows that the real value function $\omega(t) := J(u + tw)$ has a minimum at $t = 0$. Hence, if ω is differentiable at $t = 0$, then $\omega'(0) = 0$. By the mean value theorem

$$\frac{F(\nabla u + t\nabla w) - F(\nabla u)}{t} = \nabla_{\boldsymbol{\xi}} F(\nabla u + \theta t \nabla w) \cdot \nabla w,$$

and so

$$\begin{aligned}\left|\frac{F(\nabla u + t\nabla w) - F(\nabla u)}{t}\right| &\leq \|\nabla_{\boldsymbol{\xi}} F(\nabla u + \theta t \nabla w)\| \, \|\nabla w\| \\ &\leq C\left(1 + \|\nabla u + \theta t \nabla w\|^{p-1}\right)\|\nabla w\| \\ &\leq C\left(1 + \|\nabla u\|^{p-1} + \|\nabla w\|^{p-1}\right)\|\nabla w\|.\end{aligned}$$

By Hölder's inequality,

$$\int_\Omega \|\nabla u\|^{p-1}\|\nabla w\| \, d\boldsymbol{x} \leq \left(\int_\Omega \left(\|\nabla u\|^{p-1}\right)^{p/(p-1)} \, d\boldsymbol{x}\right)^{(p-1)/p} \left(\int_\Omega \|\nabla w\|^p \, d\boldsymbol{x}\right)^{1/p} < \infty$$

and so we can apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{t\to 0}\int_\Omega \frac{F(\nabla u + t\nabla w) - F(\nabla u)}{t}\,d\boldsymbol{x} = \int_\Omega \lim_{t\to 0}\frac{F(\nabla u + t\nabla w) - F(\nabla u)}{t}\,d\boldsymbol{x}$$
$$= \int_\Omega \nabla_{\boldsymbol{\xi}} F(\nabla u)\cdot \nabla w\,d\boldsymbol{x}.$$

In turn, and so

$$0 = \lim_{t\to 0}\frac{\omega(t) - \omega(0)}{t} = \lim_{t\to 0}\frac{J(u+tw) - J(u)}{t}$$
$$= \int_\Omega \nabla_{\boldsymbol{\xi}} F(\nabla u)\cdot\nabla w\,d\boldsymbol{x} - \int_\Omega wf\,d\boldsymbol{x} + \int_{\partial\Omega} g\,\mathrm{tr}\,w\,dS$$
$$= \int_\Omega \nabla_{\boldsymbol{\xi}} F(\nabla u)\cdot\nabla v\,d\boldsymbol{x} - \int_\Omega vf\,d\boldsymbol{x} + \int_{\partial\Omega} g\,\mathrm{tr}\,v\,dS,$$

where in the last identity we have used (98). ■

Remark 107 *By taking $F(\boldsymbol{\xi}) = \frac{1}{p}\|\boldsymbol{\xi}\|^p$, $1 < p < \infty$, in the previous theorem, we get a weak solution of the Neumann problem for the p-*Laplacian equation

$$\begin{cases} -\operatorname{div}\left(\|\nabla u\|^{p-2}\nabla u\right) = f & \text{in } \Omega, \\ \left(\|\nabla u\|^{p-2}\nabla u\right)\cdot \boldsymbol{n} = g & \text{on } \partial\Omega. \end{cases}$$

Wednesday, November 13, 2013

10 Regularity of Weak Solutions of the Neumann Problem

10.1 Interior Regularity of Weak Solutions

In Theorem 102 we have proved the existence of a weak solution of the Neumann problem 96. In this subsection we will show that $u \in H^2_{\mathrm{loc}}(\Omega)$.

Definition 108 *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $k \in \mathbb{N}$, with $k \geq 2$, and let $1 \leq p \leq \infty$. Define by induction the Sobolev space $W^{k,p}(\Omega)$ as*

$$W^{k,p}(\Omega) := \left\{u \in L^p(\Omega) : \nabla u \in W^{k-1,p}\left(\Omega;\mathbb{R}^N\right)\right\}.$$

For $p = 2$, we write $H^k(\Omega) := W^{k,2}(\Omega)$.

Exercise 109 *Prove that $W^{k,p}(\Omega)$ is a Banach space and that $u \in L^p(\Omega)$ belongs to $W^{k,p}(\Omega)$ if and only if for every multi-index α there exists a function $g_\alpha \in L^p(\Omega)$ such that*

$$\int_\Omega u\frac{\partial^\alpha \phi}{\partial x^\alpha}\,dx = (-1)^\alpha \int_\Omega g_\alpha \phi\,dx$$

for all $\phi \in C_c^\infty(\Omega)$.

To prove regularity, we will need the following theorem.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and for every $i = 1, \ldots, N$ and $s > 0$, let

$$\Omega_s := \{\boldsymbol{x} \in \Omega : \boldsymbol{x} + se_i \in \Omega\}.$$

Exercise 110 *Prove that if* $0 < p < 1$ *and* $a, b \geq 0$*, then*

$$(a+b)^p \leq a^p + b^p.$$

Theorem 111 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and let* $u \in W^{1,p}(\Omega)$*,* $1 \leq p < \infty$*. Then for every* $i = 1, \ldots, N$ *and* $s > 0$*,*

$$\int_{\Omega_h} \frac{|u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x})|^p}{s^p}\, d\boldsymbol{x} \leq \int_{\Omega} \left|\frac{\partial u}{\partial x_i}(\boldsymbol{x})\right|^p d\boldsymbol{x} \tag{106}$$

and

$$\lim_{s \to 0^+} \left(\int_{\Omega_s} \frac{|u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x})|^p}{s^p}\, d\boldsymbol{x}\right)^{\frac{1}{p}} = \left(\int_{\Omega} \left|\frac{\partial u}{\partial x_i}(\boldsymbol{x})\right|^p d\boldsymbol{x}\right)^{\frac{1}{p}}. \tag{107}$$

Conversely, if $u \in L^p(\Omega)$*,* $1 < p < \infty$*, is such that*

$$\liminf_{s \to 0^+} \left(\int_{\Omega_s} \frac{|u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x})|^p}{s^p}\, d\boldsymbol{x}\right)^{\frac{1}{p}} < \infty \tag{108}$$

for every $i = 1, \ldots, N$*, then* $u \in W^{1,p}(\Omega)$*.*

We will use Nirenberg's difference quotient method. To illustrate it, consider the case $\Omega = \mathbb{R}^N$ and let $u \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ be a solution of

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \text{in } \mathbb{R}^N,$$

where $f \in H^1(\mathbb{R}^N)$. Let $i \in \{1, \ldots, N\}$ and $s \neq 0$. Then

$$-\Delta u(\boldsymbol{x} + s\boldsymbol{e}_i) = f(\boldsymbol{x} + s\boldsymbol{e}_i) \quad \text{in } \mathbb{R}^N.$$

Subtracting these two equations gives

$$-\Delta u(\boldsymbol{x} + s\boldsymbol{e}_i) + \Delta u(\boldsymbol{x}) = f(\boldsymbol{x} + s\boldsymbol{e}_i) - f(\boldsymbol{x}).$$

Multiply the equation by $u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x})$ and integrating by parts gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \|\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})\|^2\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^N} (u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x}))(-\Delta u(\boldsymbol{x} + s\boldsymbol{e}_i) + \Delta u(\boldsymbol{x}))\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^N} (u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x}))(f(\boldsymbol{x} + s\boldsymbol{e}_i) - f(\boldsymbol{x}))\, d\boldsymbol{x}.
\end{aligned}$$

Dividing both sides by s^2, it follows by Hölder's inequality that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 d\boldsymbol{x} \\
&\le \left(\int_{\mathbb{R}^N} \left(\frac{u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x})}{s} \right)^2 d\boldsymbol{x} \right)^{1/2} \left(\int_{\mathbb{R}^N} \left(\frac{f(\boldsymbol{x}+s\boldsymbol{e}_i) - f(\boldsymbol{x})}{s} \right)^2 d\boldsymbol{x} \right)^{1/2} \\
&\le \left(\int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_i}(\boldsymbol{x}) \right)^2 d\boldsymbol{x} \right)^{1/2} \left(\int_{\mathbb{R}^N} \left(\frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right)^2 d\boldsymbol{x} \right)^{1/2} < \infty.
\end{aligned}
$$

where last inequality follows from the fact that $u, f \in H^1(\mathbb{R}^N)$ and Theorem 111. Again by Theorem 111, we have that $\nabla u \in H^1(\mathbb{R}^N)$ with

$$
\begin{aligned}
\int_{\mathbb{R}^N} \left\| \nabla \left(\frac{\partial u}{\partial x_i} \right)(\boldsymbol{x}) \right\|^2 d\boldsymbol{x} &= \lim_{s \to 0} \int_{\mathbb{R}^N} \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 d\boldsymbol{x} \\
&\le \left(\int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_i}(\boldsymbol{x}) \right)^2 d\boldsymbol{x} \right)^{1/2} \left(\int_{\mathbb{R}^N} \left(\frac{\partial f}{\partial x_i}(\boldsymbol{x}) \right)^2 d\boldsymbol{x} \right)^{1/2}.
\end{aligned}
$$

There are two problems in this method:

1. We should not assume that $f \in H^1(\mathbb{R}^N)$, but only that $f \in L^2(\mathbb{R}^N)$.
2. When $\mathbb{R}^N$ is replaced by Ω, the values $\boldsymbol{x} + s\boldsymbol{e}_i$ may exit the domain Ω.

To solve the first problem we use $u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x}-s\boldsymbol{e}_i) + 2u(\boldsymbol{x})$ instead of $u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x})$. The second problem is significantly more delicate. This is the reason why we will only get interior estimates and we'll need to treat the boundary separately.

Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \le p \le \infty$. The space $W_0^{1,p}(\Omega)$ is defined as the closure of the space $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ (with respect to the topology of $W^{1,p}(\Omega)$). For $p = 2$ we write $H_0^1(\Omega) := W_0^{1,2}(\Omega)$.

Theorem 112 *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $f \in L^2_{\mathrm{loc}}(\Omega)$ and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution of $-\Delta u = f$ in Ω, that is,*

$$
\int_\Omega \nabla v \cdot \nabla u \, d\boldsymbol{x} + \int_\Omega v f \, d\boldsymbol{x} = 0 \tag{109}
$$

for all $v \in H_0^1(\Omega)$. Then $u \in H^2_{\mathrm{loc}}(\Omega)$ and

$$
-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{110}
$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. Moreover, for all $U \Subset V \Subset \Omega$, there exists a constant $C(N) > 0$ such that

$$
\left\| \nabla^2 u \right\|_{L^2(U;\mathbb{R}^{N\times N})} \le C(N) \left(\|f\|_{L^2(V)} + \frac{1}{\operatorname{dist}(U, \partial V)} \|\nabla u\|_{L^2(V \setminus U;\mathbb{R}^N)} \right). \tag{111}
$$

Proof. Let $w \in H_0^1(\Omega)$ with $w = 0$ outside a compact set $K \subset V$ and let $0 < s < \operatorname{dist}(K, \partial V)$. Then the function $v(\boldsymbol{x}) := w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x})$ belongs to $H_0^1(\Omega)$ and is zero outside V and so by (109),

$$\int_\Omega \nabla w(\boldsymbol{x} - s\boldsymbol{e}_i)\cdot\nabla u(\boldsymbol{x})\, d\boldsymbol{x} - \int_\Omega \nabla w(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\, d\boldsymbol{x} + \int_V (w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x}))\, f(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Making the change of variables $\boldsymbol{y} := \boldsymbol{x} - s\boldsymbol{e}_i$ in the first integral on the left-hand side, we get

$$\int_\Omega \nabla w(\boldsymbol{y})\cdot\nabla u(\boldsymbol{y} + s\boldsymbol{e}_i)\, d\boldsymbol{y} - \int_\Omega \nabla w(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\, d\boldsymbol{x} + \int_V (w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x}))\, f(\boldsymbol{x})\, d\boldsymbol{x} = 0,$$

that is,

$$\int_\Omega \nabla w(\boldsymbol{x})\cdot(\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x}))\, d\boldsymbol{x} = -\int_V (w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x}))\, f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{112}$$

Dividing by s and using Hölder's inequality and Theorem 111 gives

$$\begin{aligned} &\int_\Omega \nabla w(\boldsymbol{x})\cdot\frac{\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\, d\boldsymbol{x} \\ &\quad\le \|f\|_{L^2(V)}\left(\int_V\left(\frac{w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x})}{s}\right)^2 d\boldsymbol{x}\right)^{1/2} \\ &\quad\le \|f\|_{L^2(\Omega)}\left(\int_V\left(\frac{\partial w}{\partial x_i}(\boldsymbol{x})\right)^2 d\boldsymbol{x}\right)^{1/2}. \end{aligned}$$

Construct a function $\eta \in C_c^\infty(\Omega; [0,1])$ such that $\eta = 1$ in U, $\eta = 0$ outside V and

$$|\nabla\eta(\boldsymbol{x})| \le \frac{C(N)}{\operatorname{dist}(U, \partial V)}$$

for all $\boldsymbol{x} \in \Omega$, and take $w(\boldsymbol{x}) := \eta^2(\boldsymbol{x})\frac{u(\boldsymbol{x}+s\boldsymbol{e}_i)-u(\boldsymbol{x})}{s}$. ■

Friday, November 15, 2013

Proof. Then

$$\nabla w(\boldsymbol{x}) = \frac{u(\boldsymbol{x} + s\boldsymbol{e}_i) - u(\boldsymbol{x})}{s}2\eta(\boldsymbol{x})\nabla\eta(\boldsymbol{x}) + \eta^2(\boldsymbol{x})\frac{\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s},$$

and so

$$\int_\Omega \eta^2(\boldsymbol{x}) \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 \, d\boldsymbol{x}$$

$$\leq \|f\|_{L^2(V)} \left(\int_K \left(\frac{u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x})}{s} 2\eta(\boldsymbol{x}) \frac{\partial \eta}{\partial x_i}(\boldsymbol{x}) \right)^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$+ \|f\|_{L^2(V)} \left(\int_V \left(\eta^2(\boldsymbol{x}) \frac{\frac{\partial u}{\partial x_i}(\boldsymbol{x}+s\boldsymbol{e}_i) - \frac{\partial u}{\partial x_i} u(\boldsymbol{x})}{s} \right)^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$- \int_K \frac{u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x})}{s} 2\eta(\boldsymbol{x}) \nabla \eta(\boldsymbol{x}) \cdot \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \, d\boldsymbol{x},$$

where $K = \operatorname{supp} \eta$. Using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, the properties of η, and Theorem 111, we have

$$\int_\Omega \eta^2(\boldsymbol{x}) \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 \, d\boldsymbol{x}$$

$$\leq \frac{C(N)}{\operatorname{dist}(U, \partial V)} \|f\|_{L^2(V)} \left(\int_V \left(\frac{\partial u}{\partial x_i}(\boldsymbol{x}) \right)^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$+ \|f\|_{L^2(V)} \left(\int_V \eta^2(\boldsymbol{x}) \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$+ \frac{1}{2} \int_\Omega \eta^2(\boldsymbol{x}) \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 \, d\boldsymbol{x}$$

$$+ 2 \int_\Omega \|\nabla \eta(\boldsymbol{x})\|^2 \left(\frac{u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x})}{s} \right)^2 \, d\boldsymbol{x}.,$$

where $K \subset W \subset V$ is an open set with $0 < s < \operatorname{dist}(K, \partial W)$. Hence,

$$\frac{1}{2} \int_\Omega \eta^2(\boldsymbol{x}) \left\| \frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s} \right\|^2 \, d\boldsymbol{x}$$

$$\leq \frac{C(N)}{\operatorname{dist}(U, \partial V)} \|f\|_{L^2(V)} \left(\int_W \left(\frac{\partial u}{\partial x_i}(\boldsymbol{x}) \right)^2 \, d\boldsymbol{x} \right)^{1/2}$$

$$+ 4 \|f\|^2_{L^2(V)} + \frac{1}{4} \int_\Omega \eta^2(\boldsymbol{x}) \|\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})\|^2 \, d\boldsymbol{x}$$

$$+ \frac{C(N)}{\operatorname{dist}^2(U, \partial V)} \int_K (u(\boldsymbol{x}+s\boldsymbol{e}_i) - u(\boldsymbol{x}))^2 \, d\boldsymbol{x}.$$

