Monday, January 13, 2014

Part I
Evolution Equations

This semester we will study second order parabolic and hyperbolic equations. We begin by introducing the Bochner integral. We refer to [DuSc88], [DieU77], [Ed95], [FL07], and [SY05] for more information on the subject and for the proofs omitted here.

1 The Bochner Integral

Definition 1 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* Y *be a Banach space. A* (measurable) simple function s *is a function* $s : X \to Y$ *of the form*

$$s = \sum_{i=1}^{\ell} c_i \chi_{E_i},$$

where $\ell \in \mathbb{N}$*,* $c_1, \ldots, c_\ell \in Y$ *are distinct and the sets* $E_i \subset X$ *are measurable and mutually disjoint.*

(i) A function $u : X \to Y$ *is said to be* strongly measurable *if there exists a sequence* $\{s_n\}$ *of (measurable) simple functions* $s_n : X \to Y$ *such that*

$$\lim_{n\to\infty} \|s_n(x) - u(x)\|_Y = 0 \quad \text{for } \mu \text{ a.e. } x \in X.$$

(ii) A function $u : X \to Y$ *is said to be* weakly measurable *if for every* $L \in Y'$ *the function*

$$x \in X \mapsto \langle L, u(x) \rangle_{Y',Y} = L(u(x)) \text{ is measurable,}$$

where $\langle \cdot, \cdot \rangle_{Y',Y} : Y' \times Y \to \mathbb{R}$ *is the duality pairing defined by*

$$\langle L, y \rangle_{Y',Y} := L(y)$$

for $L \in Y'$ *and* $y \in Y$*.*

(iii) A function $u : X \to Y'$ *is said to be* weakly star measurable *if for every* $y \in Y$ *the map*

$$x \in X \mapsto \langle u(x), y \rangle_{Y',Y} = u(x)(y) \text{ is measurable,}$$

Exercise 2 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* Y *be a Banach space. Prove that if a function* $u : X \to Y$ *is strongly measurable then the function* $x \in X \mapsto \|u(x)\|_Y$ *is measurable.*

The relation between weak and strong measurability is given by the following theorem.

Theorem 3 (Pettis) *Let $(X, \mathfrak{M}, \mu)$ be a complete measure space and let Y be a Banach space. A function $u : X \to Y$ is strongly measurable if and only if it is weakly measurable and there exists $E \in \mathfrak{M}$, with $\mu(E) = 0$, such that the set $u(X \setminus E)$ is a separable set of Y (in the norm sense).*

Proof. Step 1: Assume that $u : X \to Y$ is strongly measurable. Then there exist a sequence $\{s_n\}$ of simple functions $s_n : X \to Y$ and a set $E \in \mathfrak{M}$, with $\mu(E) = 0$, such that

$$\lim_{n\to\infty} \|s_n(x) - u(x)\|_Y = 0 \quad \text{for all } x \in X \setminus E.$$

Hence, for every $L \in Y'$ we have that

$$\lim_{n\to\infty} L(s_n(x)) = L(u(x)) \quad \text{for all } x \in X \setminus E.$$

Since the function $L \circ s_n : X \to \mathbb{R}$ is a simple function, it is measurable, and in turn, the real-valued function

$$x \in X \mapsto L(u(x))$$

is measurable as the pointwise limit of measurable functions. Hence, u is weakly measurable.

Since the range of each s_n is finite, the set $\bigcup_{n=1}^{\infty} s_n(X)$ is countable. In turn, the set

$$C := \overline{\bigcup_{n=1}^{\infty} s_n(X)}$$

is separable. If $x \in X \setminus E$, then $s_n(x) \to u(x)$, and so $u(x) \in C$. It follows that $u(X \setminus E) \subseteq C$. Using the fact that a subset of separable set (in a metric space) is separable, we have that $u(X \setminus E)$ is separable.

Step 2: Assume that u is weakly measurable and that there exists $E \in \mathfrak{M}$, with $\mu(E) = 0$, such that the set $u(X \setminus E)$ is a separable set of Y. Let $\{y_n\} \subseteq u(X \setminus E)$ be such that the set $\{y_n\}$ is dense in $u(X \setminus E)$ and for each n let $x_n \subseteq X \setminus E$ be such that $u(x_n) = y_n$. By the Hahn–Banach theorem, there exists $L_n \in Y'$ such that $L_n(y_n) = \|y_n\|_Y$ and $\|L_n\|_{Y'} = 1$. We claim that

$$\|u(x)\|_Y = \sup_n L_n(u(x))$$

for every $x \in X \setminus E$. Indeed,

$$L_n(u(x)) \le \|L_n\|_{Y'} \|u(x)\|_Y = 1 \|u(x)\|_Y$$

and so

$$\sup_n L_n(u(x)) \le \|u(x)\|_Y .$$

On the other hand, find a subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $y_{n_k} = u(x_{n_k}) \to u(x)$. Then

$$\begin{aligned}\|u(x)\|_Y &= \lim_{k\to\infty} \|u(x_{n_k})\|_Y = \lim_{k\to\infty} L_{n_k}(u(x_{n_k})) \\ &= \lim_{k\to\infty} [L_{n_k}(u(x)) + L_{n_k}(u(x_{n_k}) - u(x))] \\ &\le \sup_n L_n(u(x)) + \limsup_{k\to\infty} L_{n_k}(u(x_{n_k}) - u(x)) \\ &\le \sup_n L_n(u(x)) + \limsup_{k\to\infty} \|L_{n_k}\|_{Y'} \|u(x_{n_k}) - u(x)\|_Y \\ &= \sup_n L_n(u(x)) + \limsup_{k\to\infty} 1 \|u(x_{n_k}) - u(x)\|_Y = \sup_n L_n(u(x)) + 0.\end{aligned}$$

This proves the claim. Since u is weakly measurable, each function $x \in X \mapsto L_n(u(x))$ is measurable, and so by the claim, the function $x \in X \mapsto \|u(x)\|_Y$ is measurable. Similarly, we can show that the functions $g_n(x) := \|u(x) - y_n\|_Y$ are measurable.

For every $m \in \mathbb{N}$, consider the function $t_m : Y \to \{0, y_1, \ldots, y_m\}$ defined by

$$t_m(y) := \begin{cases} y_{n_{y,m}} & \text{if } y \in u(X \setminus E), \\ 0 & \text{otherwise.} \end{cases}$$

where for $y \in u(X \setminus E)$, $n_{y,m} \in \{1, \ldots, m\}$ is the smallest number such that

$$\left\|y - y_{n_{y,m}}\right\|_Y = \min_{1\le n\le m} \|y - y_n\|_Y .$$

Note that as $m \to \infty$ we have that $y_{n_{y,m}} \to y$ by the density of the sequence $\{y_n\}$ in $u(X \setminus E)$. Hence, $t_m(y) \to y$ for every $y \in u(X \setminus E)$. Define

$$s_m(x) := \begin{cases} t_m(u(x)) & \text{if } x \in X \setminus E, \\ 0 & \text{otherwise.} \end{cases}$$

The function s_m takes only a finite number of values. Moreover, $s_m(x) \to u(x)$ for every $x \in X \setminus E$. It remains to show that s_m is measurable. Observe that for $1 \le n \le m$,

$$\begin{aligned}&\{x \in X : \ s_m(x) = y_n\} \\ &= \left\{x \in X \setminus E : \ \|u(x) - y_n\|_Y = \min_{1\le n\le m} \|u(x) - y_n\|_Y \right\} \\ &\quad \cap \{x \in X \setminus E : \ \|u(x) - y_l\|_Y > \|u(x) - y_n\|_Y \ \text{ for all } l = 1, \ldots, n-1\}.\end{aligned}$$

This set belongs to $\mathfrak{M}$ since the function g_l is measurable for every l. This shows that s_m is a measurable simple function and conclude the proof. ■

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Example 4 *Consider the Hilbert space*

$$\ell^2([0,1]) := \left\{ f : [0,1] \to \mathbb{R} : \ \sum_{x\in[0,1]} f^2(x) < \infty \right\}$$

withe the inner product

$$(f,g)_{\ell^2([0,1])} := \sum_{x\in[0,1]} f(x)\, g(x)\,.$$

Define

$$u : [0,1] \to \ell^2([0,1])$$

by $u(x) := \chi_{\{x\}}$*. We claim that this function is weakly measurable but not strongly measurable.*

By Riesz's representation theorem we can identify $\left(\ell^2([0,1])\right)'$ *with* $\ell^2([0,1])$*, in the sense that for every* $L \in \left(\ell^2([0,1])\right)'$ *there exists a unique* $g \in \ell^2([0,1])$ *such that*

$$L(f) = (g,f)_{\ell^2([0,1])}$$

for all $f \in \ell^2([0,1])$ *and* $\|L\|_{(\ell^2([0,1]))'} = \|g\|_{\ell^2([0,1])}$*. Conversely, for every* $g \in \ell^2([0,1])$*, the functional*

$$L_g(f) := (g,f)_{\ell^2([0,1])}$$

belongs to $\left(\ell^2([0,1])\right)'$*. Hence, we can identify* L_g *with* g*.*

To prove that u *is weakly measurable, we need to show that for every* $g \in \ell^2([0,1])$*, the function*

$$x \in [0,1] \mapsto L_g(u(x)) = (g, u(x))_{\ell^2([0,1])} = \sum_{t\in[0,1]} g(t)\,\chi_{\{x\}}(t) = g(x)$$

is measurable. Since

$$\sum_{x\in[0,1]} g^2(x) < \infty,$$

we have that $g^2(x) = 0$ *for all but countably many* x*. Hence, the function* g *is equivalent to* 0 *and so it is Lebesgue measurable.*

On the other hand, given any Lebesgue measurable set $E \subset [0,1]$ *with* $\mathcal{L}^1(E) = 0$*, the set* $u([0,1]\setminus E) = \left\{\chi_{\{x\}} :\ x \in [0,1]\setminus E\right\}$ *is not separable in* $\ell^2([0,1])$*, since if* $x \neq y$*, then* $\left\|\chi_{\{x\}} - \chi_{\{y\}}\right\|_{\ell^2([0,1])} = 2$*, and so* $u([0,1]\setminus E)$ *is separable if and only if* $u([0,1]\setminus E)$ *is countable, which is not the case. Hence, by Pettis' theorem, the function* u *is not strongly measurable.*

Example 5 *Define the function* $u : [0,1] \to \ell^\infty$ *by*

$$u(x) := \left\{\frac{\operatorname{sgn}(\sin(2^n \pi x)) + 1}{2}\right\}_{n\in\mathbb{N}}, \quad x \in [0,1]\,.$$

We may identify ℓ^∞ *with the dual* $\left(\ell^1\right)'$*, so for every* $y = \{y_n\}_{n\in\mathbb{N}} \in \ell^1$ *we have that*

$$x \in [0,1] \mapsto (u(x))(y) = \sum_{n=1}^{\infty} y_n \frac{\operatorname{sgn}(\sin(2^n \pi x)) + 1}{2}$$

is well-defined. This is a measurable function; hence u is weakly star measurable. Given any Lebesgue measurable set $E \subset [0,1]$ with $\mathcal{L}^1(E) = 0$, the set $u([0,1] \setminus E)$ is not separable in ℓ^∞ (exercise). Hence by the Pettis theorem, u is not strongly measurable. Actually, it may be verified that u is not even weakly measurable, but the proof is more involved.

Definition 6 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. A (measurable) simple function $s : X \to Y$ is* (Bochner) integrable *if it has the form*

$$s = \sum_{i=1}^{\ell} c_i \chi_{E_i},$$

where $c_1, \ldots, c_\ell \in Y$, $\ell \in \mathbb{N}$, are distinct, the sets $E_i \subset X$ are mutually disjoint, and $c_i = 0$ whenever $\mu(E_i) = \infty$. For any measurable set $E \in \mathfrak{M}$ the Bochner integral *of s over E is defined by*

$$\int_E s \, d\mu := \sum_{i=1}^{\ell} c_i \mu(E_i \cap E),$$

where $c_i \mu(E_i \cap E)$ is set to be zero whenever $c_i = 0$ and $\mu(E_i \cap E) = \infty$.

Definition 7 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. A strongly measurable function $u : X \to Y$ is* (Bochner) integrable *if there exists a sequence $\{s_n\}$ of simple integrable functions such that*

$$\lim_{n \to \infty} \|s_n(x) - u(x)\|_Y = 0 \quad \text{for } \mu \text{ a.e. } x \in X$$

and

$$\lim_{n \to \infty} \int_X \|s_n - u\|_Y \, d\mu = 0.$$

For every measurable set $E \in \mathfrak{M}$ the Bochner integral *of u over E is defined by*

$$\int_E u \, d\mu := \lim_{n \to \infty} \int_E s_n \, d\mu.$$

It may be verified that this limit exists and that is independent of the particular sequence $\{s_n\}$.

Remark 8 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let Y be a Banach space, and let $u : X \to Y$ be a (Bochner) integrable function. If $J : Y \to Z$ is a continuous embedding, then the function $J \circ u : X \to Z$ is (Bochner) integrable and*

$$\int_E J \circ u \, d\mu = J\left(\int_E u \, d\mu\right)$$

for every measurable set $E \in \mathfrak{M}$. Thus, without loss of generality, we can identify $\int_E J \circ u \, d\mu$ with $\int_E u \, d\mu$.

Theorem 9 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. A strongly measurable function $u : X \to Y$ is (Bochner) integrable if and only if $\|u\|_Y$ is Lebesgue integrable over X. Moreover, if $u : X \to Y$ is (Bochner) integrable, then*

$$\left\| \int_E u \, d\mu \right\| \leq \int_E \|u\| \, d\mu$$

for every measurable set $E \in \mathfrak{M}$.

Proof. Step 1: Assume that the real-valued function $x \in X \mapsto \|u(x)\|_Y$ is integrable. Since u is strongly measurable, there exists a sequence $\{s_n\}$ of simple integrable functions such that

$$\lim_{n\to\infty} \|s_n(x) - u(x)\|_Y = 0 \quad \text{for } \mu \text{ a.e. } x \in X. \tag{1}$$

Define

$$t_n(x) := \begin{cases} s_n(x) & \text{if } \|s_n(x)\|_Y \leq \frac{3}{2}\|u(x)\|_Y, \\ 0 & \text{if } \|s_n(x)\|_Y > \frac{3}{2}\|u(x)\|_Y. \end{cases}$$

Then t_n is still a simple integrable function, with $\|t_n(x)\|_Y \leq \frac{3}{2}\|u(x)\|_Y$ for all $x \in X$ and all n. Moreover, by (1), if $\|u(x)\|_Y > 0$, taking $\varepsilon = \frac{1}{2}\|u(x)\|_Y$, we have that $\|s_n(x) - u(x)\|_Y \leq \varepsilon = \frac{1}{2}\|u(x)\|_Y$, for all n sufficiently large, which implies that $\|s_n(x)\|_Y \leq \frac{3}{2}\|u(x)\|_Y$ for all n sufficiently large. Hence, $t_n(x) = s_n(x)$ for all n large and so $t_n(x) \to u(x)$ as $n \to \infty$. On the other hand, if $\|u(x)\|_Y = 0$, then $t_n(x) = 0 = u(x)$ for all n. This shows that $\lim_{n\to\infty} \|t_n(x) - u(x)\|_Y = 0$ for μ a.e. $x \in X$. In turn,

$$\lim_{n\to\infty} \int_X \|t_n - u\|_Y \, d\mu = 0$$

by the Lebesgue dominated convergence theorem. It follows that u is Bochner integrable. ■

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Proof. Step 2. Conversely, assume that $u : X \to Y$ is Bochner integrable and consider a sequence $\{s_n\}$ of simple integrable functions such that

$$\lim_{n\to\infty} \|s_n(x) - u(x)\|_Y = 0 \quad \text{for } \mu \text{ a.e. } x \in X$$

and

$$\lim_{n\to\infty} \int_X \|s_n - u\|_Y \, d\mu = 0.$$

Then for $x \in X$ and for $\ell, n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \left| \|s_n(x)\|_Y - \|s_\ell(x)\|_Y \right| \leq \|s_n(x) - s_\ell(x)\|_Y \\ &\leq \|s_n(x) - u(x)\|_Y + \|s_\ell(x) - u(x)\|_Y, \end{aligned}$$

and so

$$\int_X \left| \|s_n\|_Y - \|s_\ell\|_Y \right| d\mu \leq \int_X \|s_n - u\|_Y \, d\mu + \int_X \|s_\ell - u\|_Y \, d\mu \to 0$$

as ℓ, $n \to \infty$. Thus $\{\|s_n\|_Y\}$ is a Cauchy sequence in $L^1(X, \mathfrak{M}, \mu)$, and so it converges in $L^1(X, \mathfrak{M}, \mu)$ (and up to a subsequence also pointwise μ a.e. in X) to a function $v : X \to \mathbb{R}$. On the other hand, for μ a.e. $x \in X$,

$$0 \leq \left|\|s_n(x)\|_Y - \|u(x)\|_Y\right| \leq \|s_n(x) - u(x)\|_Y \to 0$$

as $n \to \infty$. Hence $v(x) = \|u(x)\|_Y$ for μ a.e. $x \in X$, which shows that $\|u\|_Y$ is integrable. ■

Theorem 10 (Lebesgue Dominated Convergence Theorem) *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. Let $u_n : X \to Y$, $n \in \mathbb{N}$, and $u : X \to Y$ be strongly measurable and assume that $\|u_n(x)\| \leq g(x)$ for all $n \in \mathbb{N}$ and for μ a.e. $x \in X$, for some integrable function $g : X \to \mathbb{R}$, and that $u_n(x) \to u(x)$ as $n \to \infty$ for μ a.e. $x \in X$. Then u is (Bochner) integrable and*

$$\lim_{n\to\infty} \int_X \|u_n - u\| \, d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int_X u_n \, d\mu = \int_X u \, d\mu.$$

Theorem 11 (Fatou's Lemma) *Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and let Y be a Banach space. Let $u_n : X \to Y$, $n \in \mathbb{N}$, and $u : X \to Y$ be strongly measurable and assume that $u_n(x) \rightharpoonup u(x)$ as $n \to \infty$ for μ a.e. $x \in X$. If*

$$\sup_n \int_X \|u_n\| \, d\mu < \infty,$$

then u is (Bochner) integrable and

$$\int_X \|u\| \, d\mu \leq \liminf_{n\to\infty} \int_X \|u_n\| \, d\mu.$$

Theorem 12 (Egoroff) *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ finite, let Y be a Banach space, and let u, $u_n : X \to Y$, $n \in \mathbb{N}$, be strongly measurable functions such that*

$$\lim_{n\to\infty} \|u_n(x) - u(x)\|_Y = 0$$

for μ a.e. $x \in X$. Then for every $\varepsilon > 0$ there exists a measurable set $E \in \mathfrak{M}$, with $\mu(X \setminus E) \leq \varepsilon$, such that

$$\lim_{n\to\infty} \sup_{x\in E} \|u_n(x) - u(x)\|_Y = 0.$$

Theorem 13 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space. Let $u : X \to Y$ be (Bochner) integrable. Then for every $L \in Y'$,*

$$L\left(\int_X u \, d\mu\right) = \int_X L(u) \, d\mu.$$

Proof. We proceed as in the first part of the proof of Theorem 9 to construct a sequence $\{t_n\}$ of simple functions $t_n : X \to Y$ such that $\|t_n(x)\|_Y \leq \frac{3}{2}\|u(x)\|_Y$ for all $x \in X$ and all n, $\lim_{n\to\infty}\|t_n(x) - u(x)\|_Y = 0$ for μ a.e. $x \in X$, and

$$\lim_{n\to\infty}\int_X \|t_n - u\|_Y \, d\mu = 0.$$

Write

$$t_n = \sum_{i=1}^{m_n} c_{i,n}\chi_{E_{i,n}}.$$

Let now $L \in Y'$. Then by the continuity and linearity of L, and the fact that $\lim_{n\to\infty}\int_X t_n \, d\mu = \int_X u \, d\mu$,

$$\begin{aligned} L\left(\int_X u(x)\, d\mu\right) &= \lim_{n\to\infty} L\left(\int_X t_n(x)\, d\mu\right) \\ &= \lim_{n\to\infty}\sum_{i=1}^{m_n} L(c_{i,n})\,\mu(E_{i,n}) = \lim_{n\to\infty}\int_X L(t_n(x))\, d\mu. \end{aligned}$$

Since

$$|L(t_n(x))| \leq \|L\|_{Y'}\|t_n(x)\|_Y \leq \frac{3}{2}\|L\|_{Y'}\|u(x)\|_Y,$$

we can apply the Lebesgue dominated convergence theorem to conclude that

$$\lim_{n\to\infty}\int_X L(t_n(x))\, d\mu = \int_X \lim_{n\to\infty} L(t_n(x))\, d\mu = \int_X \lim_{n\to\infty} L(u(x))\, d\mu.$$

This concludes the proof. ■

2 L^p Spaces on Banach Spaces

Definition 14 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let Y be a Banach space, and let $1 \leq p < \infty$. Then*

$$L^p(X;Y) := \{u : X \to Y : u \text{ strongly measurable}, \\ \|u\|_{L^p(X;Y)} < \infty\},$$

where

$$\|u\|_{L^p(X;Y)} := \left(\int_X \|u\|_Y^p \, d\mu\right)^{1/p}.$$

If $p = \infty$ then

$$L^\infty(X;Y) := \{u : X \to Y : u \text{ strongly measurable}, \\ \|u\|_{L^\infty(X;Y)} < \infty\Big\},$$

where

$$\|u\|_{L^\infty(X;Y)} = \operatorname*{esssup}_{x\in X} \|u(x)\|_Y$$
$$:= \inf\{\alpha \in \mathbb{R} : \|u(x)\|_Y < \alpha \ \mu \ a.e. \ x \in X\}.$$

As in the case $Y = \mathbb{R}$ we identify functions with their equivalence classes.

Theorem 15 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let Y be a Banach space.*

(i) $L^p(X;Y)$ is a Banach space for $1 \le p \le \infty$;

(ii) the family of all integrable simple functions is dense in $L^p(X;Y)$ for $1 \le p < \infty$;

(iii) if X is a separable metric space, μ is a σ-finite Radon measure, and if Y is separable, then $L^p(X;Y)$ is separable for $1 \le p < \infty$.

A major problem in the theory of L^p spaces on Banach spaces is the identification of the dual of $L^p(X;Y)$. Here we will consider only the two important special cases in which Y is either separable or reflexive.

Definition 16 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let Y be a Banach space, and let $1 \le p \le \infty$. Then the space $L^p_w(X;Y')$ is the space of all (equivalence classes of) weakly star measurable functions $u : X \to Y'$ such that $\|u\|_{Y'} \in L^p(X;\mathbb{R})$. The space $L^p_w(X;Y')$ is endowed with the norm*

$$\|u\|_{L^p_w(X;Y')} := \left(\int_X \|u\|^p_{Y'} \, d\mu\right)^{1/p}$$

for $1 \le p < \infty$, and

$$\|u\|_{L^\infty_w(X;Y')} := \operatorname*{esssup}_{x\in X} \|u(x)\|_{Y'}$$

for $p = \infty$.

Theorem 17 (Riesz representation theorem in L^p) *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ σ-finite, let Y be a Banach space, let $1 \le p < \infty$, and let q be its Hölder conjugate exponent.*

(i) Assume that Y is separable. If $L \in (L^p(X;Y))'$ then there exists a unique $v \in L^q_w(X;Y')$ such that

$$L(u) = \int_X \langle v, u\rangle_{Y',Y} \, d\mu \tag{2}$$

for every $u \in L^p(X;Y)$. Moreover, the norm of L coincides with $\|v\|_{L^q_w(X;Y')}$. Conversely, every functional of the form (2), where $v \in L^q_w(X;Y')$, is a bounded linear functional on $L^p(X;Y)$.

(ii) Assume that Y is reflexive. Then for $L \in (L^p(X;Y))'$ there exists a unique $v \in L^q(X;Y')$ such that

$$L(u) = \int_X \langle v, u \rangle_{Y',Y} \, d\mu \tag{3}$$

for every $u \in L^p(X;Y)$. Moreover, the norm of L coincides with $\|v\|_{L^q(X;Y')}$. Conversely, every functional of the form (3), where $v \in L^q(X;Y')$, is a bounded linear functional on $L^p(X;Y)$.

Corollary 18 *Let $(X, \mathfrak{M}, \mu)$ be a measure space with μ σ-finite, let Y be a reflexive Banach space, and let $1 < p < \infty$. Then $L^p(X;Y)$ is reflexive.*

In what follows we will often use the following result, without further notice.

Theorem 19 *Let $I \subseteq \mathbb{R}$ be an open interval, let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $1 \leq p < \infty$. Then $L^p(I; L^p(\Omega))$ can be identified with $L^p(\Omega \times I)$.*

Proof. Given $u \in L^p(\Omega \times I)$, by Fubini's theorem, the function $t \in I \mapsto \int_\Omega |u(\boldsymbol{x}, t)|^p \, d\boldsymbol{x}$ is Lebesgue measurable with

$$\int_I \left(\int_\Omega |u(\boldsymbol{x}, t)|^p \, d\boldsymbol{x} \right) dt = \|u\|^p_{L^p(\Omega \times I)} < \infty.$$

It follows that $\int_\Omega |u(\boldsymbol{x}, t)|^p \, d\boldsymbol{x} < \infty$ for $\mathcal{L}^1$ a.e. $t \in I$. Hence, setting $v(t) := u(\cdot, t)$, by Theorem 9, we have that $v : I \to L^p(\Omega)$ with

$$\|v\|^p_{L^p(I;L^p(\Omega))} = \int_I \left(\|v(t)\|^p_{L^p(\Omega)} \right) dt = \int_I \int_\Omega |u(\boldsymbol{x}, t)|^p \, d\boldsymbol{x} dt = \|u\|^p_{L^p(\Omega \times I)} < \infty,$$

which shows that $v \in L^p(I; L^p(\Omega))$ with $\|v\|_{L^p(I;L^p(\Omega))} = \|u\|_{L^p(\Omega \times I)}$. Consider the linear operator

$$T : L^p(\Omega \times I) \to L^p(I; L^p(\Omega))$$
$$u \mapsto v$$

Then T preserves the norms and so it is injective. By identifying u with $T(u)$, this shows that $L^p(\Omega \times I) \subseteq L^p(I; L^p(\Omega))$.

Conversely, given $v \in L^p(I; L^p(\Omega))$, there exists a sequence $\{s_n\}$ of simple integrable functions $s_n : I \to L^p(\Omega)$ such that

$$\lim_{n \to \infty} \|s_n(t) - v(t)\|_{L^p(\Omega)} = 0 \quad \text{for } \mathcal{L}^1 \text{ a.e. } t \in I$$

and

$$\lim_{n \to \infty} \int_I \|s_n(t) - v(t)\|^p_{L^p(\Omega)} \, dt = 0.$$

Write

$$s_n = \sum_{i=1}^{m_n} u_{i,n} \chi_{E_{i,n}},$$

where $u_{i,n} \in L^p(\Omega)$ and $E_{i,n} \subseteq I$ are Lebesgue measurable pairwise disjoint sets. Define

$$t_n(\boldsymbol{x}, t) := \sum_{i=1}^{m_n} u_{i,n}(\boldsymbol{x}) \chi_{E_{i,n}}(t).$$

Then $t_n : \Omega \times I \to \mathbb{R}$ is Lebesgue measurable, since it is given by sums and products of Lebesgue measurable functions. Moreover,

$$\begin{aligned} \|s_n\|^p_{L^p(I;L^p(\Omega))} &= \int_I \left(\|s_n(t)\|^p_{L^p(\Omega)} \right) dt = \\ &= \int_I \left(\sum_{i=1}^{m_n} \chi_{E_{i,n}}(t) \int_\Omega |u_{i,n}(\boldsymbol{x})|^p \, d\boldsymbol{x} \right) dt \\ &= \|t_n\|^p_{L^p(\Omega \times I)}. \end{aligned}$$

A similar computation, shows that $\|s_n - s_m\|_{L^p(I;L^p(\Omega))} = \|t_n - t_m\|_{L^p(\Omega\times I)}$. Since $s_n \to v$ in $L^p(I; L^p(\Omega))$, we have that $\{s_n\}$ is a Cauchy sequence in $L^p(I; L^p(\Omega))$. In turn, $\{t_n\}$ is a Cauchy sequence in $L^p(\Omega \times I)$. Hence, there exists $u \in L^p(\Omega \times I)$ such that $t_n \to u$ in $L^p(\Omega \times I)$. However, from the first part of the theorem $L^p(\Omega \times I) \subseteq L^p(I; L^p(\Omega))$. Moreover, $T(t_n) = s_n$. Hence, $T(u) = v$, which shows that $L^p(\Omega \times I) \subseteq L^p(I; L^p(\Omega))$. ■

Monday, January 20, 2014

3 Sobolev Spaces with Values in Normed Spaces

Definition 20 *Let Y be a normed space, let $E \subseteq \mathbb{R}$ and let $u : E \to Y$. We say that u is* differentiable *at $t_0 \in E$ if there exists $y \in Y$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left\| \frac{u(t) - u(t_0)}{t - t_0} - y \right\|_Y \leq \varepsilon$$

for all $t \in E$ with $0 < |t - t_0| \leq \delta$. We write $y = u'(t_0)$.

Definition 21 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval and let $u : I \to Y$ be a locally integrable function (in the Bochner sense). We say that u is* weakly differentiable *if there exists a locally integrable function $v : I \to Y$ such that*

$$\int_I \varphi'(t) u(t) \, dt = -\int_I \varphi(t) v(t) \, dt$$

for every function $\varphi \in C^1_c(I)$. The function v is called the weak *or* distributional derivative *of u and is denoted u'.*

Remark 22 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval and let $u : I \to Y$ be a locally integrable function (in the Bochner sense). If $J : Y \to Z$ is a continuous embedding, and if the function $J \circ u : I \to Z$ is weakly differentiable* *with weak derivative $v : I \to Z$, then in view of Remark 8 is (Bochner) integrable and*

$$J\left(\int_I \varphi'(t)\, u(t)\ \ dt\right) = \int_I \varphi'(t)\,(J \circ u)(t)\ \ dt = -\int_I \varphi(t)\, v(t)\ \ dt.$$

Since we agreed to identify $\int_I \varphi'(t)\, u(t)\ \ dt$ with $J\left(\int_I \varphi'(t)\, u(t)\ \ dt\right)$, with an abuse of notation, we can write

$$\int_I \varphi'(t)\, u(t)\ \ dt = -\int_I \varphi(t)\, v(t)\ \ dt.$$

Next we extend the notion of Lebesgue points to Banach-valued functions.

Theorem 23 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an interval and let $v : I \to Y$ be a locally integrable function. Then for $\mathcal{L}^1$ a.e. $t \in I$,*

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|v(s) - v(t)\|\ \ ds = 0.$$

In particular,

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} v(s)\ \ ds = v(t)$$

for $\mathcal{L}^1$ a.e. $t \in I$.

Proof. It is enough to consider the case in which $I = [a, b]$ and $v : I \to Y$ is integrable. By Pettis' integrability theorem, up to a set of measure zero, the set $v([a, b])$ is separable. Hence, by replacing Y with the closure of the subspace generated by the set $v([a, b])$, without loss of generality, we may assume that Y is separable. Let $\{y_n\}$ be dense in Y. Consider the real-valued function $v_n(t) := \|v(t) - y_n\|$. Then v_n is integrable and so by Lebesgue's differentiation theorem, there exists $E_n \subset I$ with $\mathcal{L}^1(E_n) = 0$ such that for $t \in I \setminus E_n$,

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|v(s) - y_n\|\ \ ds = \|v(t) - y_n\|.$$

Let $E = \bigcup_n E_n$. Then $\mathcal{L}^1(E) = 0$ and for $t \in I \setminus E$ and every $n \in \mathbb{N}$,

$$\begin{aligned}
\limsup_{h \to 0^+} \left\| \frac{1}{h} \int_t^{t+h} v(s)\ \ ds - v(t) \right\| &\le \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|v(s) - v(t)\|\ \ ds \\
&\le \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|v(s) - y_n\|\ \ ds + \|v(t) - y_n\| \\
&= 2\,\|v(t) - y_n\|.
\end{aligned}$$

By the density of $\{y_n\}$ in Y, we can find a subsequence $y_{n_k} \to v(t)$ as $k \to \infty$. ∎

Remark 24 *Similarly, if* $v \in L^p_{\mathrm{loc}}(I;Y)$*, then for* $\mathcal{L}^1$ *a.e.* $t \in I$*,*

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}\|v(s)-v(t)\|^p\;ds=0.$$

As a consequence of this theorem, we can extend the mollification theory to Banach-valued functions. Given a nonnegative bounded function $\varphi \in L^1(\mathbb{R})$ with

$$\operatorname{supp}\varphi \subset \overline{B(0,1)}, \quad \int_{\mathbb{R}}\varphi(t)\;dt=1, \tag{4}$$

for every $\varepsilon > 0$ we define

$$\varphi_\varepsilon(t) := \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right), \quad t\in\mathbb{R}.$$

The functions φ_ε are called *mollifiers*. Note that $\operatorname{supp}\varphi_\varepsilon \subset \overline{B(0,\varepsilon)}$. Hence, given an open interval $I \subset \mathbb{R}$, a Banach space Y, and a function $u \in L^1_{\mathrm{loc}}(I;Y)$, we may define

$$u_\varepsilon(t) := (u*\varphi_\varepsilon)(t) = \int_\Omega \varphi_\varepsilon(t-s)\,u(s)\;ds \tag{5}$$

for $t \in I_\varepsilon$, where the open set I_ε is given by

$$I_\varepsilon := \{t\in I : \operatorname{dist}(t,\partial I) > \varepsilon\}.$$

The function $u_\varepsilon : I_\varepsilon \to Y$ is called a *mollification* of u.

Theorem 25 *Let* $I \subset \mathbb{R}$ *be an open interval, let* $\varphi \in L^1(I)$ *be a nonnegative bounded function satisfying (4), let* Y *be a Banach space, and let* $u \in L^1_{\mathrm{loc}}(I;Y)$*.*

(i) For every Lebesgue point $t \in I$*,* $u_\varepsilon(t) \to u(t)$ *as* $\varepsilon \to 0^+$*.*

(ii) If $1 \le p \le \infty$*, then*

$$\|u_\varepsilon\|_{L^p(I_\varepsilon;Y)} \le \|u\|_{L^p(I;Y)} \tag{6}$$

for every $\varepsilon > 0$ *and*

$$\|u_\varepsilon\|_{L^p(I_\varepsilon;Y)} \to \|u\|_{L^p(I;Y)} \quad \text{as } \varepsilon \to 0^+. \tag{7}$$

(iii) If $u \in L^p(I;Y)$*,* $1 \le p < \infty$*, then*

$$\lim_{\varepsilon\to 0^+}\left(\int_{I_\varepsilon}\|u_\varepsilon - u\|_Y^p\;dt\right)^{\frac{1}{p}} = 0.$$

In particular, for any open interval $J \subset I$ *with* $\operatorname{dist}(J,\partial I) > 0$*,* $u_\varepsilon \to u$ *in* $L^p(J;Y)$*.*

Another consequence of Theorem 23 is the following result.

Corollary 26 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval and let $v : I \to Y$ be a locally integrable function such that*

$$\int_I \varphi(t) v(t) \, dt = 0$$

for every for every function $\varphi \in C_c^\infty(I)$. Then v is equivalent to zero.

Proof. Let t_0 be a Lebesgue point of v and let $h > 0$ be such that $[t_0, t_0 + h] \subset I$. Let φ_ε be a standard mollifier. Then $\varphi_\varepsilon * \chi_{[t_0,t_0+h]} \in C_c^\infty(I)$ for all $\varepsilon > 0$ sufficiently small, and so

$$\int_I \left(\varphi_\varepsilon * \chi_{[t_0,t_0+h]}\right)(t) v(t) \, dt = 0.$$

Since $\left|\left(\varphi_\varepsilon * \chi_{[t_0,t_0+h]}\right)(t)\right| \leq 1$ and $\varphi_\varepsilon * \chi_{[t_0,t_0+h]}$ has compact support, it follows by the Lebesgue dominated convergence theorem for Bochner integrals that there exists

$$0 = \lim_{\varepsilon \to 0^+} \int_I \left(\varphi_\varepsilon * \chi_{[t_0,t_0+h]}\right)(t) v(t) \, dt = \int_I \chi_{[t_0,t_0+h]}(t) v(t) \, dt = \int_{t_0}^{t_0+h} v(t) \, dt.$$

Since this is true for all $h > 0$ small, it follows by the previous theorem that $v(t_0) = 0$. Hence, $v(t) = 0$ for $\mathcal{L}^1$ a.e. $t \in I$. ■

The next corollary will be used to study Sobolev spaces for Banach-valued functions.

Corollary 27 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval and let $u : I \to Y$ be a locally integrable function whose weak derivative is zero. Then u is equivalent to a constant function.*

Proof. Step 1: The proof is quite simple in the case in which $u \in L^1(I; Y)$ and $I = (a, b)$, and thus we begin with this case. A simple density argument shows that

$$\int_a^b \varphi'(t) u(t) \, dt = 0 \tag{8}$$

for all $\varphi \in C^\infty([a, b])$ with $\varphi(a) = \varphi(b) = 0$. We claim that

$$\int_a^b w(t) u(t) \, dx = 0 \tag{9}$$

for all $w \in C^\infty([a, b])$ with $\int_a^b w(t) \, dt = 0$. To see this, fix any such w and define the function the function

$$\varphi(t) := \int_a^t w(s) \, ds, \quad t \in I.$$

Since $\varphi(a) = 0$ and $\varphi(b) = \int_a^b w(s) \, ds = 0$, we have that φ is admissible in (8), and so the claim holds.

Let now $w \in C_c^\infty([a,b])$. Taking $w - \frac{1}{b-a}\int_a^b w\,ds$ in (9) we get

$$\int_a^b u\left(w - \frac{1}{b-a}\int_a^b w\,ds\right)\,dt = 0,$$

which can be written as

$$\int_a^b w\left(u - \frac{1}{b-a}\int_a^b u\,dt\right)\,ds = 0.$$

Since this is true for all $w \in C_c^\infty([a,b])$, it follows by the previous corollary that

$$u(s) - \frac{1}{b-a}\int_a^b u\,dt = 0$$

for $\mathcal{L}^1$ a.e. $s \in I$.

Step 2: In the general case, let (a_n, b_n) be an increasing sequence of open intervals such that $[a_n, b_n] \subset (a_{n+1}, b_{n+1})$ and

$$I = \bigcup_{n=1}^{\infty}(a_n, b_n).$$

Then for every n,

$$\int_{(a_n,b_n)} u\varphi'\,dx = 0$$

for all $\varphi \in C_c^1((a_n, b_n))$. Hence, by Step 1, there exists $c_n \in Y$ such that $u(x) = c_n$ for $\mathcal{L}^1$ a.e. $x \in (a_n, b_n)$. But since, $[a_n, b_n] \subset (a_{n+1}, b_{n+1})$, it follows that $c_n = c_{n+1}$ for all n. ■

Theorem 28 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval and let $u : I \to Y$ be a locally integrable function. Then u is weakly differentiable if and only if there exist a representative of u and a locally integrable function $v : I \to Y$ such that*

$$u(t) = u(t_0) + \int_{t_0}^{t} v(s)\ ds \tag{10}$$

for all $t, t_0 \in T$. Moreover, in this case, u is (strongly) differentiable at $\mathcal{L}^1$ a.e. $t \in I$ and its strong derivative coincides with $v(t)$.

Wednesday, January 22, 2014

Proof. Let u is given by the formula (10), then

$$\frac{u(t+h) - u(t)}{h} = \frac{1}{h}\int_t^{t+h} v(s)\ ds \to v(t)$$

as $h \to 0$ for $\mathcal{L}^1$ a.e. $t \in I$ by Theorem 23. Hence, u is strongly differentiable for $\mathcal{L}^1$ a.e. $t \in I$ with (strong) derivative given by v. Let's prove that v is also the weak derivative of u. Given $\varphi \in C_c^1(I)$, let $(a,b) \Subset I$ be such that the support of φ is contained in (a,b). Let

$$w(t) := \begin{cases} v(t) & \text{if } t \in (a,b), \\ 0 & \text{otherwise.} \end{cases}$$

Then $w \in L^1(\mathbb{R};Y)$. Hence, by Theorem 25 (with the mollifier $\chi_{(-1,0)}$)

$$\frac{1}{h}\int_t^{t+h} w(s)\ ds \to w \text{ in } L(\mathbb{R};Y).$$

On the other hand, for $t \in \operatorname{supp}\varphi$ and $h > 0$ small,

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h}\int_t^{t+h} v(s)\ ds = \frac{1}{h}\int_t^{t+h} w(s)\ ds.$$

Let $0 < h < \operatorname{dist}(\operatorname{supp}\varphi, I \setminus \{a,b\})$. Then

$$\begin{aligned} &\int_a^b \frac{\varphi(t)-\varphi(t-h)}{h} u(t)\ dt \\ &= \frac{1}{h}\int_a^b \varphi(t)\left(u(t)-u(t+h)\right)\ dt. \end{aligned}$$

Since,

$$\begin{aligned} \left\| \int_a^b \varphi(t)\left(\frac{u(t)-u(t+h)}{h} - v(t)\right)\ dt \right\|_H &= \left\| \int_a^b \varphi(t)\left(\frac{1}{h}\int_t^{t+h} w(s)\ ds - w(t)\right)\ dt \right\|_H \\ &\le \|\varphi\|_\infty \int_a^b \left\| \frac{1}{h}\int_t^{t+h} w(s)\ ds - w(t)\right\|_Y\ dt \to 0 \end{aligned}$$

By the Lebesgue dominated convergence theorem

$$\begin{aligned} \int_I \varphi'(t)\, u(t)\ dt &= \int_a^b \varphi'(t)\, u(t)\ dt \\ &= \lim_{h\to 0^+}\int_a^b \frac{\varphi(t-h)-\varphi(t)}{-h} u(t)\ dt \\ &= -\lim_{h\to 0^+}\int_a^b \varphi(t)\,\frac{u(t+h)-u(t)}{h}\ dt \\ &= -\int_a^b \varphi(t)\, v(t)\ dt. \end{aligned}$$

Conversely, assume that v is the weak derivative of u and define $w(t) = u(t_0) + \int_{t_0}^t v(s)\ ds$. Then by the first part of the proof, v is the weak derivative of w. Hence, for every $\varphi \in C_c^1(I)$,

$$\int_I \varphi'(t)\, u(t)\ dt = -\int_a^b \varphi(t)\, v(t)\ dt = -\int_a^b \varphi'(t)\, w(t)\ dt,$$

and so

$$\int_I \varphi'(t)(u(t)-w(t))\ dt = 0$$

for all $\varphi \in C_c^1(I)$. It follows by the previous corollary, that $u-w$ is constant. Since $u(t_0)-w(t_0)=0$, it follows that $u=w$. ■

We now define Sobolev spaces for Banach-valued functions.

Definition 29 *Let Y be a Banach space, let $I \subseteq \mathbb{R}$ be an open interval, and let $1 \le p < \infty$. The* Sobolev space $W^{1,p}(I;Y)$ *is defined as the space of all functions $u \in L^p(I;Y)$ that have a weak derivative u' in $L^p(I;Y)$. The norm of u is defined as*

$$\|u\|_{W^{1,p}(I;Y)} := \|u\|_{L^p(I;Y)} + \|u'\|_{L^p(I;Y)}.$$

By induction, we can define higher order Sobolev spaces $W^{k,p}(I;Y)$, $k \in \mathbb{N}$. Next we show that the embedding theorems continue to hold.

Definition 30 *Let Y be a normed space and let $u : [a,b] \to Y$. We say that u is* absolutely continuous *if there exists an integrable function $g : [a,b] \to [0,\infty)$ such that*

$$\|u(t)-u(s)\|_Y \le \int_s^t g(r)\ dr$$

for every $t,s \in [a,b]$, with $s<t$.

Exercise 31 *Prove that $u : [a,b] \to \mathbb{R}$ is absolutely continuous if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sum_{i=1}^n |u(b_i) - u(a_i)| \le \varepsilon$$

for every $[a_1,b_1]$, $\ldots$, $[a_n,b_n] \subseteq [a,b]$, $n \in \mathbb{N}$, with $(a_i,b_i) \cap (a_j,b_j) = \emptyset$ for $i \ne j$, and $\sum_{i=1}^n (b_i - a_i) \le \delta$.

If $Y=\mathbb{R}$, then an absolutely continuous function is differentiable $\mathcal{L}^1$ a.e. in $[a,b]$ and the fundamental theorem of calculus holds. This is not the case when Y is an infinite dimensional normed space.

Exercise 32 *Let $Y = L^1([a,b])$ and let $u : [a,b] \to Y$ be defined by*

$$(u(t))(x) := \begin{cases} 1 & \text{if } a \le x < t, \\ 0 & \text{if } t \le x \le b. \end{cases}$$

Then u is absolutely continuous, but it is not differentiable at any point.

The following theorem shows that this cannot happen if Y is reflexive.

Theorem 33 (Komura) *Let Y be a reflexive Banach space and let $u : [a,b] \to Y$ be absolutely continuous. Then u is differentiable for $\mathcal{L}^1$ a.e. $t \in [a,b]$, u' is (Bochner) integrable and for all $t \in [a,b]$,*

$$u(t) - u(a) = \int_a^t u'(s)\ ds.$$

Proof. By Pettis' measurability theorem and the continuity of u, we have that $u([a,b])$ is separable. Hence, by replacing Y with the closure of the subspace generated by the set $u([a,b])$, without loss of generality, we may assume that Y is reflexive and separable. Given $0 < \varepsilon < b - a$ for $a \le t \le b - \varepsilon$ we have

$$\frac{\|u(t+\varepsilon) - u(t)\|}{\varepsilon} \le \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(r)\ dr.$$

By Lebesgue's differentiation theorem, for $\mathcal{L}^1$ a.e. $t \in (a,b)$,

$$\limsup_{\varepsilon \to 0^+} \frac{\|u(t+\varepsilon) - u(t)\|}{\varepsilon} \le \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(r)\ dr = g(t) < \infty \tag{11}$$

for $\mathcal{L}^1$ a.e. $t \in (a,b)$. Set

$$E_0 := \left\{ t \in I : \ \limsup_{\varepsilon \to 0^+} \frac{\|u(t+\varepsilon) - u(t)\|}{\varepsilon} = \infty \right\}.$$

Then $\mathcal{L}^1(E_0) = 0$.

Since Y is reflexive and separable, its dual Y' is reflexive and separable. Let $\{L_n\} \subset Y'$ be dense in Y'. Consider the function

$$f_n(t) := L_n(u(t)).$$

Then

$$\begin{aligned} |f_n(t) - f_n(s)| &= |L_n(u(t)) - L_n(u(s))| = |L_n(u(t) - u(s))| \\ &\le \|L_n\|_{Y'} \|u(t) - u(s)\| \le \|L_n\|_{Y'} \int_s^t g(r)\ dr \end{aligned}$$

for all $s, t \in I$ with $s < t$, which shows that f_n is absolutely continuous. Since f_n is real-valued, it is differentiable for all $t \in I \setminus E_n$, where $\mathcal{L}^1(E_n) = 0$. By the linearity of L_n, this implies that there exists

$$\lim_{\varepsilon \to 0} L_n\left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}\right) = f_n'(t) \in \mathbb{R} \tag{12}$$

for all $\in I \setminus E_n$. Set

$$E := \bigcup_{n=0}^{\infty} E_n.$$

Then $\mathcal{L}^1(E) = 0$. Let $t_0 \in I \setminus E$. Since $t_0 \notin E_0$, there exists $M_0 > 0$ such that

$$\frac{\|u(t_0+\varepsilon) - u(t_0)\|}{\varepsilon} \le M_0$$

for all $\varepsilon > 0$ sufficiently small. Using the fact that X is reflexive, we can find a sequence $\varepsilon_k \to 0^+$ and $v(t_0) \in Y$ such that

$$\frac{u(t_0+\varepsilon_k) - u(t_0)}{\varepsilon_k} \rightharpoonup v(t_0).$$

It follows from (12) that $f_n'(t_0) = L_n(v(t_0))$. We claim that

$$\frac{u(t_0+\varepsilon) - u(t_0)}{\varepsilon} \rightharpoonup v(t_0)$$

as $\varepsilon \to 0^+$. Fix $L \in Y'$ and $\eta > 0$. By the density of $\{L_n\}$ in Y', there exists L_m such that $\|L - L_m\|_{Y'} \le \eta$. Since $f_m'(t_0) = L_m(v(t_0))$, there exists $\varepsilon_0 > 0$ such that

$$\left| L_m\left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}\right) - L_m(v(t_0))\right| \le \eta$$

for all $0 < \varepsilon \le \varepsilon_0$. In turn,

$$\begin{aligned}\left| L\left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}\right) - L(v(t_0))\right| &\le 2\|L - L_m\|_{Y'}\|v(t_0)\|_Y \\ + \left| L_m\left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}\right) - L_m(v(t_0))\right| &\le 2\eta\|v(t_0)\|_Y + \eta\end{aligned}$$

for all $0 < \varepsilon \le \varepsilon_0$. This proves the claim.

The function

$$t \in I \setminus E \mapsto v(t)$$

is weakly measurable, since each function $t \in I \mapsto \frac{u(t+\varepsilon)-u(t)}{\varepsilon}$ is strongly measurable. Since Y is separable, it follows by Pettis' theorem that it is strongly measurable. Moreover, by the lower semicontinuity of the norm with respect to weak convergence,

$$\|v(t)\|_Y \le \liminf_{\varepsilon \to 0^+} \frac{\|u(t_0+\varepsilon) - u(t_0)\|}{\varepsilon}.$$

It follows by (11) and Fatou's lemma that

$$\begin{aligned}\int_I \|v(t)\|_Y \, dt &\le \int_I \liminf_{\varepsilon \to 0^+} \frac{\|u(t_0+\varepsilon) - u(t_0)\|}{\varepsilon} \, dt \\ &\le \liminf_{\varepsilon \to 0^+} \int_I \frac{\|u(t_0+\varepsilon) - u(t_0)\|}{\varepsilon} \, dt \\ &\le \liminf_{\varepsilon \to 0^+} \int_I \left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon} g(r) \, dr\right) dt = \int_I g(t) \, dt < \infty.\end{aligned}$$

Hence, v is (Bochner) integrable.

For $L \in Y'$, by the fundamental theorem of calculus applied to the absolutely continuous function $f(t) := L(u(t))$, we have that

$$L(u(t)) - L(u(s)) = f(t) - f(s) = \int_s^t f'(r)\ dr = \int_s^t L(v(r))\ dr$$
$$= L\left(\int_s^t v(r)\ dr\right),$$

where we have used Theorem 13. Hence,

$$L\left(u(t) - u(s) - \int_s^t v(r)\ dr\right) = 0.$$

Taking the supremum over all L with $\|L\|_{Y'} \leq 1$ and using the fact that for $y \in Y$,

$$\|y\|_Y = \sup_{\|L\|_{Y'} \leq 1} L(y),$$

we get

$$u(t) = u(s) + \int_s^t v(r)\ dr.$$

This concludes the proof. ■

We conclude this section by discussing evolution triples. Let Y be a Banach space continuously embedded in a Hilbert space H. Assume that Y is dense in H. We claim that $H' \hookrightarrow Y'$ with continuous embedding. To see this, given $h \in H$, consider the linear functional $L_h : Y \to \mathbb{R}$ defined by

$$L_h(y) := (h, y)_H, \quad y \in Y.$$

Then

$$|L_h(y)| = |(h, y)_H| \leq \|h\|_H \|y\|_H \leq C \|h\|_H \|y\|_Y.$$

Hence, $L_h \in Y'$ with

$$\|L_h\|_{Y'} \leq C \|h\|_H.$$

Note that

$$(h, y)_H = \langle L_h, y \rangle_{Y',Y} \tag{13}$$

for all $y \in Y$. The mapping

$$h \in H \mapsto L_h$$

is linear, continuous. It is also injective, since if $L_h = 0$, then $(h, y)_H = 0$ for all $y \in Y$ and so, by the density of Y in H, we have that $h = 0$. The claim follows by identifying h with L_h. In particular, we can rewrite (13) as

$$(h, y)_H = \langle h, y \rangle_{Y',Y} \tag{14}$$

for all $h \in H$ and $y \in Y$.

By identifying H with its dual, we have that

$$Y \hookrightarrow H \cong H' \hookrightarrow Y'. \tag{15}$$

This is called an *evolution triple.*

Friday, January 24, 2014

Proposition 34 *Let H be a Hilbert space, let $I \subseteq \mathbb{R}$ be an open interval and let $u, v \in C^1(I; H)$. Then the function*

$$g(t) := (u(t), v(t))_H$$

is of class $C^1(I)$ and

$$g'(t) = (u'(t), v(t))_H + (u(t), v'(t))_H.$$

Proof. Let $t \in I$ and $h \neq 0$ small. Then using the bilinearity of the inner product,

$$\begin{aligned}
\frac{g(t+h) - g(h)}{h} &= \frac{(u(t+h), v(t+h))_H - (u(t), v(t))_H}{h} \\
&= \frac{(u(t+h) - u(t) + u(t), v(t+h))_H - (u(t), v(t))_H}{h} \\
&= \left(\frac{u(t+h) - u(t)}{h}, v(t+h)\right)_H + \left(u(t), \frac{v(t+h) - v(t)}{h}\right)_H.
\end{aligned}$$

Letting $h \to 0$ and using the continuity of the inner product gives the desired result. ■

Theorem 35 *Let Y be a Banach space continuously embedded in a Hilbert space H, with Y dense in H, and let $1 \leq p < \infty$. If $u \in L^p((a,b); Y)$ has a weak derivative $u' \in L^{p'}((a,b); Y')$, then $u \in C([a,b]; H)$ with*

$$\|u\|_{C([a,b];H)} \leq C\left(\|u\|_{L^p((a,b);Y)} + \|u'\|_{L^{p'}((a,b);Y')}\right).$$

Moreover, the function $t \in [a,b] \mapsto \|u(t)\|_H^2$ is absolutely continuous, with

$$\frac{d}{dt}\left(\|u(t)\|_H^2\right) = 2u'(t)(u(t)) = 2\langle u'(t), u(t)\rangle_{Y',Y}$$

for $\mathcal{L}^1$ a.e. $t \in [a,b]$.

Proof. Extend u by reflection in an interval (a', b') containing $[a,b]$ and let $u_\varepsilon := u * \varphi_\varepsilon$, where φ_ε is a standard mollifier and $0 < \varepsilon < \min\{b' - b, a - a'\}$. Then reasoning as in the real-valued case, we have that $u_\varepsilon \in C_c^\infty((a', b'); Y)$, $u_\varepsilon \to u$ in $L^p((a', b'); Y)$ and $u'_\varepsilon \to u'$ in $L^p((a', b'); Y')$. Since $u_\varepsilon \in C_c^\infty((a', b'); Y)$

and $Y \subset H$, we have that $u_\varepsilon \in C_c^\infty((a',b');H)$ and so by the previous proposition, the fundamental theorem of calculus, property (14), and Hölder's inequality

$$\begin{aligned}\|u_\varepsilon(t)\|_H^2 &= \|u_\varepsilon(a')\|_H^2 + \int_{a'}^t \frac{d}{ds}\left(\|u_\varepsilon(s)\|_H^2\right) ds \\ &= 0 + 2\int_{a'}^t (u_\varepsilon'(s), u_\varepsilon(s))_H \, ds \\ &= 0 + 2\int_{a'}^t \langle u_\varepsilon'(s), u_\varepsilon(s)\rangle_{Y',Y} \, ds\end{aligned} \tag{16}$$

By Hölder's inequality

$$\begin{aligned}\|u_\varepsilon(t)\|_H^2 &\le 2\int_{a'}^t \|u_\varepsilon'(s)\|_{Y'}\|u_\varepsilon(s)\|_Y \, ds \\ &\le 2\|u_\varepsilon'\|_{L^{p'}((a',b');Y')}\|u_\varepsilon\|_{L^p((a',b');Y)}.\end{aligned}$$

In turn,

$$\|u_\varepsilon(t) - u_\delta(t)\|_H^2 \le 2\|u_\varepsilon' - u_\delta'\|_{L^{p'}((a',b');Y')}\|u_\varepsilon - u_\delta\|_{L^p((a',b');Y)}.$$

Since $u_\varepsilon \to u$ in $L^p((a',b');Y)$ and $u_\varepsilon' \to u'$ in $L^p((a',b');Y')$, it follows that

$$\begin{aligned}\sup_{t\in(a',b')}\|u_\varepsilon(t) - u_\delta(t)\|_H^2 &\le 2\|u_\varepsilon' - u_\delta'\|_{L^{p'}((a',b');Y')}\|u_\varepsilon - u_\delta\|_{L^p((a',b');Y)} \\ &\to 0\end{aligned}$$

as $\varepsilon, \delta \to 0^+$. Hence, $\{u_\varepsilon\}$ is a Cauchy sequence in $C([a',b'];H)$ and so it converges uniformly to a function $w \in C([a',b'];H)$. On the other hand, since $u_\varepsilon(t) \to u(t)$ for every Lebesgue point of u, we have that $u(t) = w(t)$ for $\mathcal{L}^1$ a.e. $t \in [a',b']$.

Letting $\varepsilon \to 0^+$ in (16) gives

$$\|w(t)\|_H^2 = 2\int_{a'}^t \langle u'(s), u(s)\rangle_{Y',Y} \, ds$$

for all $t \in [a',b']$. In turn,

$$\|u(t)\|_H^2 = 2\int_{a'}^t \langle u'(s), u(s)\rangle_{Y',Y} \, ds$$

for $\mathcal{L}^1$ a.e. $t \in [a',b']$. This implies that the function $t \in [a,b] \mapsto \|u(t)\|_H^2$ is absolutely continuous and concludes the proof. ■

Theorem 36 (Aubin–Lions) *Let Y_0, Y, and Y_1 be Banach spaces, with Y_0 is reflexive. Assume that $Y_0 \hookrightarrow Y \hookrightarrow Y_1$, that the embedding $Y \hookrightarrow Y_1$ is continuous and that the embedding $Y_0 \hookrightarrow Y$ is compact. Let $1 < p < \infty$, $1 \le q < \infty$, and* *let $\mathcal{V}$ be the Banach space of all functions $u \in L^p((a,b);Y_0)$ whose distributional derivative u' belongs to $L^q((a,b);Y_1)$ endowed with the norm*

$$\|u\|_{\mathcal{V}} := \|u\|_{L^p((a,b);Y_0)} + \|u'\|_{L^q((a,b);Y_1)}. \tag{17}$$

Then the embedding $\mathcal{V} \hookrightarrow L^p((a,b);Y)$ is compact.

Proof. Step 1: By hypothesis there exist $C_1, C_2 > 0$ such that

$$\|y\|_Y \leq C_1 \|y\|_{Y_0} \tag{18}$$

for all $y \in Y_0$ and

$$\|y\|_{Y_1} \leq C_2 \|y\|_Y \tag{19}$$

for all $y \in Y$. We claim that for every $\eta > 0$ there exists $c_\eta > 0$ such that

$$\|y\|_Y \leq \eta \|y\|_{Y_0} + c_\eta \|y\|_{Y_1}$$

for every $y \in Y_0$. Indeed, if not, then for every n there exists $y_n \in Y_0$ such that

$$\|y_n\|_Y > \eta \|y_n\|_{Y_0} + n \|y_n\|_{Y_1}.$$

Note that by (18) the previous inequality implies that $\|y_n\|_{Y_0} > 0$. Let $w_n := \frac{v_n}{\|y_n\|_{Y_0}}$. Then

$$\|w_n\|_Y > \eta \|w_n\|_{Y_0} + n \|w_n\|_{Y_1} = \eta + n \|w_n\|_{Y_1}. \tag{20}$$

Since the sequence $\{w_n\}$ is bounded in Y_0, by (18), it is also bounded in Y. In turn, by (20), $w_n \to 0$ in Y_1. On the other hand, since the embedding $Y_0 \hookrightarrow Y$ is compact, there exist a subsequence $\{w_{n_k}\}$ and $w_0 \in Y$ such that $w_{n_k} \to w_0$ in Y. Necessarily, $w_0 = 0$ by (19). Hence, taking $n = n_k$ in (20) and letting $k \to \infty$ we get

$$0 = \lim_{k \to \infty} \|w_{n_k}\|_Y \geq \eta > 0,$$

which is a contradiction. ■

Monday, January 27, 2014

Proof. Step 2: Let $\{v_n\}$ be a bounded sequence in $\mathcal{V}$. Using (17) we obtain that

$$\{v_n\} \text{ is bounded in } L^p((a,b);Y_0), \tag{21}$$

$$\{v_n'\} \text{ is bounded in } L^1((a,b);Y_1). \tag{22}$$

Since Y_0 is reflexive and $1 < p < \infty$, by Corollary 18, the space $L^p((a,b);Y_0)$ is reflexive. By (21), extracting a subsequence (not relabeled), we have that

$$v_n \rightharpoonup v \text{ in } L^p((a,b);Y_0) \tag{23}$$

for some $v \in L^p((a,b);Y_0)$. By Step 1,

$$\|v_n - v\|_{L^p((a,b);Y)} \leq \eta \|v_n - v\|_{L^p((a,b);Y_0)} + c_\eta \|v_n - v\|_{L^p((a,b);Y_1)}$$

for every n. By (21), for every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\|v_n - v\|_{L^p((a,b);Y)} \leq \varepsilon + c_\eta \|v_n - v\|_{L^p([0,T];Y_1)}$$

for every n. Therefore, to prove that $v_n \to v$ strongly in $L^p((a,b)\,;Y)$ it is enough to show that

$$v_n \to v \text{ strongly in } L^p((a,b)\,;Y_1). \tag{24}$$

Since $\mathcal{V} \hookrightarrow W^{1,1}((a,b)\,;Y_1) \hookrightarrow C([a,b];Y_1)$ with continuous embedding, the sequence $\{v_n\}$ is bounded in $C([a,b];Y_1)$. By the dominated convergence theorem, to obtain (24) it suffices to prove that

$$v_n(t) \to v(t) \text{ strongly in } Y_1 \text{ for a.e. } t \in (a,b)\,. \tag{25}$$

For $t \in (a,b)$ and $n \in \mathbb{N}$ define

$$V_n(t) := \int_a^t \|v_n'(s)\|_{Y_1}\, ds\,.$$

By (22), $\{V_n\}$ is a bounded sequence of monotone functions. By the Helly theorem there exists a subsequence (not relabeled) that converges pointwise to a monotone function $V : (a,b) \to [0,\infty)$.

Let t_0 be a continuity point of V and a Lebesgue point of v, considered as an integrable function with values in Y_1, i.e.,

$$\lim_{s\to 0} \frac{1}{s}\int_{t_0}^{t_0+s} \|v(t) - v(t_0)\|_{Y_1}\, dt = 0.$$

We want to prove that

$$v_n(t_0) \to v(t_0) \text{ strongly in } Y_1. \tag{26}$$

Fix $\varepsilon > 0$ and $s > 0$ such that

$$V(t_0+s) - V(t_0) < \varepsilon \quad \text{and} \quad \frac{1}{s}\int_{t_0}^{t_0+s} \|v(t) - v(t_0)\|_{Y_1}\, dt < \varepsilon. \tag{27}$$

Write

$$v_n(t_0) = \frac{1}{s}\int_{t_0}^{t_0+s} v_n(t)\, dt - \frac{1}{s}\int_{t_0}^{t_0+s} (t_0+s-t) v_n'(t)\, dt =: a_n - b_n, \tag{28}$$

and define

$$a := \frac{1}{s}\int_{t_0}^{t_0+s} v(t)\, dt.$$

By (23), the sequence $\{a_n\}$ converges to a weakly in Y_0. Since the embedding $Y_0 \hookrightarrow Y$ is compact and the embedding $Y \hookrightarrow Y_1$ is continuous, we have that $\{a_n\}$ converges strongly in Y_1. Since $\|a - v(t_0)\|_{Y_1} < \varepsilon$ by (27), we obtain

$$\lim_{n\to\infty} \|a_n - v(t_0)\|_{Y_1} < \varepsilon \quad \text{for } n \text{ large enough.} \tag{29}$$

On the other hand,

$$\|b_n\|_{Y_1} \le \int_{t_0}^{t_0+s} \|v_n'(t)\|_{Y_1}\, dt = V_n(t_0+s) - V_n(t_0),$$

so that, by (27),

$$\limsup_{n\to\infty} \|b_n\|_{Y_1} \le V(t_0+s) - V(t_0) < \varepsilon. \tag{30}$$

In view of the arbitrariness of $\varepsilon > 0$, (26) follows from (28), (29), and (30). Since the continuity points for V that are Lebesgue points for v form a set of full measure, we have proved (25), which concludes the proof of the theorem. ■

Wednesday, January 29, 2014

4 Convex Functions and Subdifferentiability

Definition 37 *A function $f : Y \to [-\infty, \infty]$ is said to be*

(i) convex *if*

$$f(\theta y_1 + (1-\theta) y_2) \le \theta f(y_1) + (1-\theta) f(y_2) \tag{31}$$

for all y_1, $y_2 \in Y$ and $\theta \in (0,1)$ for which the right-hand side is well-defined;

(ii) strictly convex *if*

$$f(\theta y_1 + (1-\theta) y_2) < \theta f(y_1) + (1-\theta) f(y_2)$$

for all y_1, $y_2 \in Y$, $y_1 \neq y_2$, and $\theta \in (0,1)$ for which the right-hand side is well-defined;

(iii) proper *if it is convex, does not take the value $-\infty$, and is not identically ∞;*

(iv) concave *(respectively* strictly concave*) if $-f$ is convex (respectively strictly convex).*

In (i) and (ii) the right-hand side is not defined only when $f(y_1) = \pm\infty$ and $f(y_2) = \mp\infty$.

Let Y be a vector space. The *effective domain* of a function $f : Y \to [-\infty, \infty]$ is the set

$$\operatorname{dom}_e f := \{y \in Y : f(y) < \infty\}.$$

We observe that if f is convex, then the effective domain of f is a convex set.

Theorem 38 *Let Y be a Banach space and let $f : Y \to (-\infty, \infty]$ be a lower semicontinuous convex function. Then f is continuous over the interior of $\operatorname{dom}_e f$.*

Exercise 39 *Let $Y := L^1([0,1])$, where the underlying measure is the Lebesgue measure. Show that the functional*

$$I : L^1([0,1]) \to [0,\infty]$$

defined by

$$I(v) := \begin{cases} \int_0^1 |v(x)|^2 \, dx & \text{if } v \in L^2([0,1]), \\ \infty & \text{otherwise,} \end{cases}$$

is convex and lower semicontinuous, but the sets

$$\left\{ v \in L^1([0,1]) : \int_0^1 |u(x)|^2 \, dx \le t \right\}, \quad t > 0,$$

have empty interiors.

If Y is a normed space, an *affine continuous* function $g : Y \to \mathbb{R}$ is a function of the form

$$g(y) = \alpha + \langle y', y \rangle_{Y',Y},$$

where $y' \in Y'$ and $\alpha \in \mathbb{R}$.

Proposition 40 *Let Y be a normed space and let $f : Y \to (-\infty, \infty]$ be a convex and lower semicontinuous function. Then*

(i) there exists an affine continuous function g such that $g \le f$;

(ii) $f(y) = \sup\{g(y) : g$ affine continuous, $g \le f\}$.

Remark 41 *If f takes the value $-\infty$, then (i) fails, and thus the right-hand side in (ii) is identically $-\infty$, while there exist functions $f : Y \to \{-\infty, \infty\}$ that are convex and lower semicontinuous with $f \not\equiv -\infty$. As an example let $Y = \mathbb{R}$ and define*

$$f(z) := \begin{cases} \infty & \text{if } z > 0, \\ -\infty & \text{if } z \le 0. \end{cases}$$

Corollary 42 (Jensen's Inequality) *Let Y be a Banach space and let $f : Y \to (-\infty, \infty]$ be a convex, lower semicontinuous function. Given a probability measure μ on a measurable space $(X, \mathfrak{M})$ and a function $g \in L^1(X; Y)$, then*

$$f\left(\int_X g \, d\mu\right) \le \int_X f \circ g \, d\mu. \tag{32}$$

Corollary 43 *Let Y be a normed space and let $f : Y \to (-\infty, \infty]$ be a convex and lower semicontinuous function. Let $\{y_n\} \subset Y$ be such that $y_n \rightharpoonup y$ in Y. Then*

$$\liminf_{n\to\infty} f(y_n) \ge f(y).$$

Proof. Let g be an affine continuous function with, $g \leq f$. Then g has the form

$$g(y) = \alpha + \langle y', y \rangle_{Y',Y},$$

where $y' \in Y'$ and $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} f(y_n) &\geq \liminf_{n\to\infty} g(y_n) = \liminf_{n\to\infty} \left(\alpha + \langle y', y_n \rangle_{Y',Y}\right) \\ &= \alpha + \langle y', y \rangle_{Y',Y} = g(y).\end{aligned}$$

Taking the supremum over all such g gives the desired result. ■

Theorem 44 *Let Y be a reflexive Banach space and let $f : Y \to (-\infty, \infty]$ be convex, lower semicontinuous, and such that*

$$\lim_{\|y\|_Y \to\infty} f(y) = \infty. \tag{33}$$

Then f admits a minimizer. Moreover, if f is not identically ∞ and is strictly convex in $\mathrm{dom}_e f$*, then the minimizer is unique.*

Proof. If f is identically ∞, then every point $y \in Y$ is a minimizer. Thus, assume that f is not identically ∞ and let

$$m := \inf_{y\in Y} f(y).$$

Note that $m \in [-\infty, \infty)$, f is not identically ∞. Using the definition of infimum consider a sequence $\{y_k\} \subset Y$ such that

$$m \leq f(y_k) \leq m + \frac{1}{k}.$$

Then

$$\lim_{k\to\infty} f(y_k) = m.$$

It follows from (33) and the fact that $f(y_k) \leq m + 1$ for all k, that $\{y_k\}$ is bounded in Y. Since Y is reflexive, up to a subsequence, not relabelled, there exists $y_0 \in Y$ such that $y_k \rightharpoonup y_0$ in Y. We claim that $f(y_0) = m$. By Corollary 43,

$$m = \liminf_{k\to\infty} f(y_k) \geq f(y_0) \geq m.$$

Note that this implies that $m \in \mathbb{R}$.

Assume next that f is strictly convex and let y_1, $y_2 \in \mathrm{dom}_e f$ be two minimizers with $y_1 \neq y_2$. Then

$$m \leq f(\theta y_1 + (1-\theta) y_2) < \theta f(y_1) + (1-\theta) f(y_2) = \theta m + (1-\theta) m = m$$

for all $\theta \in (0, 1)$, which is a contradiction. ■

Corollary 45 *Let Y be a reflexive Banach space and let $f : Y \to (-\infty, \infty]$ be convex, lower semicontinuous. For $x \in Y$ and $\lambda > 0$ consider the function*

$$f_{\lambda,x}(y) := f(y) + \frac{\lambda}{2} \|y - x\|_Y^2 . \tag{34}$$

Then $f_{\lambda,x}$ admits a minimizer.

Proof. Since f is convex and lower semicontinuous, by Proposition 40, there exist $y' \in Y'$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \alpha + \langle y', y \rangle_{Y',Y}$$

for all $y \in Y$. Write

$$\begin{aligned} \langle y', y \rangle_{Y',Y} &= \langle y', y - x \rangle_{Y',Y} + \langle y', x \rangle_{Y',Y} \\ &\geq - \|y'\|_{Y'} \|y - x\|_Y + \langle y', x \rangle_{Y',Y} \\ &\geq -\frac{\lambda}{4} \|y - x\|_Y^2 - \frac{1}{\lambda} \|y'\|_{Y'}^2 + \langle y', x \rangle_{Y',Y}, \end{aligned}$$

where we have used the inequality $ab = \sqrt{\frac{\lambda}{2}} a \frac{b}{\sqrt{\frac{\lambda}{2}}} \leq \frac{\lambda}{4} a^2 + \frac{1}{\lambda} b^2$. Hence,

$$\begin{aligned} f_{\lambda,x}(y) &\geq \alpha + \langle y', y \rangle_{Y',Y} + \frac{\lambda}{2} \|y - x\|_Y^2 \\ &\geq \frac{\lambda}{4} \|y - x\|_Y^2 - \frac{1}{\lambda} \|y'\|_{Y'}^2 + \langle y', x \rangle_{Y',Y}, \end{aligned}$$

which shows that $f_{\lambda,x}(y) \to \infty$ as $\|y\|_Y \to \infty$.

Since the function $f_{\lambda,x}$ is convex (sum of two convex functions) and lower semicontinuous (sum of a lower semicontinuous function and a continuous function), we can apply the previous theorem to conclude that $f_{\lambda,x}$ admits a minimizer. ■

Friday, January 31, 2014

Definition 46 *Let Y be a normed space, let $f : Y \to [-\infty, \infty]$, and let $y_0 \in Y$ be such that $f(y_0) \in \mathbb{R}$. The function f is said to be* subdifferentiable *at y_0 if there exists $y' \in Y'$ such that*

$$f(y) \geq f(y_0) + \langle y', y - y_0 \rangle_{Y',Y} \quad \textit{for all } y \in Y.$$

The element y' is called a subgradient *of f at y_0, and the set of all subgradients at y_0 is called the* subdifferential *of f at y_0 and is denoted by $\partial f(y_0)$. Precisely,*

$$\partial f(y_0) = \{y' \in Y : f(y) \geq f(y_0) + \langle y', y - y_0 \rangle_{Y',Y} \quad \textit{for all } y \in Y\} .$$

If f is not subdifferentiable at y_0, then $\partial f(y_0) := \emptyset$.

Remark 47 *The set* $\partial f(y_0)$ *is convex and is closed with respect to weak star convergence in* Y'. *To see the latter property, note that if* $\{y_n'\} \subseteq \partial f(y_0)'$ *and* $y_n' \overset{*}{\rightharpoonup} y'$, *then*

$$f(y) \geq f(y_0) + \langle y_n', y - y_0\rangle_{Y',Y} \quad \text{for all } y \in Y.$$

Letting $n \to \infty$ *and using the fact that* $y_n' \overset{*}{\rightharpoonup} y'$, *we get that*

$$f(y) \geq f(y_0) + \langle y', y - y_0\rangle_{Y',Y} \quad \text{for all } y \in Y,$$

and so $y' \in \partial f(y_0)$.

Example 48 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set and take* H *be a Hilbert space such that*

$$H^1(\Omega) \hookrightarrow H,$$

for example, $H = L^2(\Omega)$ *or* $H = \left(H^1(\Omega)\right)'$ *(in this case, we are using* (15)*). Define the function* $\Psi : H \to [0,\infty]$ *by*

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in H \setminus H^1(\Omega). \end{cases}$$

Then Ψ *is convex and lower semicontinuous. If* Ψ *is subdifferentiable at some* $v_0 \in H$, *then* $\Psi(v_0) < \infty$ *and so* $v_0 \in H^1(\Omega)$. *Let* $g \in \partial\Psi(v_0)$. *Then*

$$\Psi(v) \geq \Psi(v_0) + (g, v - v_0)_H$$

for all $v \in H$. *Taking* $v = v_0 \pm tw$, *where* $w \in H^1(\Omega)$, *we get*

$$\frac{1}{2}\int_\Omega \|\nabla v_0 \pm t\nabla w\|^2 \, d\boldsymbol{x} \geq \frac{1}{2}\int_\Omega \|\nabla v_0\|^2 \, d\boldsymbol{x} \pm t(g,w)_H$$

that is

$$\frac{1}{2}t^2\int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} \pm t\int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} \geq \pm t(g,w)_H.$$

Dividing by $t > 0$ *and letting* $t \to 0^+$, *we have*

$$\pm\int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} \geq \pm(g,w)_H$$

for all $w \in H^1(\Omega)$, *that is,*

$$\int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} = (g,w)_H$$

for all $w \in H^1(\Omega)$. *This shows that* v_0 *is a weak solution of* $-\Delta v_0 = g \in H$ *in* Ω.

We now study the existence of the subdifferential.

Theorem 49 *Let Y be a normed space and let $f : Y \to [-\infty, \infty]$ be a convex function. If there exists $y_0 \in Y$ such that $f(y_0) \in \mathbb{R}$ and f is continuous at y_0, then $\partial f(y) \neq \emptyset$ for all y in the interior of* $\mathrm{dom}_e f$. *In particular, $\partial f(y_0) \neq \emptyset$.*

Theorem 50 (Monotonicity of the subdifferential) *Let Y be a normed space and let $f : Y \to [-\infty, \infty]$ be a convex function. Let*

$$D_f := \{y \in Y : \partial f(y) \neq \emptyset\}.$$

Then for all $y, x \in D_f$ and all $y' \in \partial f(y)$ and $x' \in \partial f(x)$,

$$\langle y' - x', y - x \rangle_{Y',Y} \geq 0.$$

Proof. By definition of subdifferentiability,

$$\begin{aligned} f(y) &\geq f(x) + \langle x', y - x \rangle_{Y',Y}, \\ f(x) &\geq f(y) - \langle y', y - x \rangle_{Y',Y}, \end{aligned}$$

and by adding these inequalities we obtain that

$$0 \geq \langle x' - y', y - x \rangle_{Y',Y},$$

which gives (iii). ■

Next we study the subdifferentiability of the sum of two convex functions.

Proposition 51 (Subdifferential of the Sum) *Let Y be a normed space and let f_1, $f_2 : Y \to (-\infty, \infty]$ be two proper convex, lower semicontinuous functions. Assume that there exists*

$$y_0 \in \mathrm{dom}_e f_1 \cap \mathrm{dom}_e f_2$$

such that f_1 is continuous at y_0. Then for every $y \in Y$,

$$\partial (f_1 + f_2)(y) = \partial f_1(y) + \partial f_2(y).$$

We now turn to the relation between subdifferentiability and (Gâteaux) differentiability.

Definition 52 *Let Y be a locally convex topological vector space. A function $f : Y \to [-\infty, \infty]$ is* Gâteaux differentiable *at $y_0 \in Y$ if $f(y_0) \in \mathbb{R}$ and there exists $y' \in Y'$ such that for every $y \in Y$,*

$$\lim_{t \to 0^+} \frac{f(y_0 + ty) - f(y_0)}{t} = \langle y', y \rangle_{Y',Y},$$

The element y' is called the Gâteaux differential *of f at y_0 and is denoted by $\nabla f(y_0)$.*

Theorem 53 *Let Y be a normed space and let $f : Y \to [-\infty, \infty]$ be a convex function. If f is Gâteaux differentiable at $y_0 \in Y$, then it is subdifferentiable at y_0 and $\partial f(y_0) = \{\nabla f(y_0)\}$. Conversely, if f is continuous and finite at $y_0 \in Y$ and the subdifferential of f at y_0 is a singleton, then f is Gâteaux differentiable at y_0.*

5 Moreau–Yosida Approximations

We begin with some auxiliary results.

Proposition 54 *Let H be a Hilbert space, let $f : H \to (-\infty, \infty]$ be convex, lower semicontinuous, not identically ∞, and let $x_0, z_0 \in H$. Consider the function*

$$g_\varepsilon(y) := f(y) - (y, z_0)_H + \frac{1}{2\varepsilon} \|y - x_0\|_H^2, \quad y \in H, \tag{35}$$

where $\varepsilon > 0$. Then g_ε admits a unique minimizer.

Proof. Existence follows from Corollary 45. It remains to prove uniqueness. If $x, y \in \operatorname{dom}_e f$ are two minimizers, by the convexity of f,

$$\begin{aligned} g_\varepsilon(x) + g_\varepsilon(y) - 2g_\varepsilon\left(\frac{x+y}{2}\right) &= f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \\ &\quad + \frac{1}{2\varepsilon}\|x - x_0\|_H^2 + \frac{1}{2\varepsilon}\|y - x_0\|_H^2 - \frac{1}{4\varepsilon}\|x - x_0 + y - x_0\|_H^2 \\ &= f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) + \frac{1}{4\varepsilon}\|x - y\|_H^2 \\ &\geq \frac{1}{4\varepsilon}\|x - y\|_H^2. \end{aligned}$$

If $g_\varepsilon(x) = g_\varepsilon(y) = m$, then since $m \leq g_\varepsilon\left(\frac{x+y}{2}\right) < \infty$, we get

$$0 = m + m - 2m \geq g_\varepsilon(x) + g_\varepsilon(y) - 2g_\varepsilon\left(\frac{x+y}{2}\right) \geq \frac{1}{4\varepsilon}\|x - y\|_H^2,$$

which implies that $x = y$. ∎

Proposition 55 *Let H be a Hilbert space, let $f : H \to [0, \infty]$ be convex, lower semicontinuous, not identically ∞, and let $x_0, y_0, z_0 \in H$. Consider the function g_ε in (35), where $\varepsilon > 0$. Then the following two statements are equivalent:*

(ii) The function g_ε admits a minimum at y_0;

(ii) $z_0 + \frac{1}{\varepsilon}(x_0 - y_0) \in \partial f(y_0)$.

Proof. Assume that $z_0 + \frac{1}{\varepsilon}(x_0 - y_0) \in \partial f(y_0)$. Then

$$\begin{aligned} f(y) - f(y_0) &\geq \left(z_0 + \frac{1}{\varepsilon}(x_0 - y_0), y - y_0\right)_H \\ &= (z_0, y - y_0)_H + \frac{1}{\varepsilon}(x_0 - y_0, y - x_0 + x_0 - y_0)_H \\ &= (z_0, y - y_0)_H + \frac{1}{\varepsilon}(x_0 - y_0, y - x_0)_H + \frac{1}{\varepsilon}\|x_0 - y_0\|_H^2 \\ &\geq (z_0, y - y_0)_H - \frac{1}{2\varepsilon}\|x_0 - y_0\|_H^2 - \frac{1}{2\varepsilon}\|y - x_0\|_H^2 + \frac{1}{\varepsilon}\|x_0 - y_0\|_H^2 \\ &= (z_0, y - y_0)_H + \frac{1}{2\varepsilon}\left(\|x_0 - y_0\|_H^2 - \|y - x_0\|_H^2\right), \end{aligned}$$

which shows that $g_\varepsilon(y) \geq g_\varepsilon(y_0)$ for every $y \in X$. ■

Monday, February 3, 2014

Proof. Conversely, assume that $g_\varepsilon(y_0) = \min_{y\in H} g_\varepsilon(y)$. Take $y = (1-t)y_0 + tz$ with $t \in (0,1)$. Then by convexity

$$(1-t) f(y_0) + t f(z) \geq f((1-t)y_0 + tz) = f(y)$$

and so

$$\begin{aligned} t(f(z) - f(y_0)) &\geq f((1-t)y_0 + tz) - f(y_0) \\ &= g_\varepsilon((1-t)y_0 + tz) - g_\varepsilon(y_0) + (z_0, (1-t)y_0 + tz)_H \\ &\quad - (z_0, y_0)_H - \frac{1}{2\varepsilon}\|(1-t)y_0 + tz - x_0\|_H^2 + \frac{1}{2\varepsilon}\|y_0 - x_0\|_H^2 \\ &\geq t(z_0, z - y_0)_H - \frac{1}{2\varepsilon}\|t(z-y_0) + y_0 - x_0\|_H^2 + \frac{1}{2\varepsilon}\|y_0 - x_0\|_H^2 \\ &= t(z_0, z-y_0)_H - \frac{1}{2\varepsilon}t^2\|z-y_0\|_H^2 + \frac{1}{\varepsilon}t(x_0 - y_0, z - y_0)_H . \end{aligned}$$

Dividing by t and letting $t \to 0^+$ we get

$$f(z) - f(y_0) \geq (z_0, z - y_0)_H + \frac{1}{\varepsilon}(x_0 - y_0, z - y_0)_H$$

for all $z \in H$, which shows that $z_0 + \frac{1}{\varepsilon}(x_0 - y_0) \in \partial f(y_0)$. ■

The following theorem allows to approximate from below convex functions with differentiable convex functions.

Theorem 56 (Moreau–Yosida) *Let H be a Hilbert space, let $f : H \to (-\infty, \infty]$ be convex, lower semicontinuous, not identically ∞. For every $\varepsilon > 0$ consider the function*

$$f_\varepsilon(x) := \inf_{y\in H}\left\{f(y) + \frac{1}{2\varepsilon}\|x-y\|_H^2\right\}, \quad x \in H.$$

Then $f_\varepsilon : H \to \mathbb{R}$ is convex, Frechet differentiable, ∇f_ε is $\frac{1}{\varepsilon}$-Lipschitz, and $f_\varepsilon \nearrow f$ as $\varepsilon \to 0^+$. Moreover, if $x \in H$ is such that $\partial f(x)$ is nonempty, then

$$\|\nabla f_\varepsilon(x)\|_H \leq \|z\|_H$$

for all $z \in \partial f(x)$.

Proof. Step 1: Fix $x \in H$ and consider the function

$$g_{\varepsilon,x}(y) := f(y) + \frac{1}{2\varepsilon}\|x-y\|_H^2, \quad y \in H.$$

By Proposition 40, there exist $z_0 \in H$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \alpha + (z_0, y)_H$$

for all $y \in H$. Hence,

$$g_{\varepsilon,x}(y) \geq \alpha + (z_0, y)_H + \frac{1}{2\varepsilon}\|x-y\|_H^2 \tag{36}$$

for all $y \in H$. By Propositions 54 and 55, $g_{\varepsilon,x}$ admits a unique minimizer $y_{\varepsilon,x} \in H$ and

$$z_{\varepsilon,x} := \frac{1}{\varepsilon}(x - y_{\varepsilon,x}) \in \partial f(y_{\varepsilon,x}). \tag{37}$$

In particular, we have that

$$f_\varepsilon(x) = \min_{y\in H} g_{\varepsilon,x}(y) = g_{\varepsilon,x}(y_{\varepsilon,x}) = f(y_{\varepsilon,x}) + \frac{\varepsilon}{2}\|z_{\varepsilon,x}\|_H^2. \tag{38}$$

Step 2: We claim that the mapping $x \in H \mapsto z_{\varepsilon,x}$ is Lipschitz, with Lipschitz constant less than or equal $\frac{1}{\varepsilon}$. Let $x_1, x_2 \in H$. Since $z_{\varepsilon,x_i} \in \partial f(y_{\varepsilon,x_i})$, $i = 1, 2$, by the monotonicity of ∂f,

$$0 \leq (z_{\varepsilon,x_2} - z_{\varepsilon,x_1}, y_{\varepsilon,x_2} - y_{\varepsilon,x_1})_H.$$

Multiplying the previous inequality by $\frac{1}{\varepsilon}$, we get,

$$\begin{aligned}
0 &\leq \left(z_{\varepsilon,x_2} - z_{\varepsilon,x_1}, \frac{1}{\varepsilon}y_{\varepsilon,x_2} - \frac{1}{\varepsilon}y_{\varepsilon,x_1}\right)_H \\
&= \left(z_{\varepsilon,x_2} - z_{\varepsilon,x_1}, \frac{1}{\varepsilon}(x_2 - x_1) - \frac{1}{\varepsilon}(x_2 - y_{\varepsilon,x_2}) + \frac{1}{\varepsilon}(x_1 - y_{\varepsilon,x_1})\right)_H \\
&= -\|z_{\varepsilon,x_2} - z_{\varepsilon,x_1}\|_H^2 + \frac{1}{\varepsilon}(z_{\varepsilon,x_2} - z_{\varepsilon,x_1}, x_2 - x_1)_H,
\end{aligned}$$

and so

$$\begin{aligned}
\|z_{\varepsilon,x_2} - z_{\varepsilon,x_1}\|_H^2 &\leq \frac{1}{\varepsilon}(z_{\varepsilon,x_2} - z_{\varepsilon,x_1}, x_2 - x_1)_H \\
&\leq \frac{1}{\varepsilon}\|z_{\varepsilon,x_2} - z_{\varepsilon,x_1}\|_H \|x_2 - x_1\|_H,
\end{aligned}$$

which gives

$$\|z_{\varepsilon,x_2} - z_{\varepsilon,x_1}\|_H \leq \frac{1}{\varepsilon}\|x_2 - x_1\|_H.$$

Step 3: We will show that f_ε is differentiable. If x, $x_1 \in H$, then $z_{\varepsilon,x_1} \in \partial f(y_{\varepsilon,x_1})$, and so

$$f(y_{\varepsilon,x}) - f(y_{\varepsilon,x_1}) \geq (z_{\varepsilon,x_1}, y_{\varepsilon,x} - y_{\varepsilon,x_1})_H.$$

In turn

$$\begin{aligned}
f_\varepsilon(x) - f_\varepsilon(x_1) &= \frac{\varepsilon}{2}\|z_{\varepsilon,x}\|_H^2 - \frac{\varepsilon}{2}\|z_{\varepsilon,x_1}\|_H^2 + f(y_{\varepsilon,x}) - f(y_{\varepsilon,x_1}) \\
&\geq \frac{\varepsilon}{2}\|z_{\varepsilon,x}\|_H^2 - \frac{\varepsilon}{2}\|z_{\varepsilon,x_1}\|_H^2 + (z_{\varepsilon,x_1}, y_{\varepsilon,x} - y_{\varepsilon,x_1})_H \\
&= \frac{\varepsilon}{2}\left[\|z_{\varepsilon,x}\|_H^2 - \|z_{\varepsilon,x_1}\|_H^2 + 2\left(z_{\varepsilon,x_1}, \frac{y_{\varepsilon,x} - x + x - y_{\varepsilon,x_1} - x_1 + x_1}{\varepsilon}\right)_H\right] \\
&= \frac{\varepsilon}{2}\left[\|z_{\varepsilon,x}\|_H^2 - \|z_{\varepsilon,x_1}\|_H^2 - 2(z_{\varepsilon,x_1}, z_{\varepsilon,x} - z_{\varepsilon,x_1})_H\right] + (z_{\varepsilon,x_1}, x - x_1)_H,
\end{aligned}$$

and so,

$$f_\varepsilon(x) - f_\varepsilon(x_1) - (z_{\varepsilon,x_1}, x - x_1)_H \geq \frac{\varepsilon}{2}\|z_{\varepsilon,x} - z_{\varepsilon,x_1}\|_H^2 \geq 0.$$

By interchanging x an x_1, we get

$$f_\varepsilon(x_1) - f_\varepsilon(x) - (z_{\varepsilon,x}, x_1 - x)_H \geq 0,$$

or, equivalently,

$$f_\varepsilon(x) - f_\varepsilon(x_1) - (z_{\varepsilon,x_1}, x - x_1)_H - (z_{\varepsilon,x} - z_{\varepsilon,x_1}, x - x_1)_H \leq 0.$$

Hence,

$$0 \leq f_\varepsilon(x) - f_\varepsilon(x_1) - (z_{\varepsilon,x_1}, x - x_1)_H \leq (z_{\varepsilon,x} - z_{\varepsilon,x_1}, x - x_1)_H,$$

and so,

$$\frac{\left|f_\varepsilon(x) - f_\varepsilon(x_1) - (z_{\varepsilon,x_1}, x - x_1)_H\right|}{\|x - x_1\|_H} \leq \|z_{\varepsilon,x} - z_{\varepsilon,x_1}\|_H \leq \frac{1}{\varepsilon}\|x - x_1\|_H \to 0$$

as $x \to x_1$. Thus, f_ε is differentiable and $\nabla f_\varepsilon(x_1) = z_{\varepsilon,x_1} \in \partial f(y_{\varepsilon,x_1})$. The convexity of f_ε is left as an exercise.