Using Theorem 111, we obtain

$$\begin{aligned}
&\frac{1}{4}\int_U \left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i)-\nabla u(\boldsymbol{x})}{s}\right\|^2\,d\boldsymbol{x}\\
&\quad\le \frac{1}{4}\int_\Omega \eta^2(\boldsymbol{x})\left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i)-\nabla u(\boldsymbol{x})}{s}\right\|^2\,d\boldsymbol{x} \qquad (113)\\
&\quad\le \frac{C(N)}{\operatorname{dist}(U,\partial V)}\|f\|_{L^2(V)}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2(W)}+4\|f\|^2_{L^2(V)}\\
&\qquad+\frac{C(N)}{\operatorname{dist}^2(U,\partial V)}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(W)}.
\end{aligned}$$

Again by Theorem 111, we conclude that $\nabla u \in H^1(U)$ with

$$\begin{aligned}
\frac{1}{4}\int_U \left\|\nabla\left(\frac{\partial u}{\partial x_i}\right)(\boldsymbol{x})\right\|^2\,d\boldsymbol{x} &= \lim_{s\to 0}\frac{1}{4}\int_U \left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i)-\nabla u(\boldsymbol{x})}{s}\right\|^2\,d\boldsymbol{x}\\
&\le 5\|f\|^2_{L^2(V)}+\frac{C(N)}{\operatorname{dist}^2(U,\partial V)}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(W)}.
\end{aligned}$$

Letting $W \searrow K$. This shows that $u \in H^2_{\text{loc}}(\Omega)$. In turn, for every $v \in C^1_c(\Omega)$, we can integrate by parts (109) to get

$$\int_\Omega (-\Delta u + f)\, v\, d\boldsymbol{x} = 0,$$

which implies that (110) holds. This concludes the proof. ■

Remark 113 *Note that if $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$, then we can take $V = \Omega$ in the previous proof.*

Next we study higher order regularity. We begin with the following results for Sobolev spaces.

Theorem 114 (Sobolev–Gagliardo–Nirenberg's embedding theorem) *Let $1 \le p < N$. Then there exists a constant $C = C(N,p) > 0$ such that for every function $u \in W^{1,p}(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{p^*}\,dx\right)^{\frac{1}{p^*}} \le C\left(\int_{\mathbb{R}^N} |\nabla u(x)|^p\,dx\right)^{\frac{1}{p}}, \qquad (114)$$

where

$$p^* := \frac{Np}{N-p}.$$

In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for all $p \le q \le p^*$.

The number p^* is called *Sobolev critical exponent.*

Exercise 115 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp < N$. Prove that*

(i) $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $W^{j,q}\left(\mathbb{R}^N\right)$ for all $j \in \mathbb{N}$ and for all $p \leq q \leq \frac{Np}{N-kp}$,

(ii) $W^{k,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q \leq \frac{Np}{N-kp}$.

Corollary 116 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp < N$. Then $W^{k,p}(\Omega)$ is contained in $L^q_{\text{loc}}(\Omega)$ for all $p \leq q \leq \frac{Np}{N-kp}$. Moreover, if Ω is bounded with Lipschitz boundary, then $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $p \leq q \leq \frac{Np}{N-kp}$.*

Theorem 117 *The space $W^{1,N}\left(\mathbb{R}^N\right)$ is continuously embedded in the space $L^q\left(\mathbb{R}^N\right)$ for all $N \leq q < \infty$.*

Exercise 118 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp = N$. Prove that*

(i) $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $W^{j,q}\left(\mathbb{R}^N\right)$ for all $j \in \mathbb{N}$ and for all $p \leq q < \infty$,

(ii) $W^{k,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q < \infty$.

Corollary 119 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp = N$. Then $W^{k,p}(\Omega)$ is contained in $L^q_{\text{loc}}(\Omega)$ for all $p \leq q < \infty$. Moreover, if Ω is bounded with Lipschitz boundary, then $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $p \leq q < \infty$.*

Theorem 120 (Morrey) *Let $N < p < \infty$. Then the space $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$. Moreover, if $u \in W^{1,p}\left(\mathbb{R}^N\right)$ and $\bar{u}$ is its representative in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, then*

$$\lim_{|x|\to\infty} \bar{u}(x) = 0.$$

Exercise 121 *Let $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp > N > (k-1)p$. Prove that $W^{k+j,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $C^{j,\alpha}\left(\mathbb{R}^N\right)$ for all $0 < \alpha \leq k - \frac{N}{p}$.*

Corollary 122 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, and $1 \leq p < \infty$ be such that $k \geq 1$ and $kp > N > (k-1)p$. Then $W^{k+j,p}(\Omega)$ is contained in $C^{j,\alpha}_{\text{loc}}(\Omega)$ for all $0 < \alpha \leq k - \frac{N}{p}$. Moreover, if Ω is bounded with Lipschitz boundary, then $W^{k+j,p}(\Omega)$ is continuously embedded in $C^{j,\alpha}(\Omega)$ for all $0 < \alpha \leq k - \frac{N}{p}$.*

Exercise 123 *Let* $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, *and* $1 \le p < \infty$ *be such that* $k \ge 1$ *and* $N = (k-1)p$. *Prove that* $W^{k+j,p}(\mathbb{R}^N)$ *is continuously embedded in* $C^{j,\alpha}(\mathbb{R}^N)$ *for all* $0 < \alpha < 1$.

Remark 124 *If* $j \in \mathbb{N}_0$, $k = N-1$, *and* $p = 1$, *then* $W^{k+j,1}(\mathbb{R}^N)$ *is continuously embedded in* $C^{j,\alpha}(\mathbb{R}^N)$ *for all* $0 < \alpha \le 1$.

Corollary 125 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set, let* $k \in \mathbb{N}$, $j \in \mathbb{N}_0$, *and* $1 \le p < \infty$ *be such that* $k \ge 1$ *and* $N = (k-1)p$. *Then* $W^{k+j,p}(\Omega)$ *is contained in* $C^{j,\alpha}_{\text{loc}}(\Omega)$ *for all* $0 < \alpha < 1$. *Moreover, if* Ω *is bounded with Lipschitz boundary, then* $W^{k+j,p}(\Omega)$ *is continuously embedded in* $C^{j,\alpha}(\Omega)$ *for all* $0 < \alpha < 1$.

Monday, November 18, 2013

Next we prove that if $f_{\text{loc}} \in H^k(\Omega)$, then $u \in H^{k+2}_{\text{loc}}(\Omega)$. The idea here is to differentiate the equation $-\Delta u = f$ with respect to x_i to get

$$-\Delta\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial f}{\partial x_i} \in L^2_{\text{loc}}(\Omega)$$

and then to apply the previous theorem with $\frac{\partial u}{\partial x_i}$ in place of u.

Theorem 126 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, let* $f \in H^k_{\text{loc}}(\Omega)$ *for* $k \in \mathbb{N}$ *and let* $u \in H^1_{\text{loc}}(\Omega)$ *be a weak solution of* $-\Delta u = f$ *in* Ω. *Then* $u \in H^{k+2}_{\text{loc}}(\Omega)$. *Moreover, for all* $U \Subset V \Subset \Omega$ *and for every multi-index* $\boldsymbol{\alpha}$ *with* $2 \le |\boldsymbol{\alpha}| \le k+2$ *there exists a constant* $C(N,\boldsymbol{\alpha}) > 0$ *such that*

$$\left\|\frac{\partial^{\boldsymbol{\alpha}} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}\right\|_{L^2(U)} \le C(N,\boldsymbol{\alpha})\left(\sum_{i=0}^{|\boldsymbol{\alpha}|-2}\sum_{\boldsymbol{\beta}\ \text{multi-index},\ |\boldsymbol{\beta}|=i}\frac{1}{\operatorname{dist}^{|\boldsymbol{\alpha}|-2-i}(U,\partial V)}\left\|\frac{\partial^{\boldsymbol{\beta}} f}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}\right\|_{L^2(V)} + \frac{1}{\operatorname{dist}^{|\boldsymbol{\alpha}|-1}(U,\partial V)}\|\nabla u\|_{L^2(V)}\right).$$

Proof. We only give the proof for $k = 1$. Given $w \in C^\infty_c(\Omega)$, take $v := \frac{\partial w}{\partial x_i} \in C^\infty_c(\Omega)$ in (109) to find

$$\int_\Omega \nabla\left(\frac{\partial w}{\partial x_i}\right)\cdot\nabla u\, d\boldsymbol{x} + \int_\Omega \frac{\partial w}{\partial x_i} f\, d\boldsymbol{x} = 0.$$

Since $u \in H^2_{\text{loc}}(\Omega)$ by Theorem 112, we can integrate by parts the previous equation to get

$$\int_\Omega \nabla w \cdot \nabla\left(\frac{\partial u}{\partial x_i}\right) d\boldsymbol{x} + \int_\Omega w\frac{\partial f}{\partial x_i}\, d\boldsymbol{x} = 0.$$

By density, this equation holds for all $w \in H^1_0(\Omega)$. Hence, $\frac{\partial u}{\partial x_i} \in H^1_{\text{loc}}(\Omega)$ is a weak solution of $-\Delta\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$ and so we are in a position to apply Theorem

112 to the function $\frac{\partial u}{\partial x_i}$ to conclude that $\frac{\partial u}{\partial x_i} \in H^2_{\text{loc}}(\Omega)$ with

$$\left\| \nabla^2 \left(\frac{\partial u}{\partial x_i} \right) \right\|_{L^2(U)} \leq C(N) \left(\left\| \frac{\partial f}{\partial x_i} \right\|_{L^2(V_1)} + \frac{1}{\operatorname{dist}(U, \partial V_1)} \left\| \nabla \left(\frac{\partial u}{\partial x_i} \right) \right\|_{L^2(V_1)} \right),$$

where $U \Subset V_1 \Subset V$. Moreover, by (111),

$$\left\| \nabla^2 \left(\frac{\partial u}{\partial x_i} \right) \right\|_{L^2(U)} \leq C(N) \left(\left\| \frac{\partial f}{\partial x_i} \right\|_{L^2(V_1)} + \frac{1}{\operatorname{dist}(U, \partial V_1)} \left(\|f\|_{L^2(V)} + \frac{1}{\operatorname{dist}(V_1, \partial V)} \|\nabla u\|_{L^2(V)} \right) \right),$$

and so by taking V_1 in such a way that

$$\operatorname{dist}(U, \partial V_1) = \frac{1}{2} \operatorname{dist}(U, \partial V), \quad \operatorname{dist}(V_1, \partial V) = \frac{1}{2} \operatorname{dist}(U, \partial V),$$

we get

$$\left\| \nabla^2 \left(\frac{\partial u}{\partial x_i} \right) \right\|_{L^2(U)} \leq C(N) \left(\left\| \frac{\partial f}{\partial x_i} \right\|_{L^2(V)} + \frac{2}{\operatorname{dist}(U, \partial V)} \|f\|_{L^2(V)} + \frac{4}{\operatorname{dist}^2(U, \partial V)} \|\nabla u\|_{L^2(V)} \right).$$

■

Wednesday, November 20, 2013
Friday, November 22, 2013

10.2 Boundary Regularity of Weak Solutions for the Neumann Problem

To prove regularity up to the boundary we need to characterize the space of traces of Sobolev functions.

Definition 127 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set such that $\partial\Omega$ is Lipschitz and let $1 \leq p < \infty$ and $0 < s < 1$. A function $u \in L^p(\partial\Omega)$ belongs to the* fractional Sobolev space $W^{s,p}(\partial\Omega)$ *if*

$$\|u\|_{W^{s,p}(\partial\Omega)} := \|u\|_{L^p(\partial\Omega)} + |u|_{W^{s,p}(\partial\Omega)} < \infty,$$

where

$$|u|_{W^{s,p}(\partial\Omega)} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N-1+sp}} \, dS(x) \, dS(y) \right)^{\frac{1}{p}}.$$

For $p = 2$, we write $H^s(\partial\Omega)$.

Theorem 128 (Gagliardo) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open set whose boundary $\partial\Omega$ is uniformly Lipschitz and let $1 < p < \infty$. Then then there exists a constant $C = C(p, N, \Omega) > 0$ such that*

$$\|\mathrm{Tr}(u)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

for all $u \in W^{1,p}(\Omega)$. Conversely, for every $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ there exists a function $u \in W^{1,p}(\Omega)$ such that $\mathrm{Tr}(u) = g$ and

$$\|u\|_{W^{1,p}(\Omega)} \leq C \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)},$$

where $C = C(N, p, \Omega) > 0$.

Given i, j, the tangential derivative $\frac{\partial}{\partial \tau_{jk}}$ is defined

$$\frac{\partial}{\partial \tau_{jk}} := \nu_j \frac{\partial}{\partial x_k} - \nu_k \frac{\partial}{\partial x_j}$$

and the tangential gradient is defined by

$$\nabla_\tau := \left(\sum_{j=1}^N \nu_j \frac{\partial}{\partial \tau_{j1}}, \ldots, \sum_{j=1}^N \nu_j \frac{\partial}{\partial \tau_{jN}} \right).$$

We define the Sobolev space

$$W^{1,p}(\partial\Omega) = \{u \in L^p(\partial\Omega) : \nabla_\tau u \in L^p(\partial\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p}(\partial\Omega)} := \|u\|_{L^p(\partial\Omega)} + \|\nabla_\tau u\|_{L^p(\partial\Omega)}.$$

For $p = 2$, we write $H^1(\partial\Omega)$

Definition 129 *For $s \in (1, 2)$ define the fractional Sobolev space $W^{s,p}(\partial\Omega)$ as all functions $u \in W^{1,p}(\partial\Omega)$ such that $\nabla_\tau u \in W^{s,p}(\partial\Omega)$ with*

$$\|u\|_{W^{s,p}(\partial\Omega)} := \|u\|_{W^{1,p}(\partial\Omega)} + |\nabla_\tau u|_{W^{s,p}(\partial\Omega)}.$$

For $p = 2$, we write $H^s(\partial\Omega)$.