Step 4: We show that $f_\varepsilon \nearrow f$ as $\varepsilon \to 0^+$. If $0 < \varepsilon_1 < \varepsilon_2$, we have that $\frac{1}{2\varepsilon_1} > \frac{1}{2\varepsilon_2}$, and so for every $x \in H$,

$$\begin{aligned} f(x) \geq f_{\varepsilon_1}(x) &= \inf_{y \in H}\left\{f(y) + \frac{1}{2\varepsilon_1}\|x - y\|_H^2\right\} \\ &\geq \inf_{y \in H}\left\{f(y) + \frac{1}{2\varepsilon_2}\|x - y\|_H^2\right\} = f_{\varepsilon_2}(x). \end{aligned}$$

Thus,

$$f(x) \geq \lim_{\varepsilon \to 0^+} f_\varepsilon(x).$$

■

Wednesday, February 5, 2014

Proof. To prove the opposite inequality, it suffices to assume that

$$\ell := \lim_{\varepsilon \to 0^+} f_\varepsilon(x) < \infty.$$

Then by (38) and (36),

$$\begin{aligned} \ell \geq f_\varepsilon(x) \geq f(y_{\varepsilon,x}) + \frac{\varepsilon}{2}\|z_{\varepsilon,x}\|_H^2 &= f(y_{\varepsilon,x}) + \frac{1}{2\varepsilon}\|x - y_{\varepsilon,x}\|_H^2 \\ \geq \alpha + (z_0, y_{\varepsilon,x})_H + \frac{1}{2\varepsilon}\|x - y_{\varepsilon,x}\|_H^2, \end{aligned}$$

which implies that $y_{\varepsilon,x} \to x$ as $\varepsilon \to 0^+$. In turn, by the lower semicontinuity of f, and the fact that $f_\varepsilon(x) \geq f(y_{\varepsilon,x})$ again by (38), we have that

$$\lim_{\varepsilon \to 0^+} f_\varepsilon(x) \geq \liminf_{\varepsilon \to 0^+} f(y_{\varepsilon,x}) \geq f(x).$$

Step 5: Let $x \in H$ be such that $\partial f(x)$ is nonempty and let $z \in \partial f(x)$. Since $z_{\varepsilon,x} := \frac{1}{\varepsilon}(x - y_{\varepsilon,x}) \in \partial f(y_{\varepsilon,x})$, by the monotonicity of ∂f,

$$\begin{aligned} 0 &\leq (z_{\varepsilon,x} - z, y_{\varepsilon,x} - x)_H = -\varepsilon \left(z_{\varepsilon,x} - z, \frac{1}{\varepsilon}(x - y_{\varepsilon,x}) \right)_H \\ &= -\varepsilon (z_{\varepsilon,x} - z, z_{\varepsilon,x})_H = -\varepsilon \|z_{\varepsilon,x}\|_H^2 + \varepsilon (z, z_{\varepsilon,x})_H , \end{aligned}$$

and so

$$\begin{aligned} \|z_{\varepsilon,x}\|_H^2 &\leq (z, z_{\varepsilon,x})_H \\ &\leq \|z_{\varepsilon,x}\|_H \|z\|_H , \end{aligned}$$

which gives

$$\|\nabla f_\varepsilon(x)\|_H = \|z_{\varepsilon,x}\|_H \leq \|z\|_H ,$$

which concludes the proof. ■

6 Minimizing Movements in Hilbert Spaces

In this section we study the existence of solutions of the following evolution problem

$$\begin{cases} f(t) - \frac{du}{dt}(t) \in \partial \Psi(u(t)), \\ u(0) = u_0, \end{cases} \tag{39}$$

where $u : [0,T] \to H$, with H a Hilbert space, $\Psi : H \to (-\infty, \infty]$ is a convex function, $u_0 \in \operatorname{dom}_e \Psi$, and $f : [0,T] \to H$.

Definition 57 *Let H be a Hilbert space, let $\Psi : H \to (-\infty, \infty]$ be, let $f : [0,T] \to H$, and let $u_0 \in H$. A* strong solution *of the Cauchy problem* (39) *is a function $u \in C([0,T]; H)$ such that u is absolutely continuous on compact sets of $(0,T)$, $u(t) \in \operatorname{dom}_e \Psi$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$ with $f(t) - \frac{du}{dt}(t) \in \partial \Psi(u(t))$, and $u(0) = u_0$.*

Theorem 58 *Let H be a Hilbert space, let $\Psi : H \to [0, \infty]$ be convex, lower semicontinuous, not identically ∞, let $f \in L^2((0,T); H)$ and let $u_0 \in \operatorname{dom}_e \Psi$. Then the Cauchy problem* (39) *admits a unique strong solution $u \in H^1((0,T); H)$. Moreover,*

$$\int_0^T \left\| \frac{du}{dt}(t) \right\|_H^2 \, dt \leq M, \tag{40}$$

$$\int_0^T \Psi(u(t)) \, dt \leq T\Psi(u_0) + \frac{4}{3} M T^{3/2},$$

where

$$M := 2\Psi(u_0) + \int_0^T \|f(t)\|_H^2 \, dt$$

Proof. Step 1: For $\ell \in \mathbb{N}$ set $\tau := \frac{T}{\ell}$ and subdivide $(0, T)$ into ℓ intervals of length τ,

$$\tau_0 := 0 < t_1 < \cdots < \tau_\ell = T,$$

where $\tau_n := n\tau$, $n = 1, \ldots, \ell$. For every $n = 1, \ldots, \ell$, let $u_n \in H$ be a solution of the minimization problem

$$\min_{v \in H} \Phi_n(v), \tag{41}$$

where

$$\Phi_n(v) := \Psi(v) - (f_n, v)_H + \frac{1}{2\tau} \|v - u_{n-1}\|_H^2,$$

with

$$f_n := \frac{1}{\tau} \int_{\tau_{n-1}}^{\tau_n} f(t)\ dt.$$

Note that u_n, Φ_n and f_n all depend on ℓ. By Proposition 54, u_n exists, and by Proposition 55,

$$f_n - \frac{u_n - u_{n-1}}{\tau} \in \partial\Psi(u_n). \tag{42}$$

■

Friday, February 7, 2014

Proof. Step 2: A priori bounds and convergence. For $t \in (\tau_{n-1}, \tau_n]$, $n = 1, \ldots, \ell$, define

$$u_\tau(t) := u_n + (t - \tau_n) \frac{u_n - u_{n-1}}{\tau}, \tag{43}$$
$$f_\tau(t) := f_n.$$

Then (42) reads as

$$f_\tau(t) - \frac{du_\tau}{dt}(t) \in \partial\Psi\left(u_\tau\left(\left\lceil \frac{t}{\tau} \right\rceil \tau\right)\right), \tag{44}$$

where $\lceil s \rceil$ is the ceiling of $s \in \mathbb{R}$.

The goal of this step is to obtain a priori bounds on u_τ. By (42),

$$\Psi(v) - \Psi(u_n) \geq (f_n, v - u_n)_H - \frac{1}{\tau}(u_n - u_{n-1}, v - u_n)_H$$

for all $v \in H$. Taking $v = u_{n-1}$ in the previous inequality, we get

$$\begin{aligned}
\tau \left\| \frac{u_n - u_{n-1}}{\tau} \right\|_H^2 &\leq \Psi(u_{n-1}) - \Psi(u_n) + (f_n, u_n - u_{n-1})_H \\
&= \Psi(u_{n-1}) - \Psi(u_n) + \tau \left(f_n, \frac{u_n - u_{n-1}}{\tau}\right)_H \\
&\leq \Psi(u_{n-1}) - \Psi(u_n) + \frac{\tau}{2} \left\| \frac{u_n - u_{n-1}}{\tau} \right\|_H^2 + \frac{\tau}{2} \|f_n\|_H^2.
\end{aligned}$$

It follows that for $t \in (\tau_{n-1}, \tau_n)$,

$$\frac{1}{2}\int_{\tau_{n-1}}^{\tau_n} \left\| \frac{du_\tau}{dt}(t) \right\|_H^2 \, dt = \frac{\tau}{2} \left\| \frac{u_n - u_{n-1}}{\tau} \right\|_H^2 \le \Psi(u_{n-1}) - \Psi(u_n) + \frac{\tau}{2} \|f_n\|_H^2 .$$

Summing from $n = 1, \ldots, \ell$, we get

$$\begin{aligned} \frac{1}{2}\int_0^T \left\| \frac{du_\tau}{dt}(t) \right\|_H^2 \, dt &= \frac{1}{2}\sum_{n=1}^{\ell} \int_{\tau_{n-1}}^{\tau_n} \left\| \frac{du_\tau}{dt}(t) \right\|_H^2 \, dt \\ &\le \sum_{n=1}^{\ell} \left(\Psi(u_{n-1}) - \Psi(u_n)\right) + \frac{\tau}{2}\sum_{n=1}^{\ell} \|f_n\|_H^2 \\ &\le \Psi(u_0) - \Psi(u_\ell) + \frac{\tau}{2}\sum_{n=1}^{\ell} \|f_n\|_H^2 \\ &\le \Psi(u_0) + \frac{\tau}{2}\sum_{n=1}^{\ell} \|f_n\|_H^2 \end{aligned}$$

Now, by Hölder's inequality,

$$\begin{aligned} \tau \sum_{n=1}^{\ell} \|f_n\|_H^2 &= \sum_{n=1}^{\ell} \frac{1}{\tau} \left\| \int_{\tau_{n-1}}^{\tau_n} f(t) \, dt \right\|_H^2 \\ &\le \sum_{n=1}^{\ell} \int_{\tau_{n-1}}^{\tau_n} \|f(t)\|_H^2 \, dt = \int_0^T \|f(t)\|_H^2 \, dt < \infty. \end{aligned}$$

It follows that

$$\int_0^T \left\| \frac{du_\tau}{dt}(t) \right\|_H^2 \, dt \le 2\Psi(u_0) + \int_0^T \|f(t)\|_H^2 \, dt =: M. \tag{45}$$

In turn, for every $0 \le t_1 < t_2 \le T$, since u_τ is absolutely continuous,

$$\begin{aligned} \|u_\tau(t_2) - u_\tau(t_1)\|_H &= \left\| \int_{t_1}^{t_2} \frac{du_\tau}{dt}(t) \, dt \right\|_H \\ &\le (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} \left\| \frac{du_\tau}{dt}(t) \right\|_H^2 \, dt \right)^{1/2} \\ &\le M^{1/2} (t_2 - t_1)^{1/2} . \end{aligned} \tag{46}$$

Taking $t_1 = 0$ and using the fact that $u_\tau(0) = u_0$ gives

$$\|u_\tau(t) - u_0\|_H \le M^{1/2} t^{1/2}. \tag{47}$$

Moreover for $0 < \eta < \tau$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u_\tau(t)-u_\eta(t)\right\|_H^2 &= \left(\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t), u_\tau(t)-u_\eta(t)\right)_H \\ &= -\left(f_\tau(t)-\frac{du_\tau}{dt}(t)-\left(f_\eta(t)-\frac{du_\eta}{dt}(t)\right), u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right)_H \\ &\quad+\left(f_\tau(t)-f_\eta(t), u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right)_H \\ &\quad+\left(\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t), u_\tau(t)-u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)\right)_H \\ &\quad+\left(\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t), -u_\eta(t)+u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right)_H .\end{aligned}$$

Using the monotonicity of the subdifferential of Ψ and (44) we have that the first term on the right-hand side is nonpositive. Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u_\tau(t)-u_\eta(t)\right\|_H^2 &\le \left(f_\tau(t)-f_\eta(t), u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right)_H \\ &\quad+\left\|\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t)\right\|_H\left\|u_\tau(t)-u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)\right\|_H \\ &\quad+\left\|\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t)\right\|_H\left\|u_\eta(t)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right\|_H \\ &\le \left\|f_\tau(t)-f_\eta(t)\right\|_H\left\|u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right\|_H \\ &\quad+\tau\left\|\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t)\right\|_H\left\|\frac{du_\tau}{dt}(t)\right\|_H \\ &\quad+\eta\left\|\frac{du_\tau}{dt}(t)-\frac{du_\eta}{dt}(t)\right\|_H\left\|\frac{du_\eta}{dt}(t)\right\|_H\end{aligned}$$

where in the last inequality we have used (43). Reasoning as in (47),

$$\begin{aligned}\left\|u_\tau\left(\left\lceil\frac{t}{\tau}\right\rceil\tau\right)-u_\eta\left(\left\lceil\frac{t}{\eta}\right\rceil\tau\right)\right\|_H &= \left\|\int_0^{\left\lceil\frac{t}{\tau}\right\rceil\tau}\frac{du_\tau}{dt}(t)\ dt-\int_0^{\left\lceil\frac{t}{\eta}\right\rceil\tau}\frac{du_\eta}{dt}(t)\ dt\right\|_H \\ &\le 2M^{1/2}T^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u_\tau(t)-u_\eta(t)\right\|_H^2 &\le 2M^{1/2}T^{1/2}\left\|f_\tau(t)-f_\eta(t)\right\|_H \\ &\quad+4\tau\left\|\frac{du_\tau}{dt}(t)\right\|_H^2+4\tau\left\|\frac{du_\eta}{dt}(t)\right\|_H^2 .\end{aligned}$$

By integrating in time and using Theorem 28 and (45) we get

$$\begin{aligned}\frac{1}{2}\left\|u_\tau(t)-u_\eta(t)\right\|_H^2 &\leq 2M^{1/2}T^{1/2}\int_0^t \left\|f_\tau(s)-f_\eta(s)\right\|_H\, ds \\ &\quad + 4\tau\int_0^t \left\|\frac{du_\tau}{dt}(t)\right\|_H^2 ds + 4\tau\int_0^t \left\|\frac{du_\eta}{dt}(t)\right\|_H^2 ds \\ &\leq 2M^{1/2}T^{1/2}\int_0^T \left\|f_\tau(s)-f_\eta(s)\right\|_H\, ds + 8\tau M.\end{aligned}$$

■

Monday, February 10, 2014

Proof. Since $f_\tau \to f$ in $L^2((0,T);H)$ (by Lebesgue points and the Lebesgue dominated convergence theorem), it follows that $\{u_t\}$ is a Cauchy sequence in the space $C([0,T];H)$. Hence, it converges uniformly to a function $u \in C([0,T];H)$. Moreover, by (45), $\left\{\frac{du_\tau}{dt}\right\}$ is bounded in $L^2((0,T);H)$. Since $L^2((0,T);H)$ is reflexive by Corollary 18, up to a subsequence, $\frac{du_\tau}{dt} \rightharpoonup v$ in $L^2((0,T);H)$. It follows that there exists the weak derivative of u and it is given by v. Hence, $u \in H^1((0,T);H) \cap C([0,T];H)$.

Step 3: Strong solution. We claim that

$$f(t) - \frac{du}{dt}(t) \in \partial\Psi(u(t))$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$. To see this, for $t \in (\tau_{n-1},\tau_n]$, $n = 1, 1, \ldots, \ell$, define the piecewise constant function

$$\tilde{u}_\tau(t) := u_n. \tag{48}$$

We claim that $\tilde{u}_\tau \to u$ in $L^\infty((0,T);H)$. Given $t \in (0,T]$ find n such that $t \in (\tau_{n-1},\tau_n]$. Then

$$\tilde{u}_\tau(t) - u_\tau(t) = u_n - u_\tau(t) = u_\tau(\tau_n) - u_\tau(t),$$

and so by (46),

$$\left\|\tilde{u}_\tau(t) - u_\tau(t)\right\|_H = \left\|u_\tau(\tau_n) - u_\tau(t)\right\|_H \leq M^{1/2}\tau^{1/2}.$$

Since $\{u_t\}$ converges uniformly to u in $C([0,T];H)$, it follows that $\{\tilde{u}_\tau\}$ converges uniformly to u in $L^\infty([0,T];H)$.

By (44) we have that

$$f_\tau(t) - \frac{du_\tau}{dt}(t) \in \partial\Psi(\tilde{u}_\tau(t)),$$

for all but countably many $t \in (0,T)$. Hence, for every $z \in H$,

$$\Psi(z) \geq \Psi(\tilde{u}_\tau(t)) + (f_\tau(t), z - \tilde{u}_\tau(t))_H - \left(\frac{du_\tau}{dt}(t), z - \tilde{u}_\tau(t)\right)_H$$

By averaging this inequality between t_0 and t_0+h with $h>0$, we get

$$\begin{aligned}\Psi(z) \geq & \frac{1}{h}\int_{t_0}^{t_0+h} \Psi(\tilde{u}_\tau(t))\ dt + \frac{1}{h}\int_{t_0}^{t_0+h} (f_\tau(t), z-\tilde{u}_\tau(t))_H\ dt \\ & -\frac{1}{h}\int_{t_0}^{t_0+h} \left(\frac{du_\tau}{dt}(t), z-\tilde{u}_\tau(t)\right)_H\ dt.\end{aligned}$$

Letting $\tau \to 0^+$ and using Fatou's lemma and the facts that $\tilde{u}_\tau \to u$ in $L^\infty((0,T);H)$, $f_\tau \to f$ in $L^2((0,T);H)$, $\frac{du_\tau}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2((0,T);H)$ and $u_\tau \to u$ in $C((0,T);H)$, we obtain

$$\begin{aligned}\Psi(z) \geq & \frac{1}{h}\int_{t_0}^{t_0+h} \Psi(u(t))\ dt + \frac{1}{h}\int_{t_0}^{t_0+h} (f(t), z-u(t))_H\ dt \\ & -\frac{1}{h}\int_{t_0}^{t_0+h} \left(\frac{du}{dt}(t), z-u(t)\right)_H\ dt.\end{aligned}$$

Let t_0 be a Lebesgue point of u' and of f (see Theorem 23 and Theorem 28). By Jensen's inequality

$$\frac{1}{h}\int_{t_0}^{t_0+h} \Psi(u(t))\ dt \geq \Psi\left(\frac{1}{h}\int_{t_0}^{t_0+h} u(t)\ dt\right).$$

Then letting $h \to 0^+$, and using the fact that Ψ is lower semicontinuous, we get

$$\begin{aligned}\Psi(z) \geq & \Psi(u(t_0)) + (f(t_0), z-u(t_0))_H \\ & -\left(\frac{du}{dt}(t_0), z-u(t_0)\right)_H\end{aligned} \tag{49}$$

for all $z \in H$. This shows that $f(t_0) - \frac{du}{dt}(t_0) \in \partial\Psi(u(t_0))$.

In turn, by Proposition 59, $\Psi \circ u$ is absolutely continuous and

$$\Psi(u(t)) = \Psi(u_0) + \int_0^t \left(f(s), \frac{du}{dt}(s)\right)_H\ ds - \int_0^t \left\|\frac{du}{dt}(s)\right\|_H^2\ ds$$

for all $t \in [0,T]$. In turn, by Young's inequality and (45) for u,

$$\begin{aligned}\Psi(u(t)) &\leq \Psi(u_0) + \frac{1}{2}\int_0^T \left\|\frac{du}{dt}(s)\right\|_H^2\ ds + \int_0^T \|f(s)\|_H^2\ ds \\ &\leq 3\Psi(u_0) + 3\int_0^T \|f(s)\|_H^2\ ds.\end{aligned}$$

Step 4: Uniqueness. To prove uniqueness, let w be another solution. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)-w(t)\|_H^2 &= \left(\frac{du}{dt}(t) - \frac{dw}{dt}(t), u(t)-w(t)\right)_H \\ &= -\left(f(t) - \frac{du}{dt}(t) - \left(f(t) - \frac{dw}{dt}(t)\right), u(t)-w(t)\right)_H \leq 0\end{aligned}$$

by the monotonicity of the subdifferential of Ψ. Since $u(0) = w(0) = u_0$, it follows that $u = w$. ■

The proof above makes use of the following result.

Proposition 59 *Let H be a Hilbert space, let $\Psi : H \to (-\infty, \infty]$ be convex, lower semicontinuous, not identically ∞. Let $u \in H^1((0,T);H)$ be such that $u(t) \in \operatorname{dom}_e \Psi$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Assume that there exists a function $g \in L^2((0,T);H)$ such that*

$$g(t) \in \partial\Psi(u(t)) \quad \text{for } \mathcal{L}^1 \text{ a.e. } t \in (0,T).$$

Then the function $\Psi \circ u$ is absolutely continuous. Moreover for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and for every $h \in \partial\Psi(u(t))$,

$$\frac{d}{dt}(\Psi \circ u)(t) = \left(h, \frac{du}{dt}(t)\right)_H.$$

Proof. Let Ψ_ε be the Moreau–Yosida approximation of Ψ (see Theorem 56). Then $\Psi_\varepsilon \circ u$ is absolutely continuous and by the chain rule

$$\Psi_\varepsilon(u(t_2)) - \Psi_\varepsilon(u(t_1)) = \int_{t_1}^{t_2} \left(\nabla\Psi_\varepsilon(u(s)), \frac{du}{ds}(s)\right) ds$$

for all $0 \le t_1 < t_2 \le T$. Hence,

$$|\Psi_\varepsilon(u(t_2)) - \Psi_\varepsilon(u(t_1))| \le \int_{t_1}^{t_2} \|\nabla\Psi_\varepsilon(u(t))\|_H \left\|\frac{du}{ds}(t)\right\|_H ds.$$

Since $g(t) \in \partial\Psi(u(t))$, by Theorem 56 we have that $\|\nabla\Psi_\varepsilon(u(t))\|_H \le \|g(t)\|_H$, and so

$$|\Psi_\varepsilon(u(t_2)) - \Psi_\varepsilon(u(t_1))| \le \int_{t_1}^{t_2} \|g(t)\|_H \left\|\frac{du}{ds}(t)\right\|_H ds.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$|\Psi(u(t_2)) - \Psi(u(t_1))| \le \int_{t_1}^{t_2} \|g(t)\|_H \left\|\frac{du}{ds}(t)\right\|_H ds.$$

It follows by Hölder's inequality that the function $\|g\|_H \left\|\frac{du}{ds}\right\|_H$ is integrable. Hence, $\Psi \circ u$ is absolutely continuous.

To prove the second part of the proposition, let $t \in (0,t)$ be such that $\Psi \circ u$ and u are differentiable at t, $u(t) \in \operatorname{dom}_e \Psi$ and $g(t) \in \partial\Psi(u(t))$. Let $h \in \partial\Psi(u(t))$. Then for all $v \in H$,

$$\Psi(v) \ge \Psi(u(t)) + (h, v - u(t))_H.$$

Taking $v = u(t+\delta)$, we get

$$\Psi(u(t+\delta)) - \Psi(u(t)) \ge (h, u(t+\delta) - u(t))_H.$$

Dividing by $\delta > 0$ and letting $\delta \to 0^+$ gives

$$\frac{d}{dt}(\Psi \circ u)(t) \geq \left(h, \frac{du}{dt}(t)\right)_H.$$

By taking $v = u(t - \delta)$, we obtain the opposite inequality. ■

Wednesday, February 12, 2014

7 Second Order Parabolic Equations

Consider a second order partial differential equation of the form

$$F\left(\boldsymbol{x}, t, u(\boldsymbol{x}, t), \nabla u(\boldsymbol{x}, t), \nabla^2 u(\boldsymbol{x}), -\frac{\partial u}{\partial t}(\boldsymbol{x}, t)\right) = 0, \tag{50}$$

where $(\boldsymbol{x}, t) \in U \subseteq \mathbb{R}^N \times \mathbb{R}$, with U an open set and where $\nabla := \nabla_{\boldsymbol{x}}$. Observe that if we are looking for a classical solution u of class C^2 in $\boldsymbol{x}$, then by the Schwartz theorem,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\boldsymbol{x}, t) = \frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}, t)$$

for all $i, j = 1, \ldots, N$. Hence, the $N \times N$ matrix $\nabla^2 u(\boldsymbol{x})$ is symmetric. Write $F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$. By replacing F with

$$G(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) := F\left(\boldsymbol{x}, t, z, \boldsymbol{p}, \frac{\boldsymbol{A} + \boldsymbol{A}^T}{2}, s\right),$$

we are not changing the PDE (in the case of classical solutions). Hence, we can assume that

$$F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) = F\left(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}^T, s\right)$$

for all $\boldsymbol{A}$. We say the PDE (50) is *parabolic* at a point $(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$ if

$$F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A} + \boldsymbol{B}, s + h) > F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$$

for every positive definite matrix $\boldsymbol{B}$ and every $h > 0$. We say the the PDE (50) is *parabolic* if it is parabolic at all points $(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$.

Observe that if F is differentiable at $(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$ and $\partial_s F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) \neq 0$, then a sufficient condition for the PDE (50) to be parabolic at $(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$ is that the $N \times N$ matrix

$$\frac{\nabla_{\boldsymbol{A}} F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)}{\partial_s F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)}$$

is positive definite. If we assume in addition that F is of class C^1, then in a neighborhood of $(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s)$ we can use the implicit function theorem to write (50) in the form

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) = H\left(\boldsymbol{x}, t, u(\boldsymbol{x}, t), \nabla u(\boldsymbol{x}, t), \nabla^2 u(\boldsymbol{x})\right),$$

where $\nabla_{\boldsymbol{A}} H(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A})$ is positive definite.

The typical example of a parabolic equation is the *heat equation*

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,\ t \geq 0,$$

where

$$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}.$$

In this case,

$$F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) = \sum_{i=1}^{N} a_{i,i} + s + f(\boldsymbol{x}, t),$$

and so

$$\nabla_{\boldsymbol{A}} F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) = I_N, \quad \partial_s F(\boldsymbol{x}, t, z, \boldsymbol{p}, \boldsymbol{A}, s) = 1.$$

8 Heat Equation

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $T > 0$. Define $\Omega_T := \Omega \times (0, T)$. For the heat equation

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega_T,$$

one usually prescribes initial conditions, that is,

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

and boundary data, that is, either a Dirichlet boundary datum

$$u(\boldsymbol{x}, t) = u_1(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T),$$

or a Neumann boundary datum

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T).$$

8.1 Existence of Weak Solutions

In this subsection we prove existence of weak solutions of the Neumann problem. For simplicity we will consider only the case $g = 0$. Thus, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T). \end{cases} \tag{51}$$

In what follows, we will identify $L^2\left((0, T); L^2(\Omega)\right)$ with $L^2(\Omega_T)$ and so, with an abuse of notation, we will write $f(t)(\boldsymbol{x}) = f(\boldsymbol{x}, t)$ and $u(t)(\boldsymbol{x}) = u(\boldsymbol{x}, t)$.

Theorem 60 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in H^1(\Omega)$, and let $f \in L^2((0,T);L^2(\Omega))$, where $0 < T < \infty$. Then there exists a unique weak solution u of (51). Moreover, $u \in L^\infty((0,T);H^1(\Omega)) \cap L^2((0,T);H^2_{\text{loc}}(\Omega))$, $\frac{\partial u}{\partial t} \in L^2((0,T);L^2(\Omega))$,*

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t)$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$, and the following estimates hold

$$\|u(\cdot,t) - u_0\|_{L^2(\Omega)} \le t^{1/2}\left(\|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))}\right), \tag{52}$$

$$\|\nabla u(\cdot,t)\|_{L^2(\Omega)} \le 2\left(\|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))}\right), \tag{53}$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))} \le \|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))} \tag{54}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and there exists a constant $C = C(N) > 0$ such that for all $U \Subset \Omega$,

$$\left\|\nabla^2 u\right\|_{L^2((0,T);L^2(U))} \le C\left(1 + \frac{1}{\operatorname{dist}(U,\partial\Omega)}\right)\left(\|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))}\right).$$

Finally, if Ω is bounded and of class C^2, then u belongs to $L^2((0,T);H^2(\Omega))$ and satisfies the Neumann boundary condition

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 \tag{55}$$

for $\boldsymbol{x} \in \partial\Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and

$$\int_\Omega u(\boldsymbol{x},t)\,d\boldsymbol{x} = \int_\Omega u_0(\boldsymbol{x})\,d\boldsymbol{x} + \int_0^t \int_\Omega f(\boldsymbol{x},s)\,d\boldsymbol{x}ds. \tag{56}$$

Moreover,

$$\left\|\nabla^2 u\right\|_{L^2((0,T);L^2(\Omega))} \le C(N,\Omega)\left(\|\nabla u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);L^2(\Omega))}\right).$$

Proof. Step 1: We apply Theorem 58 with $H = L^2(\Omega)$ and

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2\,d\boldsymbol{x} & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega). \end{cases}$$

Then $u_0 \in \operatorname{dom}_e \Psi$ and so we are in a position to apply Theorem 58 to find a unique function $u \in H^1((0,T);L^2(\Omega))$ such that $f(\cdot,t) - \frac{\partial u}{\partial t}(\cdot,t) \in \partial\Psi(u(\cdot,t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, and $u(\cdot,0) = u_0$. By Example **??**, it follows that for $\mathcal{L}^1$ a.e. $t \in (0,T)$ the function $u(\cdot,t)$ belongs to $H^1(\Omega)$ and satisfies

$$\int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x},t)\,d\boldsymbol{x} - \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x},t)\,d\boldsymbol{x} + \int_\Omega w(\boldsymbol{x}) \frac{\partial u}{\partial t}(\boldsymbol{x},t)\,d\boldsymbol{x} = 0 \tag{57}$$

for all $w \in H^1(\Omega)$. Moreover, again by Theorem 58,

$$\int_{\Omega_T} \left(\frac{\partial u}{\partial t}\right)^2 \, dt d\boldsymbol{x} \leq \int_{\Omega} \|\nabla u_0\|^2 \, d\boldsymbol{x} + \int_{\Omega_T} f^2 \, dt d\boldsymbol{x} =: M$$

$$\max_{t\in[0,T]} \int_{\Omega} \|\nabla u(\boldsymbol{x},t)\|^2 \, d\boldsymbol{x} \leq 3\int_{\Omega} \|\nabla u_0\|^2 \, d\boldsymbol{x} + 3\int_{\Omega_T} f^2 \, dt d\boldsymbol{x}.$$

Since $u \in H^1\left((0,T); L^2(\Omega)\right)$, by the fundamental theorem of calculus

$$\begin{aligned}\int_{\Omega} \left(u(\boldsymbol{x},t_2) - u(\boldsymbol{x},t_1)\right)^2 \, d\boldsymbol{x} &= \int_{\Omega} \left(\int_{t_1}^{t_2} \frac{\partial u}{\partial t}(\boldsymbol{x},t) \, dt\right)^2 \, d\boldsymbol{x} \\ &\leq (t_2 - t_1) \int_{\Omega_T} \left(\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right)^2 \, dt d\boldsymbol{x} \\ &\leq M(t_2 - t_1).\end{aligned}$$

Taking $t_1 = 0$ and using the fact that $u_\tau(\boldsymbol{x},0) = u_0(\boldsymbol{x})$ gives

$$\int_{\Omega} \left(u(\boldsymbol{x},t) - u_0(\boldsymbol{x})\right)^2 \, d\boldsymbol{x} \leq Mt$$

for all $t \in (0,T)$. In turn,

$$\int_{\Omega} \left(u(\boldsymbol{x},t)\right)^2 \, d\boldsymbol{x} \leq 2Mt + 2\int_{\Omega} u_0^2(\boldsymbol{x}) \, d\boldsymbol{x}$$

for all $t \in (0,T)$.

Step 2. **Higher regularity.** Since $f(\cdot,t), \frac{\partial u}{\partial t}(\cdot,t) \in L^2(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, by interior regularity, we have that $u(\cdot,t) \in H^2_{\text{loc}}(\Omega)$, with

$$\left\|\nabla^2 u(\cdot,t)\right\|_{L^2(U)} \leq C\left\|f(\cdot,t) - \frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)} + \frac{C}{\operatorname{dist}(U,\partial\Omega)} \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \tag{58}$$

for all $U \Subset \Omega$ and for some $C = C(N) > 0$. Integrating in time and using (53), and (54), we have

$$\int_0^T \int_U \left\|\nabla^2 u(\boldsymbol{x},t)\right\|^2 \, d\boldsymbol{x} dt \leq \left(1 + \frac{1}{\operatorname{dist}^2(U,\partial\Omega)}\right) CM.$$

It follows that $u \in L^2\left((0,T); H^2_{\text{loc}}(\Omega)\right)$.

Hence, we can assume that $u(\cdot,t) \in H^2_{\text{loc}}(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$. In particular, if $w \in C^1_c(\Omega)$, then can integrate by parts the first integral to obtain

$$0 = \int_{\Omega} w(\boldsymbol{x}) \left(-\Delta u(\boldsymbol{x},t) - f(\boldsymbol{x},t) + \frac{\partial u}{\partial t}(\boldsymbol{x},t)\right) \, d\boldsymbol{x}.$$

By the arbitrariness of w it follows that

$$-\Delta u(\boldsymbol{x},t) - f(\boldsymbol{x},t) + \frac{\partial u}{\partial t}(\boldsymbol{x},t) = 0$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. Note that the set of measure zero depends on t. Using Fubini's theorem, we get (exercise)

$$-\Delta u(\boldsymbol{x}, t) - f(\boldsymbol{x}, t) + \frac{\partial u}{\partial t}(\boldsymbol{x}, t) = 0$$

for all $\boldsymbol{x} \in \Omega \setminus E$ and for $\mathcal{L}^1$ a.e. $t \in (0, T)$.

Step 3. **Boundary regularity.** Finally, if Ω is bounded and of class C^2, since $f(\cdot, t), \frac{\partial u}{\partial t}(\cdot, t) \in L^2(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0, T)$, then by the boundary regularity of the Neumann problem

$$\begin{cases} -\Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) - \frac{\partial u}{\partial t}(\boldsymbol{x}, t) & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0 & \text{on } \partial\Omega, \end{cases}$$

we have that $u(\cdot, t)$ belongs to $H^2(\Omega)$, with

$$\left\|\nabla^2 u(\cdot, t)\right\|_{L^2(\Omega)}^2 \leq C(N, \Omega) \left\|f(\cdot, t) - \frac{\partial u}{\partial t}(\cdot, t)\right\|_{L^2(\Omega)}^2 .$$

Integrating in time and using (54) we have

$$\int_{\Omega_T} \left\|\nabla^2 u(\boldsymbol{x}, t)\right\|^2 \, d\boldsymbol{x} dt \leq C(N, \Omega) M.$$

Define

$$\psi(t) := \int_\Omega u(\boldsymbol{x}, t) \, d\boldsymbol{x}.$$

Given $\varphi \in C_c^1(0, T)$, by the divergence theorem and the fact that $\frac{\partial u}{\partial \boldsymbol{n}}(\cdot, t) = 0$ on $\partial\Omega$ for $\mathcal{L}^1$ a.e. $t \in (0, T)$,

$$\begin{aligned} \int_0^T \varphi'(t) \psi(t) \, dt &= \int_0^T \int_\Omega \varphi'(t) u(\boldsymbol{x}, t) \, d\boldsymbol{x} dt \\ &= -\int_0^T \int_\Omega \varphi(t) \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x} dt \\ &= -\int_0^T \varphi(t) \left(\int_\Omega \Delta u(\boldsymbol{x}, t) \, d\boldsymbol{x} + \int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x} \right) dt \\ &= -\int_0^T \varphi(t) \int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x} dt, \end{aligned}$$

which shows that the weak derivative of ψ is given by

$$\psi'(t) = \int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x}.$$

Hence, by the fundamental theorem of calculus,

$$\int_\Omega u(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_\Omega u_0(\boldsymbol{x}) \, d\boldsymbol{x} + \int_0^t \int_\Omega f(\boldsymbol{x}, s) \, d\boldsymbol{x} ds.$$

■

Friday, February 14, 2014

9 Second Order Evolution Equations

In this section we study the existence of solutions of the following evolution problem

$$\begin{cases} f(t) - \frac{d^2u}{dt^2}(t) \in \partial\Psi(u(t)), \\ u(0) = u_0, \quad \frac{du}{dt}(0) = u_1, \end{cases} \tag{59}$$

where $u : [0,T] \to H$, with H a Hilbert space, $\Psi : H \to (-\infty, \infty]$ is a convex function, $u_0, u_1 \in H$, and $f : [0,T] \to H$.

Theorem 61 *Let*

$$Y \hookrightarrow H \cong H' \hookrightarrow Y'$$

be an evolution triple, with Y reflexive and the embedding $Y \hookrightarrow H$ compact, let $\Psi : H \to [0,\infty]$ be convex, lower semicontinuous, not identically ∞, and such that $\operatorname{dom}_e \Psi = Y$, *$\Psi$ is bounded on bounded sets of Y, and*

$$\Psi(v) \to \infty \text{ as } \|v\|_Y \to \infty, \tag{60}$$

let $f \in L^2((0,T);H)$, and let $u_0 \in Y$ and $u_1 \in H$. Then there exists a function $u \in H^1((0,T);H) \cap C((0,T);Y)$, with $\frac{d^2u}{dt^2} \in L^2((0,T);Y')$, that such that $u(0) = u_0$, $\frac{du}{dt}(0) = u_1$, and $f(t) - \frac{d^2u}{dt^2}(t) \in \partial\Psi(u(t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Moreover,

$$\int_0^T \left\| \frac{d^2u}{dt^2}(t) \right\|_{Y'}^2 dt \le C(\Psi, M), \quad \max_{t\in[0,T]} \left\| \frac{du}{dt}(t) \right\|_H \le M^{1/2}$$

$$\max_{t\in[0,T]} \|u(t)\|_Y \le C(\Psi, M), \quad \max_{t\in[0,T]} \Psi(u(t)) \le \frac{1}{2}M.$$

where

$$M := 2\|u_1\|_H^2 + 4\Psi(u_0) + 2T\int_0^T \|f(t)\|_H^2 \, dt.$$

Also, the function $\Psi \circ u$ is absolutely continuous and the following energy identity holds

$$\left\| \frac{du}{dt}(t) \right\|_H^2 + \Psi(u(t)) = \|u_1\|_H^2 + \Psi(u_0) + \int_0^t \left(f(s), \frac{du}{dt}(s) \right)_H ds$$

for all $t \in [0,T]$.

Proof. Step 1: For $\ell \in \mathbb{N}$ set $\tau := \frac{T}{\ell}$ and subdivide $(0,T)$ into ℓ intervals of length τ,

$$\tau_0 := 0 < t_1 < \cdots < \tau_\ell = T,$$

where $\tau_n := n\tau$, $n = 1, \ldots, \ell$. Define $u_{-1} := u_0 - \tau u_1$. For every $n = 1, \ldots, \ell$, let $u_n \in H$ be a solution of the minimization problem

$$\min_{v\in H} \Phi_n(v),$$

where

$$\Phi_n(v) := \Psi(v) - (f_n, v)_H + \frac{1}{2\tau^2}\|v - 2u_{n-1} + u_{n-2}\|_H^2,$$

with

$$f_n := \frac{1}{\tau}\int_{\tau_{n-1}}^{\tau_n} f(t)\ dt.$$

Note that u_n, Φ_n and f_n all depend on ℓ. By Proposition 54, u_n exists, and by Proposition 55,

$$f_n - \frac{u_n - 2u_{n-1} + u_{n-2}}{\tau^2} \in \partial\Psi(u_n). \tag{61}$$

Step 2: A priori bounds and convergence. For $t \in (\tau_{n-1}, \tau_n]$, $n = 1, \ldots, \ell$, define

$$\begin{aligned}
u_\tau(t) &:= u_n + (t - \tau_n)\frac{u_n - u_{n-1}}{\tau}, \\
v_\tau(t) &:= \frac{u_n - u_{n-1}}{\tau} + \frac{t - \tau_n}{\tau}\left[\frac{u_n - u_{n-1}}{\tau} - \frac{u_{n-1} - u_{n-2}}{\tau}\right] \\
f_\tau(t) &:= f_n, \\
\tilde{u}_\tau(t) &:= u_n.
\end{aligned} \tag{62}$$

Then (61) reads as

$$f_\tau(t) - \frac{dv_\tau}{dt}(t) \in \partial\Psi(\tilde{u}_\tau(t)) \tag{63}$$

for all but countably many $t \in (0, T)$.