If $1 < p < \infty$ and $u \in W^{2,p}(\Omega)$, then $\nabla u \in W^{1,p}(\Omega)$ and so

$$\mathrm{Tr}\left(\frac{\partial}{\partial x_i}\right) \in W^{1-\frac{1}{p},p}(\partial\Omega).$$

If $\partial\Omega$ is of class C^2, it follows that $\mathrm{Tr}\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Moreover, one can show that $\mathrm{Tr}(u) \in W^{2-\frac{1}{p},p}(\partial\Omega)$ with

$$\nabla_\tau \mathrm{Tr}(u) = \mathrm{Tr}(\nabla_\tau u).$$

Conversely, given $f \in W^{2-\frac{1}{p},p}(\partial\Omega)$ and $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ one can find $u \in W^{2,p}(\Omega)$ such that $\operatorname{Tr}(u) = f$, $\operatorname{Tr}\left(\frac{\partial u}{\partial \boldsymbol{n}}\right) = g$ and

$$\|u\|_{W^{2,p}(\Omega)} \le C(N,\Omega)\left(\|f\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right).$$

Next we consider the case in which Ω is half a ball centered at $\mathbf{0}$ and of radius R, precisely,

$$B_R^+ := B(\mathbf{0}, R) \cap \mathbb{R}_+^N$$

Consider the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } B_R^+, \\ \frac{\partial u}{\partial x_N}(\boldsymbol{x}', 0) = 0 & \text{on } \Gamma_R, \end{cases} \tag{115}$$

where $\Gamma_R := \partial B_R^+ \cap \left\{x \in \mathbb{R}^N : \ x_N = 0\right\}$.

Theorem 130 *Let $f \in L^2\left(B_R^+\right)$ and let $u \in H^1\left(B_R^+\right)$ be a weak solution u of the Neumann problem (124). Then u belongs to $H^2(B_r^+)$ for every $0 < r < R$ with*

$$\left\|\nabla^2 u\right\|_{L^2\left(B_r^+\right)} \le C(N)\left(\|f\|_{L^2\left(B_R^+\right)} + \frac{1}{R-r}\sum_{i=1}^{N-1}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2\left(B_R^+\setminus B_r^+\right)}\right).$$

Proof. Let $0 < r < t < R$. Given $w \in H^1\left(B_R^+\right)$ with $w(\boldsymbol{x}) = 0$ if $\|\boldsymbol{x}\| > t$, extend w to be zero on $\mathbb{R}_+^N$. Then for $i \in \{1, \dots, N-1\}$ and $0 < s < t$, the function $v(\boldsymbol{x}) := w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x})$ belongs to $H^1\left(B_R^+\right)$ and so by (97),

$$\int_{B_R^+} \nabla w(\boldsymbol{x} - s\boldsymbol{e}_i)\cdot\nabla u(\boldsymbol{x})\, d\boldsymbol{x} - \int_{B_R^+} \nabla w(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\, d\boldsymbol{x} + \int_{B_R^+} (w(\boldsymbol{x} - s\boldsymbol{e}_i) - w(\boldsymbol{x}))\, f(\boldsymbol{x})\, d\boldsymbol{x} = 0. \tag{116}$$

Hence, we can continue exactly as in the proof of Theorem 112, with $U := B_r^+$ and $V := B_R^+$, to conclude that

$$\begin{aligned} &\frac{1}{4}\int_{B_r^+}\left\|\frac{\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\right\|^2 d\boldsymbol{x} \\ &\qquad \le 5\|f\|_{L^2\left(B_R^+\right)}^2 + \frac{C(N)}{(R-r)^2}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2\left(B_R^+\setminus B_r^+\right)}^2. \end{aligned}$$

Since $u \in H^2_{\mathrm{loc}}\left(B_R^+\right)$, by Theorem 111, we conclude that $\nabla\left(\frac{\partial u}{\partial x_i}\right) \in L^2\left(B_r^+\right)$ with

$$\begin{aligned} \frac{1}{4}\int_{B_r^+}\left\|\nabla\left(\frac{\partial u}{\partial x_i}\right)(\boldsymbol{x})\right\|^2 d\boldsymbol{x} &= \lim_{s\to 0}\frac{1}{4}\int_{B_r^+}\left\|\frac{\nabla u(\boldsymbol{x} + s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\right\|^2 d\boldsymbol{x} \\ &\le 5\|f\|_{L^2\left(B_R^+\right)}^2 + \frac{C(N)}{(R-r)^2}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2\left(B_R^+\setminus B_r^+\right)}^2. \end{aligned}$$

Since $-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x})$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in B_R^+$, it follows that

$$\frac{\partial^2 u}{\partial x_N^2} = -\sum_{i=1}^{N-1} \frac{\partial^2 u}{\partial x_i^2} + f \in L^2\left(B_r^+\right),$$

with

$$\int_{B_r^+} \left\| \frac{\partial^2 u}{\partial x_N^2} \right\|^2 \, d\boldsymbol{x} \le C(N) \left(\|f\|_{L^2\left(B_R^+\right)}^2 + \frac{1}{(R-r)^2} \sum_{i=1}^{N-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2\left(B_R^+ \setminus B_r^+\right)}^2 \right).$$

Step 2. Since $g \in H^{1/2}(\Gamma_R)$, there exists a function $G \in H^2\left(B_R^+\right)$ such that $\frac{\partial G}{\partial x_N} = g$ on Γ_R and

$$\|G\|_{H^2\left(B_R^+\right)} \le C(R) \|g\|_{H^{1/2}(\Gamma_R)}.$$

To find $C(R)$ one could use a rescaling argument. Then the function $w := u - G$ is a weak solution of

$$\begin{array}{ll} -\Delta w(\boldsymbol{x}) = f(\boldsymbol{x}) - \Delta G(\boldsymbol{x}) & \text{in } B_R^+, \\ \frac{\partial w}{\partial x_N}(\boldsymbol{x}', 0) = 0 & \text{on } \Gamma_R. \end{array}$$

Hence, by the previous step,

$$\begin{aligned} \left\|\nabla^2 w\right\|_{L^2\left(B_r^+\right)} &\le C(N) \left(\|f - \Delta G\|_{L^2\left(B_R^+\right)} + \frac{1}{R-r} \sum_{i=1}^{N-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2\left(B_R^+ \setminus B_r^+\right)} \right) \\ &\le C(N) \left(\|f\|_{L^2\left(B_R^+\right)} + \|\Delta G\|_{L^2\left(B_R^+\right)} + \frac{1}{R-r} \sum_{i=1}^{N-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2\left(B_R^+ \setminus B_r^+\right)} \right) \\ &\le C(N) \left(\|f\|_{L^2\left(B_r^+\right)} + C(R) \|g\|_{H^{1/2}(\Gamma_R)} + \frac{1}{R-r} \sum_{i=1}^{N-1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2\left(B_R^+ \setminus B_r^+\right)} \right). \end{aligned}$$

It follows that

$$\begin{aligned} \left\|\nabla^2 u\right\|_{L^2\left(B_r^+\right)} &\le \left\|\nabla^2 w\right\|_{L^2\left(B_r^+\right)} + \left\|\nabla^2 G\right\|_{L^2\left(B_r^+\right)} \\ &\le \left\|\nabla^2 w\right\|_{L^2\left(B_r^+\right)} + C(R) \|g\|_{H^{1/2}(\Gamma_R)} \end{aligned}$$

and the proof is complete. ■

Next we consider the following variation.

$$\left\{ \begin{array}{ll} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) - \operatorname{div} \boldsymbol{F}(\boldsymbol{x}) & \text{in } B_R^+, \\ \frac{\partial u}{\partial x_N}(\boldsymbol{x}', 0) = 0 & \text{on } \Gamma_R. \end{array} \right. \tag{117}$$

A weak solution of (117) satisfies

$$\int_{B_R^+} \nabla v \cdot \nabla u \, d\boldsymbol{x} + \int_{B_R^+} \nabla v \cdot \boldsymbol{F} \, d\boldsymbol{x} + \int_{B_R^+} v f \, d\boldsymbol{x} = 0$$

for all $v \in H^1\left(B_R^+\right)$ with with $\operatorname{tr} v = 0$ on $\partial B_R^+ \setminus \Gamma_R$.

Theorem 131 *Let $f \in L^2\left(B_R^+\right)$, $\boldsymbol{F} \in H^1\left(B_R^+;\mathbb{R}^N\right)$, and let $u \in H^1\left(B_R^+\right)$ be a weak solution of the Neumann problem* (117)*. Then u belongs to $H^2\left(B_r^+\right)$ for every $0 < r < R$ with*

$$\left\|\nabla^2 u\right\|_{L^2\left(B_r^+\right)} \leq C(N)\left(\|f\|_{L^2\left(B_R^+\right)} + \|\nabla \boldsymbol{F}\|_{L^2\left(B_R^+\right)} + \frac{1}{R-r}\sum_{i=1}^{N-1}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2\left(B_R^+\setminus B_r^+\right)}\right).$$

Proof. This follows from the estimate

$$\begin{aligned} &\frac{1}{8}\int_{B_R^+}\eta^2(\boldsymbol{x})\left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i)-\nabla u(\boldsymbol{x})}{s}\right\|^2\,d\boldsymbol{x} \\ &\leq 5\|f\|_{L^2(V)}^2 + \frac{C(N)}{(R-r)^2}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^2\left(B_R^+\right)}^2 \\ &\quad + 5\int_{B_R^+}\eta^2(\boldsymbol{x})\left\|\frac{\boldsymbol{F}(\boldsymbol{x}+s\boldsymbol{e}_i)-\boldsymbol{F}(\boldsymbol{x})}{s}\right\|^2\,d\boldsymbol{x}, \end{aligned} \tag{118}$$

which is left as an exercise. ■

Next we consider the Neumann problem

$$\begin{cases} -\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}) & \text{in } B_R^+, \\ (\boldsymbol{A}(\boldsymbol{x}',0)\nabla u(\boldsymbol{x}',0))\cdot\boldsymbol{e}_N = 0 & \text{on } \Gamma_R, \end{cases} \tag{119}$$

where

$$\boldsymbol{A}(\boldsymbol{x}) = I_N + \boldsymbol{B}(\boldsymbol{x}),$$

with $\boldsymbol{B} \in C^1\left(\overline{B_R^+};\mathbb{R}^{N\times N}\right)$. A weak solution $u \in H^1\left(B_R^+\right)$ satisfies

$$\int_{B_R^+}\nabla v(\boldsymbol{x})\cdot(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x}))\,d\boldsymbol{x} + \int_{B_R^+}v(\boldsymbol{x})f(\boldsymbol{x})\,d\boldsymbol{x} = 0$$

for all $v \in H^1\left(B_R^+\right)$ with with $\operatorname{tr} v = 0$ on $\partial B_R^+\setminus\Gamma_R$, or, equivalently,

$$\int_{B_R^+}\nabla v(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x})\,d\boldsymbol{x} + \int_{B_R^+}\nabla v(\boldsymbol{x})\cdot\boldsymbol{B}(\boldsymbol{x})\nabla u(\boldsymbol{x})\,d\boldsymbol{x} + \int_{B_R^+}v(\boldsymbol{x})f(\boldsymbol{x})\,d\boldsymbol{x} = 0.$$

Theorem 132 *Let $f \in L^2\left(B_R^+\right)$, $\boldsymbol{F} \in H^1\left(B_R^+;\mathbb{R}^N\right)$, and let $u \in H^1\left(B_R^+\right)$ be a weak solution of the Neumann problem* (119)*, where $\boldsymbol{B} \in C^1\left(\overline{B_R^+};\mathbb{R}^{N\times N}\right)$ is such that*

$$\sup_{\boldsymbol{x}\in B_R^+}\|\boldsymbol{B}(\boldsymbol{x})\| \leq \theta << 1.$$

and

$$M := \sup_{\boldsymbol{x}\in B_R^+}\|\nabla\boldsymbol{B}(\boldsymbol{x})\| < \infty.$$

Then u belongs to $H^2\left(B_r^+\right)$ for every $0 < r < R$ with

$$\left\|\nabla^2 u\right\|_{L^2\left(B_r^+\right)} \leq C(N)\left\{\|f\|_{L^2\left(B_R^+\right)} + \left(\frac{1}{R-r} + M\right)\|\nabla u\|_{L^2\left(B_R^+\setminus B_r^+\right)}\right\}.$$

Proof. Indeed, write the equation as

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) + \operatorname{div}(\boldsymbol{B}(\boldsymbol{x})\nabla u(\boldsymbol{x})),$$

which has the form (117) with $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\nabla u(\boldsymbol{x})$. It follows that

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{x}+s\boldsymbol{e}_i) - \boldsymbol{F}(\boldsymbol{x})\| &= \|\boldsymbol{B}(\boldsymbol{x}+s\boldsymbol{e}_i)\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \boldsymbol{B}(\boldsymbol{x})\nabla u(\boldsymbol{x})\| \\ &\le \|\boldsymbol{B}(\boldsymbol{x}+s\boldsymbol{e}_i)(\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x}))\| \\ &\quad + \|(\boldsymbol{B}(\boldsymbol{x}+s\boldsymbol{e}_i) - \boldsymbol{B}(\boldsymbol{x}))\nabla u(\boldsymbol{x})\| \\ &\le \theta\|\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})\| + C(N)\, sM_i\|\nabla u(\boldsymbol{x})\|.\end{aligned}$$

In turn,

$$\left\|\frac{\boldsymbol{F}(\boldsymbol{x}+s\boldsymbol{e}_i) - \boldsymbol{F}(\boldsymbol{x})}{s}\right\|^2 \le 2\theta^2\left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\right\|^2 + C(N)\, M^2\|\nabla u(\boldsymbol{x})\|^2,$$

and so

$$\begin{aligned}&\int_{B_R^+}\eta^2(\boldsymbol{x})\left\|\frac{\boldsymbol{F}(\boldsymbol{x}+s\boldsymbol{e}_i) - \boldsymbol{F}(\boldsymbol{x})}{s}\right\|^2 d\boldsymbol{x} \\ &\le 2\theta^2\int_{B_R^+}\eta^2(\boldsymbol{x})\left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\right\|^2 d\boldsymbol{x} + C(N)\, M^2\int_{B_R^+}\eta^2(\boldsymbol{x})\|\nabla u(\boldsymbol{x})\|^2\, d\boldsymbol{x}.\end{aligned}$$

It follows from (118) that

$$\begin{aligned}\left(\frac{1}{8} - 10\theta^2\right)\int_{B_R^+}\eta^2(\boldsymbol{x})\left\|\frac{\nabla u(\boldsymbol{x}+s\boldsymbol{e}_i) - \nabla u(\boldsymbol{x})}{s}\right\|^2 d\boldsymbol{x}& \\ \le 5\|f\|^2_{L^2(B_R^+)} + \frac{C(N)}{(R-r)^2}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(B_R^+)}& \\ + C(N)\, M^2\|\nabla u\|^2_{L^2(B_R^+;\mathbb{R}^N)}.&\end{aligned}$$

Hence, by Theorem 111,

$$\begin{aligned}\left(\frac{1}{8} - 10\theta^2\right)\int_{B_r^+}\left\|\nabla\left(\frac{\partial u}{\partial x_i}\right)(\boldsymbol{x})\right\|^2 d\boldsymbol{x}& \\ \le 5\|f\|^2_{L^2(B_R^+)} + \frac{C(N)}{(R-r)^2}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(B_R^+\setminus B_r^+)}& \\ + C(N)\, M^2\|\nabla u\|^2_{L^2(B_R^+;\mathbb{R}^N)}.&\end{aligned}$$

To estimate $\frac{\partial^2 u}{\partial x_N^2}$ note that since $-\operatorname{div}(\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x})$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in B_R^+$, it follows that

$$\frac{\partial^2 u}{\partial x_N^2}(1 + b_{N,N}) = f - \sum_{i=1}^{N}\frac{\partial \boldsymbol{b}^{(i)}}{\partial x_i}\cdot\nabla u - \sum_{i=1}^{N-1}\left(\boldsymbol{b}^{(i)}\cdot\nabla\frac{\partial u}{\partial x_i} + b_{i,N}\nabla\frac{\partial^2 u}{\partial x_i\partial x_N}\right).$$

If either $b_{N,N} \geq 0$ or using the fact that $\theta << 1$, we get

$$\int_{B_r^+} \left| \frac{\partial^2 u}{\partial x_N^2} \right|^2 \, d\boldsymbol{x} \leq C(N) \|f\|^2_{L^2(B_R^+)} + \frac{C(N)}{(R-r)^2} \|\nabla_{\boldsymbol{x}'} u\|^2_{L^2(B_R^+ \setminus B_r^+)} + C(N) M^2 \|\nabla u\|^2_{L^2(B_R^+)} .$$

■

Now we consider the case in which the boundary is not flat. For $s > 0$, we set $B_s := B(\mathbf{0}, s)$, $B_s^+ := B^+(\mathbf{0}, s)$, $\Gamma_s := \partial B_s^+ \cap \{x_N = 0\}$, and $B_s' := B_{N-1}(\mathbf{0}, s)$. We will use the following theorem.