The goal of this step is to obtain a priori bounds on u_τ. By (61),

$$\Psi(v) - \Psi(u_n) \geq (f_n, v - u_n)_H - \frac{1}{\tau^2}(u_n - u_{n-1} - (u_{n-1} - u_{n-2}), v - u_n)_H$$

for all $v \in H$. Taking $v = u_{n-1}$ in the previous inequality, we get

$$\left\|\frac{u_n - u_{n-1}}{\tau}\right\|_H^2 - \left(\frac{u_{n-1} - u_{n-2}}{\tau}, \frac{u_n - u_{n-1}}{\tau}\right)_H \leq \Psi(u_{n-1}) - \Psi(u_n) + (f_n, u_n - u_{n-1})_H.$$

Using the identity

$$\|a\|_H^2 - (a, b)_H = \frac{1}{2}\|a\|_H^2 + \frac{1}{2}\|a - b\|_H^2 - \frac{1}{2}\|b\|_H^2$$

we can rewrite the previous inequality as

$$\begin{aligned}
\frac{1}{2}\left\|\frac{u_n - u_{n-1}}{\tau}\right\|_H^2 - \frac{1}{2}\left\|\frac{u_{n-1} - u_{n-2}}{\tau}\right\|_H^2 + \frac{1}{2}\left\|\frac{u_n - u_{n-1}}{\tau} - \frac{u_{n-1} - u_{n-2}}{\tau}\right\|_H^2 \\
\leq \Psi(u_{n-1}) - \Psi(u_n) + (f_n, u_n - u_{n-1})_H.
\end{aligned}$$

Summing from $n = 1, \ldots, j$ and using the fact that $u_{-1} = u_0 - \tau u_1$, so that $\frac{u_0 - u_{-1}}{\tau} = u_1$, we obtain

$$\frac{1}{2}\left\|\frac{u_j - u_{j-1}}{\tau}\right\|_H^2 + \frac{1}{2}\sum_{n=0}^{j}\left\|\frac{u_n - u_{n-1}}{\tau} - \frac{u_{n-1} - u_{n-2}}{\tau}\right\|_H^2 + \Psi(u_j) \tag{64}$$
$$\leq \frac{1}{2}\|u_1\|_H^2 + \Psi(u_0) + \sum_{n=0}^{j}(f_n, u_n - u_{n-1})_H .$$

It follows that for $t \in (\tau_{j-1}, \tau_j)$,

$$\frac{1}{2}\left\|\frac{du_\tau}{dt}(t)\right\|_H^2 + \frac{\tau}{2}\int_0^{\tau_j}\left\|\frac{dv_\tau}{dt}(t)\right\|_H^2 \, dt + \Psi(\tilde{u}_\tau(t))$$
$$\leq \frac{1}{2}\|u_1\|_H^2 + \Psi(u_0) + \int_0^{\tau_j}\left(f_\tau(s), \frac{du_\tau}{dt}(s)\right)_H ds.$$

Now, by Hölder's inequality,

$$\left\|\frac{du_\tau}{dt}(t)\right\|_H^2 + \tau\int_0^{\tau_j}\left\|\frac{dv_\tau}{dt}(t)\right\|_H^2 \, dt + 2\Psi(\tilde{u}_\tau(t))$$
$$\leq \|u_1\|_H^2 + 2\Psi(u_0) + 2\left(\int_0^T \|f(t)\|_H^2 \, dt\right)^{1/2} \sup_{s\in[0,T]}\left\|\frac{du_\tau}{dt}(s)\right\|_H T^{1/2}$$
$$\leq \|u_1\|_H^2 + 2\Psi(u_0) + 4T\int_0^T \|f(t)\|_H^2 \, dt + \frac{1}{2}\sup_{s\in[0,T]}\left\|\frac{du_\tau}{dt}(s)\right\|_H^2$$

Since this is true for every $j = 1, \ldots, \ell$ and for every $t \in (\tau_{j-1}, \tau_j)$, we get

$$\sup_{t\in[0,T]}\left\|\frac{du_\tau}{dt}(t)\right\|_H^2 \leq \|u_1\|_H^2 + 2\Psi(u_0) + 4T\int_0^T \|f(t)\|_H^2 \, dt + \frac{1}{2}\sup_{s\in[0,T]}\left\|\frac{du_\tau}{dt}(s)\right\|_H^2,$$

which gives

$$\sup_{t\in[0,T]}\left\|\frac{du_\tau}{dt}(t)\right\|_H^2 \leq 2\|u_1\|_H^2 + 4\Psi(u_0) + 8T\int_0^T \|f(t)\|_H^2 \, dt =: M. \tag{65}$$

Hence, u_τ is Lipschitz with Lipschitz constant at most $M^{1/2}$. Since $u_\tau(0) = u_0$, it follows that $\{u_\tau\}$ is bounded in $W^{1,\infty}((0,T);H)$. Hence, up to a subsequence, not relabeled, we may find $u \in H^1((0,T);H)$ such that

$$u_\tau \rightharpoonup u \text{ in } H^1((0,T);H).$$

Also,

$$\begin{aligned}
&\tau \int_0^{\tau_j} \left\| \frac{dv_\tau}{dt}(t) \right\|_H^2 dt + 2\Psi(\tilde{u}_\tau(t)) \\
&\leq \|u_1\|_H^2 + 2\Psi(u_0) + T \int_0^T \|f(t)\|_H^2 \, dt + \frac{1}{2} \sup_{s\in[0,T]} \left\| \frac{du_\tau}{dt}(s) \right\|_H^2 \\
&\leq 2\|u_1\|_H^2 + 4\Psi(u_0) + 2T \int_0^T \|f(t)\|_H^2 \, dt = M,
\end{aligned}$$

and so

$$\tau \int_0^T \left\| \frac{dv_\tau}{dt}(t) \right\|_H^2 dt \leq M, \qquad \sup_{t\in[0,T]} \Psi(\tilde{u}_\tau(t)) \leq \frac{1}{2}M. \tag{66}$$

By (60) and (66) and the fact that $\operatorname{dom}_e \Psi = Y$, there exists $R > 0$ such that

$$\sup_{t\in[0,T]} \|\tilde{u}_\tau(t)\|_Y \leq R \tag{67}$$

for all τ. Since for $t \in (\tau_{n-1}, \tau_n]$,

$$u_\tau(t) = \left(1 - \frac{\tau_n - t}{\tau}\right) u_n + \left(\frac{\tau_n - t}{\tau}\right) u_{n-1},$$

we have that $u_\tau(t) \in Y$. In turn,

$$\|u_\tau(t)\|_Y \leq \left(1 - \frac{\tau_n - t}{\tau}\right) \|u_n\|_Y + \left(\frac{\tau_n - t}{\tau}\right) \|u_{n-1}\|_Y \leq R. \tag{68}$$

This shows that $\{u_\tau\}$ is bounded in $L^\infty((0,T);Y)$. Using the embedding $H \cong H' \hookrightarrow Y'$, we also have that $\left\{\frac{du_\tau}{dt}\right\}$ is bounded in $L^\infty((0,T);Y')$. Hence, we may apply the Aubin–Lions compactness theorem (see Theorem 36) to conclude, that up to a subsequence, $u_\tau \to u$ in $L^2((0,T);H)$, while $\frac{du_\tau}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2((0,T);H)$ and $u_\tau \rightharpoonup u$ in $L^2((0,T);Y)$. ■

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Proof. We claim that $\{u_\tau - \tilde{u}_\tau\}$ converges uniformly to zero in $L^\infty([0,T];H)$. Given $t \in (0,T]$ find n such that $t \in (\tau_{n-1}, \tau_n]$. Then

$$\tilde{u}_\tau(t) - u_\tau(t) = u_n - u_\tau(t) = u_\tau(\tau_n) - u_\tau(t),$$

and so, since u_τ is Lipschitz with Lipschitz constant at most $M^{1/2}$,

$$\|\tilde{u}_\tau(t) - u_\tau(t)\|_H = \|u_\tau(\tau_n) - u_\tau(t)\|_H \leq M^{1/2}\tau.$$

This proves the claim. In particular, since $u_\tau \to u$ in $L^2((0,T);H)$, it follows that $\tilde{u}_\tau \to u$ in $L^2((0,T);H)$.

By (62), (64) and (65), reasoning as in (68), we have

$$\begin{aligned}
\|v_\tau(t)\|_H &\leq \left(1 - \frac{\tau_n - t}{\tau}\right) \left\| \frac{u_n - u_{n-1}}{\tau} \right\|_H \\
&\quad + \left(\frac{\tau_n - t}{\tau}\right) \left\| \frac{u_{n-1} - u_{n-2}}{\tau} \right\|_H \leq C_Y M^{1/2}.
\end{aligned}$$

Hence, $\{v_\tau\}$ is bounded in $L^\infty((0,T);H)$.

Using (63), for every $z \in H$ we have that

$$\Psi(z) \geq \Psi(\tilde{u}_\tau(t)) + (f_\tau(t), z - \tilde{u}_\tau(t))_H - \left(\frac{dv_\tau}{dt}(t), z - \tilde{u}_\tau(t)\right)_H.$$

Take $z := \tilde{u}_\tau(t) \pm y$, where $y \in Y$ and $\|y\|_Y \leq 1$. Then

$$\Psi(\tilde{u}_\tau(t) \pm y) \geq \Psi(\tilde{u}_\tau(t)) \pm (f_\tau(t), y)_H \mp \left(\frac{dv_\tau}{dt}(t), y\right)_H,$$

and so

$$\left\|\frac{dv_\tau}{dt}(t)\right\|_{Y'} = \sup_{y\in Y,\ \|y\|_Y \leq 1} \left|\left(\frac{dv_\tau}{dt}(t), y\right)_H\right| \leq C_\Psi + C_Y \|f_\tau(t)\|_H,$$

where

$$C_\Psi := 2\sup\{\Psi(z) : \ \|z\|_Y \leq R+1\} < \infty,$$

by (67) and the fact that Ψ is bounded on bounded sets of Y. Hence,

$$\int_0^T \left\|\frac{dv_\tau}{dt}(t)\right\|_{Y'}^2 \ dt \leq 2C_\Psi^2 + 2C_Y \int_0^T \|f(t)\|_H^2 \ dt.$$

This shows that the sequence $\{v_\tau\}$ is bounded in $L^\infty((0,T);H)$ and $\left\{\frac{dv_\tau}{dt}\right\}$ in $L^2((0,T);Y')$. Hence, up to a subsequence, $v_\tau \rightharpoonup v$ in $L^2((0,T);H)$, while $\frac{dv_\tau}{dt} \rightharpoonup \frac{dv}{dt}$ in $L^2((0,T);Y')$.

Next we will show that $v = \frac{du}{dt}$. For every $0 \leq t_1 < t_2 \leq T$,

$$\begin{aligned}\|v_\tau(t_2) - v_\tau(t_1)\|_{Y'} &= \left\|\int_{t_1}^{t_2} \frac{dv_\tau}{dt}(t)\ dt\right\|_Y \\ &\leq (t_2 - t_1)^{1/2}\left(\int_{t_1}^{t_2}\left\|\frac{dv_\tau}{dt}(t)\right\|_{Y'}^2\ dt\right)^{1/2} \\ &\leq C_1(t_2 - t_1)^{1/2}.\end{aligned} \tag{69}$$

Given $t \in (0,T]$ find n such that $t \in (\tau_{n-1}, \tau_n]$. Then

$$v_\tau(t) - \frac{du_\tau}{dt}(t) = v_\tau(t) - \frac{u_n - u_{n-1}}{\tau} = v_\tau(t) - v_\tau(\tau_n),$$

and so by (69),

$$\left\|v_\tau(t) - \frac{du_\tau}{dt}(t)\right\|_{Y'} = \|v_\tau(t) - v_\tau(\tau_n)\|_{Y'} \leq C_1\tau^{1/2}.$$

It follows that $\left\{v_\tau - \frac{du_\tau}{dt}\right\}$ converges uniformly to zero in $C([0,T];Y')$ but since $\frac{du_\tau}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2((0,T);H)$ and $v_\tau \rightharpoonup v$ in $L^2((0,T);H)$, we have that $v = \frac{du}{dt}$.

Step 3: Strong solution. We claim that

$$f(t) - \frac{d^2u}{dt^2}(t) \in \partial\Psi(u(t))$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Since $\tilde{u}_\tau \to u$ in $L^2((0,T);H)$, up to a subsequence, we have that $\tilde{u}_\tau(t) \to u(t)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Using (63), for every $z \in H$ we have that

$$\begin{aligned}\Psi(z) &\geq \Psi(\tilde{u}_\tau(t)) + (f_\tau(t), z - \tilde{u}_\tau(t))_H - \left(\frac{dv_\tau}{dt}(t), z - \tilde{u}_\tau(t)\right)_H \\ &= \Psi(\tilde{u}_\tau(t)) + (f_\tau(t), z - \tilde{u}_\tau(t))_H - \left\langle\frac{dv_\tau}{dt}(t), z - \tilde{u}_\tau(t)\right\rangle_{Y',Y}.\end{aligned}$$

Averaging this inequality between t_0 and $t_0 + h$ with $h > 0$ gives

$$\begin{aligned}\Psi(z) \geq{}& \frac{1}{h}\int_{t_0}^{t_0+h} \Psi(\tilde{u}_\tau(t))\ dt + \frac{1}{h}\int_{t_0}^{t_0+h} (f_\tau(t), z - \tilde{u}_\tau(t))_H\ dt \\ &- \frac{1}{h}\int_{t_0}^{t_0+h} \left\langle\frac{dv_\tau}{dt}(t), z - \tilde{u}_\tau(t)\right\rangle_{Y',Y} dt.\end{aligned}$$

Letting $\tau \to 0^+$ and using Fatou's lemma and the facts that $\tilde{u}_\tau(t) \to u(t)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, $\tilde{u}_\tau \to u$ in $L^2((0,T);H)$, $f_\tau \to f$ in $L^2((0,T);H)$, $\frac{dv_\tau}{dt} \rightharpoonup \frac{d^2u}{dt^2}$ in $L^2((0,T);Y')$, we obtain

$$\begin{aligned}\Psi(z) \geq{}& \frac{1}{h}\int_{t_0}^{t_0+h} \Psi(u(t))\ dt + \frac{1}{h}\int_{t_0}^{t_0+h} (f(t), v - u(t))_H\ dt \\ &- \frac{1}{h}\int_{t_0}^{t_0+h} \left\langle\frac{d^2u}{dt^2}(t), z - u(t)\right\rangle_{Y',Y} dt.\end{aligned}$$

Let t_0 be a Lebesgue point of $\frac{d^2u}{dt^2}$ and of f (see Theorem 23 and Theorem 28). By Jensen's inequality

$$\frac{1}{h}\int_{t_0}^{t_0+h} \Psi(u(t))\ dt \geq \Psi\left(\frac{1}{h}\int_{t_0}^{t_0+h} u(t)\ dt\right).$$

Then letting $h \to 0^+$, and using the fact that Ψ is lower semicontinuous, we get

$$\begin{aligned}\Psi(z) \geq{}& \Psi(u(t_0)) + (f(t_0), z - u(t_0))_H \\ &- \left\langle\frac{d^2u}{dt^2}(t_0), z - u(t_0)\right\rangle_{Y',Y}\end{aligned}$$

for all $v \in H$. This shows that $f(t_0) - \frac{d^2u}{dt^2}(t_0) \in \partial\Psi(u(t_0))$ and proves the claim. ∎

Remark 62 *If $h \in Y \hookrightarrow H$, then we identify h with $L_h \in Y'$ and so in Y' the norm of h is*

$$\|L_h\|_{Y'} = \sup_{y \in Y\setminus\{0\},\ \|y\|_Y \le 1} |(h,y)_H| \le C \|h\|_H \le C' \|h\|_Y .$$

In particular, if $Y = H^1(\Omega)$ and $H = L^2(\Omega)$, then given $u \in H^1(\Omega)$, we have that u is identified with $L_u \in \left(H^1(\Omega)\right)'$, where

$$\|L_u\|_{(H^1(\Omega))'} = \sup_{v \in H^1(\Omega)\setminus\{0\},\ \|v\|_{H^1(\Omega)} \le 1} \left| \int_\Omega uv \ d\boldsymbol{x} \right| \le \|u\|_{L^2(\Omega)} < \|u\|_{H^1(\Omega)}$$

provided u is not constant on each connected component of Ω.

Remark 63 *Note in place of (60), it is enough to assume that*

$$\Psi(v_n) \to \infty \tag{70}$$

for every sequence $\{v_n\} \subset Y$ such that $\|v_n\|_Y \to \infty$ and $\{v_n\}$ is bounded in H. Indeed, since $\{u_\tau\}$ is bounded in $W^{1,\infty}((0,T);H)$ by (65) and the fact that $u_\tau(0) = u_0$, we have that

$$\sup_{t \in [0,T]} \|\tilde{u}_\tau(t)\|_H \le C.$$

Hence, from (66) and (70), we can still conclude that (67) holds.

Not done in class, please read

10 Heat Equation, Alternative Proof

Theorem 64 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in H^1(\Omega)$, and let $f \in L^2\left((0,T);L^2(\Omega)\right)$. Then there exists a unique weak solution u of (51). Moreover, $u \in L^\infty\left((0,T);H^1(\Omega)\right) \cap L^2\left((0,T);H^2_{\text{loc}}(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^2\left((0,T);L^2(\Omega)\right)$,*

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t)$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$, and the following estimates hold

$$\int_\Omega (u(\boldsymbol{x},t) - u_0(\boldsymbol{x}))^2 \ d\boldsymbol{x} \le t \left(\int_\Omega \|\nabla u_0\|^2 \ d\boldsymbol{x} + \int_{\Omega_T} f^2 \ dsd\boldsymbol{x} \right), \tag{71}$$

$$\int_\Omega \|\nabla u(\boldsymbol{x},t)\|^2 \ d\boldsymbol{x} \le 2 \int_\Omega \|\nabla u_0\|^2 \ d\boldsymbol{x} + 2 \int_{\Omega_T} f^2 \ dsd\boldsymbol{x}, \tag{72}$$

$$\int_{\Omega_T} \left(\frac{\partial u}{\partial t} \right)^2 \ dtd\boldsymbol{x} \le \int_\Omega \|\nabla u_0\|^2 \ d\boldsymbol{x} + \int_{\Omega_T} f^2 \ dtd\boldsymbol{x} \tag{73}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and there exists a constant $C = C(N) > 0$ such that for all $U \Subset \Omega$,

$$\int_0^T \int_U \left\|\nabla^2 u\right\|^2 \, d\boldsymbol{x}dt \le C\left(1 + \frac{1}{\operatorname{dist}^2(U, \partial\Omega)}\right)\left(\int_\Omega \|\nabla u_0\|^2 \, d\boldsymbol{x} + \int_{\Omega_T} f^2 \, dtd\boldsymbol{x}\right).$$

Finally, if Ω is bounded and of class C^2, then u belongs to $L^2\left((0,T); H^2(\Omega)\right)$ and satisfies the Neumann boundary condition

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T)$$

and

$$\int_\Omega u(\boldsymbol{x}, t) \, d\boldsymbol{x} = \int_\Omega u_0(\boldsymbol{x}) \, d\boldsymbol{x} + \int_0^t \int_\Omega f(\boldsymbol{x}, s) \, d\boldsymbol{x}ds. \tag{74}$$

Moreover,

$$\int_{\Omega_T} \left\|\nabla^2 u(\boldsymbol{x}, t)\right\|^2 \, d\boldsymbol{x}dt \le C(N, \Omega)\left(\int_\Omega \|\nabla u_0\|^2 \, d\boldsymbol{x} + \int_{\Omega_T} f^2 \, dtd\boldsymbol{x}\right).$$

Proof. Step 1: For $\ell \in \mathbb{N}$ set $\tau := \frac{T}{\ell}$ and subdivide $(0,T)$ into ℓ intervals of length τ,

$$\tau_0 := 0 < t_1 < \cdots < \tau_\ell = T,$$

where $\tau_n := n\tau$, $n = 1, \dots, \ell$. For every $n = 1, \dots, \ell$, let $u_n \in H^1(\Omega)$ be a solution of the minimization problem

$$\min_{v \in H^1(\Omega)} J_n(v), \tag{75}$$

where

$$J_n(v) := \frac{1}{2}\int_\Omega \|\nabla v\|^2 \, d\boldsymbol{x} - \int_\Omega v f_n \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x},$$

with

$$f_n(\boldsymbol{x}) := \frac{1}{\tau}\int_{\tau_{n-1}}^{\tau_n} f(\boldsymbol{x}, t) \, dt.$$

Note that u_n, J_n and f_n all depend on ℓ. To prove the existence of u_n, we begin by showing that J_n is bounded from below. Write

$$\begin{aligned} -\int_\Omega v f_n \, d\boldsymbol{x} &= -\int_\Omega (v - u_{n-1}) f_n \, d\boldsymbol{x} - \int_\Omega u_{n-1} f_n \, d\boldsymbol{x} \\ &\ge -\frac{1}{4\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x} - \tau\int_\Omega f_n^2 \, d\boldsymbol{x} - \int_\Omega u_{n-1} f_n \, d\boldsymbol{x}, \end{aligned}$$

where we have used the inequality $ab = \frac{a}{\sqrt{2\tau}}\sqrt{2\tau}b \le \frac{1}{4\tau}a^2 + \tau b^2$. Hence,

$$J_n(v) \ge \frac{1}{2}\int_\Omega \|\nabla v\|^2 \, d\boldsymbol{x} + \frac{1}{4\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x} - \tau\int_\Omega f_n^2 \, d\boldsymbol{x} - \int_\Omega u_{n-1} f_n \, d\boldsymbol{x}. \tag{76}$$

Next let

$$m_n := \inf_{v \in H^1(\Omega)} J_n(v)$$

and, using the definition of infimum consider a sequence $\{v_k\} \subset H^1(\Omega)$ such that

$$m_n \le J_n(v_k) \le m_n + \frac{1}{k}.$$

Then

$$\lim_{k \to \infty} J_n(v_k) = m_n.$$

It follows from (76) and the fact that $J_n(v_k) \le m_n + 1$ for all k, that $\{v_k\}$ is bounded in $H^1(\Omega)$. Hence, up to a subsequence, not relabelled, there exists $u_n \in H^1(\Omega)$ such that $\{v_k\}$ converges weakly to u_n in $L^2(\Omega)$ and strongly in $L^2_{\text{loc}}(\Omega)$, and $\{\nabla v_k\}$ converges weakly to ∇u_n in $L^2(\Omega; \mathbb{R}^N)$. We claim that $J_n(u_n) = m_n$. To see this, observe that

$$\begin{aligned}
J_n(v_k) &= \frac{1}{2}\int_\Omega \|\nabla v_k\|^2 \, d\boldsymbol{x} - \int_\Omega v f_n \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x} \\
&= \frac{1}{2}\int_\Omega \|\nabla v_k - \nabla u_n + \nabla u_n\|^2 \, d\boldsymbol{x} - \int_\Omega (v_k - u_n + u_n) f_k \, d\boldsymbol{x} \\
&\quad + \frac{1}{2\tau}\int_\Omega (v_k - u_n + u_n - u_{n-1})^2 \, d\boldsymbol{x} \\
&= J_n(u_n) + \frac{1}{2}\int_\Omega \|\nabla v_k - \nabla u_n\|^2 \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega (v_k - u_n)^2 \, d\boldsymbol{x} \\
&\quad + \int_\Omega (\nabla v_k - \nabla u_n) \cdot \nabla u_n \, d\boldsymbol{x} - \int_\Omega (v_k - u_n) f_k \, d\boldsymbol{x} \\
&\quad + \frac{1}{\tau}\int_\Omega (v_k - u_n)(u_n - u_{n-1}) \, d\boldsymbol{x}.
\end{aligned}$$

Hence,

$$\begin{aligned}
m_n &\le J_n(u_n) \le J_n(u_n) + \frac{1}{2}\int_\Omega \|\nabla v_k - \nabla u_n\|^2 \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega (v_k - u_n)^2 \, d\boldsymbol{x} \\
&= J_n(v_k) - \int_\Omega (\nabla v_k - \nabla u_n) \cdot \nabla u_n \, d\boldsymbol{x} + \int_\Omega (v_k - u_n) f_n \, d\boldsymbol{x} \\
&\quad - \frac{1}{\tau}\int_\Omega (v_k - u_n)(u_n - u_{n-1}) \, d\boldsymbol{x}.
\end{aligned}$$

Letting $k \to \infty$ and using the facts that $v_k \rightharpoonup u_n$ in $H^1(\Omega)$ and $J_n(v_k) \to m_n$, shows that

$$m_n = J_n(u_n)$$

and that $v_k \to u_n$ in $H^1(\Omega)$.

It follows that for every $w \in H^1(\Omega)$ and for every $t \in \mathbb{R}$,

$$J_n(u_n) \le J_n(u_n + tw).$$

This shows that the real value function $\omega(t) := J_n(u_n + tw)$ has a minimum at $t = 0$. Hence, $\omega'(0) = 0$. We have

$$\begin{aligned}\omega(t) = J_n(u_n) + \frac{t^2}{2}\int_\Omega \|\nabla w\|^2\ d\boldsymbol{x} + \frac{t^2}{2\tau}\int_\Omega w^2\ d\boldsymbol{x} \\ + t\int_\Omega \nabla w\cdot\nabla u_n\ d\boldsymbol{x} - t\int_\Omega w f_n\ d\boldsymbol{x} + \frac{t}{\tau}\int_\Omega w(u_n - u_{n-1})\ d\boldsymbol{x},\end{aligned}$$

and so

$$0 = \omega'(0) = \int_\Omega \nabla w\cdot\nabla u_n\ d\boldsymbol{x} - \int_\Omega w f_n\ d\boldsymbol{x} + \frac{1}{\tau}\int_\Omega w(u_n - u_{n-1})\ d\boldsymbol{x} \qquad (77)$$

for every $w \in H^1(\Omega)$. In particular, this shows that u_n is a weak solution of the equation

$$-\Delta u_n(\boldsymbol{x}) = f_n(\boldsymbol{x}) - \frac{1}{\tau}(u_n(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x})) \quad \text{in } \Omega. \qquad (78)$$

Moreover, if Ω is bounded and Lipschitz, taking $w = 1$ in (77) gives

$$0 = \int_\Omega f_n\ d\boldsymbol{x} + \frac{1}{\tau}\int_\Omega (u_n - u_{n-1})\ d\boldsymbol{x}. \qquad (79)$$

Also $\frac{\partial u_n}{\partial \boldsymbol{n}}(\boldsymbol{x}) = 0$ on $\partial\Omega$.
Step 2: A priori bounds. For $\boldsymbol{x} \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$, $n = 1, \dots, \ell$, define

$$\begin{aligned}u_\tau(\boldsymbol{x}, t) &:= u_n(\boldsymbol{x}) + (t - \tau_n)\frac{u_n(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x})}{\tau}, \\ f_\tau(\boldsymbol{x}, t) &:= f_n(\boldsymbol{x}).\end{aligned} \qquad (80)$$

The goal of this step is to obtain a priori bounds on u_τ. Taking $w = u_n - u_{n-1}$ in (77), we get

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (u_n - u_{n-1})^2\ d\boldsymbol{x} + \int_\Omega \|\nabla u_n\|^2\ d\boldsymbol{x} - \int_\Omega \nabla u_{n-1}\cdot\nabla u_n\ d\boldsymbol{x} \\ = \int_\Omega (u_n - u_{n-1}) f_n\ d\boldsymbol{x}.\end{aligned}$$

Using the fact that $\|\boldsymbol{a}\|^2 - \boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}\|\boldsymbol{a}\|^2 + \frac{1}{2}\|\boldsymbol{a} - \boldsymbol{b}\|^2 - \frac{1}{2}\|\boldsymbol{b}\|^2$, we can rewrite the previous identity as

$$\begin{aligned}\frac{1}{\tau}\int_\Omega (u_n - u_{n-1})^2\ d\boldsymbol{x} + \frac{1}{2}\int_\Omega \|\nabla u_n\|^2\ d\boldsymbol{x} - \frac{1}{2}\int_\Omega \|\nabla u_{n-1}\|^2\ d\boldsymbol{x} \\ + \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u_{n-1}\|^2\ d\boldsymbol{x} = \int_\Omega (u_n - u_{n-1}) f_n\ d\boldsymbol{x}.\end{aligned}$$

Summing from $n = 1, \ldots, \ell$, we obtain

$$\frac{1}{\tau}\sum_{n=1}^{\ell}\int_{\Omega}(u_n - u_{n-1})^2\ d\boldsymbol{x} + \frac{1}{2}\int_{\Omega}\|\nabla u_\ell\|^2\ d\boldsymbol{x} + \frac{1}{2}\sum_{n=1}^{\ell}\int_{\Omega}\|\nabla u_n - \nabla u_{n-1}\|^2\ d\boldsymbol{x}$$
$$= \frac{1}{2}\int_{\Omega}\|\nabla u_0\|^2\ d\boldsymbol{x} + \sum_{n=1}^{\ell}\int_{\Omega}(u_n - u_{n-1}) f_n\ d\boldsymbol{x}.$$

Using the inequality $ab = \frac{a}{\sqrt{\tau}}\sqrt{\tau}b \le \frac{1}{2\tau}a^2 + \frac{\tau}{2}b^2$ we get

$$\frac{1}{2\tau}\sum_{n=1}^{\ell}\int_{\Omega}(u_n - u_{n-1})^2\ d\boldsymbol{x} + \frac{1}{2}\int_{\Omega}\|\nabla u_\ell\|^2\ d\boldsymbol{x} + \frac{1}{2}\sum_{n=1}^{\ell}\int_{\Omega}\|\nabla u_n - \nabla u_{n-1}\|^2\ d\boldsymbol{x} \tag{81}$$
$$\le \frac{1}{2}\int_{\Omega}\|\nabla u_0\|^2\ d\boldsymbol{x} + \frac{\tau}{2}\sum_{n=1}^{\ell}\int_{\Omega} f_n^2\ d\boldsymbol{x}.$$

By (80), for $\boldsymbol{x} \in \Omega$ and $t \in (\tau_{n-1}, \tau_n)$,

$$\frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t) = \frac{u_n(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x})}{\tau}, \tag{82}$$
$$\nabla u_\tau(\boldsymbol{x}, t) = \nabla u_n(\boldsymbol{x}) + (t - \tau_{n-1})\frac{\nabla u_n(\boldsymbol{x}) - \nabla u_{n-1}(\boldsymbol{x})}{\tau},$$

and so (81) becomes

$$\frac{1}{2}\int_{\Omega_T}\left(\frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t)\right)^2\ dtd\boldsymbol{x} + \frac{1}{2}\int_{\Omega}\|\nabla u_\ell\|^2\ d\boldsymbol{x} + \frac{1}{2}\sum_{n=1}^{\ell}\int_{\Omega}\|\nabla u_n - \nabla u_{n-1}\|^2\ d\boldsymbol{x} \tag{83}$$
$$\le \frac{1}{2}\int_{\Omega}\|\nabla u_0\|^2\ d\boldsymbol{x} + \frac{1}{2}\int_{\Omega_T}(f_\tau(\boldsymbol{x}, t))^2\ dtd\boldsymbol{x}.$$

Now, by Hölder's inequality,

$$\int_{\Omega_T}(f_\tau(\boldsymbol{x}, t))^2\ dtd\boldsymbol{x} = \int_{\Omega}\tau\sum_{n=1}^{\ell}\frac{1}{\tau^2}\left(\int_{\tau_{n-1}}^{\tau_n} f(\boldsymbol{x}, t)\ dt\right)^2\ d\boldsymbol{x} \tag{84}$$
$$\le \int_{\Omega}\sum_{n=1}^{\ell}\int_{\tau_{n-1}}^{\tau_n} f^2(\boldsymbol{x}, t)\ dtd\boldsymbol{x} = \int_{\Omega_T} f^2(\boldsymbol{x}, t)\ dtd\boldsymbol{x} < \infty.$$

Hence, by (83),

$$\int_{\Omega_T}\left(\frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t)\right)^2\ dtd\boldsymbol{x} \le M \tag{85}$$

for every $\tau > 0$, where

$$M := \|\nabla u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega_T)}.$$

In turn, for every $0 \leq t_1 < t_2 \leq T$, since u_τ is absolutely continuous,

$$\begin{aligned} \int_\Omega \left(u_\tau(\boldsymbol{x}, t_2) - u_\tau(\boldsymbol{x}, t_1)\right)^2 \, d\boldsymbol{x} &= \int_\Omega \left(\int_{t_1}^{t_2} \frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t) \, dt\right)^2 \, d\boldsymbol{x} \\ &\leq (t_2 - t_1) \int_{\Omega_T} \left(\frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t)\right)^2 \, dt d\boldsymbol{x} \qquad (86) \\ &\leq M (t_2 - t_1). \end{aligned}$$

Taking $t_1 = 0$ and using the fact that $u_\tau(\boldsymbol{x}, 0) = u_0(\boldsymbol{x})$ gives

$$\int_\Omega \left(u_\tau(\boldsymbol{x}, t) - u_0(\boldsymbol{x})\right)^2 \, d\boldsymbol{x} \leq Mt \qquad (87)$$

for every $\tau > 0$ and all $t \in (0, T)$. In turn,

$$\int_\Omega \left(u_\tau(\boldsymbol{x}, t)\right)^2 \, d\boldsymbol{x} \leq 2Mt + 2 \int_\Omega u_0^2(\boldsymbol{x}) \, d\boldsymbol{x} \qquad (88)$$

for every $\tau > 0$ and all $t \in (0, T)$.

On the other hand, by (82), for $\boldsymbol{x} \in \Omega$ and $t \in (\tau_{n-1}, \tau_n)$,

$$\|\nabla u_\tau(\boldsymbol{x}, t)\| \leq \|\nabla u_n(\boldsymbol{x})\| + \|\nabla u_n(\boldsymbol{x}) - \nabla u_{n-1}(\boldsymbol{x})\|,$$

and so by (81) and (84),

$$\int_\Omega \|\nabla u_\tau(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} \leq 2M \qquad (89)$$

for every $\tau > 0$ and all $t \in (0, T)$.

Step 3: Convergence as $\boldsymbol{\tau \to 0^+}$. In Step 2 we have shown that $\{u_\tau\}$ is bounded in $L^2\left((0, T); H^1(\Omega)\right)$ and $\{u_\tau'\}$ in $L^2\left((0, T); L^2(\Omega)\right)$. Since these spaces are reflexive by Corollary 18, up to a subsequence, $u_\tau \rightharpoonup u$ in $L^2\left((0, T); H^1(\Omega)\right)$ and in $H^1\left((0, T); L^2(\Omega)\right)$. Using the fact that the embedding $H^1(U) \hookrightarrow L^2(U)$ is compact whenever U is open bounded set with Lipschitz boundary, it follows by the compactness theorem of Aubin and Lions (see Theorem 36) and a diagonal argument that, up to a further subsequence, $u_\tau \to u$ in $L^2\left((0, T); L^2_{\mathrm{loc}}(\Omega)\right)$. In turn, for $\mathcal{L}^1$ a.e. $t \in (0, T)$, we have that $u_\tau(\cdot, t) \to u(\cdot, t)$ in $L^2_{\mathrm{loc}}(\Omega)$. Letting $\ell \to \infty$, or equivalently, $\tau \to 0^+$ in (85), (87), (89), we deduce that (71), (72), and (73) hold.

Step 4: Next we prove that u is a weak solution of the heat equation. To see this, for $\boldsymbol{x} \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$, $n = 1, 1, \ldots, \ell$, define

$$\tilde{u}_\tau(\boldsymbol{x}, t) := u_n(\boldsymbol{x}). \qquad (90)$$

We claim that $\tilde{u}_\tau \rightharpoonup u$ in $L^2\left((0, T); H^1(\Omega)\right)$. Given $t \in (0, T]$ find n such that $t \in (\tau_{n-1}, \tau_n]$. Then

$$\tilde{u}_\tau(\boldsymbol{x}, t) - u_\tau(\boldsymbol{x}, t) = u_n(\boldsymbol{x}) - u_\tau(\boldsymbol{x}, t) = u_\tau(\boldsymbol{x}, \tau_{n-1}) - u_\tau(\boldsymbol{x}, t).$$

By (86),

$$\int_{\Omega} \|\tilde{u}_\tau(\boldsymbol{x}, t) - u_\tau(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} = \int_{\Omega} \|u_\tau(\boldsymbol{x}, \tau_{n-1}) - u_\tau(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x}$$
$$\leq M(t - \tau_{n-1}) \leq M\tau \to 0$$

as $\tau \to 0^+$. This shows that $\tilde{u}_\tau(\cdot, t) - u_\tau(\cdot, t) \to 0$ in $L^2(\Omega)$ as $\tau \to 0^+$. Given $\varphi \in L^2(\Omega \times (0, T))$, we have

$$\int_{\Omega_T} \tilde{u}_\tau(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t) \, d\boldsymbol{x}dt = \int_{\Omega_T} (\tilde{u}_\tau(\boldsymbol{x}, t) - u_\tau(\boldsymbol{x}, t))\varphi(\boldsymbol{x}, t) \, d\boldsymbol{x}dt$$
$$+ \int_{\Omega_T} u_\tau(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t) \, d\boldsymbol{x}dt.$$

By Holder's inequality we can bound in absolute value the first integral on the right-hand side by

$$\left(\int_0^T \int_{\Omega} \|\tilde{u}_\tau - u_\tau\|^2 \, d\boldsymbol{x}dt\right)^{1/2} \|\varphi\|_{L^2(\Omega_T)} \leq M^{1/2}\tau^{1/2}T^{1/2} \|\varphi\|_{L^2(\Omega_T)} \to 0$$

as $\tau \to 0^+$. Since $u_\tau \rightharpoonup u$ in $L^2((0, T); H^1(\Omega))$, the second integral converges to $\int_{\Omega_T} u(\boldsymbol{x}, t)\varphi(\boldsymbol{x}, t) \, d\boldsymbol{x}dt$, which proves that $\tilde{u}_\tau \rightharpoonup u$ in $L^2((0, T); L^2(\Omega))$. On the other hand, by (89), and again the fact that $\tilde{u}_\tau(\boldsymbol{x}, t) = u_\tau(\boldsymbol{x}, \tau_{n-1})$ for $t \in (\tau_{n-1}, \tau_n]$,

$$\int_{\Omega} \|\nabla\tilde{u}_\tau(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} \leq 2M \tag{91}$$

for every $\tau > 0$ and all $t \in (0, T)$. Hence, $\tilde{u}_\tau \rightharpoonup u$ in $L^2((0, T); H^1(\Omega))$.

By (77) for every $w \in L^2((0, T); H^1(\Omega))$,

$$0 = \int_{\Omega} \nabla w(\boldsymbol{x}, t) \cdot \nabla\tilde{u}_\tau(\boldsymbol{x}, t) \, d\boldsymbol{x} - \int_{\Omega} w(\boldsymbol{x}, t) f_\tau(\boldsymbol{x}, t) \, d\boldsymbol{x} \tag{92}$$
$$+ \int_{\Omega} w(\boldsymbol{x}, t) \frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x}.$$

Integrating in time over (t_1, t_2) gives

$$0 = \int_{t_1}^{t_2} \int_{\Omega} \nabla w(\boldsymbol{x}, t) \cdot \nabla\tilde{u}_\tau(\boldsymbol{x}, t) \, d\boldsymbol{x}dt - \int_{t_1}^{t_2} \int_{\Omega} w(\boldsymbol{x}, t) f_\tau(\boldsymbol{x}, t) \, d\boldsymbol{x}dt$$
$$+ \int_{t_1}^{t_2} \int_{\Omega} w(\boldsymbol{x}, t) \frac{\partial u_\tau}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x}dt.$$

Letting $\tau \to 0^+$ and using the facts that $\tilde{u}_\tau \rightharpoonup u$ in $L^2((0, T); H^1(\Omega))$, $u_\tau \rightharpoonup u$ in $H^1((0, T); L^2(\Omega))$, and $f_\tau \to f$ in $L^2((0, T); L^2(\Omega))$, we obtain

$$0 = \int_{t_1}^{t_2} \int_{\Omega} \nabla w(\boldsymbol{x}, t) \cdot \nabla u(\boldsymbol{x}, t) \, d\boldsymbol{x}dt - \int_{t_1}^{t_2} \int_{\Omega} w(\boldsymbol{x}, t) f(\boldsymbol{x}, t) \, d\boldsymbol{x}dt \tag{93}$$
$$+ \int_{t_1}^{t_2} \int_{\Omega} w(\boldsymbol{x}, t) \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x}dt.$$

In particular, let $\{w_k\} \subset H^1(\Omega)$ be dense. Taking w to be w_k in (93) and using the fact that $u(\cdot,t) \in H^1(\Omega)$ and $f(\cdot,t), \frac{\partial u}{\partial t}(\cdot,t) \in L^2(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, by the arbitrariness of t_1 and t_2, we find that

$$\int_\Omega \nabla w_k(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x},t)\ d\boldsymbol{x} - \int_\Omega w_k(\boldsymbol{x}) f(\boldsymbol{x},t)\ d\boldsymbol{x}dt + \int_\Omega w_k(\boldsymbol{x}) \frac{\partial u}{\partial t}(\boldsymbol{x},t)\ d\boldsymbol{x} = 0$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Note that the set of measure zero depends on k. Since $\{w_k\}$ is countable, we can find a set $E \subset (0,T)$ with $\mathcal{L}^1(E) = 0$ such that the previous equality holds for all $t \in (0,T) \setminus E$ and all k. In turn, by the density of $\{w_k\} \subset H^1(\Omega)$ and again the fact that $u(\cdot,t) \in H^1(\Omega)$ and $f(\cdot,t), \frac{\partial u}{\partial t}(\cdot,t) \in L^2(\Omega)$ for $t \in (0,T) \setminus E$, we get that

$$\int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x},t)\ d\boldsymbol{x} - \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x},t)\ d\boldsymbol{x}dt + \int_\Omega w(\boldsymbol{x}) \frac{\partial u}{\partial t}(\boldsymbol{x},t)\ d\boldsymbol{x} = 0 \tag{94}$$

for all $t \in (0,T) \setminus E$ and all $w \in H^1(\Omega)$. Note that if Ω has finite measure, taking $w = 1$ gives

$$-\int_\Omega f(\boldsymbol{x},t)\ d\boldsymbol{x}dt + \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x},t)\ d\boldsymbol{x} = 0 \tag{95}$$

for all $t \in (0,T) \setminus E$.

The remaining of the proof is the same as the one of Theorem 60. ■

Wednesday, February 19, 2014

11 Wave Equation

11.1 Existence of Weak Solutions

In this subsection we prove existence of weak solutions of the Neumann problem for the wave equation. For simplicity we will consider only the case when the Neumann datum is zero. Thus, we consider the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \frac{\partial u}{\partial t}(\boldsymbol{x},0) = u_1(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases} \tag{96}$$

Theorem 65 *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, let $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, and let $f \in L^2((0,T); L^2(\Omega))$, where $0 < T < \infty$. Then there exists a unique weak function $u \in H^1((0,T); L^2(\Omega))$, with $\frac{\partial^2 u}{\partial t^2} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$, such that*

$$\int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x},t)\ d\boldsymbol{x} - \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x},t)\ d\boldsymbol{x} + \left\langle \frac{\partial^2 u}{\partial t^2}(\cdot,t), w \right\rangle_{(H^1(\Omega))', H^1(\Omega)} = 0 \tag{97}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and for all $w \in H^1(\Omega)$. Moreover, the following estimates hold

$$\int_\Omega \|\nabla u(\boldsymbol{x},t)\|^2 \, d\boldsymbol{x} \le \frac{1}{2}M, \quad \int_\Omega \left\|\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right\|^2 \, d\boldsymbol{x} \le M,$$
$$\int_0^T \left\|\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2_{(H^1(\Omega))'} dt \le MT + 2\int_{\Omega_T} f^2 \, d\boldsymbol{x}dt$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where

$$M := 2\int_\Omega u_1^2 \, d\boldsymbol{x} + 2\int_\Omega \|\nabla u_0\|^2 \, d\boldsymbol{x} + 2T\int_{\Omega_T} f^2 \, dtd\boldsymbol{x}.$$

Proof. Step 1: We apply Theorem 61 with $H = L^2(\Omega)$, $V = H^1(\Omega)$, and

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega). \end{cases}$$

Note that Ψ satisfies (70) but not (60). Hence, we are in a position to apply Theorem 61 to find a function $u \in H^1\left((0,T); H^1(\Omega)\right)$ with $\frac{\partial^2 u}{\partial t^2} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$ such that $f(\cdot,t) - \frac{\partial^2 u}{\partial t^2}(\cdot,t) \in \partial\Psi(u(\cdot,t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, $u(\cdot,0) = u_0$, and $\frac{\partial u}{\partial t}(\cdot,0) = u_1$. Reasoning as in Example 48, it follows that (97) holds. Moreover, again by Theorem 61,

$$\int_{\Omega_T} \left(\frac{\partial^2 u}{\partial^2 t}\right)^2 dtd\boldsymbol{x} \le M, \quad \max_{t\in[0,T]} \int_\Omega \left\|\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right\|^2 \, d\boldsymbol{x} \le M,$$
$$\int_\Omega \|\nabla u(\boldsymbol{x},t)\|^2 \, d\boldsymbol{x} \le \frac{1}{2}M,$$

where

$$M := 2\int_\Omega u_1^2 \, d\boldsymbol{x} + 2\int_\Omega \|\nabla u_0\|^2 \, d\boldsymbol{x} + 2T\int_{\Omega_T} f^2 \, dtd\boldsymbol{x}.$$

From (97) we have

$$\left|\left\langle \frac{\partial^2 u}{\partial t^2}(\cdot,t), w\right\rangle_{(H^1(\Omega))', H^1(\Omega)}\right| \le \left(\|\nabla u(\cdot,t)\|_{L^2(\Omega)} + \|f(\cdot,t)\|_{L^2(\Omega)}\right)\|w\|_{H^1(\Omega)}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and for all $w \in H^1(\Omega)$. Hence,

$$\left\|\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|_{(H^1(\Omega))'} \le \|\nabla u(\cdot,t)\|_{L^2(\Omega)} + \|f(\cdot,t)\|_{L^2(\Omega)}$$

and so

$$\left\|\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2_{(H^1(\Omega))'} \le 2\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} + 2\|f(\cdot,t)\|.$$

Integrating in time, we get

$$\int_0^T \left\| \frac{\partial^2 u}{\partial t^2}(\cdot,t) \right\|^2_{(H^1(\Omega))'} dt \le 2\int_{\Omega_T} \|\nabla u\|^2 \ d\boldsymbol{x}dt + 2\int_{\Omega_T} f^2 \ d\boldsymbol{x}dt$$

$$\le MT + 2\int_{\Omega_T} f^2 \ d\boldsymbol{x}dt.$$

Step 2. **Uniqueness.** Let v_1 and v_2 be two weak solutions. Then $v := v_1 - v_2$ satisfies the equation

$$\int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x},t) \ d\boldsymbol{x} + \left\langle \frac{\partial^2 v}{\partial t^2}(\cdot,t), w \right\rangle_{(H^1(\Omega))', H^1(\Omega)} = 0$$

for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Define

$$w(\boldsymbol{x},t) := \begin{cases} \int_t^s v(\boldsymbol{x},r) \ dr & \text{if } 0 \le t \le s, \\ 0 & \text{if } t > s \end{cases}$$

and take $w(\cdot,t)$ a test function in the previous identity. Upon integrating in time, we get

$$0 = \int_0^T \int_\Omega \nabla w(\boldsymbol{x},t) \cdot \nabla v(\boldsymbol{x},t) \ d\boldsymbol{x}dt + \int_0^T \left\langle \frac{\partial^2 v}{\partial t^2}(\cdot,t), w(\cdot,t) \right\rangle_{(H^1(\Omega))', H^1(\Omega)} dt$$

$$= \int_0^s \int_\Omega \nabla w(\boldsymbol{x},t) \cdot \nabla v(\boldsymbol{x},t) \ d\boldsymbol{x}dt + \int_0^s \left\langle \frac{\partial^2 v}{\partial t^2}(\cdot,t), w(\cdot,t) \right\rangle_{(H^1(\Omega))', H^1(\Omega)} dt.$$

Using Fubini's theorem and integrating by parts, we have

$$\int_0^s \left\langle \frac{\partial^2 v}{\partial t^2}(\cdot,t), w(\cdot,t) \right\rangle_{(H^1(\Omega))', H^1(\Omega)} dt = -\int_0^s \int_\Omega \frac{\partial w}{\partial t}(\boldsymbol{x},t) \frac{\partial v}{\partial t}(\boldsymbol{x},t) \ d\boldsymbol{x}dt$$

$$+ \left[\left\langle \frac{\partial v}{\partial t}(\cdot,t), w(\cdot,t) \right\rangle_{(H^1(\Omega))', H^1(\Omega)} \right]_{t=0}^{t=s} = \int_0^s \int_\Omega v(\boldsymbol{x},t) \frac{\partial v}{\partial t}(\boldsymbol{x},t) \ d\boldsymbol{x}dt$$

$$= \frac{1}{2}\int_0^s \frac{d}{dt}\left(\int_\Omega v^2(\boldsymbol{x},t) \ d\boldsymbol{x} \right) dt$$

$$= \frac{1}{2}\int_\Omega v^2(\boldsymbol{x},s) \ d\boldsymbol{x} - 0$$

where we used the facts that $\frac{\partial w}{\partial t} = -v$, $w(\cdot,s) = 0$, $v(\cdot,0) = 0$, and $\frac{\partial v}{\partial t}(\cdot,0) = 0$. On the other hand,

$$\int_0^s \int_\Omega \nabla w(\boldsymbol{x},t) \cdot \nabla v(\boldsymbol{x},t) \ d\boldsymbol{x}dt = -\int_0^s \int_\Omega \nabla w(\boldsymbol{x},t) \cdot \nabla \frac{\partial w}{\partial t}(\boldsymbol{x},t) \ d\boldsymbol{x}dt =$$

$$= -\frac{1}{2}\int_0^s \frac{d}{dt}\left(\int_\Omega \|\nabla w(\boldsymbol{x},t)\|^2 \ d\boldsymbol{x} \right) dt$$

$$= 0 + \frac{1}{2}\int_\Omega \|\nabla w(\boldsymbol{x},0)\|^2 \ d\boldsymbol{x}$$

since $\frac{\partial w}{\partial t} = -v$ and $w(\cdot, s) = 0$. Hence, we have

$$\int_\Omega v^2(\boldsymbol{x}, s)\ d\boldsymbol{x} + \int_\Omega \|\nabla w(\boldsymbol{x}, 0)\|^2\ d\boldsymbol{x} = 0,$$

which shows that $v(\cdot, s) = 0$ for all s. ■

12 First Order Evolution Equations, Continued

Next we study the case in which the initial datum u_0 in Theorem 58 does not belong to the effective domain of the function Ψ.

Theorem 66 *Let H be a Hilbert space, let $\Psi : H \to [0, \infty]$ be convex, lower semicontinuous, not identically ∞, and such that there exists $\min \Psi = 0$, let $f \in L^2((0,T); H)$ and let $u_0 \in \overline{\operatorname{dom}_e \Psi}$. Then the Cauchy problem (39) admits a unique strong solution u, which satisfies*

$$\int_0^T t \left\| \frac{du}{dt}(t) \right\|_H^2\ dt \le \|u_0 - v\|_H^2 + \left(\int_0^T \|f(t)\|_H\ dt \right)^2 + \int_0^T t\, \|f(t)\|_H^2\ dt,$$

$$2 \int_0^t \Psi(u(s))\ ds + \|u(t) - v\|_H^2 \le 2\, \|u_0 - v\|_H^2 + \left(\int_0^T \|f(t)\|_H\ dt \right)^2$$

for all $v \in \Psi^{-1}(\{0\})$ and all $t \in [0,T]$. Moreover,

$$\int_\delta^T \left\| \frac{du}{dt}(t) \right\|_H^2\ dt \le \frac{2}{\delta} \|u_0 - v\|_H^2 + \frac{2}{\delta} \int_0^\delta \|f(t)\|_H\ dt + \int_0^T \|f(t)\|_H^2\ dt,$$

for every $0 < \delta < T$ and for all $v \in \Psi^{-1}(\{0\})$.

The proof makes use of the following Gronwall's type inequality.

Lemma 67 *Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, $\beta : I \to [0, \infty)$ be continuous functions, and let $\alpha \ge 0$. Assume that*

$$f^2(t) \le \alpha^2 + \int_{t_0}^t \beta(s)\, |f(s)|\ ds$$

for some $t_0 \in I$ and all $t \ge t_0$, $t \in I$. Then

$$|f(t)| \le \alpha + \frac{1}{2} \int_{t_0}^t \beta(s)\ ds$$

for all $t \ge t_0$, $t \in I$.

Proof. For $\varepsilon > 0$ let

$$g_\varepsilon (t) := (\alpha + \varepsilon)^2 + \int_{t_0}^{t} \beta (s) |f (s)| \ ds.$$

Then by the chain rule,

$$\begin{aligned} g_\varepsilon' (t) &= \beta (t) |f (t)| \leq \beta (t) \sqrt{\alpha^2 + \int_{t_0}^{t} \beta (s) |f (s)| \ ds} \\ &\leq \beta (t) \sqrt{g_\varepsilon (t)}. \end{aligned}$$

Since $g_\varepsilon (t) \geq \varepsilon^2$ for all $t \geq t_0$ and g_ε is absolutely continuous, it follows that

$$\frac{d}{dt} \sqrt{g_\varepsilon (t)} = \frac{1}{2\sqrt{g_\varepsilon (t)}} g_\varepsilon' (t) \leq \frac{1}{2\sqrt{g_\varepsilon (t)}} \beta (t) \sqrt{g_\varepsilon (t)} = \frac{\beta (t)}{2}.$$

Upon integration, for $t \geq t_0$, we get

$$\sqrt{g_\varepsilon (t)} \leq \sqrt{g_\varepsilon (t_0)} + \frac{1}{2} \int_{t_0}^{t} \beta (s) \ ds.$$

Hence,

$$\begin{aligned} |f (t)| &\leq \sqrt{\alpha^2 + \int_{t_0}^{t} \beta (s) |f (s)| \ ds} \leq \sqrt{g_\varepsilon (t)} \\ &\leq \sqrt{g_\varepsilon (t_0)} + \frac{1}{2} \int_{t_0}^{t} \beta (s) \ ds \\ &= \sqrt{(\alpha + \varepsilon)^2} + \frac{1}{2} \int_{t_0}^{t} \beta (s) \ ds. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we obtain that

$$|f (t)| \leq \alpha + \frac{1}{2} \int_{t_0}^{t} \beta (s) \ ds$$

for all $t \geq t_0$. This concludes the proof. ■

Friday, February 21, 2014

We turn to the proof of Theorem 66.

Proof. Step 1: Since $u_0 \in \overline{\operatorname{dom}_e \Psi}$, there exists a sequence $\{u_{0,n}\} \subseteq \operatorname{dom}_e \Psi$ such that $u_{0,n} \to u_0$ as $n \to \infty$. Let $u_n \in H^1 ((0, T) ; H)$ be the unique strong solution of the Cauchy problem

$$\begin{cases} f (t) - \frac{dw}{dt} (t) \in \partial \Psi (w (t)), \\ w (0) = u_{0,n}, \end{cases}$$

which exists by Theorem 58. Then

$$\frac{1}{2}\frac{d}{dt}\left\|u_n(t)-u_m(t)\right\|_H^2 = \left(\frac{du_n}{dt}(t)-\frac{du_m}{dt}(t), u_n(t)-u_m(t)\right)_H$$
$$= -\left(f(t)-\frac{du_n}{dt}(t)-\left(f(t)-\frac{du_m}{dt}(t)\right), u_n(t)-u_m(t)\right)_H \leq 0$$

by the monotonicity of the subdifferential of Ψ. Hence, upon integration,

$$\left\|u_n(t)-u_m(t)\right\|_H^2 \leq \left\|u_{0,n}-u_{0,m}\right\|_H^2$$

and so $\{u_n\}$ converges uniformly to a function $u \in C([0,T];H)$ with $u(0) = u_0$.

To prove that u is a strong solution, let $K := \{v \in H : \Psi(v) = 0\}$. Since $f(t) - \frac{du_n}{dt}(t) \in \partial\Psi(u_n(t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, we have that

$$\Psi(v) \geq \Psi(u_n(t)) + (f(t), v-u_n(t))_H$$
$$-\left(\frac{du_n}{dt}(t_0), v-u_n(t)\right)_H$$

for all $v \in K$. In turn.

$$\Psi(u_n(t)) + \frac{1}{2}\frac{d}{dt}\left\|u_n(t)-v\right\|_H^2 \leq \left\|f(t)\right\|_H \left\|u_n(t)-v\right\|_H .$$

Integrating in time, it follows that

$$2\int_0^t \Psi(u_n(s))\ ds + \left\|u_n(t)-v\right\|_H^2$$
$$\leq \left\|u_{0,n}-v\right\|_H^2 + \int_0^t 2\left\|f(s)\right\|_H \left\|u_n(s)-v\right\|_H\ ds$$
$$\leq \left\|u_{0,n}-v\right\|_H^2 + \int_0^t 2\left\|f(s)\right\|_H \left(\int_0^s 2\Psi(u_n(r))\ dr + \left\|u_n(t)-v\right\|_H^2\right)^{1/2}\ dt.$$

It follows by Lemma 67 that

$$2\int_0^t \Psi(u_n(s))\ ds+\left\|u_n(t)-v\right\|_H^2 \leq 2\left\|u_{0,n}-v\right\|_H^2+\left(\int_0^t \left\|f(s)\right\|_H\ dt\right)^2. \quad (98)$$

Since $\{u_n\}$ converges uniformly to u, letting $n \to \infty$ in the previous inequality and using Fatou's lemma gives

$$2\int_0^t \Psi(u(s))\ ds + \left\|u(t)-v\right\|_H^2 \leq 2\left\|u_0-v\right\|_H^2 + \left(\int_0^t \left\|f(s)\right\|_H\ dt\right)^2$$

for all $t \in [0,T]$.

On the other hand, by Theorem 58,

$$\frac{d}{dt}(\Psi \circ u_n)(t) + \left\|\frac{du_n}{dt}(t)\right\|_H^2 = \left(f(t), \frac{du_n}{dt}(t)\right)_H \quad (99)$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Multiplying the previous identity by t and integrating over $(0,T)$, we get

$$\int_0^T t \frac{d}{dt} (\Psi \circ u_n)(t) \, dt + \int_0^T t \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt = \int_0^T t \left(f(t), \frac{du_n}{dt}(t) \right)_H \, dt.$$

Integration by parts and (98) yield

$$\begin{aligned}
&T\Psi(u_n(T)) - 0 + \int_0^T t \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt \\
&= \int_0^T \Psi(u_n(t)) \, dt + \int_0^T t \left(f(t), \frac{du_n}{dt}(t) \right)_H \, dt \\
&\le \|u_{0,n} - v\|_H^2 + \left(\int_0^T \|f(t)\|_H \, dt \right)^2 + \frac{1}{2} \int_0^T t \|f(t)\|_H^2 \, dt + \frac{1}{2} \int_0^T t \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt,
\end{aligned}$$

which gives

$$\int_0^T t \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt \le \|u_{0,n} - v\|_H^2 + \left(\int_0^T \|f(t)\|_H \, dt \right)^2 + \int_0^T t \|f(t)\|_H^2 \, dt.$$

By taking $\delta > 0$, it follows from the previous inequality that

$$\begin{aligned}
\delta \int_\delta^T \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt &\le \int_0^T t \left\| \frac{du_n}{dt}(t) \right\|_H^2 \, dt \\
&\le \|u_{0,n} - v\|_H^2 + \left(\int_0^T \|f(t)\|_H \, dt \right)^2 + \int_0^T t \|f(t)\|_H^2 \, dt,
\end{aligned}$$

which shows that the sequence $\{u_n\}$ is bounded in $H^1((\delta,T);H)$. Since $H^1((\delta,T);H)$ is reflexive, up to a subsequence, not relabeled, $\{u_n\}$ converges weakly to some function $v \in H^1((\delta,T);H)$. On the other hand, since $\{u_n\}$ converges uniformly to u in $C([0,T];H)$, we must have that $u = v$. This shows that $u \in H^1((\delta,T);H)$ for all $\delta > 0$.

Reasoning as in the proof of (49) we conclude that u satisfies $f(t) - \frac{du}{dt}(t) \in \partial\Psi(u(t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$.

By Theorem 58, $\Psi \circ u_n$ is absolutely continuous, and so by the mean value theorem for every fixed $\delta > 0$ there exists $t_n \in (0,\delta)$ such that

$$\Psi(u_n(t_n)) = \frac{1}{\delta} \int_0^\delta \Psi(u_n(t)) \, dt.$$

Integrating (99) over over (t_n, T), we get $\frac{d}{dt}(\Psi \circ u_n)(t) + \left\| \frac{du_n}{dt}(t) \right\|_H^2 = \left(f(t), \frac{du_n}{dt}(t) \right)_H$

$$\Psi(u_n(T)) + \int_{t_n}^T \left\| \frac{du_n}{dt}(t) \right\|_H^2 dt = \Psi(u_n(t_n)) + \int_{t_n}^T \left(f(t), \frac{du_n}{dt}(t) \right)_H dt$$
$$\leq \frac{1}{\delta} \int_0^\delta \Psi(u_n(t)) \, dt + \frac{1}{2} \int_{t_n}^T \left\| \frac{du_n}{dt}(t) \right\|_H^2 dt$$
$$+ \frac{1}{2} \int_{t_n}^T \|f(t)\|_H^2 \, dt.$$

Hence,

$$\frac{1}{2} \int_\delta^T \left\| \frac{du_n}{dt}(t) \right\|_H^2 dt \leq \frac{1}{\delta} \int_0^\delta \Psi(u_n(t)) \, dt + \frac{1}{2} \int_0^T \|f(t)\|_H^2 \, dt$$
$$\leq \frac{1}{\delta} \|u_{0,n} - v\|_H^2 + \frac{1}{\delta} \int_0^\delta \|f(s)\|_H \, dt + \frac{1}{2} \int_0^T \|f(t)\|_H^2 \, dt,$$

where we used (98). It suffices to let $n \to \infty$.

Step 2: Uniqueness. To prove uniqueness, let v be another solution. Then

$$\frac{1}{2} \frac{d}{dt} \|u(t) - v(t)\|_H^2 = \left(\frac{du}{dt}(t) - \frac{dv}{dt}(t), u(t) - v(t) \right)_H$$
$$= -\left(f(t) - \frac{du}{dt}(t) - \left(f(t) - \frac{dv}{dt}(t) \right), u(t) - v(t) \right)_H \leq 0$$

by the monotonicity of the subdifferential of Ψ. Since u and v are absolutely continuous on compact sets of $(0, T)$, we have that for all $0 < s < t < T$,

$$\|u(t) - v(t)\|_H^2 \leq \|u(s) - v(s)\|_H^2 .$$

But $u, v \in C([0, T]; H)$ and $u(0) = v(0) = u_0$. Hence, by letting $s \to 0^+$, the right-hand side of the previous inequality goes to zero. it follows that $u = v$. ■

Monday, February 24, 2014

Theorem 68 (Existence of Weak Solutions) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in L^2(\Omega)$, let $f \in L^2((0, T); L^2(\Omega))$, where $0 < T < \infty$. Then there exists a unique function $u \in C([0, T]; L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ such that $\sqrt{t} \frac{\partial u}{\partial t} \in L^2((0, T); L^2(\Omega))$, $u(\cdot, 0) = u_0$ and*

$$\int_\Omega \nabla u(\boldsymbol{x}, t) \cdot \nabla w(\boldsymbol{x}) \, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x}, t) w(\boldsymbol{x}) \, d\boldsymbol{x} + \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t) w(\boldsymbol{x}) \, d\boldsymbol{x} = 0$$

for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0, T)$. Moreover,

$$\|\nabla u\|_{L^2((0,T);L^2(\Omega))} + \max_{t \in [0,T]} \|u(\cdot, t)\|_{L^2(\Omega)} \leq 2 \|u_0\|_{L^2(\Omega)} + 2 \|f\|_{L^1((0,T);L^2(\Omega))}$$

$$\left\| \sqrt{t} \frac{\partial u}{\partial t} \right\|_{L^2((0,T);L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)} + \|f\|_{L^1((0,T);L^2(\Omega))} + \left\| \sqrt{t} f \right\|_{L^2((0,T);L^2(\Omega))} .$$

Proof. We apply Theorem 66 with $H = L^2(\Omega)$ and

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} & \text{if } v \in H^1(\Omega), \\ \infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega). \end{cases}$$

to find a unique function $u \in C\left([0,T]; L^2(\Omega)\right)$ such that u is absolutely continuous on compact sets of $(0,T)$, $f(\cdot,t) - \frac{\partial u}{\partial t}(\cdot,t) \in \partial\Psi(u(\cdot,t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, and $u(\cdot,0) = u_0$. By Example 48, it follows that for $\mathcal{L}^1$ a.e. $t \in (0,T)$ the function $u(\cdot,t)$ belongs to $H^1(\Omega)$ and satisfies

$$\int_\Omega \nabla u(\boldsymbol{x},t) \cdot \nabla w(\boldsymbol{x}) \, d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x},t) w(\boldsymbol{x}) \, d\boldsymbol{x} + \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x},t) w(\boldsymbol{x}) \, d\boldsymbol{x} = 0 \tag{100}$$

for all $w \in H^1(\Omega)$.

Moreover, since $\Psi(0) = 0$, again by Theorem 66, with $v = 0$,

$$\int_0^T \int_\Omega t \left|\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right|^2 d\boldsymbol{x}dt \le \|u_0\|^2_{L^2(\Omega)} + \left(\int_0^T \left(\int_\Omega f^2(\boldsymbol{x},t) \, d\boldsymbol{x}\right)^{1/2} dt\right)^2 + \int_0^T \int_\Omega t f^2(\boldsymbol{x},t) \, d\boldsymbol{x}dt$$

and

$$\begin{aligned} \int_0^t \int_\Omega \|\nabla u(\boldsymbol{x},s)\|^2 \, d\boldsymbol{x}ds + \|u(\cdot,t)\|^2_{L^2(\Omega)} &\le 2\|u_0\|^2_{L^2(\Omega)} \\ &+ \left(\int_0^T \left(\int_\Omega f^2(\boldsymbol{x},t) \, d\boldsymbol{x}\right)^{1/2} dt\right)^2 \end{aligned} \tag{101}$$

for all $t \in [0,T]$. ■

Exercise 69 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in L^2(\Omega)$, and let $f \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$, where $0 < T < \infty$. Prove that there exists a unique function $u \in L^2\left((0,T); H^1(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$ such that $u(\cdot,t) = u_0$ and*

$$\int_\Omega \nabla u(\boldsymbol{x},t) \cdot \nabla w(\boldsymbol{x}) \, d\boldsymbol{x} - \langle f(\cdot,t), w\rangle_{(H^1(\Omega))', H^1(\Omega)} + \left\langle \frac{\partial u}{\partial t}(\cdot,t), w\right\rangle_{(H^1(\Omega))', H^1(\Omega)} = 0 \tag{102}$$

for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$.

13 Higher Order Regularity

Consider the Neumann problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases} \tag{103}$$

So far we have seen that for bounded smooth domains Ω,

$$\begin{array}{lcl} u_0 \in L^2(\Omega) & & u \in L^2\left((0,T);H^1(\Omega)\right) \\ f \in L^2\left((0,T);\left(H^1(\Omega)\right)'\right) & \Longrightarrow & \frac{\partial u}{\partial t} \in L^2\left((0,T);\left(H^1(\Omega)\right)'\right) \\ u_0 \in H^1(\Omega) & & u \in L^2\left((0,T);H^2(\Omega)\right) \\ f \in L^2\left((0,T);L^2(\Omega)\right) & \Longrightarrow & \frac{\partial u}{\partial t} \in L^2\left((0,T);L^2(\Omega)\right) \end{array}$$

We want to find necessary and sufficient conditions for $u \in L^2\left((0,T);H^3(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T);H^1(\Omega)\right)$.

Theorem 70 (H^3 Regularity) *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with C^3 boundary. Assume that $u_0 \in H^2(\Omega)$, $f \in L^2\left((0,T);H^1(\Omega)\right)$, with $\frac{\partial f}{\partial t} \in L^2\left((0,T);\left(H^1(\Omega)\right)'\right)$. Let u be the unique weak solution of (103). Then a necessary and sufficient condition for u to belong to $L^2\left((0,T);H^3(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T);H^1(\Omega)\right)$ is that*

$$\frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega. \tag{104}$$

In either case, $\frac{\partial^2 u}{\partial t^2} \in L^2\left((0,T);\left(H^1(\Omega)\right)'\right)$ and the following estimates hold

$$\|u\|_{L^2((0,T);H^3(\Omega))} \le C(\Omega,T)\left(\|\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}\right).$$

and

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))} \le c_T\left(\|\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}\right), \tag{105}$$

$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2((0,T);(H^1(\Omega))')} \le c_T\left(\|\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}\right), \tag{106}$$

Proof. Step 1: Assume that (104) holds. If formally we differentiate the equation with respect to t we obtain

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \Delta\frac{\partial u}{\partial t}(\boldsymbol{x},t) = \frac{\partial f}{\partial t}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ \frac{\partial u}{\partial t}(\boldsymbol{x},0) = \Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0) & \boldsymbol{x} \in \Omega, \\ \frac{\partial}{\partial \boldsymbol{n}}\left(\frac{\partial u}{\partial t}\right)(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T), \end{cases}$$

where we used the fact that for smooth solutions,

$$\begin{aligned} \lim_{t\to 0^+}\frac{\partial u}{\partial t}(\boldsymbol{x},t) &= \lim_{t\to 0^+}\left(\Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t)\right) = \Delta u(\boldsymbol{x},0) + f(\boldsymbol{x},0) \\ &= \Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0). \end{aligned}$$

Hence, we consider the Neumann problem

$$\begin{cases} \frac{\partial v}{\partial t}(\boldsymbol{x},t) - \Delta v(\boldsymbol{x},t) = \frac{\partial f}{\partial t}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ v(\boldsymbol{x},0) = \Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0) & \boldsymbol{x} \in \Omega, \\ \frac{\partial v}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases} \tag{107}$$

Note that by Theorem 35, $f \in C\left([0,T]; L^2(\Omega)\right)$ and so $f(\cdot,0) \in L^2(\Omega)$. Hence, by Theorem 66 and Exercise 69 this problem admits a unique strong solution $v \in L^2\left((0,T); H^1(\Omega)\right)$ with $\frac{\partial v}{\partial t} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$ such that $v(\cdot,0) = \Delta u_0 + f(\cdot,0)$ and

$$\int_\Omega \nabla v(\boldsymbol{x},t)\cdot\nabla w(\boldsymbol{x})\ d\boldsymbol{x} - \left\langle \frac{\partial f}{\partial t}(\cdot,t), w \right\rangle_{(H^1(\Omega))',H^1(\Omega)} + \left\langle \frac{\partial v}{\partial t}(\cdot,t), w \right\rangle_{(H^1(\Omega))',H^1(\Omega)} = 0 \tag{108}$$

for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Moreover,

$$\|\nabla v\|_{L^2((0,T);L^2(\Omega))} + \max_{t\in[0,T]} \|v(\cdot,t)\|_{L^2(\Omega)} \le 2\|\Delta u_0 + f(\cdot,0)\|_{L^2(\Omega)} + 2\left\|\frac{\partial f}{\partial t}\right\|_{L^1((0,T);(H^1(\Omega))')}$$

$$\left\|\frac{\partial v}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')} \le \|\Delta u_0 + f(\cdot,0)\|_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial t}\right\|_{L^1((0,T);L^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}.$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Integrating (108) with respect to time and using Fubini's theorem, we get

$$\int_\Omega \int_0^t \nabla v(\boldsymbol{x},s)\ ds\cdot\nabla w(\boldsymbol{x})\ d\boldsymbol{x} - \int_\Omega (f(\boldsymbol{x},t) - f(\boldsymbol{x},0))\, w(\boldsymbol{x})\ d\boldsymbol{x} + \int_\Omega (v(\boldsymbol{x},t) - v(\boldsymbol{x},0))\, w(\boldsymbol{x})\ d\boldsymbol{x} = 0$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and all $w \in H^1(\Omega)$. Using $v(\boldsymbol{x},0) = \Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0)$, we get

$$\begin{aligned} &\int_\Omega \left(\nabla u_0(\boldsymbol{x}) + \int_0^t \nabla v(\boldsymbol{x},s)\ ds\right)\cdot\nabla w(\boldsymbol{x})\ d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x},t)\, w(\boldsymbol{x})\ d\boldsymbol{x} \\ &+ \int_\Omega v(\boldsymbol{x},t)\, w(\boldsymbol{x})\ d\boldsymbol{x} = 0, \end{aligned}$$

where we have used the divergence theorem and (104). Define

$$u(\boldsymbol{x},t) := u_0(\boldsymbol{x}) + \int_0^t v(\boldsymbol{x},s)\ ds.$$

Then $\frac{\partial u}{\partial t} = v$ and $\nabla u(\boldsymbol{x},t) = \nabla u_0(\boldsymbol{x}) + \int_0^t \nabla v(\boldsymbol{x},s)\ ds$. Hence, the previous equation becomes

$$\int_\Omega \nabla u(\boldsymbol{x},t)\ ds\cdot\nabla w(\boldsymbol{x})\ d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x},t)\, w(\boldsymbol{x})\ d\boldsymbol{x} + \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x},t)\, w(\boldsymbol{x})\ d\boldsymbol{x} = 0.$$

Then by uniqueness of weak solutions it follows that u is the solution of our problem (103). Moreover, since $\frac{\partial u}{\partial t} = v$, we have that (105) and (106) hold, using the inequality

$$\max_{t\in[0,T]} \|f(\cdot,t)\|_{L^2(\Omega)} \leq \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}$$

(see Theorem 35). In turn,

$$-\Delta u(\cdot,t) = f(\cdot,t) - \frac{\partial u}{\partial t}(\cdot,t) \in H^1(\Omega)$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, and so by elliptic regularity $u(\cdot,t) \in H^3(\Omega)$ with

$$\|u(\cdot,t)\|_{H^3(\Omega)} \leq C(\Omega)\left(\|f(\cdot,t)\|_{H^1(\Omega)} + \left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{H^1(\Omega)}\right). \tag{109}$$

By squaring the previous inequality and integrating with respect to time, we get

$$\begin{aligned}\|u\|_{L^2((0,T);H^3(\Omega))} &\leq C(\Omega)\left(\|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))}\right)\\ &\leq C(\Omega,T)\left(\|\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}\right).\end{aligned}$$

Step 2: Conversely, assume that the unique weak solution u of (103) belongs to $u \in L^2\left((0,T);H^3(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T);H^1(\Omega)\right)$. Then by Theorem 35, $u \in C\left([0,T];H^2(\Omega)\right)$ and so $\nabla u \in C\left([0,T];H^1\left(\Omega;\mathbb{R}^N\right)\right)$. In turn, $\frac{\partial u}{\partial \boldsymbol{n}} \in C\left([0,T];H^{1/2}(\partial\Omega)\right)$, and so by (55),

$$0 = \lim_{t\to 0^+} \frac{\partial u}{\partial \boldsymbol{n}}(\cdot,t) = \frac{\partial u_0}{\partial \boldsymbol{n}}(\cdot) \text{ in } H^{1/2}(\partial\Omega)$$

This shows that the compatibility condition (104) is necessary. ■

Remark 71 *If Ω is an unbounded open set, in place of (109), we can use interior regularity for elliptic problems to conclude that*

$$\begin{aligned}\left\|\nabla^3 u(\cdot,t)\right\|_{L^2(U)} \leq{}& \frac{C(N)}{\operatorname{dist}^{1/2}(U,\partial\Omega)}\left(\|f(\cdot,t)\|_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)}\right)\\ &+ C(N)\left(\|\nabla f(\cdot,t)\|_{L^2(\Omega)} + \left\|\nabla\left(\frac{\partial u}{\partial t}\right)(\cdot,t)\right\|_{L^2(\Omega)}\right)\\ &+ \frac{C(N)}{\operatorname{dist}(U,\partial\Omega)}\|\nabla u(\cdot,t)\|_{L^2(\Omega)}\end{aligned}$$

for every $U \Subset \Omega$. By squaring the previous inequality and integrating with respect to time, we get

$$\left\|\nabla^3 u\right\|_{L^2((0,T);L^2(U))} \leq \frac{C(N)}{\operatorname{dist}^{1/2}(U,\partial\Omega)}\left(\|f\|_{L^2((0,T);L^2(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}\right)$$
$$+ C(N)\left(\|\nabla f\|_{L^2((0,T);L^2(\Omega))} + \left\|\nabla\left(\frac{\partial u}{\partial t}\right)\right\|_{L^2((0,T);L^2(\Omega))}\right)$$
$$+ \frac{C(N)}{\operatorname{dist}(U,\partial\Omega)}\|\nabla u\|_{L^2((0,T);L^2(\Omega))}.$$

In turn, using (105) and (106) we get

$$\left\|\nabla^3 u\right\|_{L^2((0,T);L^2(U))} \leq \frac{c_T}{\operatorname{dist}^{1/2}(U,\partial\Omega)}\left(\|u_0\|_{H^2(\Omega)} + \|f\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')}\right).$$

Wednesday, February 26, 2014

Next we consider H^4 regularity in space.

Theorem 72 (H^4 Regularity) *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with C^4 boundary. Assume that $u_0 \in H^3(\Omega)$, $f \in L^2((0,T);H^2(\Omega))$, with $\frac{\partial f}{\partial t} \in L^2((0,T);L^2(\Omega))$. Assume that* (104) *holds. Let u be the unique weak solution of* (103)*. Then u to belong to $L^2((0,T);H^4(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2((0,T);H^2(\Omega))$ and the following estimates hold*

$$\left\|\nabla^4 u\right\|_{L^2((0,T);L^2(\Omega))} \leq C(N,\Omega)\left(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}\right),$$
$$\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2((0,T);L^2(\Omega))} \leq \|\nabla\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}.$$

Proof. Consider the solution v of (107). Since the initial datum $\Delta u_0 + f(\cdot,0) \in H^1(\Omega)$, by Theorem 60 we have that v belongs to $L^2((0,T);H^2(\Omega))$ with

$$\left\|\nabla^2 v\right\|_{L^2((0,T);L^2(\Omega))} \leq C(N,\Omega)\left(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|\nabla f(\cdot,0)\|_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}\right),$$
$$\left\|\frac{\partial v}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))} \leq \|\nabla\Delta u_0\|_{L^2(\Omega)} + \|\nabla f(\cdot,0)\|_{L^2(\Omega)} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}.$$

Since $\frac{\partial u}{\partial t} = v$, it follows that

$$-\Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) - \frac{\partial u}{\partial t}(\boldsymbol{x},t) \in H^2(\Omega).$$

Hence, by elliptic regularity, $u(\cdot,t) \in H^4(\Omega)$ with

$$\left\|\nabla^4 u(\cdot,t)\right\|_{L^2(\Omega)} \leq C(N,\Omega)\left(\|f(\cdot,t)\|_{H^2(\Omega)} + \left\|\frac{\partial u}{\partial t}(\cdot,0)\right\|_{H^2(\Omega)}\right).$$

By squaring and integrating in time, we get

$$\begin{aligned}\left\|\nabla^4 u\right\|_{L^2((0,T);L^2(\Omega))} &\leq C(N,\Omega)\left(\|f\|_{L^2((0,T);H^2(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);H^2(\Omega))}\right), \\ &\leq C(N,\Omega)\left(\|\nabla\Delta u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}\right),\end{aligned}$$

where we have used the inequality

$$\max_{t\in[0,T]}\|\nabla f(\cdot,t)\|_{L^2(\Omega)} \leq \|f\|_{L^2((0,T);H^2(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);L^2(\Omega))}$$

(see Theorem 35). ■

Remark 73 *If Ω is an open unbounded set, then we can use interior regularity for elliptic problems to conclude that $u \in L^2\left((0,T);H^4_{\mathrm{loc}}(\Omega)\right)$. The details are similar to Remark 71 and we omit them.*

We want to find necessary and sufficient conditions for $u \in L^2\left((0,T);H^5(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in u \in L^2\left((0,T);H^3(\Omega)\right)$.

Theorem 74 (H^5 Regularity) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^5 boundary. Assume that $u_0 \in H^4(\Omega)$, $f \in L^2\left((0,T);H^3(\Omega)\right)$, with $\frac{\partial f}{\partial t} \in L^2\left((0,T);H^1(\Omega)\right)$ and $\frac{\partial^2 f}{\partial t^2} \in L^2\left((0,T);\left(H^1(\Omega)\right)'\right)$. Let u be the unique weak solution of (103). Then a necessary and sufficient condition for u to belong to $L^2\left((0,T);H^5(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T);H^3(\Omega)\right)$ is that*

$$\frac{\partial u_0}{\partial \boldsymbol{n}} = 0 \text{ and } \frac{\partial}{\partial \boldsymbol{n}}\left(\Delta u_0 + f(\cdot,0)\right) = 0 \text{ on } \partial\Omega. \tag{110}$$

In either case, $\sqrt{t}\frac{\partial^2 u}{\partial t^2} \in L^2\left((0,T);L^2(\Omega)\right)$ and the following estimates hold

$$\|u\|_{L^2((0,T);H^5(\Omega))} \leq C(\Omega,T)\Bigg(\left\|\Delta^2 u_0\right\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^3(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2((0,T);(H^1(\Omega))}$$

and

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))} &\leq c_T\Bigg(\left\|\Delta^2 u_0\right\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^3(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2((0,T);(H^1(\Omega))'} \\ \left\|\sqrt{t}\frac{\partial^2 u}{\partial t^2}\right\|_{L^2((0,T);L^2(\Omega))} &\leq c_T\Bigg(\left\|\Delta^2 u_0\right\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);H^3(\Omega))} + \left\|\frac{\partial f}{\partial t}\right\|_{L^2((0,T);H^1(\Omega))} + \left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2((0,T);(H^1(\Omega))'}\end{aligned}$$

Proof. Step 1: Assume that (110) holds. Recall that (107) is

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) - \Delta \frac{\partial u}{\partial t}(\boldsymbol{x},t) = \frac{\partial f}{\partial t}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ \frac{\partial u}{\partial t}(\boldsymbol{x},0) = \Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0) & \boldsymbol{x} \in \Omega, \\ \frac{\partial}{\partial \boldsymbol{n}}\left(\frac{\partial u}{\partial t}\right)(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

If formally we differentiate the equation with respect to t we obtain

$$\begin{cases} \frac{\partial^3 u}{\partial t^3}(\boldsymbol{x},t) - \Delta \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) = \frac{\partial^2 f}{\partial t^2}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},0) = \Delta^2 u_0(\boldsymbol{x}) + \Delta f(\boldsymbol{x},0) + \frac{\partial f}{\partial t}(\boldsymbol{x},0) & \boldsymbol{x} \in \Omega, \\ \frac{\partial}{\partial \boldsymbol{n}}\left(\frac{\partial^2 u}{\partial t^2}\right)(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

where we used the fact that for smooth solutions,

$$\begin{aligned} \lim_{t\to 0^+} \frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) &= \lim_{t\to 0^+}\left(\Delta \frac{\partial u}{\partial t}(\boldsymbol{x},t) + \frac{\partial f}{\partial t}(\boldsymbol{x},t)\right) = \Delta \frac{\partial u}{\partial t}(\boldsymbol{x},0) + \frac{\partial f}{\partial t}(\boldsymbol{x},0) \\ &= \Delta\left(\Delta u_0(\boldsymbol{x}) + f(\boldsymbol{x},0)\right) + \frac{\partial f}{\partial t}(\boldsymbol{x},0). \end{aligned}$$

Hence, we consider the Neumann problem

$$\begin{cases} \frac{\partial z}{\partial t}(\boldsymbol{x},t) - \Delta z(\boldsymbol{x},t) = \frac{\partial^2 f}{\partial t^2}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ z(\boldsymbol{x},0) = \Delta^2 u_0(\boldsymbol{x}) + \Delta f(\boldsymbol{x},0) + \frac{\partial f}{\partial t}(\boldsymbol{x},0) & \boldsymbol{x} \in \Omega, \\ \frac{\partial z}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

We can now proceed as in the proof of Theorem By Theorem 70. We omit the details. ∎

Remark 75 *If Ω is an open unbounded set, then we can use interior regularity for elliptic problems to conclude that $u \in L^2\left((0,T); H^5_{\text{loc}}(\Omega)\right)$. The details are similar to Remark 71 and we omit them.*

Friday, February 28, 2014

14 Global Existence and Asymptotic Stability

Theorem 76 *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, let $u_0 \in L^2(\Omega)$. Let $f \in L^2\left((0,\infty); \left(H^1(\Omega)\right)'\right)$ then the solution of*

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t > 0. \end{cases}$$

exists for all times and

$$\lim_{t\to\infty} \|u(\cdot,t) - m(t)\|_{L^2(\Omega)} = 0,$$

where

$$\begin{aligned} m(t) &:= \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(\boldsymbol{x},t)\ d\boldsymbol{x} \\ &= \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u_0(\boldsymbol{x})\ d\boldsymbol{x} + \frac{1}{\mathcal{L}^N(\Omega)} \int_0^t \langle f(\cdot,t), 1\rangle_{(H^1(\Omega))', H^1(\Omega)}\, ds. \end{aligned}$$

Proof. Existence follows by applying Exercise 69 for every $T > 0$. Since Ω has finite measure, constant functions are in $H^1(\Omega)$. Hence, by taking $w = 1$ in (102), we get

$$\left\langle \frac{\partial u}{\partial t}(\cdot,t), 1\right\rangle_{(H^1(\Omega))', H^1(\Omega)} - \langle f(\cdot,t), 1\rangle_{(H^1(\Omega))', H^1(\Omega)} = 0.$$

Given $\varphi \in C_c^1(0,\infty)$,

$$\begin{aligned} \int_0^\infty \varphi'(t)\, m(t)\ dt &= \frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \int_\Omega \varphi'(t)\, u(\boldsymbol{x},t)\ d\boldsymbol{x}\ dt \\ &= -\frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \varphi(t) \left\langle \frac{\partial u}{\partial t}(\cdot,t), 1\right\rangle_{(H^1(\Omega))', H^1(\Omega)} dt \\ &= -\frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \varphi(t) \langle f(\cdot,t), 1\rangle_{(H^1(\Omega))', H^1(\Omega)}\ dt, \end{aligned}$$

which shows that the weak derivative of m is given by

$$m'(t) = \frac{1}{\mathcal{L}^N(\Omega)} \langle f(\cdot,t), 1\rangle_{(H^1(\Omega))', H^1(\Omega)}.$$

By Theorem 35, for $\mathcal{L}^1$ a.e. $t > 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t) - m(t)\|_{L^2(\Omega)}^2\right) &= \left\langle \frac{\partial u}{\partial t}(\cdot,t) - m'(t), u(\cdot,t) - m(t)\right\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ &= \left\langle \frac{\partial u}{\partial t}(\cdot,t), u(\cdot,t) - m(t)\right\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ &\quad - \langle m'(t), u(\cdot,t) - m(t)\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ &= -\int_\Omega \|\nabla u(\boldsymbol{x},t)\|^2\ d\boldsymbol{x} - \langle f(\cdot,t), u(\cdot,t) - m(t)\rangle_{(H^1(\Omega))', H^1(\Omega)}, \end{aligned}$$

where we used the fact that

$$\begin{aligned} \langle m'(t), u(\cdot,t) - m(t)\rangle_{(H^1(\Omega))', H^1(\Omega)} &= m'(t)\langle 1, u(\cdot,t) - m(t)\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ &= m'(t) \int_\Omega (u(\boldsymbol{x},t) - m(t))\ d\boldsymbol{x} = 0. \end{aligned}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}$$
$$\leq\left\|f\left(\cdot,t\right)\right\|_{\left(H^{1}\left(\Omega\right)\right)'}\left\|u\left(\cdot,t\right)-m\left(t\right)\right\|_{H^{1}\left(\Omega\right)}$$
$$\leq\frac{\delta}{2}\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}+\frac{\delta}{2}\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}+\frac{1}{2\delta}\left\|f\left(\cdot,t\right)\right\|_{\left(H^{1}\left(\Omega\right)\right)'}^{2}.$$

In turn, for $0<\delta\leq 1$,

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\frac{1}{2}\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}$$
$$\leq\frac{\delta}{2}\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}+\frac{1}{2\delta}\left\|f\left(\cdot,t\right)\right\|_{\left(H^{1}\left(\Omega\right)\right)'}^{2}.$$

By Poincaré's inequality,

$$\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\leq c\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}$$

for some $c=c\left(\Omega\right)\geq 1$, and so

$$\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\left(\frac{1}{c}-\delta\right)\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}$$
$$\leq\frac{1}{\delta}\left\|f\left(\cdot,t\right)\right\|_{\left(H^{1}\left(\Omega\right)\right)'}^{2}.$$

Let $\delta=\frac{1}{2c}$. It follows that

$$\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\frac{1}{2c}\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}$$
$$\leq\frac{1}{\delta}\left\|f\left(\cdot,t\right)\right\|_{\left(H^{1}\left(\Omega\right)\right)'}^{2}$$

for $\mathcal{L}^{1}$ a.e. $t\in\left(0,\infty\right)$. Set

$$h\left(t\right):=\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}.$$

Then we can apply Lemma 77 to obtain the desired result.

****

In class we did the following simplified version: Assume that Ω is of class C^{2}, $u_{0}\in H^{1}\left(\Omega\right)$, and $f\in L^{2}\left(\left(0,T\right);L^{2}\left(\Omega\right)\right)$, so that $u\in L^{2}\left(\Omega\times\left(0,T\right);H^{2}\left(\Omega\right)\right)$ by Theorem 60. Define

$$m\left(t\right):=\frac{1}{\mathcal{L}^{N}\left(\Omega\right)}\int_{\Omega}u\left(\boldsymbol{x},t\right)\ d\boldsymbol{x}.$$

For $\mathcal{L}^1$ a.e. $t \in (0, \infty)$ we have

$$\int_\Omega \nabla w(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}, t) \, d\boldsymbol{x} - \int_\Omega w(\boldsymbol{x}) f(\boldsymbol{x}, t) \, d\boldsymbol{x} + \int_\Omega w(\boldsymbol{x}) \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x} = 0$$

for all $w \in H^1(\Omega)$. Since Ω has finite measure, constant functions are in $H^1(\Omega)$. Taking $w = 1$, we get

$$-\int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x} + \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x} = 0.$$

Hence, given $\varphi \in C_c^1(0, \infty)$, integrating by parts,

$$\begin{aligned}
\int_0^\infty \varphi'(t) m(t) \, dt &= \frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \int_\Omega \varphi'(t) u(\boldsymbol{x}, t) \, d\boldsymbol{x} dt \\
&= -\frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \varphi(t) \int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t) \, d\boldsymbol{x} dt \\
&= -\frac{1}{\mathcal{L}^N(\Omega)} \int_0^\infty \varphi(t) \int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x} dt,
\end{aligned}$$

which shows that the weak derivative of m is given by

$$m'(t) = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega f(\boldsymbol{x}, t) \, d\boldsymbol{x}.$$

In turn, by Theorem 35,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u(\boldsymbol{x}, t) - m(t))^2 \, d\boldsymbol{x}\right) = \int_\Omega \left(\frac{\partial u}{\partial t}(\boldsymbol{x}, t) - m'(t)\right)(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} \\
&\quad = \int_\Omega (\Delta u(\boldsymbol{x}, t) + f(\boldsymbol{x}, t) - m'(t))(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} \\
&\quad = \int_\Omega \Delta u(\boldsymbol{x}, t)(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} + \int_\Omega (f(\boldsymbol{x}, t) - m'(t))(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} \\
&\quad = -\int_\Omega \|\nabla u(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} + \int_\Omega f(\boldsymbol{x}, t)(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} \\
&\qquad + m'(t) \int_\Omega (u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} \\
&\quad = -\int_\Omega \|\nabla u(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} + \int_\Omega f(\boldsymbol{x}, t)(u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x},
\end{aligned}$$

where we used the divergence theorem and the fact that $\int_\Omega (u(\boldsymbol{x}, t) - m(t)) \, d\boldsymbol{x} = 0$. Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u(\boldsymbol{x}, t) - m(t))^2 \, d\boldsymbol{x}\right) + \int_\Omega \|\nabla u(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} \\
&\leq \|f(\cdot, t)\|_{L^2(\Omega)} \|u(\cdot, t) - m(t)\|_{L^2(\Omega)} \\
&\leq \frac{\delta}{2} \int_\Omega (u(\boldsymbol{x}, t) - m(t))^2 \, d\boldsymbol{x} + \frac{1}{2\delta} \|f(\cdot, t)\|_{L^2(\Omega)}^2 .
\end{aligned}$$

In turn, for $0 < \delta \leq 1$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}\\&\leq\frac{\delta}{2}\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}+\frac{1}{2\delta}\left\|f\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}.\end{aligned}$$

By Poincaré's inequality,

$$\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\leq c\int_{\Omega}\left\|\nabla u\left(\boldsymbol{x},t\right)\right\|^{2}\ d\boldsymbol{x}$$

for some $c = c(\Omega) \geq 1$, and so

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\left(\frac{1}{c}-\delta\right)\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\\&\leq\frac{1}{2\delta}\left\|f\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}.\end{aligned}$$

Let $\delta = \frac{1}{2c}$. It follows that

$$\begin{aligned}&\frac{d}{dt}\left(\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\right)+\frac{1}{c}\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}\\&\leq\frac{1}{2\delta}\left\|f\left(\cdot,t\right)\right\|_{L^{2}(\Omega)}^{2}\end{aligned}$$

for $\mathcal{L}^1$ a.e. $t \in (0, \infty)$. Set

$$h\left(t\right):=\int_{\Omega}\left(u\left(\boldsymbol{x},t\right)-m\left(t\right)\right)^{2}\ d\boldsymbol{x}.$$

Then we can apply Lemma 77 to obtain that $\lim_{t\to\infty} h(t) = 0$. Recalling that by (56),

$$m\left(t\right)=\frac{1}{\mathcal{L}^{N}\left(\Omega\right)}\int_{\Omega}u\left(\boldsymbol{x},t\right)\ d\boldsymbol{x}=\frac{1}{\mathcal{L}^{N}\left(\Omega\right)}\int_{\Omega}u_{0}\left(\boldsymbol{x}\right)\ d\boldsymbol{x}+\frac{1}{\mathcal{L}^{N}\left(\Omega\right)}\int_{0}^{t}\int_{\Omega}f\left(\boldsymbol{x},s\right)\ d\boldsymbol{x}ds.$$

we obtain the desired result. ■

Lemma 77 *Let $h : [0, \infty) \to [0, \infty)$ be locally absolutely continuous, integrable, and such that*

$$h'\left(t\right)+ah\left(t\right)\leq g\left(t\right),$$

where $g : [0, \infty) \to [0, \infty)$ is integrable and $a > 0$. Then

$$\lim_{t\to\infty}h\left(t\right)=0.$$

Proof. Multiply the inequality by e^{at} to get

$$\left(h(t)\,e^{at}\right)' \le e^{at}g(t).$$

By integrating in $[r,t]$ we obtain

$$h(t)\,e^{at} \le h(r)\,e^{ar} + \int_r^t e^{as}g(s)\;ds,$$

and so

$$\begin{aligned} h(t) &\le h(r)\,e^{a(r-t)} + \int_r^t e^{a(s-t)}g(s)\;ds \\ &\le h(r) + \int_r^\infty g(s)\;ds. \end{aligned}$$

Since h is integrable, we have that

$$\liminf_{t\to\infty} h(t) = 0$$

and so there exists a sequence $r_n \to \infty$ such that $h(r_n) \to 0$. On the other hand, since g is integrable,

$$\lim_{r\to\infty}\int_r^\infty g(s)\;ds = 0,$$

and so given $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\int_t^\infty g(s)\;ds \le \varepsilon$$

for all $t \ge t_\varepsilon$. Find n_ε such that $r_n \ge t_\varepsilon$ and $h(r_n) \le \varepsilon$ for all $n \ge n_\varepsilon$. Then for all $t \ge r_n$,

$$0 \le h(t) \le h(r_n) + \int_{r_n}^\infty g(s)\;ds \le 2\varepsilon.$$

This concludes the proof. ■

Monday, March 03, 2014

15 Maximum and Comparison Principles

Given a set $U \subseteq \mathbb{R}^N \times \mathbb{R}$, the *parabolic boundary* $\partial_p U$ of U is the set of all $(\boldsymbol{x},t) \in \partial U$ such that for every $r > 0$ the cylinder

$$B_N(\boldsymbol{x},r) \times (t-r,t)$$

contains points not in U. For example, if $U = \Omega \times (0,T)$, then $\partial_p U$ is given by the boundary minus the top, to be precise,

$$\partial_p U = \overline{\Omega} \times [0,T] \setminus (\Omega \times (0,T]).$$

Theorem 78 (Weak Maximum Principle) *Let $U = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $T > 0$, and let $u : \overline{U} \to \mathbb{R}$ be continuous and such that $\frac{\partial u}{\partial t}$ exists in U, ∇u and $\nabla^2 u$ exist in U and are continuous, and*

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) \leq 0 \quad \text{in } U.$$

Then the maximum of u occurs on $\partial_p U$.