Theorem 133 (Change of variables) *Let $\Omega, \Omega' \subset \mathbb{R}^N$ be open sets, let $\Psi : \Omega' \to \Omega$ be invertible, with Ψ and Ψ^{-1} Lipschitz functions, and let $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then $u \circ \Psi \in W^{1,p}(\Omega')$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $y \in \Omega'$,*

$$\frac{\partial (u \circ \Psi)}{\partial y_i}(y) = \sum_{j=1}^N \frac{\partial u}{\partial x_j}(\Psi(y)) \frac{\partial \Psi_j}{\partial y_i}(y) .$$

Theorem 134 *Let $\Omega \subset \mathbb{R}^N$ be an open set with $\mathbf{0} \in \partial\Omega$. Assume that there exist $R > 0$ and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^2, such that $h(\mathbf{0}) = 0$, $\nabla_{\boldsymbol{x}'} h(\mathbf{0}) = \mathbf{0}$, and*

$$\Omega \cap B_R = \{\boldsymbol{x} \in B_r : x_N > h(\boldsymbol{x}')\} .$$

Let $f \in L^2(\Omega \cap B_R)$, let $g \in H^{1/2}(\partial\Omega \cap B_R)$ and let $u \in H^1(\Omega \cap B_R)$ be a weak solution of the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega \cap B_R, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega \cap B_R, \end{cases}$$

that is,

$$\int_{\Omega \cap B_R} \nabla\varphi \cdot \nabla u \, d\boldsymbol{x} + \int_{\Omega \cap B_R} \varphi f \, d\boldsymbol{x} = \int_{\partial\Omega \cap B_R} g \operatorname{tr}\varphi \, dS \tag{120}$$

for all $\varphi \in H^1(\Omega)$ with $\operatorname{tr}\varphi = 0$ on $(\partial\Omega \cap B_R) \setminus (\partial\Omega \cap B_R)$. Then,

$$\left\|\nabla^2 u\right\|_{L^2(\Omega \cap B_r)} \leq C(N) \Big\{ \|f\|_{L^2(\Omega \cap B_R)} + C(\Omega, R, r) \|g\|_{H^{1/2}(\partial\Omega \cap B_R)} + \left(\frac{1}{R-r} + \left\|\nabla^2 h\right\|_{L^\infty(B_R^+)} \right) \|\nabla u\|_{L^2(\Omega \cap B_R)} \Big\}$$

for all $0 < r < \min\left\{ R, \frac{\theta}{2\sqrt{N}\|\nabla^2 h\|_{L^\infty(B_R^+)}} \right\}$.

Proof. Step 1: Assume that $g = 0$. Consider the transformation $\boldsymbol{\Psi} : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\boldsymbol{\Psi}(\boldsymbol{x}) := (\boldsymbol{x}', x_N - h(\boldsymbol{x}')) := \boldsymbol{y},$$

and the function

$$v(\boldsymbol{y}) := u\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right) = u(\boldsymbol{y}', y_N + h(\boldsymbol{y}')), \quad \psi(\boldsymbol{y}) := \varphi\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right).$$

Then $u(\boldsymbol{x}) = v(\boldsymbol{\Psi}(\boldsymbol{x})) = v(\boldsymbol{x}', x_N - h(\boldsymbol{x}'))$ and so for $i, j = 1, \ldots, N-1$ and $\boldsymbol{y} = \boldsymbol{\Psi}(\boldsymbol{x})$,

$$\begin{aligned} \frac{\partial u}{\partial x_i}(\boldsymbol{x}) &= \frac{\partial v}{\partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) - \frac{\partial h}{\partial x_i}(\boldsymbol{x}') \frac{\partial v}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x})), \\ \frac{\partial u}{\partial x_N}(\boldsymbol{x}) &= \frac{\partial v}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x})), \end{aligned} \tag{121}$$

We claim that v is a weak solution of the Neumann problem

$$\begin{cases} -\operatorname{div}_{\boldsymbol{y}}(A(\boldsymbol{y}')\nabla_{\boldsymbol{y}} v(\boldsymbol{y})) = F(\boldsymbol{y}) & \text{in } B_R^+, \\ (A(\boldsymbol{y}')\nabla_{\boldsymbol{y}} v(\boldsymbol{y}', 0)) \cdot \boldsymbol{e}_N = 0 & \text{on } \Gamma_R, \end{cases}$$

where

$$\begin{aligned} A(\boldsymbol{y}') &:= \begin{pmatrix} & & & -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') \\ & I_{N-1} & & \vdots \\ & & & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') \\ -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') & \cdots & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') & 1 + \sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}(\boldsymbol{y}')\right)^2 \end{pmatrix} \\ &= I_N + \begin{pmatrix} & & & -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') \\ & 0_{N-1} & & \vdots \\ & & & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') \\ -\frac{\partial h}{\partial x_1}(\boldsymbol{y}') & \cdots & -\frac{\partial h}{\partial x_{N-1}}(\boldsymbol{y}') & \sum_{i=1}^{N-1}\left(\frac{\partial h}{\partial x_i}(\boldsymbol{y}')\right)^2 \end{pmatrix} =: I_N + B(\boldsymbol{y}') \end{aligned}$$

and $F(\boldsymbol{y}) := f\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right)$. Note that

$$\|\boldsymbol{B}(\boldsymbol{y}')\| = 2\,\|\nabla_{\boldsymbol{x}'} h(\boldsymbol{y}')\|.$$

We need to show that for all $\psi \in H^1\left(B_R^+\right)$ with $\operatorname{tr}\psi = 0$ on $\partial B_R^+ \setminus \Gamma_R$,

$$\int_{B_R^+} \nabla_{\boldsymbol{y}}\psi(\boldsymbol{y}) \cdot (\boldsymbol{A}(\boldsymbol{y})\nabla_{\boldsymbol{y}} v(\boldsymbol{y}))\, d\boldsymbol{y} + \int_{B_R^+} \psi(\boldsymbol{y}) F(\boldsymbol{y})\, d\boldsymbol{y} = 0.$$

Fix any such ψ and take $\varphi(\boldsymbol{x}) := \psi(\boldsymbol{\Psi}(\boldsymbol{x}))$. Then by (120) and (121),

$$\begin{aligned} &\int_{\Omega \cap B_R} \sum_{i=1}^{N-1} \left(\frac{\partial \psi}{\partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) - \frac{\partial h}{\partial x_i}(\boldsymbol{x}') \frac{\partial \psi}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x}))\right) \cdot \left(\frac{\partial v}{\partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) - \frac{\partial h}{\partial x_i}(\boldsymbol{x}') \frac{\partial v}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x}))\right) d\boldsymbol{x} \\ &\quad + \int_{\Omega \cap B_R} \frac{\partial \psi}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x})) \frac{\partial v}{\partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x})) \; + \int_{\Omega \cap B_R} \psi(\boldsymbol{\Psi}(\boldsymbol{x})) f(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}$$

Consider the change of variables $\boldsymbol{y} = \boldsymbol{\Psi}(\boldsymbol{x})$ and recalling that $\det \nabla \boldsymbol{\Psi}(\boldsymbol{x}) = 1$, we get

$$\begin{aligned}
&\int_{B_R^+} \sum_{i=1}^{N-1} \left(\frac{\partial \psi}{\partial y_i}(\boldsymbol{y}) - \frac{\partial h}{\partial y_i}(\boldsymbol{y}') \frac{\partial \psi}{\partial y_N}(\boldsymbol{y}) \right) \cdot \left(\frac{\partial v}{\partial y_i}(\boldsymbol{y}) - \frac{\partial h}{\partial y_i}(\boldsymbol{y}') \frac{\partial v}{\partial y_N}(\boldsymbol{y}) \right) \, d\boldsymbol{y} \\
&\quad + \int_{B_R^+} \frac{\partial \psi}{\partial y_N}(\boldsymbol{y}) \frac{\partial v}{\partial y_N}(\boldsymbol{y}) \quad + \int_{B_r^+} \psi(\boldsymbol{y}) f\left(\boldsymbol{\Psi}^{-1}(\boldsymbol{y})\right) \, d\boldsymbol{y} = 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_{B_R^+} \nabla \psi(\boldsymbol{y}) \cdot \nabla v(\boldsymbol{y}) \, d\boldsymbol{y} \\
&\quad - \int_{B_R^+} \sum_{i=1}^{N-1} \left(\frac{\partial \psi}{\partial y_i}(\boldsymbol{y}) \frac{\partial h}{\partial y_i}(\boldsymbol{y}') \frac{\partial v}{\partial y_N}(\boldsymbol{y}) + \frac{\partial h}{\partial y_i}(\boldsymbol{y}') \frac{\partial \psi}{\partial y_N}(\boldsymbol{y}) \frac{\partial v}{\partial y_i}(\boldsymbol{y}) \right) \, d\boldsymbol{y} \\
&\quad + \int_{B_R^+} \frac{\partial \psi}{\partial y_N}(\boldsymbol{y}) \frac{\partial v}{\partial y_N}(\boldsymbol{y}) \sum_{i=1}^{N-1} \left(\frac{\partial h}{\partial y_i}(\boldsymbol{y}') \right)^2 \, d\boldsymbol{y} + \int_{B_R^+} \psi(\boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} = 0,
\end{aligned}$$

which proves the claim. Since $\nabla_{\boldsymbol{x}'} h(\boldsymbol{0}) = \boldsymbol{0}$, by the mean value theorem, for all $\boldsymbol{y}' \in B_r^{'}$,

$$\begin{aligned}
\left| \frac{\partial h}{\partial x_i}(\boldsymbol{y}') \right| &= \left| \frac{\partial h}{\partial x_i}(\boldsymbol{y}') - \frac{\partial h}{\partial x_i}(\boldsymbol{0}) \right| \leq \left\| \nabla^2 h \right\|_{L^\infty(B_R^+)} \|\boldsymbol{y}' - \boldsymbol{0}\|_{N-1} \\
&\leq \left\| \nabla^2 h \right\|_{L^\infty(B_R^+)} r,
\end{aligned} \tag{122}$$

and so

$$\sup_{\boldsymbol{x} \in B_R^+} \|\boldsymbol{B}(\boldsymbol{x})\| = 2 \sup_{\boldsymbol{y}' \in B_r'} \|\nabla_{\boldsymbol{x}'} h(\boldsymbol{y}')\| \leq 2\sqrt{N} \left\| \nabla^2 h \right\|_{L^\infty(B_R^+)} r \leq \theta,$$

where θ is the number given in the previous theorem, provided

$$0 < r < \min \left\{ R, \frac{\theta}{2\sqrt{N} \|\nabla^2 h\|_{L^\infty(B_R^+)}} \right\}.$$

It follows by the previous theorem that $v \in H^2(B_r^+)$, for every $0 < r < R$ with

$$\left\| \nabla^2 v \right\|_{L^2(B_r^+)} \leq C(N) \left\{ \|F\|_{L^2(B_R^+)} + \left(\frac{1}{R - r} + \left\| \nabla^2 h \right\|_{L^\infty(B_R^+)} \right) \|\nabla v\|_{L^2(B_R^+ \setminus B_r^+)} \right\}. \tag{123}$$

Since

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) = {} & \frac{\partial^2 v}{\partial y_j \partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) - \frac{\partial h}{\partial x_j}(\boldsymbol{x}')\frac{\partial^2 v}{\partial y_N \partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) \\ & - \frac{\partial^2 h}{\partial x_j \partial x_i}(\boldsymbol{\Psi}(\boldsymbol{x}))\frac{\partial u}{\partial x_N}(\boldsymbol{x}) - \frac{\partial h}{\partial x_i}(\boldsymbol{x}')\frac{\partial^2 v}{\partial y_j \partial y_N}(\boldsymbol{\Psi}(\boldsymbol{x})) + \frac{\partial h}{\partial x_j}(\boldsymbol{x}')\frac{\partial h}{\partial x_i}(\boldsymbol{x}')\frac{\partial^2 v}{\partial y_N^2}(\boldsymbol{\Psi}(\boldsymbol{x})), \\ \frac{\partial^2 u}{\partial x_N \partial x_i}(\boldsymbol{x}) = {} & \frac{\partial^2 v}{\partial y_N \partial y_i}(\boldsymbol{\Psi}(\boldsymbol{x})) - \frac{\partial h}{\partial x_i}(\boldsymbol{x}')\frac{\partial^2 u}{\partial x_N^2}(\boldsymbol{x}), \\ \frac{\partial^2 u}{\partial x_N^2}(\boldsymbol{x}) = {} & \frac{\partial^2 v}{\partial y_N^2}(\boldsymbol{\Psi}(\boldsymbol{x})).\end{aligned}$$

Using the facts that $\left|\frac{\partial h}{\partial x_j}\right| \leq 1$ and $\det \nabla\boldsymbol{\Psi}(\boldsymbol{x}) = 1$, we get

$$\begin{aligned}\int_{\Omega\cap B_r} \left\|\nabla^2 u(\boldsymbol{x})\right\|^2 \, d\boldsymbol{x} & \leq C(N)\left(1 + \left\|\nabla^2 h\right\|^2_{L^\infty(B_R^+)}\right)\int_{\Omega\cap B_r}\left\|\nabla^2 v(\boldsymbol{\Psi}(\boldsymbol{x}))\right\|^2 \, d\boldsymbol{x} \\ & = C(N)\left(1 + \left\|\nabla^2 h\right\|^2_{L^\infty(B_R^+)}\right)\int_{\Omega\cap B_r}\left\|\nabla^2 v(\boldsymbol{\Psi}(\boldsymbol{x}))\right\|^2 \left|\det\nabla\boldsymbol{\Psi}(\boldsymbol{x})\right| \, d\boldsymbol{x} \\ & = C(N)\left(1 + \left\|\nabla^2 h\right\|^2_{L^\infty(B_R^+)}\right)\int_{B_r}\left\|\nabla^2 v(\boldsymbol{y})\right\|^2 \, d\boldsymbol{y}.\end{aligned}$$