Proof. Let

$$M := \sup_{\partial_p U} u.$$

By replacing u with $u - M$, without loss of generality, we may assume that $u \leq 0$ on $\partial_p U$. We want to prove that $u \leq 0$ in $\overline{U}$. For $\varepsilon > 0$, let

$$u_\varepsilon(\boldsymbol{x}, t) := u(\boldsymbol{x}, t) - \frac{\varepsilon}{T - t}, \quad (\boldsymbol{x}, t) \in \overline{\Omega} \times [0, T).$$

Assume by contradiction that u_ε takes positive values. Let $L := \max_{\overline{\Omega} \times [0,T]} u$. Then

$$u_\varepsilon(\boldsymbol{x}, t) \leq L - \frac{\varepsilon}{T - t} < 0$$

for all $t \in \left(T - \frac{\varepsilon}{L}, T\right)$ and so there exists

$$0 < M_\varepsilon := \sup_{\overline{\Omega} \times [0,T)} u_\varepsilon = \max_{\overline{\Omega} \times \left[0, T - \frac{\varepsilon}{L}\right]} u_\varepsilon = u_\varepsilon(\boldsymbol{x}_\varepsilon, t_\varepsilon).$$

Since $u \leq 0$ on $\partial_p U$ and $u_\varepsilon \leq u$, it follows that $(\boldsymbol{x}_\varepsilon, t_\varepsilon)$ cannot belong to $\partial_p U$. Hence, $(\boldsymbol{x}_\varepsilon, t_\varepsilon) \in \Omega \times (0, T)$. In turn,

$$0 = \frac{\partial u_\varepsilon}{\partial t}(\boldsymbol{x}_\varepsilon, t_\varepsilon) = \frac{\partial u}{\partial t}(\boldsymbol{x}_\varepsilon, t_\varepsilon) - \frac{\varepsilon}{(T - t_\varepsilon)^2},$$
$$0 = \frac{\partial u_\varepsilon}{\partial x_i}(\boldsymbol{x}_\varepsilon, t_\varepsilon) = \frac{\partial u}{\partial x_i}(\boldsymbol{x}_\varepsilon, t_\varepsilon).$$

Moreover, since

$$u_\varepsilon(\boldsymbol{x}_\varepsilon, t_\varepsilon) = \max_{\overline{\Omega}} u_\varepsilon(\cdot, t_\varepsilon),$$

we must have that $\nabla^2 u_\varepsilon(\boldsymbol{x}_\varepsilon, t_\varepsilon)$ is semipositive definite, which means that

$$\left(\nabla^2 u_\varepsilon(\boldsymbol{x}_\varepsilon, t_\varepsilon)\, \boldsymbol{y}^T\right) \cdot \boldsymbol{y} \leq 0 \tag{111}$$

for all $\boldsymbol{y} \in \mathbb{R}^N$. Taking $\boldsymbol{y} = \boldsymbol{e}_i$, gives $\frac{\partial^2 u_\varepsilon}{\partial x_i^2}(\boldsymbol{x}_\varepsilon, t_\varepsilon) \leq 0$. In turn, $\Delta u_\varepsilon(\boldsymbol{x}_\varepsilon, t_\varepsilon) \leq 0$, which is a contradiction, since

$$0 < \frac{\varepsilon}{(T - t_\varepsilon)^2} \leq \frac{\partial u}{\partial t}(\boldsymbol{x}_\varepsilon, t_\varepsilon) - \Delta u(\boldsymbol{x}_\varepsilon, t_\varepsilon) \leq 0.$$

This proves that $u_\varepsilon \leq 0$ in $\overline{\Omega} \times [0,T)$. Letting $\varepsilon \to 0^+$ in the previous inequality shows that $u \leq 0$ in $\overline{\Omega} \times [0,T)$, and by continuity, $u \leq 0$ in $\overline{\Omega} \times [0,T]$. This shows that

$$\max_{\overline{\Omega}\times[0,T]} u = \sup_{\Gamma_T} u.$$

■

Remark 79 *The same proof continues to work for a set*

$$U := \left\{(\boldsymbol{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \ 0 < t < T, \ \ \|\boldsymbol{x}\|_N < g(t)\right\},$$

where $g : (0,T) \to (0,\infty)$ is a continuous function. Indeed, for (111) to hold, we only need $\boldsymbol{x}_\varepsilon$ to be a point of local maximum for the function $u_\varepsilon(\cdot, t_\varepsilon)$.

Remark 80 (Physical Interpretation) *The equation $\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t)$ describes the evolution of the density u of physical quantitites such as heat. In particular, if a body occupies a region Ω which contains a heat source f, and if the body is held at zero temperature at the boundary, that is, $u = 0$ on $\partial\Omega$, an its initial temperature is u_0, then the weak maximum principle says that if $f \leq 0$, which corresponds to a heat sink, then no hot spot can develop spontaneously in the interior in the absence of heat sources. Thus,*

$$\max_{\overline{\Omega}\times[0,T]} u \leq \max_{\partial\Omega} u_0.$$

Using this theorem, we can prove a comparison principle.

Theorem 81 (Comparison Principle) *Let $U = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $T > 0$, and let $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ and $v : \overline{\Omega} \times [0,T] \to \mathbb{R}$ be continuous and such that $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ exist in $\Omega \times (0,T)$, ∇u , ∇v, and $\nabla^2 u$, $\nabla^2 v$ exist in $\Omega \times (0,T)$ and are continuous, and*

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) \geq \frac{\partial v}{\partial t}(\boldsymbol{x},t) - \Delta v(\boldsymbol{x},t) \quad \text{in } \Omega \times (0,T).$$

If $v \leq u$ in $\partial_p U$, then $v \leq u$ in $\overline{\Omega} \times [0,T]$.

Proof. Let $w := v - u$. Then

$$\frac{\partial w}{\partial t}(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) \leq 0 \quad \text{in } \Omega \times (0,T)$$

and $w \leq 0$ in $\partial_p U$. It follows by the previous theorem that $w \leq 0$ in $\overline{\Omega} \times [0,T]$. ■

Another consequence of the weak maximum principle is the following theorem.

Theorem 82 *Let* $U = \Omega \times (0,T)$*, where* $\Omega \subset \mathbb{R}^N$ *is an open bounded set,* $T > 0$*, and let* $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *be continuous and such that* $\frac{\partial u}{\partial t}$ *exists in* $\Omega \times (0,T)$*,* ∇u *and* $\nabla^2 u$ *exist in* $\Omega \times (0,T)$ *and are continuous, and*

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) \quad \text{in } \Omega \times (0,T),$$

where $f : \Omega \times (0,T) \to \mathbb{R}$ *is continuous and bounded. Then*

$$\max_{\overline{\Omega}\times[0,T]} |u| \leq T \sup_{\Omega\times(0,T)} |f| + \sup_{\partial_p U} |u|.$$

Proof. Let

$$L := \sup_{\Omega\times(0,T)} |f|, \quad M := \sup_{\partial_p U} |u|.$$

and consider the functions

$$v^{\pm}(\boldsymbol{x},t) := u(\boldsymbol{x},t) \pm (M + tL), \quad (\boldsymbol{x},t) \in \overline{\Omega} \times [0,T)$$

Then

$$\begin{aligned} \frac{\partial v^{\pm}}{\partial t}(\boldsymbol{x},t) - \Delta v^{\pm}(\boldsymbol{x},t) &= \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) \pm L \\ &= f(\boldsymbol{x},t) \pm L \gtrless 0, \end{aligned}$$

while $v^{\pm} \gtrless 0$ on $\partial_p U$. It follows by the weak maximum principle that $v^{\pm} \gtrless 0$ on $\overline{\Omega} \times [0,T]$, that is,

$$-M - TL \leq -M - tL \leq u(\boldsymbol{x},t) \leq M + tL \leq M + TL,$$

which gives the desired result. ■

The next theorem, which is due to Nirenberg, significantly improves the weak maximum principle.

Theorem 83 (Strong Maximum Principle) *Let* $U = \Omega \times (0,T)$*, where* $\Omega \subset \mathbb{R}^N$ *is an open bounded connected set,* $T > 0$*, and let* $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *be continuous and such that* $\frac{\partial u}{\partial t}$ *exists in* $\Omega \times (0,T)$*,* ∇u *and* $\nabla^2 u$ *exist in* $\Omega \times (0,T)$ *and are continuous, and*

$$\frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) \leq 0 \quad \text{in } \Omega \times (0,T).$$

If the maximum of u *occurs at some point* $(\boldsymbol{x}^*, t^*)$ *in* $\overline{U} \setminus \partial_p U$*, then* u *is constant in* $\overline{\Omega} \times [0, t^*]$*.*

We begin with a preliminary result.

Lemma 84 *Let* $\alpha > 0$*,* $R > 0$*, and let*

$$U := \left\{ (\boldsymbol{x},t) \in \mathbb{R}^N \times \mathbb{R} : \ \|\boldsymbol{x}\|_N < R, \ -\alpha < t < 0 \right\}$$

and let $u : \overline{U} \to \mathbb{R}$ be continuous, nonnegative, and such that $\frac{\partial u}{\partial t}$ exists in Q, ∇u and $\nabla^2 u$ exist in U and are continuous, and

$$\frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) \geq 0 \quad \text{in } U$$

and $u \geq 0$ in U. Let $h > 0$, $0 < \varepsilon < 1$ and assume that

$$u(\boldsymbol{x}, -\alpha) \geq h \quad \text{for all } \boldsymbol{x} \in B_N(\mathbf{0}, \varepsilon R).$$

Then there exists $c = c(\alpha, \varepsilon, N, R) > 0$ such that

$$u(\boldsymbol{x}, 0) \geq c \quad \text{for all } \boldsymbol{x} \in B_N\left(\mathbf{0}, \frac{R}{2}\right).$$

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Proof. We want to use the comparison principle. Let

$$\psi_0(t) := a(t + \alpha) + b, \quad \psi_1(\boldsymbol{x}, t) := \psi_0(t) - \|\boldsymbol{x}\|_N^2$$
$$v(\boldsymbol{x}, t) := c(\psi_0(t))^{-2q} \psi_1^2(\boldsymbol{x}, t),$$

where $a, b, c > 0$ and $q \geq 1$ are to be chosen. Consider the set

$$V := \left\{(\boldsymbol{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \ \|\boldsymbol{x}\|_N^2 < \psi_0(t), \ -\alpha < t < 0\right\}.$$

Then $\psi_0 > 0$ in $[-\alpha, 0]$ and so v is continuous in $\overline{V}$ and $\frac{\partial v}{\partial t}$, ∇v, and $\nabla^2 v$ exist in V and are continuous. The parabolic boundary of V is given by the set

$$\partial_p V = \left\{(\boldsymbol{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \ \|\boldsymbol{x}\|_N^2 < b, \ t = -\alpha\right\}$$
$$\cup \left\{(\boldsymbol{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \ \|\boldsymbol{x}\|_N = \psi_0(t), \ -\alpha < t < 0\right\}.$$

Take $b := (\varepsilon R)^2$. Then if $\|\boldsymbol{x}\|_N < \varepsilon R$ and $t = -\alpha$, we have that

$$u(\boldsymbol{x}, -\alpha) \geq h \geq v(\boldsymbol{x}, -\alpha) = cb^{-2q}\left(b - \|\boldsymbol{x}\|_N^2\right)^2,$$

provided

$$h \geq c(\varepsilon R)^{4-4q}.$$

Take $c := hb^{2q-2} = h(\varepsilon R)^{4q-4}$. On the other hand, if $\|\boldsymbol{x}\|_N = \psi_0(t)$ and $-\alpha < t < 0$, then $u \geq 0 = v$, provided $\psi_0(t) \leq R$, that is,

$$a\alpha + \varepsilon^2 R^2 \leq R^2 \Leftrightarrow a \leq \frac{\left(1 - \varepsilon^2\right) R^2}{\alpha}.$$

Take $a := \frac{\left(1-\varepsilon^2\right)R^2}{\alpha}$. This shows that $u \geq v$ on $\partial_p V$. It remains to show that $\frac{\partial v}{\partial t}(\boldsymbol{x}, t) - \Delta v(\boldsymbol{x}, t) \leq 0$ in V. Since In V we have

$$c(\psi_0(t))^{-2q} \psi_1^2(\boldsymbol{x}, t)$$

and so

$$\begin{aligned}\frac{\partial v}{\partial t}(\boldsymbol{x},t) &= -2qc\left(\psi_0(t)\right)^{-2q-1}\psi_0'(t)\,\psi_1^2(\boldsymbol{x},t) + 2c\left(\psi_0(t)\right)^{-2q}\psi_1(\boldsymbol{x},t)\frac{\partial\psi_1}{\partial t}(\boldsymbol{x},t)\\ &= \left(\psi_0(t)\right)^{1-2q}c\left[-2qa\frac{\psi_1^2(\boldsymbol{x},t)}{\psi_0^2(t)} + 2a\frac{\psi_1(\boldsymbol{x},t)}{\psi_0(t)}\right],\end{aligned}$$

where we used the fact that $\psi_0'(t) = a$. On the other hand,

$$\begin{aligned}-\Delta v(\boldsymbol{x},t) &= -c\left(\psi_0(t)\right)^{-2q}\operatorname{div}\nabla\left(\psi_1^2(\boldsymbol{x},t)\right) = 4c\left(\psi_0(t)\right)^{-2q}\operatorname{div}\left(\psi_1(\boldsymbol{x},t)\,\boldsymbol{x}\right)\\ &= 4c\left(\psi_0(t)\right)^{-2q}\left(\psi_1(\boldsymbol{x},t)\,N - 2\|\boldsymbol{x}\|_N^2\right)\\ &= 4c\left(\psi_0(t)\right)^{-2q}\left(\psi_1(\boldsymbol{x},t)\,N + 2\psi_0(t) - 2\|\boldsymbol{x}\|_N^2 - 2\psi_0(t)\right)\\ &= 4c\left(\psi_0(t)\right)^{1-2q}\left(\frac{\psi_1(\boldsymbol{x},t)}{\psi_0(t)}N + 2\frac{\psi_1(\boldsymbol{x},t)}{\psi_0(t)} - 2\right)\end{aligned}$$

Hence, setting $\xi(\boldsymbol{x},t) := \frac{\psi_1(\boldsymbol{x},t)}{\psi_0(t)}$, we have

$$\frac{\partial v}{\partial t}(\boldsymbol{x},t) - \Delta v(\boldsymbol{x},t) = -\left(\psi_0(t)\right)^{1-2q}2c\left[qa\xi^2(\boldsymbol{x},t) - (a+2N+4)\,\xi(\boldsymbol{x},t) + 4\right].$$

The expression between square brackets is quadratic in ξ and so it is always positive, provided

$$(a+2N+4)^2 - 16qa \le 0,$$

and so it suffices to take

$$q := \frac{(a+2N+4)^2}{16a} = \frac{\left(\frac{(1-\varepsilon^2)R^2}{\alpha} + 2N + 4\right)^2}{16\frac{(1-\varepsilon^2)R^2}{\alpha}}.$$

It follows by the comparison principle that $u \ge v$ in V. In particular,

$$\begin{aligned}u(\boldsymbol{x},0) &\ge v(\boldsymbol{x},0)\\ &= c\left(\psi_0(0)\right)^{-2q}\psi_1^2(\boldsymbol{x},0)\\ &= cR^{-4q}\left(R^2 - \|\boldsymbol{x}\|_N^2\right)^2\\ &\ge cR^{-4q}\left(R^2 - \frac{R^2}{4}\right)^2\\ &= \frac{9}{16}h\left(\varepsilon R\right)^{4q-4}R^{-4q}R^4 = \frac{9}{16}h\varepsilon^{4q-4}\end{aligned}$$

for all $\boldsymbol{x} \in B_N\left(\boldsymbol{0}, \frac{R}{2}\right)$, where we used the fact that

$$\psi_0(0) = a\alpha + b = R^2.$$

This completes the proof. ■

We now turn to the proof of Theorem 83.

Proof of Theorem 83.. Step 1: Let $L := \max_{\overline{U}} u$. We claim that if $(\boldsymbol{x}_0, t_0) \in \overline{U} \setminus \partial_p U$ is such that $B_N(\boldsymbol{x}_0, r) \subseteq \Omega$ and if $u(\boldsymbol{x}_1, t_1) < L$ for some $(\boldsymbol{x}_1, t_1) \in B_N\left(\boldsymbol{x}_0, \frac{r}{4}\right) \times (0, t_0)$, then $u(\boldsymbol{x}_0, t_0) < L$. By translating time and space, without loss of generality, assume that $\boldsymbol{x}_1 = \boldsymbol{0}$ and $t_0 = 0$. Take $R := 2r$ and $h := \frac{1}{2}(L - u(\boldsymbol{x}_1, t_1)) > 0$. Since $L - u(\boldsymbol{0}, t_1) \geq 2h$, by continuity there is $0 < \varepsilon < 1$ such that $L - u(\boldsymbol{x}, t_1) \geq h$ for all $\boldsymbol{x} \in B_N(\boldsymbol{0}, \varepsilon R)$. Hence, taking $\alpha = -t_1$, by the previuos lemma applied to $L - u$ we have that

$$L - u(\boldsymbol{x}, 0) \geq hc \quad \text{for all } \boldsymbol{x} \in B_N\left(\boldsymbol{0}, \frac{R}{2}\right).$$

Since

$$\|\boldsymbol{x}_0 - \boldsymbol{0}\|_N = \|\boldsymbol{x}_0 - \boldsymbol{x}_1\|_N < \frac{r}{4} = \frac{R}{2},$$

it follows that $L - u(\boldsymbol{x}_0, t_0) > 0$, which proves the claim.

Step 2: By assumption, the maximum of u occurs at some point $(\boldsymbol{x}^*, t^*) \in \overline{U} \setminus \partial_p U$. It follows that $\boldsymbol{x}^* \in \Omega$ and so there exists $B_N(\boldsymbol{x}^*, r) \subseteq \Omega$. In view of the previous step, we have that $u \equiv L$ in $B_N\left(\boldsymbol{x}^*, \frac{r}{4}\right) \times (0, t^*)$. Since Ω is connected, it follows that $u \equiv L$ in $\Omega \times (0, t^*)$ (exercise). ■

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Next we prove a version of the maximum principle for weak solutions. Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u \in W^{1,p}(\Omega)$. We want to make sense of the inequality $u \leq 0$ on $\partial\Omega$. If Ω is sufficiently regular, then u has a trace on $\partial\Omega$ and so $u \leq 0$ on $\partial\Omega$ simply means that the trace of u is nonnegative. However, for an arbitrary open sense Ω we cannot talk about the trace of u. In this case, we say that $u \leq 0$ on $\partial\Omega$ if $u^+ \in W^{1,p}(\Omega)$, where $s^+ := \max\{s, 0\}$ is the positive part of a number $s \in \mathbb{R}$. We say that $u \geq 0$ on $\partial\Omega$ if $-u \leq 0$ on $\partial\Omega$, and for $u, v \in W^{1,p}(\Omega)$, we say that $u \leq v$ on $\partial\Omega$ if $u - v \leq 0$ on $\partial\Omega$. Finally, we define

$$\sup_{\partial\Omega} u := \inf\{r \in \mathbb{R} : \ u \leq r \text{ on } \partial\Omega\}, \quad \inf_{\partial\Omega} u := -\sup_{\partial\Omega}(-u).$$

Theorem 85 (Maximum Principle for Weak Solutions) *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in L^2(\Omega)$, let $u_1 \in L^2\left((0,T); H^1(\Omega)\right)$, and let u be a weak solution of the Dirichlet problem*

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) \leq 0 & (\boldsymbol{x}, t) \in \Omega \times (0, T), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = u_1(\boldsymbol{x}, t) & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T), \end{cases}$$

that is, $u \in C\left([0,T]; L^2(\Omega)\right) \cap H^1_{\mathrm{loc}}\left((0,T); L^2(\Omega)\right) \cap L^2\left((0,T); H^1(\Omega)\right)$, with

$$\int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t)\, v(\boldsymbol{x})\ d\boldsymbol{x} + \int_\Omega \nabla u(\boldsymbol{x}, t) \cdot \nabla v(\boldsymbol{x})\ d\boldsymbol{x} \leq 0$$

for all $v \in H^1_0(\Omega)$ with $v \geq 0$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Then

$$\operatorname*{essup}_{(0,T)} \operatorname*{essup}_{\Omega} u \leq \operatorname*{essup}_{\Omega} (u_0)^+ + \sup_{(0,T)} \sup_{\partial\Omega} (u_1)^+ =: M.$$

Proof. If $M = \infty$, there is nothing to prove. Thus, asssume that $M < \infty$ and let $k > M$. Consider the function $v(\boldsymbol{x}, t) := (u(\boldsymbol{x}, t) - k)^+$. Since the function $g(t) := (t - k)^+$ is Lipschitz, by the chain rule, we have that

$$\nabla (u(\boldsymbol{x}, t) - k)^+ = \begin{cases} \nabla u(\boldsymbol{x}, t) & \text{if } u(\boldsymbol{x}, t) > k, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and $\mathcal{L}^1$ a.e. $t \in (0, T)$. Hence, $v(\cdot, t) \in H^1(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0, T)$. Moreover, since $k > \sup_{\partial\Omega} (u_1(\cdot, t))^+$, we have that $v(\cdot, t) \in H_0^1(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0, T)$. For any such t we have

$$\int_\Omega (u(\boldsymbol{x}, t) - k)^+ \frac{\partial u}{\partial t}(\boldsymbol{x}, t)\ d\boldsymbol{x} + \int_\Omega \nabla (u(\boldsymbol{x}, t) - k)^+ \cdot \nabla u(\boldsymbol{x}, t)\ d\boldsymbol{x} \leq 0.$$

Observe that

$$\begin{aligned} \int_\Omega (u(\boldsymbol{x}, t) - k)^+ \frac{\partial u}{\partial t}(\boldsymbol{x}, t)\ d\boldsymbol{x} &= \frac{1}{2} \int_\Omega \frac{\partial}{\partial t} \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} \\ &= \frac{1}{2} \frac{d}{dt} \int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x}, \end{aligned}$$

while

$$\int_\Omega \nabla (u(\boldsymbol{x}, t) - k)^+ \cdot \nabla u(\boldsymbol{x}, t)\ d\boldsymbol{x} = \int_\Omega \left\| \nabla (u(\boldsymbol{x}, t) - k)^+ \right\|^2\ d\boldsymbol{x}.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} + \int_\Omega \left\| \nabla (u(\boldsymbol{x}, t) - k)^+ \right\|^2\ d\boldsymbol{x} \leq 0.$$

It follows that

$$\frac{d}{dt} \int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} \leq 0.$$

Integrating between $0 < s < t$, we have

$$\int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} \leq \int_\Omega \left((u(\boldsymbol{x}, s) - k)^+ \right)^2\ d\boldsymbol{x}.$$

Since $u \in C\left([0, T]; L^2(\Omega)\right)$, letting $s \to 0^+$ gives

$$\int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} \leq \int_\Omega \left((u_0(\boldsymbol{x}) - k)^+ \right)^2\ d\boldsymbol{x} = 0,$$

since $k > \operatorname{essup}_\Omega (u_0)^+$. Hence,

$$\int_\Omega \left((u(\boldsymbol{x}, t) - k)^+ \right)^2\ d\boldsymbol{x} \leq 0,$$

which shows that $(u(\boldsymbol{x}, t) - k)^+ = 0$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$. ■

Theorem 86 *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set, let $f \in L^\infty(\Omega \times (0,T))$, let $u_0 \in L^\infty(\Omega)$, let $u_1 \in L^2\left((0,T); H^1(\Omega)\right) \cap L^\infty(\Omega \times (0,T))$, and let u be a weak solution of the Dirichlet problem*

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega \times (0,T), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = u_1(\boldsymbol{x},t) & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega\times(0,T))} \le \|u_0\|_{L^\infty(\Omega)} + \|u_1\|_{L^\infty(\Omega\times(0,T))} + C(N,\Omega,T)\|f\|_{L^\infty(\Omega\times(0,T))}.$$

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Proof. Since u is a weak solution, we have that

$$\int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x},t)\, v(\boldsymbol{x})\; d\boldsymbol{x} + \int_\Omega \nabla u(\boldsymbol{x},t)\cdot\nabla v(\boldsymbol{x})\; d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x},t)\, v(\boldsymbol{x})\; d\boldsymbol{x}$$

for all $v \in H_0^1(\Omega)$. Reasoning as in the previous theorem, with $k > M$ and $v(\boldsymbol{x},t) := (u(\boldsymbol{x},t) - k)^+$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 d\boldsymbol{x} + \int_\Omega \left\|\nabla (u(\boldsymbol{x},t)-k)^+\right\|^2 d\boldsymbol{x} = \int_\Omega (u(\boldsymbol{x},t)-k)^+ f(\boldsymbol{x},t)\; d\boldsymbol{x}.$$

Hence,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 d\boldsymbol{x} + \int_\Omega \left\|\nabla (u(\boldsymbol{x},t)-k)^+\right\|^2 d\boldsymbol{x} \\ &\le \frac{\varepsilon}{2}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 d\boldsymbol{x} + \frac{1}{2\varepsilon}\int_{E_k(t)} f^2(\boldsymbol{x},t)\; d\boldsymbol{x} \end{aligned}$$

where

$$E_k(t) := \{\boldsymbol{x} \in \Omega :\ u(\boldsymbol{x},t) > k\}.$$

Step 1: Assume that $N \ge 3$. Then by the Sobolev–Gagliardo–Nirenberg embedding theorem, we have that

$$\|w\|_{L^{2^*}(\Omega)} \le C(N)\|\nabla w\|_{L^2(\Omega)}$$

for all $w \in H_0^1(\Omega)$, where $2^* := \frac{2N}{N-2}$. In turn, by Hölder's inequality with $p = \frac{2^*}{2} = \frac{N}{N-2}$ and

$$p' = \frac{p}{p-1} = \frac{\frac{N}{N-2}}{\frac{N}{N-2}-1} = \frac{N}{2},$$

we have

$$\begin{aligned} \int_{E_k(t)} (u(\boldsymbol{x},t)-k)^2\; d\boldsymbol{x} &\le \left(\int_{E_k(t)} (u(\boldsymbol{x},t)-k)^{2^*}\; d\boldsymbol{x}\right)^{2/2^*} \left(\int_{E_k(t)} 1\; d\boldsymbol{x}\right)^{1/(2^*/2)'} \qquad (112) \\ &\le C(N)\int_\Omega \left\|\nabla (u(\boldsymbol{x},t)-k)^+\right\|^2 d\boldsymbol{x}\,\left(\mathcal{L}^N(E_k(t))\right)^{2/N}. \end{aligned}$$

Taking

$$\frac{\varepsilon}{2} = \frac{1}{C(N)\left(\mathcal{L}^N(E_k(t))\right)^{2/N}},$$

we get that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} + \frac{1}{C\left(\mathcal{L}^N(E_k(t))\right)^{2/N}}\int_{E_k(t)} (u(\boldsymbol{x},t)-k)^2 \, d\boldsymbol{x} \\
&\leq \frac{1}{2}\frac{d}{dt}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} + \int_\Omega \left\|\nabla (u(\boldsymbol{x},t)-k)^+\right\|^2 \, d\boldsymbol{x} \\
&\leq \frac{1}{C\left(\mathcal{L}^N(E_k(t))\right)^{2/N}}\int_{E_k(t)} (u(\boldsymbol{x},t)-k)^2 \, d\boldsymbol{x} + C(N)\left(\mathcal{L}^N(E_k(t))\right)^{2/N}\int_{E_k(t)} f^2(\boldsymbol{x},t) \, d\boldsymbol{x}
\end{aligned}$$

and so

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} &\leq C(N)\left(\mathcal{L}^N(E_k(t))\right)^{2/N}\int_{E_k(t)} f^2(\boldsymbol{x},t) \, d\boldsymbol{x}. \\
&\leq C(N)\left(\mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))} \\
&\leq C(N)\left(\sup_{0<t<T}\mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))}.
\end{aligned}$$

Integrating in time between $0 < s < t$ gives

$$\begin{aligned}
\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} \leq \int_\Omega \left((u(\boldsymbol{x},s)-k)^+\right)^2 \, d\boldsymbol{x} \\
+ C(N)\,T\left(\sup_{0<t<T}\mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))}.
\end{aligned}$$

Since $u \in C\left([0,T];L^2(\Omega)\right)$, letting $s \to 0^+$ gives

$$\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} \leq C(N)\,T\left(\sup_{0<t<T}\mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))},$$

where we have used the fact that $k > \operatorname{essup}_\Omega (u_0)^+$.

Now for $h > k > M$, we have

$$\begin{aligned}
\int_\Omega \left((u(\boldsymbol{x},t)-k)^+\right)^2 \, d\boldsymbol{x} &= \int_{E_k(t)} (u(\boldsymbol{x},t)-k)^2 \, d\boldsymbol{x} \\
&\geq \int_{E_h(t)} (u(\boldsymbol{x},t)-k)^2 \, d\boldsymbol{x} \\
&\geq (h-k)^2\,\mathcal{L}^N(E_h(t)).
\end{aligned}$$

Hence,

$$(h-k)^2\,\mathcal{L}^N(E_h(t)) \leq C(N)\,T\left(\sup_{0<t<T}\mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))}$$

for all $h > k > M$ and for all $t \in (0,T)$ (recall that $u \in C\left([0,T]; L^2(\Omega)\right)$). It follows that

$$(h-k)^2 \sup_{0<t<T} \mathcal{L}^N(E_h(t)) \le C(N)\,T \left(\sup_{0<t<T} \mathcal{L}^N(E_k(t))\right)^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))}.$$

Define

$$g(s) := \sup_{0<t<T} \mathcal{L}^N(E_s(t)).$$

Then

$$(h-k)^2\, g(h) \le C(N)\,T\,(g(k))^{1+\frac{2}{N}} \|f\|^2_{L^\infty(\Omega\times(0,T))}.$$

Note that the function g is decreasing. It now follows from the following lemma with $\alpha = 2$ and $\beta = 1 + \frac{2}{N}$ that

$$g(M+d) = 0,$$

where

$$\begin{aligned} d &= \left(C(N)\,T\,\|f\|^2_{L^\infty(\Omega\times(0,T))} (g(M))^{\beta-1}\, 2^{2\beta/(\beta-1)}\right)^{1/2} \\ &\le C_1(N)\,T^{1/2} \|f\|_{L^\infty(\Omega\times(0,T))} \left(\mathcal{L}^N(\Omega)\right)^{\frac{\beta-1}{2}}. \end{aligned}$$

Hence, recalling the definition of g,

$$\begin{aligned} u(\boldsymbol{x},t) &\le M + d \\ &\le \operatorname*{essup}_{\Omega}(u_0)^+ + \sup_{(0,T)} \sup_{\partial\Omega} (u_1)^+ + C_1 T^{1/2} \left(\mathcal{L}^N(\Omega)\right)^{\frac{1}{N}} \|f\|_{L^\infty(\Omega\times(0,T))}. \end{aligned}$$

By applying the same proof to $-u$, we get the desired inequality.
Step 2: If $N = 2$, then $H^1_0(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \le q < \infty$. Fix $q > 1$. Then

$$\|w\|_{L^{2q}(\Omega)} \le C(N,q)\,\|w\|_{H^1(\Omega)}$$

for all $w \in H^1_0(\Omega)$. On the other hand, by the Poincaré inequality

$$\|w\|_{L^2(\Omega)} \le C(N)\operatorname{diam}\Omega\,\|\nabla w\|_{L^2(\Omega)},$$

and so

$$\|w\|_{L^{2q}(\Omega)} \le C(N,q)\operatorname{diam}\Omega\,\|\nabla w\|_{L^2(\Omega)}$$

for all $w \in H^1_0(\Omega)$. In turn, (112) becomes

$$\begin{aligned} \int_{E_k(t)} (u(\boldsymbol{x},t)-k)^2 \, d\boldsymbol{x} &\le \left(\int_{E_k(t)} (u(\boldsymbol{x},t)-k)^{2q} \, d\boldsymbol{x}\right)^{1/q} \left(\int_{E_k(t)} 1 \, d\boldsymbol{x}\right)^{1/q'} \\ &\le C(N,q) \int_\Omega \left\|\nabla (u(\boldsymbol{x},t)-k)^+\right\|^2 \, d\boldsymbol{x} \left(\mathcal{L}^N(E_k(t))\right)^{1/q'}. \end{aligned}$$

We can now continue as before, with the only difference that the exponent $1+\frac{2}{N}$ should be replaced by $1+\frac{1}{q'}$ everywhere.
Step 3: The case $N = 1$ is even simpler, since $H^1_0(\Omega) \hookrightarrow C\left(\overline{\Omega}\right)$. We omit the details. ■

Remark 87 *The theorem continues to hold if in place of* $f \in L^{\infty}(\Omega \times (0,T))$, *we require the weaker hypothesis that* $f \in L^{p}((0,T); L^{q}(\Omega))$, *where*

$$\frac{1}{p} + \frac{N}{2q} < 1.$$

The idea of the proof is similar, with a careful use of Hölder's inequality. This case is due to Aronson (1963), while the case $p = q = \infty$ *was proved by Ladyženskaja and Ural'ceva (1962) using different argument. The truncation argument and Lemma 88 below are due to Stampacchia (1960).*

Lemma 88 *Let* $g : [s_0, \infty) \to [0, \infty)$ *be a decreasing function such that*

$$g(t) \leq \frac{c}{(t-s)^{\alpha}} (g(s))^{\beta} \tag{113}$$

for all $s_0 \leq s < t$, *where* $c > 0$, $\alpha > 0$, $\beta > 1$. *Then*

$$g(s_0 + d) = 0,$$

where

$$d := \left(c (g(s_0))^{\beta - 1} 2^{\alpha\beta/(\beta-1)} \right)^{1/\alpha}. \tag{114}$$

Proof. Let

$$s_n := s_0 + d - \frac{d}{2^n}.$$

By (113), with $t = s_{n+1}$ and $s = s_n$, we get

$$\begin{aligned} g(s_{n+1}) &\leq \frac{c}{(s_{n+1} - s_n)^{\alpha}} (g(s_n))^{\beta} = \frac{c}{\left(-\frac{d}{2^{n+1}} + \frac{d}{2^n}\right)^{\alpha}} (g(s_n))^{\beta} \\ &= \frac{c 2^{(n+1)\alpha}}{d^{\alpha}} (g(s_n))^{\beta}. \end{aligned} \tag{115}$$

We claim that

$$g(s_n) \leq \frac{g(s_0)}{2^{n\mu}}, \quad \mu := \frac{\alpha}{\beta - 1}. \tag{116}$$

The proof is by induction on n. It is true for $n = 0$. Assume that the result is true for n and let's prove it for $n + 1$. By (115) and (116), we have

$$\begin{aligned} g(s_{n+1}) &\leq \frac{c 2^{(n+1)\alpha}}{d^{\alpha}} (g(s_n))^{\beta} \leq \frac{c 2^{(n+1)\alpha}}{d^{\alpha}} \left(\frac{g(s_0)}{2^{n\mu}} \right)^{\beta} \\ &= \frac{c 2^{(n+1)\alpha}}{c (g(s_0))^{\beta-1} 2^{\alpha\beta/(\beta-1)}} \left(\frac{g(s_0)}{2^{n\mu}} \right)^{\beta} = \frac{g(s_0)}{2^{(n+1)(\mu\beta - \alpha)}} = \frac{g(s_0)}{2^{(n+1)\mu}}. \end{aligned}$$

This proves the claim. Letting $n \to \infty$ in (116) we get

$$g_{-}(s_0 + d) = \lim_{n \to \infty} g\left(s_0 + d - \frac{d}{2^n} \right) = \lim_{n \to \infty} g(s_n) = 0.$$

Since g is decreasing, it follows that $g(s) = 0$ for all $s \geq s_0 + d$. ■

Friday, March 05, 2014

16 Quasilinear Equations: Global Existence and Blow-Up

In this section we discuss blow-up and global existence of solutions of the quasilinear equation

$$\frac{\partial u}{\partial t} - \Delta u = f(u).$$

We begin with the Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(u(\boldsymbol{x}, t)) & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, \infty). \end{cases} \tag{117}$$

Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(z) > 0$ for $z > 0$. Then a necessary condition for blow-up is the Osgood condition

$$\int^{\infty} \frac{1}{f(z)}\, dz < \infty. \tag{118}$$

To understand this condition, consider the differential equation

$$\begin{cases} \frac{dv}{dt}(t) = f(v(t)), \\ v(0) = v_0 > 0. \end{cases} \tag{119}$$

Since f is continuous, there exists a maximal solution $v : [0, T) \to \mathbb{R}$ for some $T > 0$ possibly infinite. Moreover, since $f(z) > 0$ for $z > 0$ and $v_0 > 0$, we have that v must be strictly increasing. Hence, there exists

$$\lim_{t \to T^-} v(t) = \ell > v_0.$$

Since T is the maximal time of existence, if T is finite, necessarily, $\ell = \infty$.

Since $f(v(t)) > 0$, we can consider

$$\frac{1}{f(v(t))} \frac{dv}{dt}(t) = 1.$$

Integrating both sides in time, we get

$$\int_0^t \frac{1}{f(v(s))} \frac{dv}{dt}(s)\, ds = t - 0.$$

Consider the change of variables $z = v(s)$. Then $dz = \frac{dv}{dt}(s)\, ds$ and so

$$\int_{v_0}^{v(t)} \frac{1}{f(z)}\, dz = t - 0.$$

Now if (118) holds, then T cannot be infinite, since otherwise, letting $t \to \infty$ in the previous identity, we would get

$$\int_{v_0}^{\ell} \frac{1}{f(z)}\, dz = \infty,$$

which is a contradiction. Hence, $T < \infty$ and $\ell = \infty$. On the other hand, if (118) fails, then we have global existence. Indeed, if $T < \infty$, then

$$\int_{v_0}^{\ell} \frac{1}{f(z)}\, dz = T < \infty,$$

which implies that $\ell < \infty$. But then we could extend the solution by considering the initial value problem

$$\begin{cases} \frac{dv}{dt}(t) = f(v(t)), \\ v(T) = \ell > 0. \end{cases}$$

Hence, we have shown that for the Cauchy problem (119), (118) is a necessary and sufficient condition for blow-up.

Next assume that $u_0 \geq 0$ and is bounded from above. We claim that (118) is a necessary condition for blow-up of solutions of (117), under some additional hypotheses on f.

Theorem 89 (Global Existence) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let $u_0 \in L^\infty(\Omega)$ be nonnegative, and let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and such that f is positive for $z > 0$ and $f(0) = 0$. If*

$$\int^{\infty} \frac{1}{f(z)}\, dz = \infty \tag{120}$$

then there exists a unique weak solution $u \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\Omega))$ of (117). Moreover,

$$0 \leq u(\boldsymbol{x}, t) \leq v(t)$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where v is the solution of (119) with $v_0 := \sup_\Omega u_0$.

Proof. Fix $T > 0$, let

$$M \geq \max_{t \in [0,T]} v(t),$$

and define

$$f_M(z) := \begin{cases} f(z) & \text{if } 0 \leq z \leq M, \\ 0 & \text{if } z < 0, \\ f(M) & \text{if } z \geq M. \end{cases}$$

Note that f_M is (globally) Lipschitz and bounded. By Theorem 92 there exists a unique weak solution of the Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f_M(u(\boldsymbol{x}, t)) & (\boldsymbol{x}, t) \in \Omega \times (0,T), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases} \tag{121}$$

Since $-f_M \leq 0$, by the maximum principle (see Theorem 85) applied to $-u$, we have that

$$\operatorname*{essup}_{\Omega \times (0,T)} (-u(\boldsymbol{x}, t)) \leq \operatorname*{essup}_{\Omega} (-u_0(\boldsymbol{x})) \leq 0,$$

and so $u(\boldsymbol{x},t) \geq 0$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Next we claim that

$$u(\boldsymbol{x},t) \leq v(t)$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Since $u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \leq v_0$ and $0 = u(\boldsymbol{x},t) < v(t)$ for $\boldsymbol{x} \in \partial\Omega$, $t \in (0,\infty)$. Hence, $u \leq v$ on $\partial_p U$. Consider the function $w(\boldsymbol{x},t) := v(t) - u(\boldsymbol{x},t)$. It is a weak solution of the Dirichlet problem

$$\begin{cases} \frac{\partial w}{\partial t}(\boldsymbol{x},t) - \Delta w(\boldsymbol{x},t) = f_M(v(t)) - f_M(u(\boldsymbol{x},t)) & (\boldsymbol{x},t) \in \Omega \times (0,T), \\ w(\boldsymbol{x},0) = v_0 - u_0(\boldsymbol{x}) \geq 0 & \boldsymbol{x} \in \Omega, \\ w(\boldsymbol{x},t) = v(t) > 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Let $L_M > 0$ be the Lipschitz constant of f_M in $[-M.M]$. Then

$$|f_M(v(t)) - f_M(u(\boldsymbol{x},t))| \leq L_M |v(t) - u(\boldsymbol{x},t)| = L_M |w(\boldsymbol{x},t)|$$

for all $\boldsymbol{x} \in \Omega$ and all $0 < t \leq T$.

Since $w^-(\boldsymbol{x},t) = 0$ for $\boldsymbol{x} \in \partial\Omega$ and $0 < t \leq T$, we have that $w^-(\cdot,t) \in H_0^1(\Omega)$ for $\mathcal{L}^1$ a.e. $0 < t \leq T$. Using $w^-(\cdot,t)$ as a test function we get

$$\begin{aligned} \int_\Omega w^-(\boldsymbol{x},t) \frac{\partial w}{\partial t}(\boldsymbol{x},t)\ d\boldsymbol{x} + \int_\Omega \nabla w^-(\boldsymbol{x},t) \cdot \nabla w(\boldsymbol{x},t)\ d\boldsymbol{x} &\leq L_M \int_\Omega w^-(\boldsymbol{x},t) |w(\boldsymbol{x},t)|\ d\boldsymbol{x} \\ &= L_M \int_\Omega \left(w^-(\boldsymbol{x},t)\right)^2\ d\boldsymbol{x} \end{aligned}$$

for $\mathcal{L}^1$ a.e. $0 < t \leq T$. Reasoning as in the proof of Theorem 85 gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w^-(\boldsymbol{x},t)\right)^2\ d\boldsymbol{x} + \int_\Omega \left\|\nabla w^-(\boldsymbol{x},t)\right\|^2\ d\boldsymbol{x} \leq L_M \int_\Omega \left(w^-(\boldsymbol{x},t)\right)^2\ d\boldsymbol{x}.$$

Hence,

$$\begin{aligned} \frac{1}{2}\frac{dh}{dt}(t) &\leq \frac{1}{2}\frac{dh}{dt}(t) + \int_\Omega \left\|\nabla w^-(\boldsymbol{x},t)\right\|^2\ d\boldsymbol{x} \\ &\leq L_M h(t), \end{aligned}$$

where

$$h(t) := \int_\Omega \left(w^-(\boldsymbol{x},t)\right)^2\ d\boldsymbol{x}.$$

Multiplying by $e^{-2L_M t}$ and integrating over $0 < s \leq t < T$ gives

$$e^{-2L_M t} h(t) \leq e^{-2L_M s} h(s).$$

Since $u_0 \geq 0$, we have that $w(0) = 0$. Hence, letting $s \to 0^+$ gives $h(t) = 0$ for all $t > 0$. This implies that $\int_\Omega \left(w^-(\boldsymbol{x},t)\right)^2\ d\boldsymbol{x} = 0$ for all $0 < t < T$. It follows that $w^-(\boldsymbol{x},t) = 0$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$.

In conclusion, we have shown that

$$0 \leq u(\boldsymbol{x},t) \leq v(t)$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. It follows that

$$f_M(u(\boldsymbol{x},t)) = f(u(\boldsymbol{x},t))$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$. Hence, u is a weak solution of (117) in $\Omega \times (0,T)$.

Given the arbitrariness of T, and the uniqueness of weak solutions of (117), we have that u is weak solution of (117) in $\Omega \times (0,\infty)$. ■

Remark 90 *Using a bootstrap argument, one can improve the regularity of the solution.*

Remark 91 *If we don't assume (120), the previous proof also shows that if the solution v of (119) with $v_0 := \sup_\Omega u_0$ exists in a maximal time interval $[0,T^*)$, then there exists a weak solution of (117) in $[0,T^*)$. The only change is that one should take $T < T^*$.*

Theorem 92 *Let H be a Hilbert space, let $\Psi : H \to [0,\infty]$ be convex, lower semicontinuous, and such that there exists $\min \Psi = 0$, let $u_0 \in H$, and let $B : [0,T] \times H \to H$ be such that*

$$\|B(t,z_1) - B(t,z_2)\|_H \leq L\|z_1 - z_2\|_H \tag{122}$$

for all $t \in [0,T]$ and all $z_1, z_2 \in H$ and for all $z \in H$ the function $B(\cdot,z) \in L^2((0,T);H)$. Then the evolution problem

$$\begin{cases} -\frac{du}{dt}(t) + B(t,u(t)) \in \partial\Psi(u(t)), \\ u(0) = u_0, \end{cases}$$

admits a unique strong solution u. Moreover,

$$\int_0^T t\left\|\frac{du}{dt}(t)\right\|_H^2 dt \leq \|u_0 - v\|_H^2 + 2T\int_0^T t\|B(t,u(t))\|_H^2 \, dt, \tag{123}$$

$$2\int_0^t \Psi(u(s)) \, ds + \|u(t) - v\|_H^2 \leq 2\|u_0 - v\|_H^2 + T\int_0^t \|B(s,u(s))\|_H \, ds \tag{124}$$

for all $t \in [0,T]$.

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Proof. If $v \in L^2((0,T);H)$ and $v_0 \in H$, then by (122),

$$\|B(t,v(t))\|_H \leq L\|v(t) - v_0\|_H + \|B(t,u_0)\|_H. \tag{125}$$

Hence,

$$\int_0^t \|B(s,v(s))\|_H^2 \, ds \leq 2L^2\int_0^t \|v(s) - v_0\|_H^2 \, ds + 2\int_0^t \|B(s,v_0)\|_H^2 \, ds \tag{126}$$

for all $t \in [0, T]$. It follows that the function $B(\cdot, v(\cdot)) \in L^2((0,T);H)$.

Define $u_0(t) \equiv u_0$ and inductively let u_n be the unique solution of the Cauchy problem

$$\begin{cases} -\frac{du_n}{dt}(t) + B(t, u_{n-1}(t)) \in \partial\Psi(u_n(t)), \\ u_n(0) = u_0. \end{cases} \tag{127}$$

Note that u_n exists in view of Theorem 66 and satisfies

$$\int_0^t s\left\|\frac{du_n}{dt}(s)\right\|_H^2 ds \le \|u_0 - v\|_H^2 + \left(\int_0^t \|B(s, u_{n-1}(s))\|_H \, ds\right)^2 + \int_0^t s\,\|B(s, u_{n-1}(s))\|_H^2 \, ds, \tag{128}$$
$$\le \|u_0 - v\|_H^2 + 2t\int_0^t \|B(s, u_{n-1}(s))\|_H^2 \, ds,$$

and

$$2\int_0^t \Psi(u_n(s))\, ds + \|u_n(t) - v\|_H^2 \le 2\|u_0 - v\|_H^2 + \left(\int_0^t \|B(s, u_{n-1}(s))\|_H \, ds\right)^2 \tag{129}$$
$$\le 2\|u_0 - v\|_H^2 + t\int_0^t \|B(s, u_{n-1}(s))\|_H^2 \, ds$$

for all $v \in \Psi^{-1}(\{0\})$ and all $t \in [0, T]$, where we have used Hölder's inequality.

By Theorem 28, for $t > 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_{n+1}(t) - u_n(t)\|_H^2 &= \left(\frac{du_{n+1}}{dt}(t) - \frac{du_n}{dt}(t), u_{n+1}(t) - u_n(t)\right)_H \\ &= -\left(B(t, u_n(t)) - \frac{du_{n+1}}{dt}(t) - \left(B(t, u_{n-1}(t)) - \frac{du_n}{dt}(t)\right), u_{n+1}(t) - u_n(t)\right)_H \\ &\quad + (B(t, u_n(t)) - B(t, u_{n-1}(t)), u_{n+1}(t) - u_n(t))_H \\ &\le (B(t, u_n(t)) - B(t, u_{n-1}(t)), u_{n+1}(t) - u_n(t))_H \\ &\le L\|u_n(t) - u_{n-1}(t)\|_H \|u_{n+1}(t) - u_n(t)\|_H . \end{aligned} \tag{130}$$

by the monotonicity of the subdifferential of Ψ. Integrating in (r, t), it follows that

$$\|u_{n+1}(t) - u_n(t)\|_H^2 \le \|u_{n+1}(r) - u_n(r)\|_H^2 + 2L\int_r^t \|u_n(s) - u_{n-1}(s)\|_H \|u_{n+1}(s) - u_n(s)\|_H \, ds.$$

Since $u_{n+1}(0) = u_n(0) = u_0$, letting $r \to 0^+$ gives

$$\|u_{n+1}(t) - u_n(t)\|_H^2 \le 2L\int_0^t \|u_n(s) - u_{n-1}(s)\|_H \|u_{n+1}(s) - u_n(s)\|_H \, ds.$$

Hence, by Lemma 67,

$$\left\|u_{n+1}(t)-u_n(t)\right\|_H \leq L\int_0^t \left\|u_n(s)-u_{n-1}(s)\right\|_H \, ds.$$

By iterating this inequality, we get

$$\left\|u_{n+1}(t)-u_n(t)\right\|_H \leq \frac{(Lt)^n}{n!}\left\|u_1-u_0\right\|_{C([0,T];H)}.$$

Hence, the sequence $\{u_n\}$ converges uniformly to a function $u \in C([0,T];H)$. In turn, by (126), the sequence $\{B(\cdot,u_n(\cdot))\}$ is bounded in $L^2((0,T);H)$ by some constant $M_0>0$. It follows from (128) and (129) that

$$\int_0^t s\left\|\frac{du_n}{dt}(s)\right\|_H^2 \, ds \leq \left\|u_0-v\right\|_H^2+2t\int_0^t \left\|B(s,u_{n-1}(s))\right\|_H^2 \, ds \leq C,$$

$$2\int_0^t \Psi(u_n(s)) \, ds+\left\|u_n(t)-v\right\|_H^2 \leq 2\left\|u_0-v\right\|_H^2+t\int_0^t \left\|B(s,u_{n-1}(s))\right\|_H^2 \, ds \leq C$$

for all $v \in \Psi^{-1}(\{0\})$ and all $t \in [0,T]$. This shows that $\{u_n\}$ is bounded in $H^1_{\mathrm{loc}}((0,T);H)$ and so it converges weakly to u in $H^1_{\mathrm{loc}}((0,T);H)$. By the lower semicontinuity of the norm and of Ψ, Fatou's lemma, (125), and the Lebesgue dominated convergence theorem, we get

$$\int_0^t s\left\|\frac{du}{dt}(s)\right\|_H^2 \, ds \leq \left\|u_0-v\right\|_H^2+2t\int_0^t \left\|B(s,u(s))\right\|_H^2 \, ds \leq C, \tag{131}$$

$$2\int_0^t \Psi(u(s)) \, ds+\left\|u(t)-v\right\|_H^2 \leq 2\left\|u_0-v\right\|_H^2+t\int_0^t \left\|B(s,u(s))\right\|_H^2 \, ds \leq C \tag{132}$$

for all $v \in \Psi^{-1}(\{0\})$ and all $t \in [0,T]$. We now show that u is a strong solution. By (127),

$$-\frac{du_n}{dt}(t)+B(t,u_{n-1}(t)) \in \partial\Psi(u_n(t))$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and all n. Hence, for every $v \in H$,

$$\Psi(v)-\Psi(u_n(t)) \geq (B(t,u_{n-1}(t)),v-u_n(t))_H-\left(\frac{du_n}{dt}(t),v-u_n(t)\right)_H.$$

Reasoning as in the proof of (49) we get

$$\Psi(v)-\Psi(u(t)) \geq (B(t,u(t)),v-u(t))_H-\left(\frac{du}{dt}(t),v-u(t)\right)_H,$$

which shows that $-\frac{du}{dt}(t)+B(t,u(t)) \in \partial\Psi(u(t))$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$.

Finally, to prove uniqueness, we proceed as in (130) to conclude that if w is another solution, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u(t)-w(t)\right\|_H^2 &\leq \left(B(t,u(t))-B(t,w(t)),u(t)-w(t)\right)_H \\ &\leq L\left\|u(t)-w(t)\right\|_H^2,\end{aligned}$$

which implies that $\left\|u(t)-w(t)\right\|_H=0$ for all $t>0$. ■

Remark 93 *If $u_0 \in \operatorname{dom}_e \Psi$, then we can assume that $B:[0,T]\times\operatorname{dom}_e\Psi\to H$. In this case, applying Theorem 58 in place of Theorem 66, we obtain a solution $u\in H^1((0,T);H)$.*

To prove the a priori bounds, we will use Gronwall's inequality.

Theorem 94 (Gronwall's Inequality) *Let $I\subseteq\mathbb{R}$ be an interval, let $f:I\to\mathbb{R}$, $\beta:I\to[0,\infty)$ be continuous functions, and let $\alpha:I\to[0,\infty)$ be a measurable function. Assume that*

$$f(t)\leq\alpha(t)+\int_{t_0}^t\beta(s)f(s)\ ds$$

for some $t_0\in I$ and all $t\geq t_0$, $t\in I$. Then

$$f(s)\leq\alpha(t)+\int_{t_0}^t\alpha(s)\beta(s)\exp\left(\int_s^t\beta(r)\ dr\right)\ ds$$

for all $t\geq t_0$, $t\in I$. Moreover, if α is increasing, then

$$f(s)\leq\alpha(t)\exp\left(\int_{t_0}^t\beta(r)\ dr\right)$$

for all $t\geq t_0$, $t\in I$.

Proof. Let

$$g(t):=\int_{t_0}^t\beta(s)f(s)\ ds\exp\left(-\int_{t_0}^t\beta(r)\ dr\right).$$

Then by the chain rule,

$$\begin{aligned}g'(t)&=\beta(t)\left(f(t)-\int_{t_0}^t\beta(r)\ dr\right)\exp\left(-\int_{t_0}^t\beta(r)\ dr\right)\\ &\leq\alpha(t)\beta(t)\exp\left(-\int_{t_0}^t\beta(r)\ dr\right).\end{aligned}$$

Since $g(t_0)=0$, it follows by integration that

$$g(t)-0\leq\int_{t_0}^t\alpha(s)\beta(s)\exp\left(-\int_{t_0}^s\beta(r)\ dr\right)\ ds.$$

In turn,

$$\begin{aligned}\int_{t_0}^{t} \beta(s) f(s)\ ds &= g(t) \exp\left(\int_{t_0}^{t} \beta(r)\ dr\right) \\ &\leq \exp\left(\int_{t_0}^{t} \beta(r)\ dr\right) \int_{t_0}^{t} \alpha(s) \beta(s) \exp\left(-\int_{t_0}^{s} \beta(r)\ dr\right)\ ds \\ &= \int_{t_0}^{t} \alpha(s) \beta(s) \exp\left(\int_{s}^{t} \beta(r)\ dr\right)\ ds,\end{aligned}$$

and so

$$\begin{aligned}f(t) &\leq \alpha(t) + \int_{t_0}^{t} \beta(s) f(s)\ ds \\ &\leq \alpha(t) + \int_{t_0}^{t} \alpha(s) \beta(s) \exp\left(\int_{s}^{t} \beta(r)\ dr\right)\ ds.\end{aligned}$$

This concludes the first part of the proof.

If α is increasing, then

$$\begin{aligned}f(t) &\leq \alpha(t) + \int_{t_0}^{t} \alpha(s) \beta(s) \exp\left(\int_{s}^{t} \beta(r)\ dr\right)\ ds \\ &\leq \alpha(t) + \alpha(t) \int_{t_0}^{t} \beta(s) \exp\left(\int_{s}^{t} \beta(r)\ dr\right)\ ds \\ &= \alpha(t) + \alpha(t) \int_{t_0}^{t} \frac{d}{ds}\left(-\exp\left(\int_{s}^{t} \beta(r)\ dr\right)\right)\ ds \\ &= \alpha(t) + \alpha(t)\left[-1 + \exp\left(\int_{t_0}^{t} \beta(r)\ dr\right)\right].\end{aligned}$$

This concludes the proof. ■

The hypothesis that B is globally Lipschitz is very strong. In the following theorem we treat the case in which B is locally Lipschitz.

Theorem 95 *Let H be a Hilbert space, let $\Psi : H \to [0, \infty]$ be convex, lower semicontinuous, not identically ∞, and such that there exists $\min \Psi = 0$, let $u_0 \in H$, and let $B : [0, T] \times H \to H$ be such that for every $M > 0$ there exists $L_M > 0$ such that*

$$\|B(t, z_1) - B(t, z_2)\|_H \leq L_M \|z_1 - z_2\|_H \tag{133}$$

for all $t \in [0, T]$ and all $z_1, z_2 \in H$ with $\|z_1\|_H \leq M$, $\|z_2\|_H \leq M$, and for all $z \in H$ the function $B(\cdot, z) \in L^2((0, T); H)$. Then there exists $0 < T_0 \leq T$ depending only on u_0 (and B, Ψ) such that the evolution problem

$$\begin{cases} -\frac{du}{dt}(t) + B(t, u(t)) \in \partial\Psi(u(t)), \\ u(0) = u_0, \end{cases}$$

admits a unique strong solution u. Moreover u satisfies (123) *and* (124)*.*

First Proof. If $v \in L^{\infty}((0,T);H)$ and $v_0 \in H$, then taking

$$M_v := \|v\|_{L^{\infty}((0,T);H)} + \|v_0\|_H$$

and letting $L_v := L_{M_v}$ be the corresponding constant in (133), by (126),

$$\begin{aligned}\int_0^t \|B(s,v(s))\|_H^2 \, ds &\le 2L_v^2 \int_0^t \|v(s)-v_0\|_H^2 \, ds + 2\int_0^t \|B(s,v_0)\|_H^2 \, ds \\ &\le 8L_v^2 M_v^2 t + 2\int_0^t \|B(s,v_0)\|_H^2 \, ds \qquad (134)\end{aligned}$$

for all $t \in [0,T]$. It follows that the function $B(\cdot, v(\cdot)) \in L^2((0,T);H)$.

Since the solutions given in Theorem 66 are in $C([0,T];H)$, by (134), we can apply Theorem 66 inductively to define u_n as in the previous theorem. Let

$$M := 4(\|u_0\|_H + \|v\|_H + 1),$$

where $v \in \Psi^{-1}(\{0\})$ has been fixed, and let L_M be the corresponding constant in (133). We claim that

$$\|u_n\|_{L^{\infty}((0,T_0);H)} \le M \qquad (135)$$

for all n provided

$$T_0 \le \left(8L_M^2 M^2 \min\{T,1\} + 2\int_0^{\min\{T,1\}} \|B(t,u_0)\|_H^2 \, dt + 1\right)^{-1}. \qquad (136)$$

The proof is by induction on n. Assume that u_{n-1} satisfies (135). By (129) and (134) applied to u_{n-1},

$$\begin{aligned}\|u_n(t)-v\|_H^2 &\le 2\|u_0-v\|_H^2 + t\int_0^t \|B(s,u_{n-1}(s))\|_H^2 \, ds \\ &\le 2\|u_0-v\|_H^2 + t\left(8L_M^2 M^2 t + 2\int_0^t \|B(s,u_0)\|_H^2 \, ds\right) \\ &\le 2\|u_0-v\|_H^2 + 1\end{aligned}$$

for all $t \in [0,T_0]$. It follows that (135) holds for u_n.This proves the claim.

We can now continue as in the previous theorem to show that $\{u_n\}$ converges uniformly to a function u. ■

We present a second proof (suggested by Adrian)

Second Proof. Let $M > \|u_0\|_H + 1$ and define

$$B_M(t,z) := \begin{cases} B(t,z) & \text{if } \|z\| \le M, \\ B\left(t, \frac{z}{\|z\|}M\right) & \text{if } \|z\| > M. \end{cases}$$

Then B_M is Lipschitz continuous with Lipschitz constant at most $2L_M$. Indeed, let $z_1, z_2 \in H$. If $\|z_1\| \le M$ and $\|z_2\| \le M$, then there is nothing to prove. If

$\|z_1\| > M$ and $\|z_2\| > M$, then

$$\begin{aligned}\|B_M(t,z_1) - B_M(t,z_2)\|_H &= \left\| B\left(t, \frac{z_1}{\|z_1\|}M\right) - B\left(t, \frac{z_2}{\|z_2\|}M\right)\right\|_H \le L_M \left\| \frac{z_1}{\|z_1\|}M - \frac{z_2}{\|z_2\|}M \right\|_H \\ &\le L_M M \left\| \frac{z_1}{\|z_1\|} - \frac{z_1}{\|z_2\|}\right\|_H + L_M M \left\| \frac{z_1}{\|z_2\|} - \frac{z_2}{\|z_2\|}\right\|_H \\ &\le L_M \frac{M\|z_1\|}{\|z_1\|\|z_2\|} \left|\|z_1\| - \|z_2\|\right| + L_M \frac{M}{\|z_2\|}\|z_1 - z_2\|_H \\ &\le 2L_M \|z_1 - z_2\|_H ,\end{aligned}$$

while if $\|z_1\| \le M$ and $\|z_2\| > M$, consider w a point on the segment joining z_1 with z_2 with $\|w\| = M$. Then $\|z_1 - w\|_H + \|w - z_2\|_H = \|z_1 - z_2\|_H$ and so, by the other two cases[1]

$$\begin{aligned}\|B_M(t,z_1) - B_M(t,z_2)\|_H &\le \|B_M(t,z_1) - B_M(t,w)\|_H + \|B_M(t,w) - B_M(t,z_2)\|_H \\ &\le \|B(t,z_1) - B(t,w)\|_H + \|B_M(t,w) - B_M(t,z_2)\|_H \\ &\le L\|z_1 - w\|_H + 2L_M\|z_2 - w\|_H \le 2L_M \|z_2 - z_1\|_H .\end{aligned}$$

Applying the previous theorem to B_M we get a strong solution of the evolution problem

$$\begin{cases} -\frac{du}{dt}(t) + B_M(t,u(t)) \in \partial\Psi(u(t)), \\ u(0) = u_0. \end{cases}$$

Since $u \in C([0,T];H)$ by continuity there exists $T_0 > 0$ such that

$$\|u(t) - u_0\|_H \le 1$$

for all $t \in [0,T_0]$. In turn, $\|u(t)\|_H \le M$ for all $t \in [0,T_0]$ and since $B_M(t,z) = B(t,z)$ for $\|z\| \le M$, we have that u is a solution of a our problem in $[0,T_0]$. ■

Remark 96 *If $u_0 \in \operatorname{dom}_e \Psi$, then we can assume that $B : [0,T] \times \operatorname{dom}_e \Psi \to H$. Moreover, in this case we have that u is more regular, that is, $u \in H^1([0,T_0];H)$ instead of just $u \in C([0,T_0];H)$ with $H^1_{\mathrm{loc}}((0,T_0);H)$.*

Wednesday, March 26, 2014

Corollary 97 *Assume that in the previous theorem $B : [0,\infty) \times H \to H$ satisfies (133) for all $t \in [0,\infty)$ and all $z_1, z_2 \in H$ with $\|z_1\|_H \le M$, $\|z_2\|_H \le M$, and for all $z \in H$ the function $B(\cdot,z) \in L^2_{\mathrm{loc}}([0,T);H)$. Then the solution u can be extended to an interval $[0,T_{\max})$ and either $T_{\max} = \infty$ or $T_{\max} < \infty$ and*

$$\lim_{t\to T^-_{\max}} \|u(t)\|_H = \infty.$$

[1] In a Banach space, we could use the fact that $\|z_1 - w\|_H \le \|z_1 - z_2\|_H$ and $\|z_2 - w\|_H \le \|z_1 - z_2\|_H$, to get a Lipschitz constant less than or equal to $3L_M$.

Proof. Assume that there are two solutions $u_1 : [0, T_1) \to H$ and $u_2 : [0, T_2) \to H$ with $T_1 \le T_2 \le \infty$. Then by uniqueness, $u_1 = u_2$ in $[0, T_1)$. Thus, u_2 is an extension of u_1 in the interval $[0, T_2)$ and we say that $u_1 \preceq u_2$. Since $\preceq$ is a partial order, we can apply Zorn's lemma to find $T_{\max} \in (0, \infty]$ such that $[0, T_{\max})$ is the maximal interval in which a solution u exists. If $T_{\max} = \infty$, there is nothing to prove. Thus, assume that $T_{\max} < \infty$. We claim that

$$\liminf_{t \to T_{\max}^-} \|u(t)\|_H = \infty.$$

Indeed, if not, then

$$\liminf_{t \to T_{\max}^-} \|u(t)\|_H < \infty,$$

and so we can find $t_n \nearrow T_{\max}$ and $K > 0$ such that

$$\|u(t_n)\|_H \le K \tag{137}$$

for all n. In view of (124), we have that $u(t) \in \operatorname{dom}_e \Psi$ for $\mathcal{L}^1$ a.e. $t \in (0, T_{\max})$. On the other hand, since $u \in H^1_{\mathrm{loc}}((0, T_{\max}); H)$, it follows from Proposition 59 that $\Psi \circ u$ is absolutely continuous in each closed subinterval of $(0, T_{\max})$. In particular, $u(t_n) \in \operatorname{dom}_e \Psi$ for all n. Hence, by the previous theorem and Remark 96, the evolution problem

$$\begin{cases} -\frac{dw}{dt}(t) + B(t + t_n, w(t)) \in \partial\Psi(w(t)), \\ w(0) = u(t_n), \end{cases}$$

admits a unique strong solution $u_n \in H^1((0, T_n); H)$, where by (136),

$$T_n = \left(8L_{K+1}^2 (K+1)^2 + 2\int_0^1 \|B(t + t_n, u(t_n))\|_H^2 \, dt + 1\right)^{-1}.$$

We claim that $T_n \ge S > 0$ for all n and some $S > 0$. To see this, note that by the change of variable $s = t + t_n$,

$$\begin{aligned} \int_0^1 \|B(t + t_n, u(t_n))\|_H^2 \, dt &= \int_{t_n}^{1+t_n} \|B(s, u(t_n))\|_H^2 \, ds \le \int_0^{1+T_{\max}} \|B(s, u(t_n))\|_H^2 \, ds \\ &\le \int_0^{1+T_{\max}} \left(\|B(s, u(t_1))\|_H + L_{K+1} \|u(t_n) - u(t_1)\|_H\right)^2 \, ds \\ &\le \int_0^{1+T_{\max}} \left(\|B(s, u(t_1))\|_H + 2L_{K+1} K\right)^2 \, ds. \end{aligned}$$

This proves the claim. Since $t_n \nearrow T_{\max}$, we can find n large enough that $t_n + S > T_{\max}$. Fix any such n and consider the function

$$v(t) := \begin{cases} u(t) & \text{for } 0 \le t \le t_n, \\ u_n(t - t_n) & \text{for } t_n \le t < t_n + S. \end{cases}$$

Since $u \in H^1((\delta, t_n); H)$ and $u_n \in H^1((0, T_n); H)$, v satisfies the fundamental theorem of calculus in $[\delta, t_n + S]$ and so $v \in H^1((\delta, t_n + S); H)$ for every $\delta > 0$. Moreover, for $\mathcal{L}^1$ a.e. $t \in (0, S)$,

$$-\frac{du_n}{dt}(t) + B(t + t_n, u_n(t)) \in \partial\Psi(u_n(t))$$

and so

$$-\frac{du_n}{dt}(t - t_n) + B(t, u_n(t - t_n)) \in \partial\Psi(u_n(t - t_n))$$

for $\mathcal{L}^1$ a.e. $t \in (t_n, t_n + S)$. In turn, $-\frac{dv}{dt}(t) + B(t, v(t)) \in \partial\Psi(v(t))$ for $\mathcal{L}^1$ a.e. $t \in (t_n, t_n + S)$. Thus, v is a strong solution in $[0, t_n + S)$, which is a contradictions since $T_{\max} < t_n + S$. ■

Friday, March 28, 2014

Next we investigate non existence and blow-up. The following theorem gives an example of an evolution equation for which there is non-existence of global solutions without blow-up.

Theorem 98 (Ball) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^∞ boundary. Then there exists $v_0 \in L^2(\Omega)$ such thatthe backward parabolic equation*

$$\begin{cases} \frac{\partial v}{\partial t}(\boldsymbol{x}, t) + \Delta v(\boldsymbol{x}, t) = \left(\int_\Omega v^2(\boldsymbol{y}, t)\ d\boldsymbol{y}\right) v(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ v(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ v(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, \infty) \end{cases} \tag{138}$$

admits a weak solution $v \in [0, 1) \times \Omega \to \mathbb{R}$ that cannot be extended in time to any $[0, T)$ for $T > 1$ but such that

$$\lim_{t \to 1^-} \|v(\cdot, t)\|_{L^2(\Omega)} < \infty.$$

Proof. Let $u_0 \in L^2(\Omega) \setminus H^1(\Omega)$ and consider the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = -\left(\int_\Omega u^2(\boldsymbol{y}, t)\ d\boldsymbol{y}\right) u(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, \infty). \end{cases} \tag{139}$$

Let $B : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$B(w)(\boldsymbol{x}) := -\left(\int_\Omega w^2(\boldsymbol{y})\ d\boldsymbol{y}\right) w(\boldsymbol{x}).$$

Given $M > 0$ and w_1, w_2 with $\|w_1\|_{L^2(\Omega)}, \|w_2\|_{L^2(\Omega)} \leq M$, we have that

$$\begin{aligned}\|B(w_1) - B(w_2)\|_{L^2(\Omega)} &= \left\| \|w_1\|^2_{L^2(\Omega)} w_1 - \|w_2\|^2_{L^2(\Omega)} w_2 \right\|_{L^2(\Omega)} \\ &\leq \left| \|w_1\|^2_{L^2(\Omega)} - \|w_2\|^2_{L^2(\Omega)} \right| \|w_1\|_{L^2(\Omega)} \\ &\quad + \|w_2\|^2_{L^2(\Omega)} \|w_1 - w_2\|_{L^2(\Omega)} \\ &\leq 2M^2 \left| \|w_1\|_{L^2(\Omega)} - \|w_2\|_{L^2(\Omega)} \right| \\ &\quad + M^2 \|w_1 - w_2\|_{L^2(\Omega)} \\ &\leq 3M^3 \|w_1 - w_2\|_{L^2(\Omega)}.\end{aligned}$$

Hence, we are in a position to apply Theorem 95 and Corollary 97 to find a solution

$$u \in C\left([0, T_{\max}); L^2(\Omega)\right) \cap H^1_{\text{loc}}\left((0, T_{\max}); L^2(\Omega)\right) \cap L^2_{\text{loc}}\left((0, T_{\max}); H^1_0(\Omega)\right)$$

and either $T_{\max} = \infty$ or $T_{\max} < \infty$ and

$$\lim_{t \to T^-_{\max}} \|u(\cdot, t)\|_{L^2(\Omega)} = \infty.$$

Then

$$\int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t) w(\boldsymbol{x}) \, d\boldsymbol{x} + \int_\Omega \nabla u(\boldsymbol{x}, t) \cdot \nabla w(\boldsymbol{x}) \, d\boldsymbol{x} = -\left(\int_\Omega u^2(\boldsymbol{y}, t) \, d\boldsymbol{y}\right) \int_\Omega u(\boldsymbol{x}, t) w(\boldsymbol{x}, t) \, d\boldsymbol{x}$$

for all $w \in H^1_0(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in [0, T_{\max})$.

By a bootstrap argument, it can be shown that $u \in C^\infty\left(\overline{\Omega} \times (0, T_{\max})\right)$ and that $u(\cdot, t) \in H^1_0(\Omega)$ for all $t \in (0, T_{\max})$. Taking $w = u(\cdot, t)$, we have that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u(\boldsymbol{x}, t))^2 \, d\boldsymbol{x} + \int_\Omega \|\nabla u(\boldsymbol{x}, t)\|^2 \, d\boldsymbol{x} = -\left(\int_\Omega u^2(\boldsymbol{x}, t) \, d\boldsymbol{x}\right)^2 \leq 0.$$

It follows that

$$\frac{1}{2}\int_\Omega (u(\boldsymbol{x}, t))^2 \, d\boldsymbol{x} + \int_s^t \int_\Omega \|\nabla u(\boldsymbol{x}, r)\|^2 \, d\boldsymbol{x} dr \leq \frac{1}{2}\int_\Omega (u(\boldsymbol{x}, s))^2 \, d\boldsymbol{x}$$

for all $0 < s < t$. Letting $s \to 0^+$ gives

$$\frac{1}{2}\int_\Omega (u(\boldsymbol{x}, t))^2 \, d\boldsymbol{x} + \int_0^t \int_\Omega \|\nabla u(\boldsymbol{x}, r)\|^2 \, d\boldsymbol{x} dr \leq \frac{1}{2}\int_\Omega (u_0(\boldsymbol{x}))^2 \, d\boldsymbol{x}$$

for all $t \in [0, T_{\max})$. This implies that $T_{\max} = \infty$.

Now define $v_0 := u(\cdot, 1)$. Then the function $v(\boldsymbol{x}, t) := u(\boldsymbol{x}, 1 - t)$ is a weak solution of (138) for $t \in (0, 1)$. Moreover, since $u \in C\left([0, \infty); L^2(\Omega)\right)$, we have that

$$\lim_{t \to 1^-} \|v(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} < \infty.$$

However, the function v cannot be extended to a weak solution of (138) in some interval $(0,T)$ with $T>1$, since, otherwise, taking $1<T_1<T$, the function $u_1(\boldsymbol{x},t):=v(\boldsymbol{x},T_1-t)$ would be a weak solution of (139), which is a contradiction, since by the regularizing effects of (139), we would have that $u_1(\cdot,t)\in H_0^1(\Omega)$ for all $t\in(0,T_1)$ but

$$u_1(\boldsymbol{x},T_1-1)=v(\boldsymbol{x},T_1-(T_1-1))=v(\boldsymbol{x},1)=u(\boldsymbol{x},0)=u_0(\boldsymbol{x})\notin H_0^1(\Omega).$$

This completes the proof. ■

Monday, March 31, 2014

No class

Wednesday, April 2, 2014

The previus proof shows that for some initial data v_0 there is non-existence of global solutions without blow-up of the L^2 norm (compare this result with Corollary 97). One could have proved non-existence of global soutions by a blow-up argument, that could be misleading.

Theorem 99 *Let $v_0\in L^2(\Omega)$, with $v_0\neq 0$. Then (138) admits no global weak solutions.*

Proof. Assume by contradiction that there exists a global weak solution v. Then

$$\int_\Omega \frac{\partial v}{\partial t}(\boldsymbol{x},t)\,w(\boldsymbol{x})\;d\boldsymbol{x}-\int_\Omega \nabla v(\boldsymbol{x},t)\cdot\nabla w(\boldsymbol{x})\;d\boldsymbol{x}=\left(\int_\Omega v^2(\boldsymbol{y},t)\;d\boldsymbol{y}\right)\int_\Omega v(\boldsymbol{x},t)\,w(\boldsymbol{x},t)\;d\boldsymbol{x}$$

for all $w\in H_0^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t\in[0,\infty)$. Take $w:=v(\cdot,t)$ in the previous identity to get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2(\boldsymbol{x},t)\;d\boldsymbol{x}=\int_\Omega \|\nabla v(\boldsymbol{x},t)\|^2\;d\boldsymbol{x}+\left(\int_\Omega v^2(\boldsymbol{x},t)\;d\boldsymbol{x}\right)^2$$

for $\mathcal{L}^1$ a.e. $t\in[0,\infty)$. It follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2(\boldsymbol{x},t)\;d\boldsymbol{x}\geq\left(\int_\Omega v^2(\boldsymbol{x},t)\;d\boldsymbol{x}\right)^2,$$

that is, the function $G(t):=\int_\Omega v^2(\boldsymbol{x},t)\;d\boldsymbol{x}$ satisfies the differential inequality

$$g'(t)\geq 2g^2(t)$$

for $\mathcal{L}^1$ a.e. $t\in[0,\infty)$. This implies in particular that g is increasing, so $g(t)\geq g(0)>0$ for all $t>0$. In turn,

$$\frac{g'(t)}{g^2(t)}\geq 2\quad\Leftrightarrow\quad\int_{g(0)}^{g(t)}\frac{1}{r^2}\;dr\geq 2t\quad\Leftrightarrow\quad 0>-\frac{1}{g(t)}\geq 2t-\frac{1}{g(0)}.$$

Note that the last inequality implies that $t<\frac{1}{2g(0)}$ and for these t,

$$g(t)\geq\frac{g(0)}{1-2g(0)\,t}.$$

It follows that

$$\lim_{t\to\left(\frac{1}{2g(0)}\right)^{+}} g(t) = \infty.$$

Thus (138) admits no global weak solutions. ■

Remark 100 *The previous proof does not show that there is blow-up of solutions. Actually, it gives very little information. It could be that there are no local solutions, or that there are local solutions who cease to exist without blow-up of the L^2 norm, or that there are weak solutions that cease to exist because there is blow-up of the L^2 norm.*

Next we study non-existence of global solutions.

Theorem 101 *Let*

$$Y \hookrightarrow H \cong H' \hookrightarrow Y'$$

be an evolution triple, let $\Psi : H \to [0,\infty]$ be convex, lower semicontinuous and such that $\mathrm{dom}_e\,\Psi = Y$*, $\Psi : Y \to [0,\infty)$ is of class C^1 and*

$$q\Psi(y) \geq \langle \nabla\Psi(y), y\rangle_{Y',Y} \tag{140}$$

for all $y \in Y$ and for some $q > 0$. Let $F : H \to \mathbb{R}$ be of class C^1, with $F(0) = 0$, and such that for every $\tau > 0$ there exists $c > 0$ and $p > 2$ such that

$$(\nabla F(z), z)_H - qF(z) \geq c\,(F(z))^{1/p'}\,\|z\|_H \tag{141}$$

for all $z \in H$ such that $F(z) \geq \tau$. Then for every $u_0 \in Y$ with

$$\Psi(u_0) - F(u_0) < 0$$

such that any strong solution of the evolution problem

$$\begin{cases} -\frac{du}{dt}(t) + \nabla F(u(t)) \in \partial\Psi(u(t)), \\ u(0) = u_0, \end{cases} \tag{142}$$

does not exist globally.

Proof. Assume by contradiction that (142) admits a global solution u. Consider the potential energy

$$E(t) := (\Psi \circ u)(t) - (F \circ u)(t).$$

By Proposition 59, $\Psi \circ u$ is absolutely continuous with

$$\begin{aligned} \frac{d}{dt}(\Psi \circ u)(t) + \left\|\frac{du}{dt}(t)\right\|_H^2 &= \left(\nabla F(u(t)), \frac{du}{dt}(t)\right)_H \\ &= \frac{d}{dt}(F \circ u)(t). \end{aligned}$$

Hence,

$$E'(t) = -\left\| \frac{du}{dt}(t) \right\|_H^2$$

and so

$$E(t) = E(0) - \int_0^t \left\| \frac{du}{dt}(s) \right\|_H^2 ds \le E(0) = \Psi(u_0) - F(u_0) =: -\tau < 0.$$

Since $\Psi \ge 0$, this implies that $(F \circ u)(t) \ge \tau$ for all $t > 0$. In particular, $u(t) \ne 0$ for all $t > 0$ because $F(0) = 0$.

Since $u(t) \in Y$ for all $t > 0$ and $\Psi : Y \to [0, \infty)$ is of class C^1, we have that $\partial\Psi(u(t)) = \{\nabla\Psi(u(t))\}$, so that

$$-\frac{du}{dt}(t) + \nabla F(u(t)) = \nabla\Psi(u(t)).$$

In turn, by (140) and (141),

$$\begin{aligned} 0 &= -\langle \nabla\Psi(u(t)), u(t)\rangle_{Y',Y} - \left(\frac{du}{dt}(t), u(t)\right)_H + (\nabla F(u(t)), u(t))_H \\ &\quad + q((\Psi \circ u)(t) - (F \circ u)(t) - E(t)) \\ &\ge -\left(\frac{du}{dt}(t), u(t)\right)_H + c(F(u(t)))^{1/p'} \|u(t)\|_H. \end{aligned}$$

It follows that

$$\left\| \frac{du}{dt}(t) \right\|_H \|u(t)\|_H \ge \left(\frac{du}{dt}(t), u(t)\right)_H \ge c(F(u(t)))^{1/p'} \|u(t)\|_H,$$

and so, using the fact that $u(t) \ne 0$ for all $t > 0$,

$$-E'(t) = \left\| \frac{du}{dt}(t) \right\|_H^2 \ge c(F(u(t)))^{2/p'} \ge c(-E(t))^{2/p'}.$$

Since $p > 2$, we have that $2/p' > 1$. Hence, $-E$ will blow up in finite time. ■

Remark 102 *If ∇F is locally Lipschitz, then we are in a position to apply Corollary 97 to conclude that there exists a unique solution u of (142) defined in a maximal interval $[0, T_{\max})$, where $T_{\max} < \infty$ and*

$$\lim_{t \to T_{\max}^-} \|u(t)\|_H = \infty.$$

Friday, April 02, 2014

Next we extend the previous theorem to more general evolution problems of the form

$$\begin{cases} Q\left(t, \frac{du}{dt}(t)\right) + \nabla\Psi(u(t)) = \nabla F(u(t)), \\ u(0) = u_0. \end{cases} \tag{143}$$

Here X, Y, W are Banach spaces containing a common subspace Z, with X continuously embedded in W, $Q : [0,\infty) \times W \to W'$ is continuous, $\Psi : Y \to [0,\infty)$ is of class C^1, $F : X \to \mathbb{R}$ is of class C^1, and $u_0 \in Z$.

Given $T \leq \infty$, consider the space K_T of all functions $\varphi : [0,T) \to Z$ such that

$$\varphi \in C([0,T);X) \cap C([0,T);Y) \cap AC((0,T);W)$$

By a strong solution of (143) in some interval $[0,T)$, with $T \leq \infty$, we mean a function $u \in K_T$ such that $u(0) = u_0$ and

$$\int_0^t \left[\left\langle Q\left(s, \frac{du}{dt}\right), \varphi \right\rangle_{W',W} + \langle \nabla\Psi(u), \varphi \rangle_{Y',Y} - \langle \nabla F(u), \varphi \rangle_{X',X} \right] ds = 0 \tag{144}$$

for all $\varphi \in K_T$ and all $t \in [0,T)$ and satisfying the energy identity

$$E(t) = E(0) - \int_0^t \left\langle Q\left(s, \frac{du}{dt}(s)\right), \frac{du}{dt}(s) \right\rangle_{W',W} ds$$

for all $t \in [0,T)$, where

$$E(t) := (\Psi \circ u)(t) - (F \circ u)(t).$$

Theorem 103 *Let X, Y, W be Banach spaces containing a common subspace Z, with X continuously embedded in W. Let $Q : [0,\infty) \times W \to W'$ is continuous and such that*

$$\|Q(t,w)\|_{W'} \leq g(t)\, \|w\|_W^{m-1} \tag{145}$$

for all $t \geq 0$, for all $w \in W$, and for some $m > 1$ and some positive and locally bounded function g such that

$$\int^{\infty} g^{-1/(m-1)}(t)\ dt = \infty, \tag{146}$$

and

$$\|Q(t,w)\|_{W'}\, \|w\|_W \leq \gamma \langle Q(t,w), w \rangle_{W',W} \tag{147}$$

for all $t \geq 0$, for all $w \in W$, and for some $\gamma \geq 1$. Let $\Psi : Y \to [0,\infty)$ be of class C^1 and

$$q\Psi(y) \geq \langle \nabla\Psi(y), y \rangle_{Y',Y} \tag{148}$$

for all $y \in Y$ and for some $q > 0$. Let $F : X \to \mathbb{R}$ be of class C^1, with $F(0) = 0$, and such that for every $\tau > 0$ there exist $c > 0$ and $p > m$ such that

$$\langle \nabla F(x), x \rangle_{X',X} - qF(x) \geq c\,(F(x))^{1/p'}\, \|x\|_X \tag{149}$$

for all $x \in X$ such that $F(x) \geq \tau$. Then for every $u_0 \in Z$ with

$$\Psi(u_0) - F(u_0) < 0$$

the evolution problem (143) admits no strong global solution.

Proof. Assume by contradiction that (143) admits a global solution u. By the energy identity and (147),

$$E(t) = E(0) - \int_0^t \left\langle Q\left(s, \frac{du}{dt}(s)\right), \frac{du}{dt}(s) \right\rangle_{W',W} ds \le E(0) = \Psi(u_0) - F(u_0) =: -\tau < 0.$$

Since $\Psi \ge 0$, this implies that $(F \circ u)(t) \ge \tau$ for all $t > 0$. In particular, $u(t) \ne 0$ for all $t > 0$ because $F(0) = 0$.