Together with (123), this shows that $u \in H^2(\Omega \cap B_r)$, with

$$\left\|\nabla^2 u\right\|_{L^2(\Omega\cap B_r)} \leq C(N)\left\{\|f\|_{L^2(\Omega\cap B_R)} + \left(\frac{1}{R-r} + \left\|\nabla^2 h\right\|_{L^\infty(B_R^+)}\right)\|\nabla u\|_{L^2(\Omega\cap B_R)}\right\}.$$

■

Monday, November 25, 2013

Proof. Step 2: Since $g \in H^{1/2}(\partial\Omega \cap B_R)$, there exists a function $G \in H^2(\Omega \cap B_R)$ such that $\frac{\partial G}{\partial \boldsymbol{n}} = g$ on $\partial\Omega \cap B_R$ and

$$\|G\|_{H^2(\Omega\cap B_R)} \leq C(\Omega, R)\|g\|_{H^{1/2}(\partial\Omega\cap B_R)}.$$

Then the function $w := u - G$ is a weak solution of

$$\begin{aligned} -\Delta w(\boldsymbol{x}) &= f(\boldsymbol{x}) - \Delta G(\boldsymbol{x}) && \text{in } \Omega \cap B_R, \\ \frac{\partial w}{\partial \boldsymbol{n}}(\boldsymbol{x}) &= 0 && \text{on } \partial\Omega \cap B_R. \end{aligned}$$

Hence, by the previous step,

$$\begin{aligned}\left\|\nabla^2 w\right\|_{L^2(\Omega\cap B_r)} & \leq C(N)\left(\|f - \Delta G\|_{L^2(\Omega\cap B_R)} + \frac{1}{R-r}\|\nabla w\|_{L^2(\Omega\cap B_R)}\right) \\ & \leq C(N)\left(\|f\|_{L^2(\Omega\cap B_R)} + \|\Delta G\|_{L^2(\Omega\cap B_R)} + \frac{1}{R-r}\left(\|\nabla u\|_{L^2(\Omega\cap B_R)} + \|\nabla G\|_{L^2(\Omega\cap B_R)}\right)\right) \\ & \leq C(N)\left(\|f\|_{L^2(B_r^+)} + C(\Omega, R, r)\|g\|_{H^{1/2}(\partial\Omega\cap B_R)} + \frac{1}{R-r}\|\nabla u\|_{L^2(\Omega\cap B_R)}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\nabla^2 u\right\|_{L^2(\Omega\cap B_r)} &\le \left\|\nabla^2 w\right\|_{L^2(\Omega\cap B_r)} + \left\|\nabla^2 G\right\|_{L^2(\Omega\cap B_r)} \\ &\le \left\|\nabla^2 w\right\|_{L^2(\Omega\cap B_r)} + C(R)\|g\|_{H^{1/2}(\partial\Omega\cap B_R)}\end{aligned}$$

and the proof is complete. ■

Exercise 135 *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $u \in C^2(\Omega)$, and let $\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{R}\boldsymbol{x}$, where $\boldsymbol{R}$ is an orthogonal matrix. Prove that*

$$\Delta(u \circ \boldsymbol{\Phi}) = (\Delta u) \circ \boldsymbol{\Phi}$$

on $\boldsymbol{\Phi}^{-1}(\Omega)$.

Consider the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = g(\boldsymbol{x}) & \text{on } \partial\Omega. \end{cases} \tag{124}$$

Theorem 136 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^2 boundary. Let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ be such that (98) holds and let $u \in H^1(\Omega)$ be a weak solution u of the Neumann problem (124). Then u belongs to $H^2(\Omega)$. Moreover,*

$$\left\|\nabla^2 u\right\|_{L^2(\Omega)} \le C(\Omega)\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)} + \|u\|_{H^1(\Omega)}\right).$$

Proof. Fix $0 < \theta < 1$. Since $\partial\Omega$ is of class C^2 for every $\boldsymbol{x}_0 \in \partial\Omega$ there exist $R_{\boldsymbol{x}_0} > 0$, $i \in \{1, \ldots, N\}$ and a function $h_{\boldsymbol{x}_0} : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^2 such that

$$\Omega \cap B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) : x_i > h_{\boldsymbol{x}_0}(\boldsymbol{x}_i)\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, R_{\boldsymbol{x}_0}) : x_i < h_{\boldsymbol{x}_0}(\boldsymbol{x}_i)\}.$$

Since $h_{\boldsymbol{x}_0}$ is of class C^2, reasoning as in (122), we have that

$$\left\|\nabla_{\boldsymbol{x}^i} h_{\boldsymbol{x}_0}(\boldsymbol{x}_i) - \nabla_{\boldsymbol{x}^i} h_{\boldsymbol{x}_0}\left(\boldsymbol{x}_0^i\right)\right\|_{N-1} \le \sqrt{N}\left\|\nabla^2_{\boldsymbol{x}^i} h_{\boldsymbol{x}_0}\right\|_{L^\infty\left(B_{N-1}\left(\boldsymbol{x}_0', R_{\boldsymbol{x}_0}\right)\right)} r \le \frac{\theta}{2}, \tag{125}$$

provided

$$0 \le r \le \min\left\{R_{\boldsymbol{x}_0}, \frac{\theta}{2\sqrt{N}\left\|\nabla^2_{\boldsymbol{x}^i} h_{\boldsymbol{x}_0}\right\|_{L^\infty\left(B_{N-1}\left(\boldsymbol{x}_0', R_{\boldsymbol{x}_0}\right)\right)}}\right\} =: r_{\boldsymbol{x}_0}.$$

Since the family $\left\{B\left(\boldsymbol{x}_0, \frac{1}{2} r_{\boldsymbol{x}_0}\right)\right\}_{\boldsymbol{x}_0 \in \partial\Omega}$ covers the compact set $\partial\Omega$, there exist ℓ such that

$$\partial\Omega \subset \bigcup_{i=1}^{\ell} B\left(\boldsymbol{x}_i, \frac{1}{2} r_{\boldsymbol{x}_i}\right).$$

Let $r := \frac{1}{4} \min r_{\boldsymbol{x}_i}$ and

$$\Omega_t := \{\boldsymbol{x} \in \Omega : \ \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > t\}.$$

Then

$$\overline{\Omega} \subset \bigcup_{i=1}^{\ell} B\left(\boldsymbol{x}_i, \frac{1}{2} r_{\boldsymbol{x}_i}\right) \cup \Omega_{\frac{r}{2}}.$$

Define

$$\phi_i := \varphi_{\frac{1}{4} r_{\boldsymbol{x}_i}} * \chi_{B\left(\boldsymbol{x}_i, \frac{3}{4} r_{\boldsymbol{x}_i}\right)}, \quad \phi_0 := \varphi_{\frac{1}{4} r} * \chi_{\Omega_{\frac{3r}{4}}}.$$

Then

$$\psi_i := \frac{\phi_i}{\sum_{k=0}^{\ell} \phi_k}$$

is a partition of unity subordinated to $\{B(\boldsymbol{x}_i, r_{\boldsymbol{x}_i})\}_{i=1}^{\ell} \cup \{\Omega_r\}$. Since

$$\sum_{n=0}^{\ell} \psi_n = 1$$

in $\overline{\Omega}$ and $u \in H^2_{\mathrm{loc}}(\Omega)$ by Theorem 112, we have that

$$\begin{aligned}
\left\|\nabla^2 u\right\|_{L^2(\Omega)} &= \left\|\nabla^2 \left(\sum_{n=0}^{\ell} \psi_n u\right)\right\|_{L^2(\Omega)} = \left\|\sum_{n=0}^{\ell} \nabla^2 (\psi_n u)\right\|_{L^2(\Omega)} \\
&\leq \sum_{n=0}^{\ell} \left\|\nabla^2 (\psi_n u)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, to prove that u belongs $H^2(\Omega)$, it suffices to show that $\psi_n u \in H^2(\Omega)$. Since

$$\begin{aligned}
\Delta(\psi_n u) &= \operatorname{div}(\nabla(\psi_n u)) = \operatorname{div}(\psi_n \nabla u + u \nabla \psi_n) \\
&= \psi_n \Delta u + u \Delta \psi_n + 2\nabla u \cdot \nabla \psi_n \\
&= \psi_n f + u \Delta \psi_n + 2\nabla u \cdot \nabla \psi_n =: f_n
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial(\psi_n u)}{\partial \boldsymbol{n}} &= \nabla(\psi_n u) \cdot \boldsymbol{n} = (\psi_n \nabla u + u \nabla \psi_n) \cdot \boldsymbol{n} \\
&= \psi_n g + u \nabla \psi_n \cdot \boldsymbol{n} =: g_n,
\end{aligned} \tag{126}$$

we have that $\psi_n u$ solves the Neumann problem

$$\begin{cases} -\Delta(\psi_n u) = f_n & \text{in } \Omega, \\ \frac{\partial(\psi_n u)}{\partial \boldsymbol{n}} = g_n & \text{on } \partial\Omega. \end{cases}$$

There are now two cases. For $n = 0$, by Theorem 112 with $U := \Omega_r$ there exists a constant $C(N) > 0$ such that

$$\left\|\nabla^2 (\psi_0 u)\right\|_{L^2(\Omega_r)} \le C(N) \left(\|f_0\|_{L^2(\Omega)} + \frac{1}{\operatorname{dist}(\Omega_r, \partial\Omega)} \|\nabla (\psi_0 u)\|_{L^2(\Omega\setminus\Omega_r)} \right)$$
$$\le C \|f\|_{L^2(\Omega_r)} + C \|\Delta\psi_0\|_{L^\infty(\Omega_r)} \|u\|_{L^2(\Omega_r)} + 2C \|\nabla\psi_0\|_{L^\infty(\Omega_r;\mathbb{R}^N)} \|\nabla u\|_{L^2(\Omega_r)}$$
$$+ \frac{C}{\operatorname{dist}(\Omega_r, \partial\Omega)} \left(\|\nabla\psi_0\|_{L^\infty(\Omega\setminus\Omega_r)} \|u\|_{L^2(\Omega\setminus\Omega_r)} + \|\nabla u\|_{L^2(\Omega\setminus\Omega_r)} \right).$$

It follows that

$$\left\|\nabla^2 (\psi_0 u)\right\|_{L^2(\Omega_r)} \le C(N, \Omega, r) \left\{ \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right\}.$$

It remains to study the case $n \ge 1$. Assume $\operatorname{supp}\psi_n \subset B(\boldsymbol{x}_0, 2r)$, where $\boldsymbol{x}_0 \in \partial\Omega$ and there exist $r > 0$, $k \in \{1, \ldots, N\}$ and a function $h : \mathbb{R}^{N-1} \to \mathbb{R}$ of class C^2 such that either

$$\Omega \cap B(\boldsymbol{x}_0, 2r) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, 2r) : \, x_k > h(\boldsymbol{x}_k)\}$$

or

$$\Omega \cap B(\boldsymbol{x}_0, 2r) = \{\boldsymbol{x} \in B(\boldsymbol{x}_0, r) : \, x_k < h(\boldsymbol{x}_k)\}.$$

Without loss of generality, we take $k = N$ and assume that the first case holds. Also, by a translation, a rotation (see Exercise 135), and by (125), we can assume that $\boldsymbol{x}_0 = \boldsymbol{0}$ and $\nabla_{\boldsymbol{x}'} h(\boldsymbol{0}) = \boldsymbol{0}$, where as usual $\boldsymbol{x}' := \boldsymbol{x}_N = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. Set $B_r := B(\boldsymbol{x}_0, r)$. Hence, we are in a position to apply Theorem 134 to conclude that

$$\left\|\nabla^2 (\psi_n u)\right\|_{L^2(\Omega\cap B_r)} \le C(N, \Omega, r) \|f_n\|_{L^2(\Omega\cap B_{2r})}$$
$$+ \|g_n\|_{H^{1/2}(\partial\Omega\cap B_{2r})} + \|\nabla (\psi_n u)\|_{L^2(\Omega\cap B_{2r})} \Big\}.$$

Now, we leave as an exercise to show that

$$\|\psi_n g\|_{H^{1/2}(\partial\Omega\cap B_{2r})} \le |g|_{H^{1/2}(\partial\Omega\cap B_{2r})} + C(\Omega) \|g\|_{L^2(\partial\Omega\cap B_{2r})},$$

and similarly,

$$\|u \nabla\psi_n \cdot \boldsymbol{n}\|_{H^{1/2}(\partial\Omega\cap B_{2r})} \le |u|_{H^{1/2}(\partial\Omega\cap B_{2r})} + C(\Omega) \|u\|_{L^2(\partial\Omega\cap B_{2r})}$$
$$\le C(\Omega) \|u\|_{H^1(\Omega\cap B_{2r})},$$

where in the last inequality we have used the continuity of the trace operator. Hence,

$$\left\|\nabla^2 (\psi_n u)\right\|_{L^2(\Omega\cap B_r)} \le C(N, \Omega, r) \Big\{ \|f\|_{L^2(\Omega)} + |g|_{H^{1/2}(\partial\Omega)}$$
$$+ \|u\|_{H^1(\Omega)} \Big\}.$$

■

Monday, December 2, 2013

11 BMO Regularity

11.1 BMO Spaces

In what follows $Q(\boldsymbol{x}_0, r) := \boldsymbol{x}_0 + \left(-\frac{r}{2}, \frac{r}{2}\right)^N$ and we use the notation

$$u_E := \frac{1}{\mathcal{L}^N(E)} \int_E u(\boldsymbol{x})\, d\boldsymbol{x}.$$

Definition 137 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that a function $u \in L^1_{\text{loc}}(\Omega)$ has* bounded mean oscillation*, and we write $u \in BMO(\Omega)$, if*

$$|u|_{BMO(\Omega)} := \sup \frac{1}{r^N} \int_{Q(\boldsymbol{x}_0, r)} \left| u(\boldsymbol{x}) - u_{Q(\boldsymbol{x}_0, r)} \right| d\boldsymbol{x} < \infty, \tag{127}$$

where the supremum is taken over all cubes $Q(\boldsymbol{x}_0, r) \subseteq \Omega$.

Remark 138 *For an arbitrary open set Ω, $|\cdot|_{BMO(\Omega)}$ is a seminorm. However, if Ω is connected, then $|u|_{BMO(\Omega)} = 0$ if and only if $u \equiv$const. Therefore $|\cdot|_{BMO(\Omega)}$ is a norm in the quotient space*

$$BMO(\Omega)/\mathbb{R}.$$

It can be shown that $BMO(\Omega)/\mathbb{R}$ is a Banach space.

Proposition 139 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then $L^\infty(\Omega) \subset BMO(\Omega)$.*

Example 140 *To see that the inclusion is strict, observe that the function* $\log|x|$ *belongs to $BMO(\mathbb{R})$ and is unbounded.*

Theorem 141 (John-Nirenberg Inequalities) *Let $u \in BMO(\mathbb{R}^N)$. Then*

(i) for every $1 \le p < \infty$ the function u belongs to $L^p_{\text{loc}}(\mathbb{R}^N)$ and there exists a constant $C_p > 0$ such that for every cube $Q(\boldsymbol{x}_0, r)$,

$$\frac{1}{r^N} \int_{Q(\boldsymbol{x}_0, r)} \left| u(\boldsymbol{x}) - u_{Q(\boldsymbol{x}_0, r)} \right|^p d\boldsymbol{x} \le C_p |u|^p_{BMO(\mathbb{R}^N)};$$

(ii) there exists a constant $C_1, C_2 > 0$ such that for every $\lambda > 0$ and every cube $Q(\boldsymbol{x}_0, r)$,

$$\frac{1}{r^N} \mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q(\boldsymbol{x}_0, r) : \left| u(\boldsymbol{x}) - u_{Q(\boldsymbol{x}_0, r)} \right| > \lambda \right\}\right) \le C_1 e^{-C_2 \lambda / |u|_{BMO(\mathbb{R}^N)}};$$

(iii) for all $\lambda < C_2$ there exists a constant $C = C(\lambda, u) > 0$ such that

$$\frac{1}{r^N} \int_{Q(\boldsymbol{x}_0, r)} e^{-\lambda \left| u(\boldsymbol{x}) - u_{Q(\boldsymbol{x}_0, r)} \right|} d\boldsymbol{x} \le C$$

for every cube $Q(\boldsymbol{x}_0, r)$.

Remark 142 *It follows by property (ii) and Hölder's inequality that*

$$\sup_{Q(\boldsymbol{x}_0,r)}\left(\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}\left|u(\boldsymbol{x})-u_{Q(\boldsymbol{x}_0,r)}\right|^p\,d\boldsymbol{x}\right)^{1/p}$$

is an equivalent seminorm in BMO.

Theorem 143 (Cacciopoli's Inequality) *Let $\Omega\subseteq\mathbb{R}^N$ be an open set and let $u:\Omega\to\mathbb{R}$ be an harmonic function. Then for every $\boldsymbol{x}_0\in\Omega$ and every $Q(\boldsymbol{x}_0,R)\Subset\Omega$,*

$$\int_{Q(\boldsymbol{x}_0,r)}\left|u-u_{Q(\boldsymbol{x}_0,r)}\right|^2\,d\boldsymbol{x}\le C(N)\left(\frac{r}{R}\right)^{N+2}\int_{Q(\boldsymbol{x}_0,R)}\left|u-u_{Q(\boldsymbol{x}_0,R)}\right|^2\,d\boldsymbol{x}$$

for all $0<r\le R$.

Proof. Set $Q_r:=Q(\boldsymbol{x}_0,r)$. By a rescaling argument, it suffices to take $R=1$. Also, since $u-u_{Q_1}$ is harmonic, without loss of generality, we may assume that $u_{Q_1}=0$. Suppose first that $r<\frac{1}{2}$. Since $u\in C^\infty(\Omega)$, by the mean value theorem, there exists $\boldsymbol{x}_1\in Q_r$ such that $u_{Q_r}=u(\boldsymbol{x}_1)$, and so by the mean value theorem, for every $\boldsymbol{x}_1\in Q_r$,

$$|u(\boldsymbol{x})-u_{Q_r}|=|u(\boldsymbol{x})-u(\boldsymbol{x}_1)|\le\|\nabla u(\boldsymbol{z})\|\,r$$

for some $\boldsymbol{z}$ in the segment joining $\boldsymbol{x}$ to $\boldsymbol{x}_0$. It follows from Theorem 61, that

$$\begin{aligned}|u(\boldsymbol{x})-u_{\boldsymbol{x}_0,r}|&\le\|\nabla u(\boldsymbol{z})\|\,r\le 4rN\max_{\partial B(\boldsymbol{z},1/4)}|u|\\&=4rN\,|u(\boldsymbol{w})|=rC(N)\left|\int_{B\left(\boldsymbol{w},\frac{1}{4}\right)}u\,d\boldsymbol{y}\right|\le rC(N)\int_{Q_1}|u|\,d\boldsymbol{y}\\&\le rC(N)\left(\int_{Q_1}|u|^2\,d\boldsymbol{y}\right)^{1/2},\end{aligned}$$

where we have used the mean value formula and Hölder's inequality. Squaring both sides and integrating over Q_r, we get

$$\int_{Q_r}|u-u_{Q_r}|^2\,d\boldsymbol{x}\le C(N)\,r^{N+2}\int_{Q_1}|u|^2\,d\boldsymbol{y}=C(N)\,r^{N+2}\int_{Q_1}|u-u_{Q_1}|^2\,d\boldsymbol{y}.$$

On the other hand, if $\frac{1}{2}<r\le1$, then using the fact that

$$\min_{c\in\mathbb{R}}\int_{Q_r}|u-c|^2\,d\boldsymbol{x}=\int_{Q_r}|u-u_{Q_r}|^2\,d\boldsymbol{x},\tag{128}$$

which is left as an exercise, we have that

$$\begin{aligned}\int_{Q_r}|u-u_{Q_r}|^2\,d\boldsymbol{x}&\le\int_{Q_r}|u-u_{Q_1}|^2\,d\boldsymbol{x}\le\int_{Q_1}|u-u_{Q_1}|^2\,d\boldsymbol{x}\\&\le2^{n+2}r^{n+2}\int_{Q_1}|u-u_{Q_1}|^2\,d\boldsymbol{x},\end{aligned}$$

where we used the fact that $1\le2r$. ■

Theorem 144 (Poincaré's Inequality) *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set, let $1 \le p < \infty$. Then*

$$\int_\Omega |u(\boldsymbol{x})|^p \, d\boldsymbol{x} \le C(N,p)(\operatorname{diam}\Omega)^p \int_\Omega |\nabla u(\boldsymbol{x})|^p \, d\boldsymbol{x}$$

for all $u \in W_0^{1,p}(\Omega)$.

Theorem 145 (Interior BMO Regularity) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $f \in L^\infty(\Omega)$ and let $u \in H^1_{\text{loc}}(\Omega)$ be a weak solution of the Poisson's equation*

$$-\Delta u = f \quad \text{in } \Omega.$$

where $f \in L^\infty(\Omega)$. Then $\nabla^2 u \in BMO_{\text{loc}}(\Omega)$. Moreover, for all $U \Subset V \Subset \Omega$, with U Lipschitz, there exists a constant $C(N) > 0$ such that

$$\left|\nabla^2 u\right|^2_{BMO(U)} \le C(N)\|f\|^2_{L^\infty(V)} + \frac{C(N)}{\operatorname{dist}^{N+2}(U,\partial V)}\|\nabla u\|^2_{L^2(V)}. \tag{129}$$

Lemma 146 *Let $\omega : (0, R_0] \to \mathbb{R}$ be an increasing function such that*

$$\omega(r) \le A\left(\frac{r}{R}\right)^\beta \omega(R) + BR^\alpha$$

for all $0 < r \le R \le R_0$, and for some $A, B \ge 0$ and $0 < \alpha < \beta$. Then

$$\omega(r) \le C(A,\alpha,\beta)\left[\left(\frac{r}{R}\right)^\alpha \omega(R) + Br^\alpha\right]$$

for all $0 < r \le R_0$.

Proof. Without loss of generality we can assume that $A > 1$. Fix $\gamma \in (\alpha, \beta)$ and let $\tau > 0$ so small that be such that $A\tau^\beta = \tau^\gamma$. Then

$$\begin{aligned}\omega(\tau R) &\le A\left(\frac{\tau R}{R}\right)^\beta \omega(R) + BR^\alpha \\ &= \tau^\gamma \omega(R) + BR^\alpha,\end{aligned}$$

and in turn,

$$\begin{aligned}\omega\left(\tau^2 R\right) &\le \tau^\gamma \omega(\tau R) + B(\tau R)^\alpha \\ &\le \tau^\gamma\left(\tau^\gamma \omega(R) + BR^\alpha\right) + B(\tau R)^\alpha \\ &= \tau^{2\gamma}\omega(R) + BR^\alpha \tau^\alpha\left(1 + \tau^{\gamma-\alpha}\right).\end{aligned}$$

By induction,

$$\begin{aligned}\omega(\tau^n R) &\le \tau^{n\gamma}\omega(R) + BR^\alpha \tau^{(n-1)\alpha}\sum_{i=0}^{n-1}\tau^{(\gamma-\alpha)i} \\ &= \tau^{n\gamma}\omega(R) + BR^\alpha \tau^{(n-1)\alpha}\frac{1-\tau^{(\gamma-\alpha)n}}{1-\tau^{\gamma-\alpha}} \\ &\le \tau^{n\alpha}\omega(R) + BR^\alpha \tau^{(n-1)\alpha}\frac{1}{1-\tau^{\gamma-\alpha}},\end{aligned}$$

where we have used the fact that $0 \le \tau \le 1$ and $\alpha \le \gamma$. Now given $0 < r \le R \le R_0$, find $n \in \mathbb{N}_0$ such that $\tau^{n+1}R < r \le \tau^n R$. Since ω is increasing and $\tau \le 1$, we have

$$\begin{aligned}\omega(r) \le \omega(\tau^n R) &\le \tau^{n\alpha}\omega(R) + BR^\alpha \tau^{(n-1)\alpha}\frac{1-\tau^{(\gamma-\alpha)n}}{1-\tau^{\gamma-\alpha}} \\ &\le \frac{1}{\tau^\gamma}\left(\tau^{n+1}\right)^\alpha \omega(R) + BR^\alpha \left(\tau^{n+1}\right)^\alpha \frac{1}{\tau^{2\alpha}\left(1-\tau^{\gamma-\alpha}\right)} \\ &\le \frac{1}{\tau^\gamma}\left(\frac{r}{R}\right)^\alpha \omega(R) + BR^\alpha \left(\frac{r}{R}\right)^\alpha \frac{1}{\tau^{2\alpha}\left(1-\tau^{\gamma-\alpha}\right)}.\end{aligned}$$

This concludes the proof. ■

Wednesday, December 4, 2013

We prove the theorem on interior regularity in BMO.

Proof. Let $0 < 2R = \operatorname{dist}(U, \partial V)$. Fix $\boldsymbol{x}_0 \in U$ and write $u = v + w$, where v solves the Dirichlet's problem

$$\begin{cases} -\Delta v = 0 & \text{in } Q_{2R}, \\ v = u & \text{on } \partial Q_{2R}. \end{cases}$$

By Cacciopoli's inequality applied to the harmonic function $\frac{\partial^2 v}{\partial x_i \partial x_j}$, for all $0 < r \le R$,

$$\int_{Q_r} \left\|\nabla^2 v - \left(\nabla^2 v\right)_{Q_r}\right\|^2 \, d\boldsymbol{x} \le C(N)\left(\frac{r}{R}\right)^{N+2} \int_{Q_R} \left|\nabla^2 v - \left(\nabla^2 v\right)_{Q_R}\right|^2 \, d\boldsymbol{x}.$$

On the other hand, since w is a solution of the problem

$$\begin{cases} -\Delta w = f & \text{in } Q_{2R}, \\ w = 0 & \text{on } \partial Q_{2R}, \end{cases}$$

by Theorem 112,

$$\left\|\nabla^2 w\right\|^2_{L^2(Q_R)} \le C(N)\left(\|f\|^2_{L^2(Q_{2R})} + \frac{1}{R^2}\|\nabla w\|^2_{L^2(Q_{2R})}\right).$$

Multiplying the equation by w and integrating by parts, we get

$$\begin{aligned}\int_{Q_{2R}} \|\nabla w\|^2 \, d\boldsymbol{x} &= \int_{Q_{2R}} wf \, d\boldsymbol{x} \\ &\le \|f\|_{L^2(Q_{2R})}\|w\|_{L^2(Q_{2R})} \\ &\le C(N)\, R\, \|f\|_{L^2(Q_{2R})}\|\nabla w\|_{L^2(Q_{2R})},\end{aligned}$$

where we have used the Poincaré's inequality above. It follows that

$$\|\nabla w\|_{L^2(Q_{2R})} \le C(N)\, R\, \|f\|_{L^2(Q_{2R})},$$

and in turn,

$$\left\|\nabla^2 w\right\|^2_{L^2(Q_R)} \le C(N)\|f\|^2_{L^2(Q_{2R})} \le C(N)\, R^N \|f\|^2_{L^\infty(Q_{2R})}.$$

Hence, by (128),

$$\begin{aligned}\int_{Q_r}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_r}\right\|^2\,d\boldsymbol{x} &\le 2\int_{Q_r}\left\|\nabla^2 v-\left(\nabla^2 v\right)_{Q_r}\right\|^2\,d\boldsymbol{x}\\ &\quad+2\int_{Q_r}\left\|\nabla^2 w-\left(\nabla^2 w\right)_{Q_r}\right\|^2\,d\boldsymbol{x}\\ &\le C(N)\left(\frac{r}{R}\right)^{N+2}\int_{Q_R}\left\|\nabla^2 v-\left(\nabla^2 v\right)_{Q_R}\right\|^2\,d\boldsymbol{x}\\ &\quad+2\int_{Q_R}\left\|\nabla^2 w-\left(\nabla^2 w\right)_{Q_R}\right\|^2\,d\boldsymbol{x}.\end{aligned}$$

Using the fact that $v=u-w$, the right hand side of the previous inequality is bounded by

$$\begin{aligned}&C(N)\left(\frac{r}{R}\right)^{N+2}\int_{Q_R}\left|\nabla^2 u-\left(\nabla^2 u\right)_{Q_R}\right|^2\,d\boldsymbol{x}+C(N)\int_{Q_R}\left\|\nabla^2 w\right\|^2\,d\boldsymbol{x}\\ &\le C(N)\left(\frac{r}{R}\right)^{N+2}\int_{Q_R}\left|\nabla^2 u-\left(\nabla^2 u\right)_{\boldsymbol{x}_0,r}\right|^2\,d\boldsymbol{x}+C(N)\,R^N\,\|f\|^2_{L^\infty(Q_{2R})}.\end{aligned}$$

Hence,

$$\int_{Q_r}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_r}\right\|^2\,d\boldsymbol{x}\le C(N)\left(\frac{r}{R}\right)^{N+2}\int_{Q_R}\left|\nabla^2 u-\left(\nabla^2 u\right)_{\boldsymbol{x}_0,r}\right|^2\,d\boldsymbol{x}+C(N)\,R^N\,\|f\|^2_{L^\infty(Q_{2R})}.$$

Define the function

$$\omega(r):=\int_{Q_r}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_r}\right\|^2\,d\boldsymbol{x},$$

$0<r\le R$. Note that ω is increasing, since by (128), for $r_1\le r_2$,

$$\omega(r_1)\le\int_{Q_{r_1}}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_{r_2}}\right\|^2\,d\boldsymbol{x}\le\omega(r_2).$$

Moreover,

$$\omega(r)\le C(N)\left[\left(\frac{r}{R}\right)^{N+2}\omega(R)+\|f\|^2_{L^\infty(B_{2R})}\,R^N\right].$$

It follows by Lemma 146 that

$$\omega(r)\le C(N)\left[\left(\frac{r}{R}\right)^{N}\omega(R)+\|f\|^2_{L^\infty(B_{2R})}\,r^N\right],$$

and so

$$\begin{aligned}\frac{1}{r^N}\int_{Q_r}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_r}\right\|^2\,d\boldsymbol{x} &\le \frac{C(N)}{R^N}\int_{Q_R}\left\|\nabla^2 u-\left(\nabla^2 u\right)_{Q_R}\right\|^2\,d\boldsymbol{x}\\ &\quad+C(N)\,\|f\|^2_{L^\infty(Q_{2R})}\\ &\le\frac{C(N)}{R^N}\int_{Q_R}\left\|\nabla^2 u\right\|^2\,d\boldsymbol{x}+C(N)\,\|f\|^2_{L^\infty(Q_{2R})}.\end{aligned}$$

On the other hand, by Theorem 112,

$$\int_{Q_R} \left\|\nabla^2 u\right\|^2 \, d\boldsymbol{x} \le C(N)\left(\|f\|^2_{L^2(Q_{2R})} + \frac{1}{R^2}\int_{Q_{2R}} \|\nabla u\|^2 \, d\boldsymbol{x}\right),$$

which gives

$$\begin{aligned}\frac{1}{r^N}\int_{Q_r} \left\|\nabla^2 u - \left(\nabla^2 u\right)_{Q_r}\right\|^2 \, d\boldsymbol{x} &\le \frac{C(N)}{R^N}\int_{Q_R} \left\|\nabla^2 u - \left(\nabla^2 u\right)_{Q_R}\right\|^2 \, d\boldsymbol{x} \\ &\quad + C(N)\|f\|^2_{L^\infty(Q_{2R})} \\ &\le \frac{C(N)}{R^{N+2}}\int_{Q_{2R}} \|\nabla u\|^2 \, d\boldsymbol{x} + C(N)\|f\|^2_{L^\infty(Q_{2R})}.\end{aligned}$$

Finally, by Hölder's inequality,

$$\frac{1}{r^N}\int_{Q_r} \left\|\nabla^2 u - \left(\nabla^2 u\right)_{Q_r}\right\| \, d\boldsymbol{x} \le \frac{r^{N/2}}{r^N}\left(\int_{Q_r} \left\|\nabla^2 u - \left(\nabla^2 u\right)_{Q_r}\right\|^2 \, d\boldsymbol{x}\right)^{1/2}$$

and so

$$\begin{aligned}\frac{1}{r^N}\int_{Q_r} \left\|\nabla^2 u - \left(\nabla^2 u\right)_{Q_r}\right\| \, d\boldsymbol{x} &\le C(N)\|f\|^2_{L^\infty(V)} \\ &\quad + \frac{C(N)}{\operatorname{dist}^{N+2}(U, \partial V)}\|\nabla u\|^2_{L^2(V)}\end{aligned}$$

for all $0 < r < R$. On the other hand if $R \le r < 2R$, then we can reason as in the second part of the proof of Cacciopoli's inequality, to show that the previous inequality continues to hold. Hence, we have shown that the previous inequality holds for all cubes $Q(\boldsymbol{x}_0, r) \subseteq U$, and so (129) is satisfied. ■

We omit the proof of the next theorem.

Theorem 147 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^2 boundary. Let $f \in L^\infty(\Omega)$ and $g \in W^{1,\infty}(\partial\Omega)$ be such that (98) and let $u \in H^1(\Omega)$ be a weak solution u of the Neumann problem (124). Then $\nabla^2 u$ belongs to $BMO(\Omega)$ and*

$$\|u\|_{H^1(\Omega)} + \left\|\nabla^2 u\right\|_{BMO(\Omega)} \le C(N, \Omega)\left(\|f\|_{L^\infty(\Omega)} + \|g\|_{W^{1,\infty}(\partial\Omega)}\right).$$

Proof. The strategy of the proof of this theorem is similar to the one in Subsections 8.2 and 10.2. We omit the details. ■

Friday, December 6, 2013

2 hours. Make-up class.

12 L^p Theory

Theorem 148 (Stampacchia Interpolation Theorem) *Let $Q \subset \mathbb{R}^N$ be a cube and let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $1 \le p < \infty$, and let $T : L^p(Q) \to L^p(\Omega)$ be a linear operator such that T is continuous with respect to the norms $(L^\infty(Q), BMO(\Omega))$ and $(L^p(Q), L^p(\Omega))$. Then T is continuous with respect to the norms $(L^q(Q), L^q(Q))$ for all $p \le q < \infty$.*

To prove Stampacchia's theorem, we need a few preliminary results. Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces, let $L^0(X) := \{u : X \to \mathbb{R} \text{ measurable}\}$, $L^0(Y) := \{v : Y \to \mathbb{R} \text{ measurable}\}$, let $V \subseteq L^0(X)$ be a subspace and let $T : V \to L^0(Y)$. We say that T is *sublinear* if

$$\begin{aligned} |T(u_1 + u_2)(x)| &\le |T(u_1)(x)| + |T(u_2)(x)|, \\ |T(\lambda u)(x)| &= |\lambda| \, |Tu(x)| \end{aligned}$$

for all u, u_1, $u_2 \in V$, all $\lambda \in \mathbb{R}$, and for μ-a.e. $x \in X$. We say that V is of *closed by truncation* if for every $u \in V$ and $0 \le r_1 < r_2$, the function

$$v(x) := \begin{cases} u(x) & \text{if } r_1 \le |u(x)| \le r_2, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to V. We say that T is of *strong type* (p, q) if it is bounded from $L^p(X) \cap V$ into $L^q(Y)$, while it is of *weak type* (p, q) if it is bounded from $L^p(X) \cap V$ into $L^q_{\mathrm{w}}(Y)$, where $L^q_{\mathrm{w}}(Y)$ is the defined as the space of all measurable functions $v : Y \to \mathbb{R}$ such that

$$\|v\|_{L^q_{\mathrm{w}}(Y)} := \sup_{\lambda > 0} \lambda \nu^{1/q} (\{y \in Y : \ |v(y)| > \lambda\}) < \infty.$$

Theorem 149 (Marcinkiewicz Interpolation Theorem) *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces, with μ and ν σ-finite, let $1 \le p_1 < p_2 \le \infty$, let V be a subspace of $L^{p_1}(X) + L^{p_2}(X)$ closed by truncation and let $T : V \to L^0(Y)$ be a sublinear operator of weak type (p_1, p_1) and (p_2, p_2). Then T is of strong type (p, p) for all $p_1 < p < p_2$.*

Proof. Assume that $p_2 < \infty$. Given $\lambda > 0$ and $u \in L^p(X) \cap V$, write $u = u_1 + u_2$, where $u_1 := u\chi_{\{|u| > \lambda\}}$ and $u_2 := u\chi_{\{|u| \le \lambda\}}$. Since T is closed by truncation, $u_2 \in V$. In turn, since V is a subspace and $u, u_2 \in V$, it follows that u_1 also belongs to V. Then by the sublinearlity of T,

$$\{y \in Y : \ |Tu(y)| > \lambda\} \subseteq \left\{y \in Y : \ |Tu_1(y)| > \frac{\lambda}{2}\right\} \cup \left\{y \in Y : \ |Tu_2(y)| > \frac{\lambda}{2}\right\},$$

and so, since T is of weak type (p_1, p_1) and (p_2, p_2),

$$\begin{aligned} \nu(\{y \in Y : \ |Tu(y)| > \lambda\}) &\le \nu\left(\left\{y \in Y : \ |Tu_1(y)| > \frac{\lambda}{2}\right\}\right) + \nu\left(\left\{y \in Y : \ |Tu_2(y)| > \frac{\lambda}{2}\right\}\right) \\ &\le \left(\frac{2A_1}{\lambda} \|u_1\|_{L^{p_1}(X)}\right)^{p_1} + \left(\frac{2A_2}{\lambda} \|u_2\|_{L^{p_2}(X)}\right)^{p_2}. \end{aligned}$$

In turn, by Tonelli's theorem,

$$\begin{aligned}\|Tu\|_{L^p(Y)}^p &= p\int_0^\infty \lambda^{p-1}\nu\left(\{y\in Y:\ |Tu(y)|>\lambda\}\right)\,d\lambda\\ &\le p\,(2A_1)^{p_1}\int_0^\infty \lambda^{p-1-p_1}\int_{\{|u|>\lambda\}}|u(x)|^{p_1}\,d\mu(x)\,d\lambda\\ &\quad+p\,(2A_2)^{p_2}\int_0^\infty \lambda^{p-1-p_2}\int_{\{|u|\le\lambda\}}|u(x)|^{p_2}\,d\mu(x)\,d\lambda\\ &= p\,(2A_1)^{p_1}\int_X |u(x)|^{p_1}\int_0^{|u(x)|}\lambda^{p-1-p_1}\,d\lambda d\mu(x)\\ &\quad+p\,(2A_2)\int_X |u(x)|^{p_2}\int_{|u(x)|}^\infty \lambda^{p-1-p_2}\,d\lambda d\mu(x)\\ &= \frac{p\,(2A_1)^{p_1}}{p-p_1}\int_X |u(x)|^p\,d\mu(x)+\frac{p\,(2A_2)^{p_2}}{p_2-p}\int_X|u(x)|^p\,d\mu(x),\end{aligned}$$

which proves that T is of strong type (p,p). ■

If $u\in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ we define

$$M^{\#}u(\boldsymbol{x}):=\sup_{Q(\boldsymbol{x}_0,r):\ \boldsymbol{x}\in Q(\boldsymbol{x}_0,r)}\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}\left|u(\boldsymbol{x})-u_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x}.$$

Note that $u\in BMO\left(\mathbb{R}^N\right)$ if and only if $M^{\#}u\in L^\infty\left(\mathbb{R}^N\right)$. We also define the dyadic maximal operator

$$M^d u(\boldsymbol{x}):=\sup_{Q(\boldsymbol{x}_0,r)\text{ dyadic: }\boldsymbol{x}\in Q(\boldsymbol{x}_0,r)}\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}|u(\boldsymbol{x})|\,d\boldsymbol{x},$$

A dyadic cube Q has the form $Q=2^k\left(\boldsymbol{j}+[0,1)^N\right)$, where $k\in\mathbb{Z}$ and $\boldsymbol{j}\in\mathbb{Z}^N$.

Theorem 150 *If $1<p\le q<\infty$ and $u\in L^p\left(\mathbb{R}^N\right)$, $u\ge 0$, then*

$$\left\|M^d u\right\|_{L^q(\mathbb{R}^N)}\le C(N,q)\left\|M^{\#}u\right\|_{L^q(\mathbb{R}^N)}.$$

Proof. Step 1: Let $t,\delta>0$. We claim that

$$\begin{aligned}&\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(x)>2t,\ M^{\#}u(x)\le\delta t\right\}\right)\\ &\quad\le 2^N\delta\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(x)>t\right\}\right).\end{aligned}$$

Let $E_t:=\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(x)>t\right\}$. If $\boldsymbol{x}\in E_t$, then there exists a dyadic cube Q such that $\boldsymbol{x}\in Q$ and $u_Q>t$. Let Q be the largest such dyadic cube containing $\boldsymbol{x}$ such that $u_Q>t$. Note that such a cube exists, since by Hölder's inequality

$$u_{Q(\boldsymbol{x}_0,r)}=\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}|u|\,d\boldsymbol{y}\le\frac{r^{\frac{N}{p'}}}{r^N}\|u\|_{L^p(Q(\boldsymbol{x}_0,r))}\le\frac{1}{r^{\frac{N}{p}}}\|u\|_{L^p(\mathbb{R}^N)}\to 0$$

as $r \to \infty$. Hence, $u_{Q(\boldsymbol{x}_0,r)} \le t$ for all r sufficiently large.

Note that if $\boldsymbol{y} \in Q$, then $M^d u(\boldsymbol{y}) > t$ and Q is the largest such dyadic cube containing $\boldsymbol{y}$ such that $u_Q > t$ (a larger cube would contain also $\boldsymbol{x}$. This shows that $Q \subseteq E_t$ and that E_t can be written as a countable disjoint union of maximal dyadic cubes.

Fix one of these cubes, say Q. It is enough to show that

$$\mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q : \ M^d u(\boldsymbol{x}) > 2t,\ M^{\#}u(\boldsymbol{x}) \le \delta t\right\}\right) \le \delta 2^N \mathcal{L}^N(Q).$$

If the right-hand side of the previous inequality is zero, then there is nothing to prove. Thus, assume that there exists $\boldsymbol{x}_1 \in Q$ such that $M^d u(\boldsymbol{x}_1) > 2t$ and $M^{\#}u(\boldsymbol{x}_1) \le \delta t$. Consider the dyadic cube $\tilde{Q}$ containing Q and of twice the side-length of Q. By the maximality of Q and the definition of $M^d u$, we have that $u_{\tilde{Q}} \le t$. We claim that if $\boldsymbol{x} \in Q$ and $M^d u(\boldsymbol{x}) > 2t$, then $M^d(u\chi_Q)(\boldsymbol{x}) > 2t$. To see this, note that by the maximality of Q, if for every dyadic cube $Q(\boldsymbol{x}_0, r)$ containing Q, we must have $u_Q \ge u_{Q(\boldsymbol{x}_0,r)}$. Hence, in the definition of $M^d u(\boldsymbol{x})$ it suffices to consider dyadic cubes contained in Q. But if $Q(\boldsymbol{x}_0, r) \subseteq Q$, then in $Q(\boldsymbol{x}_0, r)$, we have that $u\chi_Q = u$, and so, $u_{Q(\boldsymbol{x}_0,r)} = (\chi_Q u)_{Q(\boldsymbol{x}_0,r)}$, which shows that $M^d(u\chi_Q)(\boldsymbol{x}) = M^d u(\boldsymbol{x})$. This proves the claim.

Hence,

$$M^d\left(\left(u - u_{\tilde{Q}}\right)\chi_Q\right)(\boldsymbol{x}) \ge M^d(u\chi_Q)(\boldsymbol{x}) - u_Q > 2t - t = t,$$

where we used the fact the sublinearity of M^d and the fact that $M^d(c) = c$. This shows that if $M^d u(\boldsymbol{x}) > 2t$, then $M^d\left(\left(u - u_{\tilde{Q}}\right)\chi_Q\right)(\boldsymbol{x}) > t$. By the weak L^1 inequality for M^d,

$$\begin{aligned}
&\mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q : \ M^d u(\boldsymbol{x}) > 2t,\ M^{\#}u(\boldsymbol{x}) \le \delta t\right\}\right) \\
&\le \mathcal{L}^N\left(\boldsymbol{x} \in Q : \ M^d\left(\left(u - u_{\tilde{Q}}\right)\chi_Q\right)(\boldsymbol{x}) > t\right) \\
&\le \frac{1}{t}\int_Q \left|u - u_{\tilde{Q}}\right| \, d\boldsymbol{x} = \frac{\mathcal{L}^N\left(\tilde{Q}\right)}{t} \frac{1}{\mathcal{L}^N\left(\tilde{Q}\right)} \int_{\tilde{Q}} \left|u - u_{\tilde{Q}}\right| \, d\boldsymbol{x} \\
&\le \frac{\mathcal{L}^N\left(\tilde{Q}\right)}{t} M^{\#}u(\boldsymbol{x}_1) \le 2^N \delta \mathcal{L}^N(Q).
\end{aligned}$$

Hence, the claim is proved.

Step 2: Since $u \in L^p\left(\mathbb{R}^N\right)$, $M^d u \in L^p\left(\mathbb{R}^N\right)$ and

$$\left\|M^d u\right\|_{L^p(\mathbb{R}^N)}^p = p\int_0^\infty \lambda^{p-1}\mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : \ M^d u(\boldsymbol{x}) > \lambda\right\}\right) \, d\lambda < \infty.$$

In particular, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
I_n &:= q\int_0^n \lambda^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : \ M^d u(\boldsymbol{x}) > \lambda\right\}\right) \, d\lambda \\
&\le q n^{q-p}\int_0^n \lambda^{p-1}\mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : \ M^d u(\boldsymbol{x}) > \lambda\right\}\right) \, d\lambda < \infty.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
I_n &:= q\int_0^n \lambda^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(\boldsymbol{x})>\lambda\right\}\right)\,d\lambda\\
&= 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(\boldsymbol{x})>2t\right\}\right)\,dt\\
&\le 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(\boldsymbol{x})>2t,\ M^{\#}u(x)\le\delta t\right\}\right)\,dt\\
&\quad + 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N: M^{\#}u(x)>\delta t\right\}\right)\,dt\\
&\le \delta 2^{q+N} q\int_0^{n/2} t^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N:\ M^d u(x)>t\right\}\right)\,dt\\
&\quad + \frac{2^q}{\delta^q} q\int_0^{\delta n/2} s^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N: M^{\#}u(x)>s\right\}\right)\,ds\\
&= \delta 2^{q+N} I_n + \frac{2^q}{\delta^q} q\int_0^{\delta n/2} s^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N: M^{\#}u(x)>s\right\}\right)\,ds.
\end{aligned}
$$

Let δ be such that $\delta 2^{q+N}=\frac{1}{2}$. Since $I_n<\infty$, it follows that

$$
\begin{aligned}
\frac{1}{2}I_n &\le \frac{2^q}{\delta^q} q\int_0^{\delta n/2} s^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N: M^{\#}u(x)>s\right\}\right)\,ds\\
&\le \frac{2^q}{\delta^q} q\int_0^{\infty} s^{q-1}\mathcal{L}^N\left(\left\{\boldsymbol{x}\in\mathbb{R}^N: M^{\#}u(x)>s\right\}\right)\,ds\\
&= \frac{2^q}{\delta^q}\left\|M^{\#}u\right\|^q_{L^q(\mathbb{R}^N)}.
\end{aligned}
$$

Letting $n\to\infty$, we get the desired result. ■

Proof of Stampacchia's Theorem. We only prove the theorem in the case when Q and Ω are replaced by $\mathbb{R}^N$ and $T:V\to L^0\left(\mathbb{R}^N\right)$ is continuous with respect to the norms $\left(L^\infty\left(\mathbb{R}^N\right), BMO\left(\mathbb{R}^N\right)\right)$ and $\left(L^p\left(\mathbb{R}^N\right), L^p\left(\mathbb{R}^N\right)\right)$, where V is a subspace of $L^\infty_c\left(\mathbb{R}^N\right)+L^p_c\left(\mathbb{R}^N\right)$ closed by truncation. Define $T_1:=M^{\#}\circ T$. Then T_1 is sublinear. Indeed, for u_1, $u_2\in V$ we have

$$
\left(T\left(u_1+u_2\right)\right)_{Q(\boldsymbol{x}_0,r)}=\left(Tu_1+Tu_2\right)_{Q(\boldsymbol{x}_0,r)}=\left(Tu_1\right)_{Q(\boldsymbol{x}_0,r)}+\left(Tu_2\right)_{Q(\boldsymbol{x}_0,r)}
$$

and

$$
\begin{aligned}
&\int_{Q(\boldsymbol{x}_0,r)}\left|T\left(u_1+u_2\right)(\boldsymbol{x})-\left(T\left(u_1+u_2\right)\right)_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x}\\
&=\int_{Q(\boldsymbol{x}_0,r)}\left|Tu_1(\boldsymbol{x})+Tu_2(\boldsymbol{x})-\left(Tu_1\right)_{Q(\boldsymbol{x}_0,r)}-\left(Tu_2\right)_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x}\\
&\le\int_{Q(\boldsymbol{x}_0,r)}\left|Tu_1(\boldsymbol{x})-\left(Tu_1\right)_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x}+\int_{Q(\boldsymbol{x}_0,r)}\left|Tu_2(\boldsymbol{x})-\left(Tu_2\right)_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x},
\end{aligned}
$$

which shows that

$$\left(M^{\#}\circ T\right)(u_1+u_2)(\boldsymbol{x})\leq\left(M^{\#}\circ T\right)(u_1)(\boldsymbol{x})+\left(M^{\#}\circ T\right)(u_2)(\boldsymbol{x}),$$

while by the linearity of T and of integration $|T(\lambda u)(\boldsymbol{x})|=|\lambda|\,|Tu(\boldsymbol{x})|$.

Moreover, for $f\in L_c^\infty\left(\mathbb{R}^N\right)\cap V$,

$$\sup_{\boldsymbol{x}\in\mathbb{R}^N}\left|\left(M^{\#}\circ T\right)(u)(\boldsymbol{x})\right|=|T(u)|_{BMO(\mathbb{R}^N)}\leq C\,\|u\|_{L^\infty(\mathbb{R}^N)},$$

which shows that $M^{\#}\circ T:L_c^\infty\left(\mathbb{R}^N\right)\cap V\to L^\infty\left(\mathbb{R}^N\right)$ is bounded. On the other hand, by Hölder's inequality,

$$\int_{Q(\boldsymbol{x}_0,r)}\left|Tu(\boldsymbol{x})-(Tu)_{Q(\boldsymbol{x}_0,r)}\right|\,d\boldsymbol{x}\leq 2\int_{Q(\boldsymbol{x}_0,r)}|Tu(\boldsymbol{x})|\,d\boldsymbol{x},$$

and so

$$\begin{aligned}\left|\left(M^{\#}\circ T\right)(u)(\boldsymbol{x})\right|^p&\leq 2^p\left(\sup_{Q(\boldsymbol{x}_0,r):\ \boldsymbol{x}\in Q(\boldsymbol{x}_0,r)}\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}|Tu(\boldsymbol{x})|\,d\boldsymbol{x}\right)^p\\&=2^p\left(\left(\left(M'\circ T\right)(u)\right)(\boldsymbol{x})\right)^p\end{aligned}$$

where

$$M'v(\boldsymbol{x}):=\sup_{Q(\boldsymbol{x}_0,r):\ \boldsymbol{x}\in Q(\boldsymbol{x}_0,r)}\frac{1}{r^N}\int_{Q(\boldsymbol{x}_0,r)}|v(\boldsymbol{x})|\,d\boldsymbol{x}$$

is the uncentered maximal function. Since $\|M'v\|_{L^p(\mathbb{R}^N)}\leq C(p,n)\,\|v\|_{L^p(\mathbb{R}^N)}$ for $p>1$, we get

$$\begin{aligned}\left\|\left(M^{\#}\circ T\right)(u)\right\|_{L^p(\mathbb{R}^N)}&\leq 2\,\|(M'\circ T)(u)\|_{L^p(\mathbb{R}^N)}\\&\leq C(p,n)\,\|Tu\|_{L^p(\mathbb{R}^N)}\\&\leq C(p,n)\,\|u\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Hence, we are in a position to apply Marcinkiewicz interpolation theorem to conclude that $M^{\#}\circ T$ is bounded in $L^q\left(\mathbb{R}^N\right)$ for all $p<q<\infty$.

Let $u\in L_c^q\left(\mathbb{R}^N\right)\cap V$ with compact support. Then $u\in L_c^p\left(\mathbb{R}^N\right)$, and so $T(u)\in L^p\left(\mathbb{R}^N\right)$. By Theorem 150, and the fact that $|v|\leq M^dv\ \mathcal{L}^N$ a.e. in $\mathbb{R}^N$,

$$\begin{aligned}\left\|Tu^\pm\right\|_{L^q(\mathbb{R}^N)}&\leq\left\|M^d\circ\left(Tu^\pm\right)\right\|_{L^q(\mathbb{R}^N)}\leq C(N,q)\left\|M^{\#}\circ\left(Tu^\pm\right)\right\|_{L^q(\mathbb{R}^N)}\\&\leq C(N,q)\left\|u^\pm\right\|_{L^q(\mathbb{R}^N)}\leq C(N,q)\,\|u\|_{L^q(\mathbb{R}^N)}.\end{aligned}$$

This concludes the proof. ■

Theorem 151 *Let $\Omega\subset\mathbb{R}^N$ be an open set of class C^2, for which there exists a bi-Lipschitz homeomorphism $\boldsymbol{\Psi}:\Omega\to Q$, where Q is a cube, let $f\in L^p(\Omega)$ and $g\in W^{1-\frac{1}{p},p}(\partial\Omega)$, $2<p<\infty$, be such that* (98) *holds and let $u\in H^1(\Omega)$ be* *a weak solution u of the Neumann problem* (124)*. Then u belongs to* $W^{2,p}(\Omega)$*. Moreover,*

$$\|u\|_{W^{2,p}(\Omega)} \le C(\Omega, N, p)\left(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right).$$

Proof. Step 1: Assume that $g = 0$. By hypothesis there exist a cube Q and a bi-Lipschitz homeomorphism $\mathbf{\Psi} : \Omega \to Q$. For every $f \in L^2(Q)$, consider $f \circ \mathbf{\Psi} = f_1 \in L^2(\Omega)$ and solve the Neumann problem (124) with f replaced by $f_1 - (f_1)_\Omega$ and $g = 0$. By Theorem 136, the solution u belongs to $H^2(\Omega)$ with

$$\begin{aligned}\left\|\nabla^2 u\right\|_{L^2(\Omega)} &\le C(N,\Omega)\left\|f_1 - (f_1)_\Omega\right\|_{L^2(\Omega)} \\ &\le C(N,\Omega)\left\|f \circ \mathbf{\Psi}\right\|_{L^2(\Omega)} \le C(N,\Omega)\|f\|_{L^2(Q)}.\end{aligned}$$

Similarly, for every $f \in L^\infty(Q)$, by Theorem 147, we have

$$\begin{aligned}\left\|\nabla^2 u\right\|_{BMO(\Omega)} &\le C(N,\Omega)\left\|f_1 - (f_1)_\Omega\right\|_{L^\infty(\Omega)} \\ &\le C(N,\Omega)\left\|f \circ \mathbf{\Psi}\right\|_{L^\infty(\Omega)} \le C(N,\Omega)\|f\|_{L^\infty(Q)}.\end{aligned}$$

For every $i, j = 1, \ldots, N$ define the linear operator $T : L^2(Q) \to L^2(\Omega)$ by

$$T_{i,j}(f) := \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Then

$$\|T_{i,j}(f)\|_{L^2(\Omega)} \le C(N,\Omega)\|f\|_{L^2(Q)}$$

while

$$\|T_{i,j}(f)\|_{BMO(\Omega)} \le C(N,\Omega)\|f\|_{L^\infty(Q)}.$$

Hence, by the previous theorem,

$$\|T_{i,j}(f)\|_{L^p(\Omega)} \le C(N,p,\Omega)\|f\|_{L^p(Q)}.$$

In particular, given any $f_2 \in L^p(\Omega)$ with $(f_2)_\Omega = 0$, the function $f := f_2 \circ \mathbf{\Psi}^{-1} \in L^p(Q)$ and so the solution of (124) with f replaced by f_2 and $g = 0$ belongs to $W^{2,p}$ and

$$\left\|\frac{\partial^2 u}{\partial x_i \partial x_j}\right\|_{L^p(\Omega)} \le C(N,p,\Omega)\|f\|_{L^p(Q)} = C(N,p,\Omega)\left\|f_2 \circ \mathbf{\Psi}^{-1}\right\|_{L^p(Q)} \le C(N,p,\Omega)\|f_2\|_{L^p(\Omega)}.$$

■

The next theorem can be used to prove interior regularity in $L^p(\Omega)$ for $1 < p < 2$.

Theorem 152 *Let Q be a cube, let $f \in L^p(Q)$, $1 < p < 2$, and let $u \in H^1(Q)$ be a weak solution u of the Poisson's problem $-\Delta u = f$ in Q. Assume that u has compact support. Then u belongs to $W^{2,p}(Q)$ and*

$$\|u\|_{W^{2,p}(Q)} \le C(N,p)\|f\|_{L^p(Q)}.$$

Proof. By interior regularity, $u \in H^2_{\text{loc}}\left(Q\mathbb{R}^N\right)$. By duality, we can write

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^p_{L^p(Q)} = \sup_{\|\varphi\|_{L^{p'}\left(\mathbb{R}^N\right)} \leq 1} \int_Q \frac{\partial^2 u}{\partial x_i \partial x_j} \varphi \, d\boldsymbol{x}.$$

Moreover, by density of smooth functions in $L^{p'}(Q)$, we can assume that $\varphi \in C_c^\infty(Q)$. Given any such $\varphi \in C_c^\infty(Q)$ with $\|\varphi\|_{L^{p'}(\mathbb{R}^N)} \leq 1$, construct a solution of the Poisson's problem

$$\Delta w = \varphi \quad \text{in } Q.$$

To do so, you solve either the Neumann or Dirichlet problem in Q. Then $w \in C^\infty(Q)$. Since $\varphi \in C_c^\infty(Q)$, integrating by parts twice ad using the fact that u and its derivatives of any order are zero on ∂Q gives

$$\begin{aligned} \int_Q \frac{\partial^2 u}{\partial x_i \partial x_j} \varphi \, d\boldsymbol{x} &= \int_Q \frac{\partial^2 u}{\partial x_i \partial x_j} \Delta w \, d\boldsymbol{x} = \int_Q u \Delta \frac{\partial^2 w}{\partial x_i \partial x_j} \, d\boldsymbol{x} \\ &= \int_Q \Delta u \frac{\partial^2 w}{\partial x_i \partial x_j} \, d\boldsymbol{x} \\ &= \int_Q f \frac{\partial^2 w}{\partial x_i \partial x_j} \, d\boldsymbol{x} \leq C \|f\|_{L^p(Q)} \|\varphi\|_{L^{p'}(Q)} \\ &\leq C \|f\|_{L^p(Q)}. \end{aligned}$$

In turn,

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(Q)} = \sup_{\|\varphi\|_{L^{p'}(Q)} \leq 1} \int_Q \frac{\partial^2 u}{\partial x_i \partial x_j} \varphi \, d\boldsymbol{x} \leq C(N,p) \|f\|_{L^p(Q)}.$$

■

Remark 153 *This theorem will be used to obtain L^p regularity in the interior for the problem $-\Delta u = f$ in Ω and it will be applied to a function ϕu where $\phi \in C_c^\infty(\Omega)$ is zero outside a cube Q. Note that ϕu has compact support in Q and*

$$\Delta(\phi u) = \phi f + 2\nabla\phi \cdot \nabla u + u\Delta\phi =: f_1 \in L_c^p(Q).$$