Taking $\varphi = u$ in (144), then by differentiating we have that for $\mathcal{L}^1$ a.e. $t > 0$,

$$\begin{aligned}
0 = &- \left\langle Q\left(t, \frac{du}{dt}(t)\right), u(t) \right\rangle_{W',W} - \langle \nabla\Psi(u(t)), u(t)\rangle_{Y',Y} + \langle \nabla F(u(t)), u(t)\rangle_{X',X} \\
&+ q\left((\Psi \circ u)(t) - (F \circ u)(t) - E(t)\right) \\
\ge &- \left\langle Q\left(t, \frac{du}{dt}(t)\right), u(t) \right\rangle_{W',W} + c\left(F(u(t))\right)^{1/p'} \|u(t)\|_X ,
\end{aligned}$$

where we used (148), (149). It follows that

$$\begin{aligned}
C\left\|Q\left(t, \frac{du}{dt}(t)\right)\right\|_{W'} \|u(t)\|_X &\ge \left\|Q\left(t, \frac{du}{dt}(t)\right)\right\|_{W'} \|u(t)\|_W \\
&\ge \left\langle Q\left(t, \frac{du}{dt}(t)\right), u(t) \right\rangle_{W',W} \ge c\left(F(u(t))\right)^{1/p'} \|u(t)\|_X ,
\end{aligned}$$

where we used the fact that X is continuously embedded in W. Since $u(t) \ne 0$ for all $t > 0$, it follows that

$$\left\|Q\left(t, \frac{du}{dt}(t)\right)\right\|_{W'} \ge c\left(F(u(t))\right)^{1/p'} \ge c\left(-E(t)\right)^{1/p'} .$$

On the other hand, by (145), (147),

$$\begin{aligned}
-E'(t) &= \left\langle Q\left(t, \frac{du}{dt}\right), \frac{du}{dt} \right\rangle_{W',W} \ge \frac{1}{\gamma}\left\|Q\left(t, \frac{du}{dt}\right)\right\|_{W'} \left\|\frac{du}{dt}\right\|_W \\
&\ge \frac{1}{\gamma}\left\|Q\left(t, \frac{du}{dt}\right)\right\|_{W'} \left\|\frac{du}{dt}\right\|_W \\
&\ge \frac{1}{\gamma}\left\|Q\left(t, \frac{du}{dt}\right)\right\|_{W'} \left\|Q\left(t, \frac{du}{dt}\right)\right\|_{W'}^{1/(m-1)} g^{-1/(m-1)}(t) \\
&= \frac{1}{\gamma} g^{-1/(m-1)}(t) \left\|Q\left(t, \frac{du}{dt}\right)\right\|_{W'}^{1+1/(m-1)}
\end{aligned}$$

and so

$$-E'(t) \ge c g^{-1/(m-1)}(t) \left(-E(t)\right)^{m'/p'} .$$

Since $p > m$, we have that $m'/p' > 1$. Hence, with $h(t) := -E(t)$,

$$\frac{h'(t)}{h^{m'/p'}(t)} \geq c g^{-1/(m-1)}(t) \quad \Leftrightarrow \quad \int_{h(0)}^{h(t)} \frac{1}{r^{m'/p'}} \, dr \geq c \int_0^t g^{-1/(m-1)}(s) \, ds$$

$$\Leftrightarrow \quad 0 > -\frac{1}{\frac{m'}{p'}-1} \frac{1}{(h(t))^{m'/p'-1}} \geq -\frac{1}{\frac{m'}{p'}-1} \frac{1}{(h(0))^{m'/p'-1}} + c \int_0^t g^{-1/(m-1)}(s) \, ds.$$

In view of (146) we have that t cannot go to infinity. ■

Exercise 104 *Prove that if*

$$\int^{\infty} g^{-1/(m-1)}(t) \, dt < \infty,$$

then for every $u_0 \in Z$ with energy sufficiently negative

$$\Psi(u_0) - F(u_0) \ll 0$$

the evolution problem (143) admits no strong global solution.

Monday, April 7, 2014

Example 105 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and consider the Dirichlet problem*

$$\begin{cases} \left|\frac{\partial u}{\partial t}(\boldsymbol{x}, t)\right|^{m-2} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \operatorname{div}\left(\|\nabla u(\boldsymbol{x}, t)\|^{q-2} \nabla u(\boldsymbol{x}, t)\right) = a\,|u(\boldsymbol{x}, t)|^{p-2} u(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t > 0, \end{cases}$$

where $m, q, p > 1$ and $a > 0$. To apply the previous theorem, take $W := L^m(\Omega)$ and $Q : L^m(\Omega) \to L^{m'}(\Omega)$ to be the operator

$$Q(w)(\boldsymbol{x}) := |w(\boldsymbol{x})|^{m-2} w(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Then

$$\int_\Omega |Q(w)(\boldsymbol{x})|^{m'} \, d\boldsymbol{x} = \int_\Omega \left| |w(\boldsymbol{x})|^{m-1} \right|^{\frac{m}{m-1}} \, d\boldsymbol{x} = \int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x} < \infty,$$

so Q is well-defined. Note that

$$\|Q(w)\|_{L^{m'}(\Omega)} = \left(\int_\Omega |Q(w)(\boldsymbol{x})|^{m'} \, d\boldsymbol{x}\right)^{\frac{m-1}{m}} = \left(\int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x}\right)^{\frac{m-1}{m}} = \|w\|_{L^m(\Omega)}^{m-1},$$

so that (145) and (146) are satified with $g = 1$, while

$$\begin{aligned} \langle Q(w), w\rangle_{L^{m'}(\Omega), L^m(\Omega)} &= \int_\Omega Q(w)(\boldsymbol{x}) w(\boldsymbol{x}) \, d\boldsymbol{x} \\ &= \int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x} = \left(\int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x}\right)^{\frac{m-1}{m}} \left(\int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x}\right)^{\frac{1}{m}} \\ &= \left(\int_\Omega |Q(w)(\boldsymbol{x})|^{m'} \, d\boldsymbol{x}\right)^{\frac{m-1}{m}} \left(\int_\Omega |w(\boldsymbol{x})|^m \, d\boldsymbol{x}\right)^{\frac{1}{m}}, \end{aligned}$$

so (147) holds with $\gamma = 1$.

Next we take with $Y = W_0^{1,q}(\Omega)$ *and*

$$\Psi(v) := \frac{1}{q}\int_\Omega \|\nabla v(\boldsymbol{x})\|^q \ d\boldsymbol{x}.$$

Since $q > 1$*, it can be verified that* Ψ *is of class* C^1 *with*

$$\nabla\Psi : W_0^{1,q}(\Omega) \to \left(W_0^{1,q}(\Omega)\right)' = W^{-1,q'}(\Omega)$$

given by

$$\langle \nabla\Psi(v), \varphi\rangle_{W^{-1,q'}(\Omega),W_0^{1,q}(\Omega)} = \int_\Omega \|\nabla v(\boldsymbol{x})\|^{q-2}\,\nabla v(\boldsymbol{x})\cdot\nabla\varphi(\boldsymbol{x}) \ d\boldsymbol{x}$$

for all $\varphi \in W_0^{1,q}(\Omega)$*. In particular, (148) is an equality.*

Finally, we take with $X = L^p(\Omega)$ *and*

$$F(v) := \frac{a}{p}\int_\Omega |v(\boldsymbol{x})|^p \ d\boldsymbol{x}.$$

Since $p > 1$*, it can be verified that* F *is of class* C^1 *with*

$$\nabla F : L^p(\Omega) \to L^{p'}(\Omega)$$

given by

$$\langle \nabla F(v), \varphi\rangle_{L^{p'}(\Omega),L^p(\Omega)} = a\int_\Omega |v(\boldsymbol{x})|^{p-2}\,v(\boldsymbol{x})\,\varphi(\boldsymbol{x}) \ d\boldsymbol{x}$$

for all $\varphi \in L^p(\Omega)$*. In particular,*

$$\begin{aligned}\langle \nabla F(v), v\rangle_{L^{p'}(\Omega),L^p(\Omega)} - qF(v) &= a\int_\Omega |v(\boldsymbol{x})|^p \ d\boldsymbol{x} - \frac{aq}{p}\int_\Omega |v(\boldsymbol{x})|^p \ d\boldsymbol{x} \\ &= c\left(1-\frac{q}{p}\right)\left(\frac{a}{p}\int_\Omega |v(\boldsymbol{x})|^p \ d\boldsymbol{x}\right)^{1/p'}\left(\int_\Omega |v(\boldsymbol{x})|^p \ d\boldsymbol{x}\right)^{1/p}\end{aligned}$$

and so (149) holds provided

$$p > q.$$

We are now in position to apply the previous theorem to conclude non-existence of global solutions when

$$p > m$$

and $u_0 \in Z := L^p(\Omega) \cap W_0^{1,q}(\Omega)$ *is such that*

$$\frac{1}{q}\int_\Omega \|\nabla u_0(\boldsymbol{x})\|^q \ d\boldsymbol{x} - \frac{a}{p}\int_\Omega |u_0(\boldsymbol{x})|^p \ d\boldsymbol{x} < 0.$$

Instead, when $p \le m$ *and* $q > 1$, ***if there is local existence and continuation of energy bounded solutions****, one expects global existence.*

Remark 106 *The differential operator* $\operatorname{div}\left(\|\nabla v\|^{q-2}\nabla v\right)$ *is called the* q-Laplacian operator*.*

We now study blow-up of solution. We will see that for the Dirichlet problem (118) is not enough to guarantee blow-up of solutions. Indeed, one has to distinguish between large initial data and small initial data. An important role is played here by the potential energy

$$E(t)=\frac{1}{2}\int_{\Omega}\|\nabla u(\boldsymbol{x},t)\|^{2}\ d\boldsymbol{x}-\int_{\Omega}F(u(\boldsymbol{x},t))\ d\boldsymbol{x},$$

where F is a potential of f, that is,

$$F(z):=\int_{0}^{z}f(s)\ ds.$$

If $E(0)$ is sufficiently small, then one expects global solutions, while if $E(0)$ is large, then one expects non existence of solutions and blow-up in finite time. For simplicity, consider the function

$$f(z)=|z|^{p-2}z,$$

where $p>1$. In this case, (118) becomes

$$\int_{z_0}^{\infty}\frac{1}{f(z)}\ dz=\int_{z_0}^{\infty}z^{-p+1}\ dz<\infty$$

if and only if $p>2$. Hence, if $p=2$, we can apply Theorem **??** to obtain existence of global solutions. For $p<2$ the function f is not locally Lipschitz near zero. We leave this case as an exercise.

It remains to consider the case $p>2$. We consider first the Dirichlet problem

$$\begin{cases}\frac{\partial u}{\partial t}(\boldsymbol{x},t)-\Delta u(\boldsymbol{x},t)=|u(\boldsymbol{x},t)|^{p-2}u(\boldsymbol{x},t) & (\boldsymbol{x},t)\in\Omega\times(0,\infty),\\ u(\boldsymbol{x},0)=u_0(\boldsymbol{x}) & \boldsymbol{x}\in\Omega,\\ u(\boldsymbol{x},t)=0 & \boldsymbol{x}\in\partial\Omega,\quad t\in(0,\infty).\end{cases}\tag{150}$$

In view of Example 105, if $u_0\in H_0^1(\Omega)$ and $u_0\in L^p(\Omega)\cap H_0^1(\Omega)$ is such that

$$\frac{1}{2}\int_{\Omega}\|\nabla u_0(\boldsymbol{x})\|^{2}\ d\boldsymbol{x}-\frac{1}{p}\int_{\Omega}|u_0(\boldsymbol{x})|^{p}\ d\boldsymbol{x}<0,$$

then there is no global existence of weak solutions. But what about local existence?

Instead, when

$$\frac{1}{2}\int_{\Omega}\|\nabla u_0(\boldsymbol{x})\|^{2}\ d\boldsymbol{x}-\frac{1}{p}\int_{\Omega}|u_0(\boldsymbol{x})|^{p}\ d\boldsymbol{x}>0,\tag{151}$$

then there can be global existence.

Theorem 107 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set of class C^∞. Let $p > 2$. Then there exists $u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$, satisfying (151) such that the Dirichlet problem*

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = |u(\boldsymbol{x},t)|^{p-2} u(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,\infty) \end{cases} \tag{152}$$

has a global solution.

Proof. Consider the eigenvalue problem

$$\begin{cases} -\Delta \phi(\boldsymbol{x}) = \lambda_1 \phi(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \phi(\boldsymbol{x}) = 0 & \boldsymbol{x} \in \partial\Omega, \end{cases}$$

where

$$\lambda_1 := \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \|\nabla\psi\|^2 \, d\boldsymbol{x}}{\int_\Omega |\psi|^2 \, d\boldsymbol{x}} > 0.$$

Let $\phi_1 \in H_0^1(\Omega)$ be a solution. By the maximum principle, we can assume that $\phi_1 > 0$ in Ω. Also, by rescaling, we can assume that $\int_\Omega \phi_1 \, d\boldsymbol{x} = 1$. Finally by regularity, we have that $\phi_1 \in C^\infty(\overline{\Omega})$. Let $M \geq \max_{\overline{\Omega}} \phi_1$. Then

$$\begin{aligned} -\Delta(\varepsilon\phi_1)(\boldsymbol{x}) - \varepsilon^{p-1}\phi_1^{p-1}(\boldsymbol{x}) &= \lambda_1 \varepsilon\phi_1(\boldsymbol{x}) - \varepsilon^{p-1}\phi_1^{p-1}(\boldsymbol{x}) \\ &= \varepsilon\phi_1(\boldsymbol{x})\left(\lambda_1 - \varepsilon^{p-2}\phi_1^{p-2}(\boldsymbol{x})\right) \\ &\geq \varepsilon\phi_1(\boldsymbol{x})\left(\lambda_1 - \varepsilon^{p-2}M^{p-2}\right) > 0 \end{aligned}$$

in Ω provided $0 < \varepsilon < \lambda_1^{1/(p-2)}/M$. Let $u_0 \in C_c^\infty(\Omega)$ be such that $0 \leq u_0 < \varepsilon\phi_1$ in Ω. Define

$$f(z) := \begin{cases} z^{p-1} & \text{if } 0 \leq z \leq \varepsilon M, \\ 0 & \text{if } z < 0, \\ (\varepsilon M)^{p-1} & \text{if } z \geq M. \end{cases}$$

Note that f is (globally) Lipschitz and bounded. By Theorem 92 there exists a unique weak solution of the Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(u(\boldsymbol{x},t)) & (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,\infty). \end{cases}$$

Since $v = \varepsilon\phi_1$ satisfies

$$\begin{cases} \frac{\partial v}{\partial t}(\boldsymbol{x},t) - \Delta v(\boldsymbol{x},t) \geq f(v(\boldsymbol{x},t)) & (\boldsymbol{x},t) \in \Omega \times (0,\infty), \\ v(\boldsymbol{x},0) \geq u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ v(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,\infty), \end{cases}$$

by the comparison principle it follows that

$$0 \leq u(\boldsymbol{x},t) \leq \varepsilon\phi_1(\boldsymbol{x})$$

for all $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$. Hence, $f(u(\boldsymbol{x}, t)) = (u(\boldsymbol{x}, t))^{p-1}$ for all $(\boldsymbol{x}, t) \in \Omega \times (0, \infty)$, which shows that u is a global solution of (152). ■

Wednesday, April 9, 2014

The situation for the Neumann problem is rather different.

Theorem 108 (Non Existence) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let $u_0 \in L^\infty(\Omega)$, and let $f : \mathbb{R} \to \mathbb{R}$ be convex, such that f is positive for $z > 0$ and*

$$\int^\infty \frac{1}{f(z)}\, dz < \infty.$$

If $\int_\Omega u_0\, d\boldsymbol{x} > 0$, then every weak solution of the Neumann problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(u(\boldsymbol{x}, t)) & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t > 0, \end{cases}$$

does not exist for all times.

Proof. Assume by contradiction that a weak solution u exists for all times. Then

$$\int_\Omega \frac{\partial u}{\partial t}(\boldsymbol{x}, t)\, w(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega \nabla u(\boldsymbol{x}, t) \cdot \nabla w(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f(u(\boldsymbol{x}, t))\, w(\boldsymbol{x}, t)\, d\boldsymbol{x}$$

for all $w \in C^1(\overline{\Omega})$. Taking $w := \frac{1}{\mathcal{L}^N(\Omega)}$ gives

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(\boldsymbol{x}, t)\, d\boldsymbol{x}\right) + 0 &= \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega f(u(\boldsymbol{x}, t))\, d\boldsymbol{x} \\ &\geq f\left(\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(\boldsymbol{x}, t)\, d\boldsymbol{x}\right), \end{aligned}$$

where we have used Jensen's inequality. Setting $v(t) := \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u(\boldsymbol{x}, t)\, d\boldsymbol{x}$ we have that

$$\begin{cases} v'(t) \geq f(v(t)), \\ v(0) = v_0, \end{cases}$$

where

$$v_0 := \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega u_0(\boldsymbol{x})\, d\boldsymbol{x} > 0.$$

Reasoning as in (119), we have that

$$\int_{v_0}^{v(t)} \frac{1}{f(z)}\, dz \geq t.$$

Letting $t \to \infty$, we obtain a contradiction. Hence, v exists only in some interval $[0, T)$. ■

17 Quasilinear Equations: Vanishing Viscosity

In this section, we discuss a few applications of fixed point theorems to nonlinear parabolic equations. Consider the equation

$$u_t - \Delta u + \sum_{i=1}^{N} b_i(u) \frac{\partial u}{\partial x_i} = h(\boldsymbol{x}, t, u), \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^N,\ t \geq 0.$$

To prove existence we consider first the following problem

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) - \operatorname{div} \boldsymbol{F}(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T), \end{cases}$$

Exercise 109 *Consider an evolution triple, with Y reflexive.*

(i) Prove that $H \cong H'$ is dense in Y'.

(ii) Prove that for every $f \in L^2((0,T); Y')$ there exists a sequence $\{f_n\} \subset L^2((0,T); H)$ such that $f_n \to f$ in $f \in L^2((0,T); Y')$.

Exercise 110 *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, let $u_0 \in L^2(\Omega)$, and let $f \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$, where $0 < T < \infty$.*

*(i) **(Uniqueness)** Prove that there exists at most one function $u \in L^2\left((0,T); H^1(\Omega)\right)$ with $\frac{\partial u}{\partial t} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$ such that $u(\cdot, t) = u_0$ and*

$$\int_\Omega \nabla u(\boldsymbol{x}, t) \cdot \nabla w(\boldsymbol{x})\, d\boldsymbol{x} - \langle f(\cdot, t), w \rangle_{(H^1(\Omega))', H^1(\Omega)} + \left\langle \frac{\partial u}{\partial t}(\cdot, t), w \right\rangle_{(H^1(\Omega))', H^1(\Omega)} = 0 \tag{153}$$

for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$.

*(ii) **(A priori estimates)** Prove that there if $u \in L^2\left((0,T); H^1(\Omega)\right)$, with $\frac{\partial u}{\partial t} \in L^2\left((0,T); \left(H^1(\Omega)\right)'\right)$, is such that $u(\cdot, t) = u_0$, and (153) holds for all $w \in H^1(\Omega)$ and for $\mathcal{L}^1$ a.e. $t \in (0,T)$, then*

$$\|u\|_{L^2((0,T);H^1(\Omega))} \leq c\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);(H^1(\Omega))')}\right),$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')} \leq c\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);(H^1(\Omega))')}\right).$$

Moreover,

$$\max_{t \in [0,T]} \|u(\cdot, t)\|_{L^2(\Omega)} \leq c\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T);(H^1(\Omega))')}\right). \tag{154}$$

(iii) Use Exercise 109 to prove existence of a solution.

Note that (154) follows from Theorem 35.

Exercise 111 *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open set, let* $u_0 \in L^2(\Omega)$*, let* $f \in L^2\left((0,T);L^2(\Omega)\right)$*, and let* $\boldsymbol{F} \in L^2\left((0,T);L^2\left(\Omega;\mathbb{R}^N\right)\right)$*.*

(i) Prove that there exists a unique weak solution u *of*

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x},t) - \Delta u(\boldsymbol{x},t) = f(\boldsymbol{x},t) - \operatorname{div}\boldsymbol{F}(\boldsymbol{x},t) & (\boldsymbol{x},t) \in \Omega_T, \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T), \end{cases}$$

where $\operatorname{div} := \operatorname{div}_{\boldsymbol{x}}$*.*

(ii) Prove that following estimates hold

$$\begin{aligned} \|\nabla u\|_{L^2(\Omega_T)} &\le C\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_T)} + \|\boldsymbol{F}\|_{L^2(\Omega_T)}\right), \\ \left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);(H^1(\Omega))')} &\le C\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega_T)} + \|\boldsymbol{F}\|_{L^2(\Omega_T)}\right), \\ \max_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} &\le C\left(\|u_0\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega_T)} + \|\boldsymbol{F}\|_{L^2(\Omega_T)}\right). \end{aligned}$$

Solution 112 *Assume first that* $\boldsymbol{F} \in L^2\left((0,T);H^1\left(\Omega;\mathbb{R}^N\right)\right)$*. Let's prove that* $\operatorname{div}\boldsymbol{F} \in L^2\left((0,T);H^{-1}(\Omega)\right)$*. For* $\mathcal{L}^1$ *a.e.* $t \in (0,T)$ *and and all* $w \in H_0^1(\Omega)$ *we have*

$$\left|\int_\Omega w(\boldsymbol{x}) \operatorname{div}\boldsymbol{F}(\boldsymbol{x},t)\, d\boldsymbol{x}\right| = \left|\int_\Omega \nabla w(\boldsymbol{x}) \cdot \boldsymbol{F}(\boldsymbol{x},t)\, d\boldsymbol{x}\right| \le \|\boldsymbol{F}(\cdot,t)\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)}.$$

Hence,

$$\begin{aligned} \|\operatorname{div}\boldsymbol{F}(\cdot,t)\|_{H^{-1}(\Omega)} &= \sup_{w\in H_0^1(\Omega)\setminus\{0\}} \frac{\left|\int_\Omega w(\boldsymbol{x}) \operatorname{div}\boldsymbol{F}(\boldsymbol{x},t)\, d\boldsymbol{x}\right|}{\|w\|_{H^1(\Omega)}} \\ &\le \|\boldsymbol{F}(\cdot,t)\|_{L^2(\Omega)}. \end{aligned}$$

It follows that

$$\|\operatorname{div}\boldsymbol{F}\|_{L^2((0,T);H^{-1}(\Omega))} \le \|\boldsymbol{F}\|_{L^2(\Omega_T)}.$$

Hence, we are in a position to apply Exercise 110.

Note that by density of smooth functions, given $\boldsymbol{F} \in L^2\left(\Omega_T;\mathbb{R}^N\right)$*, we can find a sequence* $\{\boldsymbol{F}_n\} \subset L^2\left((0,T);H^1\left(\Omega;\mathbb{R}^N\right)\right)$ *such that* $\boldsymbol{F}_n \to \boldsymbol{F}$ *in* $L^2\left(\Omega_T;\mathbb{R}^N\right)$*. In turn, by the previous estimates*

$$\|\operatorname{div}\boldsymbol{F}_n - \operatorname{div}\boldsymbol{F}_m\|_{L^2((0,T);H^{-1}(\Omega))} \le \|\boldsymbol{F}_n - \boldsymbol{F}_m\|_{L^2(\Omega_T)} \to 0$$

as $n,m \to \infty$*. Hence,* $\operatorname{div}\boldsymbol{F}_n \to \boldsymbol{G}$ *in* $L^2\left((0,T);H^{-1}(\Omega)\right)$*. Since for* $\mathcal{L}^1$ *a.e.* $t \in (0,T)$ *and and all* $w \in H_0^1(\Omega)$ *we have*

$$\langle \operatorname{div}\boldsymbol{F}_n(\cdot,t), w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega w(\boldsymbol{x}) \operatorname{div}\boldsymbol{F}_n(\boldsymbol{x},t)\, d\boldsymbol{x} = -\int_\Omega \nabla w(\boldsymbol{x})\cdot\boldsymbol{F}_n(\boldsymbol{x},t)\, d\boldsymbol{x},$$

letting $n \to \infty$ *gives*

$$\langle \boldsymbol{G}(\cdot,t), w\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_\Omega \nabla w(\boldsymbol{x}) \cdot \boldsymbol{F}(\boldsymbol{x},t)\ d\boldsymbol{x},$$

which shows that $\boldsymbol{G}(\cdot,t) = \operatorname{div} \boldsymbol{F}(\cdot,t)$.

Friday, April 11, 2014

Carnival, make-up class

18 Other Parabolic Equations

18.1 The Space $H^{-1}(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. In view of Poincaré's inequality in the space $H_0^1(\Omega)$ we can consider the equivalent inner product

$$(v,w)_{H_0^1(\Omega)} := \int_\Omega \nabla v(\boldsymbol{x}) \cdot \nabla w(\boldsymbol{x})\ d\boldsymbol{x}$$

and let $H^{-1}(\Omega)$ denote the corresponding dual space. The norm in $H^{-1}(\Omega)$ is given by

$$\|L\|_{H^{-1}(\Omega)} = \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{|L(v)|}{\|v\|_{H_0^1(\Omega)}} = \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{|L(v)|}{\|\nabla v\|_{L^2(\Omega)}}.$$

Given $L \in H^{-1}(\Omega)$, consider the Dirichlet problem

$$\begin{cases} -\Delta u = L & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{155}$$

A weak solution of this problem is a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u\ d\boldsymbol{x} = L(v) \tag{156}$$

for all $v \in H_0^1(\Omega)$.

Theorem 113 *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set and let* $L \in H^{-1}(\Omega)$. *Then there exists a unique weak solution* $u \in H_0^1(\Omega)$ *of* (155). *Moreover,*

$$\|u\|_{H_0^1(\Omega)} = \|L\|_{H^{-1}(\Omega)}. \tag{157}$$

We write that $u = \Delta^{-1}(L)$.

First proof. Step 1: Let's prove uniqueness. Let u_1 and u_2 be two weak solutions of (155). Then

$$\int_\Omega \nabla v \cdot \nabla u_1\ d\boldsymbol{x} - L(v) = 0,$$
$$\int_\Omega \nabla v \cdot \nabla u_2\ d\boldsymbol{x} - L(v) = 0$$

for all $v \in H_0^1(\Omega)$. By subtracting the two equations, we get

$$\int_\Omega \nabla v \cdot \nabla (u_1 - u_2)\ d\boldsymbol{x} = 0,$$

and taking $v = u_1 - u_2$ gives $\|u_1 - u_2\|_{H_0^1(\Omega)} = 0$. It follows that $u_1 = u_2$.
Step 2: Next we will prove that a weak solution u satisfies (157). Taking $v = u$ in (156) gives

$$\|u\|_{H_0^1(\Omega)}^2 = \int_\Omega \|\nabla u\|^2\ d\boldsymbol{x} = L(u) \leq \|L\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)}.$$

Hence,

$$\|u\|_{H_0^1(\Omega)} \leq \|L\|_{H^{-1}(\Omega)}.$$

On the other hand, by (156) and Hölder's inequality, for $v \in H_0^1(\Omega) \setminus \{0\}$,

$$|L(v)| = \left|\int_\Omega \nabla v \cdot \nabla u\ d\boldsymbol{x}\right| \leq \|\nabla v\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} = \|v\|_{H_0^1(\Omega)} \|u\|_{H_0^1(\Omega)},$$

and so diving by $\|v\|_{H_0^1(\Omega)}$ and taking the supremum over all v gives

$$\|L\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{|L(v)|}{\|v\|_{H_0^1(\Omega)}} \leq \|u\|_{H_0^1(\Omega)}.$$

Step 3: We will prove the existence of a weak solution of (155). Consider the functional

$$J(v) := \frac{1}{2}\int_\Omega \|\nabla v\|^2\ d\boldsymbol{x} - L(v)$$

defined on the space $H_0^1(\Omega)$. We begin by showing that J is bounded from below. Indeed,

$$J(v) \geq \frac{1}{2}\|v\|_{H_0^1(\Omega)}^2 - \|L\|_{H^{-1}(\Omega)} \|v\|_{H_0^1(\Omega)} \tag{158}$$

for all $u \in X$. It suffices to observe that the function

$$t \in \mathbb{R} \mapsto \frac{1}{2}t^2 - \|L\|_{H^{-1}(\Omega)}\, t$$

is bounded from below.

Next let

$$m := \inf_{u \in X} J(u)$$

and, using the definition of infimum consider a sequence $\{u_n\} \subset X$ such that

$$m \leq J(u_n) \leq m + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} J(u_n) = m.$$

It follows from (158) and the fact that $J(u_n) \leq m+1$ for all n, that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and so, up to a subsequence, not relabelled, there exists $u \in H_0^1(\Omega)$ such that $\{u_n\}$ converges weakly to u in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. We claim that $J(u) = m$. To see this, observe that

$$
\begin{aligned}
J(u_n) &= \frac{1}{2}\int_\Omega \|\nabla u_n\|^2 \, d\boldsymbol{x} - L(u_n) \\
&= \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u + \nabla u\|^2 \, d\boldsymbol{x} - L(u_n) \\
&= J(u) + \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u\|^2 \, d\boldsymbol{x} \\
&\quad + \int_\Omega (\nabla u_n - \nabla u)\cdot \nabla u \, d\boldsymbol{x} - L(u_n).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
m \leq J(u) &\leq J(u) + \frac{1}{2}\int_\Omega \|\nabla u_n - \nabla u\|^2 \, d\boldsymbol{x} \\
&= J(u_n) - \int_\Omega (\nabla u_n - \nabla u)\cdot \nabla u \, d\boldsymbol{x} - L(u_n).
\end{aligned}
$$

Letting $n \to \infty$ and using the facts that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, and $J(u_n) \to m$, shows that

$$
m = J(u)
$$

and that $\nabla u_n \to \nabla u$ in $L^2(\Omega; \mathbb{R}^N)$.

Next we will show that u is a weak solution of (**??**). To see this, let $w \in H_0^1(\Omega)$. Then for every $t \in \mathbb{R}$,

$$
J(u) \leq J(u + tw).
$$

This shows that the real value function $\omega(t) := J(u+tw)$ has a minimum at $t = 0$. Hence, if ω is differentiable at $t = 0$, then $\omega'(0) = 0$. We have

$$
\begin{aligned}
J(u+tw) &= \frac{1}{2}\int_\Omega \|\nabla u + t\nabla w\|^2 \, d\boldsymbol{x} - L(u+tw) \\
&= J(u) + \frac{t^2}{2}\int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} \\
&\quad + t\int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - tL(w),
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{\omega(t) - \omega(0)}{t} &= \frac{J(u+tw) - J(u)}{t} \\
&= \frac{t}{2}\int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} + \int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - L(w).
\end{aligned}
$$

Letting $t \to 0^+$ it follows that ω is differentiable, which implies that $\omega'(0) = 0$, so that

$$0 = \int_\Omega \nabla w \cdot \nabla u \, d\boldsymbol{x} - L(w).$$

This conclude the proof. ■

The second proof relies on the Riesz representation theorem.

Second proof. Given L, by the Riesz representation theorem in the Hilbert space $H_0^1(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$L(v) = (u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, d\boldsymbol{x}$$

for all $v \in H_0^1(\Omega)$. Thus (156) holds. ■

Now consider the operator

$$J : H_0^1(\Omega) \to H^{-1}(\Omega)$$

defined by

$$(J(u))(v) := (u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) \, d\boldsymbol{x}$$

for all $u, v \in H_0^1(\Omega)$. We have that J is an isomorphism. Indeed, given $u \in H_0^1(\Omega)$ we have

$$\|u\|_{H_0^1(\Omega)}^2 = \int_\Omega \|\nabla u\|^2 \, d\boldsymbol{x} = (J(u))(u) \le \|J(u)\|_{H^{-1}(\Omega)} \|u\|_{H_0^1(\Omega)}.$$

Hence,

$$\|u\|_{H_0^1(\Omega)} \le \|J(u)\|_{H^{-1}(\Omega)}.$$

On the other hand, by Hölder's inequality, for $v \in H_0^1(\Omega) \setminus \{0\}$,

$$|(J(u))(v)| = \left| \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} \right| \le \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} = \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)},$$

and so diving by $\|v\|_{H_0^1(\Omega)}$ and taking the supremum over all v gives

$$\|J(u)\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{|L(v)|}{\|v\|_{H_0^1(\Omega)}} \le \|u\|_{H_0^1(\Omega)}.$$

Hence, $H_0^1(\Omega)$ is isomorphic to $J\left(H_0^1(\Omega)\right)$. But by the previous theorem given $L \in H^{-1}(\Omega)$ there exists a unique $v \in H_0^1(\Omega)$ such that $J(v) = L$, that is,

$$\int_\Omega \nabla v(\boldsymbol{x}) \cdot \nabla w(\boldsymbol{x}) \, d\boldsymbol{x} = L(w)$$

for all $w \in H_0^1(\Omega)$, which says that w is a weak solution of the equation $-\Delta v = L$ in $H^{-1}(\Omega)$. Hence, $J\left(H_0^1(\Omega)\right) = H^{-1}(\Omega)$ and $J(v) = -\Delta$. Given $L_1, L_2 \in H^{-1}(\Omega)$, we can define the inner product

$$(L_1, L_2)_{H^{-1}(\Omega)} = \left(J^{-1}(L_1), J^{-1}(L_2)\right)_{H_0^1(\Omega)} = \int_\Omega \nabla\left(J^{-1}(L_1)\right)(\boldsymbol{x})\cdot\nabla\left(J^{-1}(L_2)\right)(\boldsymbol{x})\ d\boldsymbol{x}. \tag{159}$$

Note that this inner product is compatible with the norm in $H^{-1}(\Omega)$.

In particular, given a function $f_1 \in L^2(\Omega)$, consider the functional

$$L_1(w) := \int_\Omega f_1(\boldsymbol{x})\, w(\boldsymbol{x})\ d\boldsymbol{x}, \quad w \in H_0^1(\Omega).$$

Then $u_1 := J^{-1}(L_1)$ solves the equation $-\Delta u_1 = f_1$, and so

$$\begin{aligned}(L_1, L_2)_{H^{-1}(\Omega)} &= \int_\Omega \nabla u_1(\boldsymbol{x}) \cdot \nabla\left(J^{-1}(L_2)\right)(\boldsymbol{x})\ d\boldsymbol{x} \\ &= -\int_\Omega \Delta u_1(\boldsymbol{x})\, J^{-1}(L_2)(\boldsymbol{x})\ d\boldsymbol{x} \\ &= \int_\Omega f_1(\boldsymbol{x})\, J^{-1}(L_2)(\boldsymbol{x})\ d\boldsymbol{x}.\end{aligned} \tag{160}$$

18.2 The Porous Media Equation

We consider the porous media equation with Dirichlet boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta\left(|u|^{\ell-2} u\right) = f & \text{in } \Omega_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0, T). \end{cases} \tag{161}$$

We want to apply Theorem 58 with $H := H^{-1}(\Omega)$. When $N \geq 3$, in order to have $L^\ell(\Omega) \subset H^{-1}(\Omega)$, we need $H_0^1(\Omega) \subset L^{\ell'}(\Omega)$, which holds if

$$\ell' \leq 2^* = \frac{2N}{N-2} \quad \Leftrightarrow \quad \frac{\ell}{\ell-1} \leq \frac{2N}{N-2} \quad \Leftrightarrow \quad \frac{2N}{N+2} \leq \ell.$$

Define

$$\Psi(v) := \begin{cases} \frac{1}{\ell}\int_\Omega |v|^\ell(\boldsymbol{x})\ d\boldsymbol{x} & \text{if } v \in L^\ell(\Omega), \\ \infty & \text{otherwise in } H^{-1}(\Omega). \end{cases}$$

Monday, April 14, 2014

We continue the topics began on Wednesday, April 9

Theorem 114 *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $u_0 \in H_0^1(\Omega)$, let $b_i : \mathbb{R} \to \mathbb{R}$ be bounded Borel functions, $i = 1, \ldots, N$, and let $h : \Omega \times (0, T) \times \mathbb{R} \to \mathbb{R}$ be a Borel function such that $h(\boldsymbol{x}, t, 0) = 0$ for all $\boldsymbol{x} \in \Omega$ and $t \in (0, T)$, and*

$$|h(\boldsymbol{x}, t, z_1) - h(\boldsymbol{x}, t, z_2)| \leq L\,|z_1 - z_2|$$

for all for all $\boldsymbol{x} \in \Omega$, $t \in (0,T)$, *and* $z_1, z_2 \in \mathbb{R}$, *and for some* $L > 0$. *Then the Dirichlet problem has a unique weak solution*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \sum_{i=1}^{N} b_i(u) \frac{\partial u}{\partial x_i} = h(\boldsymbol{x}, t, u) & (\boldsymbol{x}, t) \in \Omega_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Proof. We will apply the Banach fixed point theorem in the space $X := L^\infty\left((0,T); L^2(\Omega)\right)$. Given $v \in X$ define

$$f(\boldsymbol{x}, t) := h(\boldsymbol{x}, t, v(\boldsymbol{x}, t))$$
$$F_i(\boldsymbol{x}, t) = \int_0^{v(\boldsymbol{x},t)} b_i(s)\ ds.$$

Then

$$|f(\boldsymbol{x}, t)| = |h(\boldsymbol{x}, t, v(\boldsymbol{x}, t)) - h(\boldsymbol{x}, t, 0)| \le L\, |v(\boldsymbol{x}, t) - 0|$$

and so $f \in L^\infty\left((0,T); L^2(\Omega)\right)$, while

$$|F_i(\boldsymbol{x}, t)| \le \|b_i\|_\infty\, |v(\boldsymbol{x}, t) - 0|$$

so that $F_i \in L^\infty\left((0,T); L^2(\Omega)\right)$. Let u be a solution of the problem (see Exercise 111)

$$\begin{cases} \frac{\partial u}{\partial t}(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) - \operatorname{div} \boldsymbol{F}(\boldsymbol{x}, t) & (\boldsymbol{x}, t) \in \Omega_T, \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (0,T). \end{cases}$$

Consider the operator

$$S : X \to X$$
$$v \mapsto u$$

Given $v, w \in X$, the function $\xi := S(v) - S(w)$ satisfies the equation

$$\xi_t - \Delta \xi = g - \operatorname{div} \boldsymbol{G},$$

where

$$g(\boldsymbol{x}, t) := h(\boldsymbol{x}, t, v(\boldsymbol{x}, t)) - h(\boldsymbol{x}, t, w(\boldsymbol{x}, t)),$$
$$G_i(\boldsymbol{x}, t) := \int_{w(\boldsymbol{x},t)}^{v(\boldsymbol{x},t)} b_i(s)\ ds.$$

with $\xi(\boldsymbol{x}, 0) = 0$ for $\boldsymbol{x} \in \Omega$. Then by Exercise 111, $\xi \in X$ with

$$\int_\Omega \xi^2(\boldsymbol{x}, t)\ d\boldsymbol{x} \le \int_0^T \int_\Omega |g(\boldsymbol{x}, t)|^2\ d\boldsymbol{x}dt + \int_0^T \int_\Omega \|\boldsymbol{G}(\boldsymbol{x}, t)\|^2\ d\boldsymbol{x}dt, .$$

Now

$$|g(\boldsymbol{x},t)| \leq L\,|v(\boldsymbol{x},t) - w(\boldsymbol{x},t)|$$

while

$$|G_i(\boldsymbol{x},t)| = \left|\int_{w(\boldsymbol{x},t)}^{v(\boldsymbol{x},t)} b_i(s)\ ds\right| \leq \|b_i\|_\infty\,|v(\boldsymbol{x},t) - w(\boldsymbol{x},t)|$$

and so

$$\int_\Omega \xi^2(\boldsymbol{x},t)\ d\boldsymbol{x} \leq \left(L^2 + \|\boldsymbol{b}\|_\infty^2\right) T \sup_s \int_\Omega |v(\boldsymbol{x},s) - w(\boldsymbol{x},s)|^2\ d\boldsymbol{x}.$$

It follows that if

$$\left(L^2 + \|\boldsymbol{b}\|_\infty^2\right) T \leq \frac{1}{2}, \tag{162}$$

then

$$\|S(v) - S(w)\|_X \leq \frac{1}{2}\|v - w\|_X\,,$$

which shows that S is a contraction. Hence, by Banach's fixed point theorem, there exists a fixed point u of S. In particular, again by Exercise 111, we have that $u \in L^2\left((0,T);H^1(\Omega)\right)$. Hence, $u(\cdot,t) \in H^1(\Omega)$ for $\mathcal{L}^1$ a.e. t. By the chain rule in Sobolev spaces

$$\frac{\partial F_i}{\partial x_j}(\boldsymbol{x},t) = b_i(u(\boldsymbol{x},t))\frac{\partial u}{\partial x_j}(\boldsymbol{x},t)\,.$$

If T does not satisfy (162), let T_1 be such that $\left(L^2 + \|\boldsymbol{b}\|_\infty^2\right) T_1 \leq \frac{1}{2}$. Then by the first part of the proof, there exists a solution in $[0,T_1]$. Set $u_1(\boldsymbol{x}) := u(\boldsymbol{x},T_1)$ and consider the Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \sum_{i=1}^N b_i(u)\frac{\partial u}{\partial x_i} = h(\boldsymbol{x},t,u) & (\boldsymbol{x},t) \in \Omega \times (T_1, 2T_2)\,, \\ u(\boldsymbol{x},T_1) = u_1(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x},t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t \in (T_1, 2T_2)\,. \end{cases}$$

We continue this process in each of the intervals $((n-1)T_1, nT_2)$ until we reach T. ■

As a corollary we get the existence of vanishing viscosity solutions

$$\begin{cases} u_t^\varepsilon + f'(u^\varepsilon)\,u_x^\varepsilon = \varepsilon u_{xx}^\varepsilon, & x \in \mathbb{R},\ t > 0, \\ u^\varepsilon(x,0) = g(x)\,, & x \in \mathbb{R}, \end{cases}$$

where $f \in C_c^\infty(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. We studied these equations last semester.

Wednesday, April 16, 2014

We now go back to the porous media equation.

For $\ell \geq 1$ the function Ψ is convex and lower semicontinuous. If Ψ is subdifferentiable at some $L_0 \in H$, then $\Psi(L_0) < \infty$ and so $L_0 = L_{v_0}$ for some $v_0 \in L^\ell(\Omega)$. Let $L \in \partial\Psi(L_0)$. Then

$$\Psi(M) \geq \Psi(L_{v_0}) + (L, M - L_{v_0})_{H^{-1}(\Omega)}$$

for all $M \in H^{-1}(\Omega)$. Taking $M = M_v$, where $v = v_0 \pm tw$ and $w \in L^\ell(\Omega)$, we get

$$\frac{1}{\ell}\int_\Omega |v_0 \pm tw|^\ell \, d\boldsymbol{x} \geq \frac{1}{\ell}\int_\Omega |v_0|^\ell \, d\boldsymbol{x} \pm t\,(L, L_w)_{H^{-1}(\Omega)}$$

Dividing by $t > 0$ gives

$$\frac{\frac{1}{\ell}\int_\Omega |v_0 \pm tw|^\ell \, d\boldsymbol{x} - \frac{1}{\ell}\int_\Omega |v_0|^\ell \, d\boldsymbol{x}}{t} \geq \pm (L, L_w)_{H^{-1}(\Omega)}.$$

Letting $t \to 0^+$, it follows by the Lebesgue dominated convergence theorem that

$$\pm \int_\Omega |v_0|^{\ell-2} v_0 w \, d\boldsymbol{x} \geq \pm (L, L_w)_{H^{-1}(\Omega)}$$

for all $w \in L^\ell(\Omega)$, that is,

$$\int_\Omega |v_0|^{\ell-2} v_0 w \, d\boldsymbol{x} = (L, L_w)_{H^{-1}(\Omega)} \tag{163}$$

for all $w \in L^\ell(\Omega)$. On the other hand, by (**??**),

$$(L, L_w)_{H^{-1}(\Omega)} = \int_\Omega J^{-1}(L)(\boldsymbol{x})\, w(\boldsymbol{x}) \, d\boldsymbol{x},$$

and so

$$\int_\Omega |v_0|^{\ell-2}(\boldsymbol{x})\, v_0(\boldsymbol{x})\, w(\boldsymbol{x}) \, d\boldsymbol{x} = \int_\Omega J^{-1}(L)(\boldsymbol{x})\, w(\boldsymbol{x}) \, d\boldsymbol{x}$$

for all $w \in L^\ell(\Omega)$, which shows that $J^{-1}(L) = |v_0|^{\ell-2} v_0$, that is,

$$L = -\Delta\left(|v_0|^{\ell-2} v_0\right) \quad \text{in } H^{-1}(\Omega).$$

By Theorem 113, $|v_0|^{\ell-2} v_0 \in H_0^1(\Omega)$ and $\partial\Psi(v_0) = \left\{-\Delta\left(|v_0|^{\ell-2} v_0\right)\right\}$.

Hence, given $f \in L^2\left((0,T); H^{-1}(\Omega)\right)$ and $u_0 \in L^\ell(\Omega)$, by Theorem 58, there exists a function $u \in H^1\left((0,T); H^{-1}(\Omega)\right)$ such that $u(\cdot,t) \in L^\ell(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$ and

$$f(\cdot,t) - \frac{\partial u}{\partial t}(\cdot,t) \in \partial\Psi(u(\cdot,t)) \quad \text{in } H^{-1}(\Omega)$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$,

$$\lim_{t\to 0^+} u(\cdot,t) = u_0 \quad \text{in } H^{-1}(\Omega)$$

and

$$\|u(\cdot,t)\|_{L^\ell(\Omega)} \leq C\left(\|u_0\|_{L^\ell(\Omega)} + \|f\|_{L^2((0,T);H^{-1}(\Omega))}\right),$$

$$\left\|\frac{\partial u}{\partial t}\right\|_{L^2((0,T);H^{-1}(\Omega))} \leq C\left(\|u_0\|_{L^\ell(\Omega)} + \|f\|_{L^2((0,T);H^{-1}(\Omega))}\right).$$

Now if $f(\cdot,t)-\frac{\partial u}{\partial t}(\cdot,t)\in\partial\Psi(u(\cdot,t))$ and $f(\cdot,t)-\frac{\partial u}{\partial t}(\cdot,t)\in L^{\ell}(\Omega)$ is a function, then by (**??**) and (163),

$$\begin{aligned}\int_{\Omega}\left(f(\boldsymbol{x},t)-\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right)J^{-1}(w)(\boldsymbol{x})\ d\boldsymbol{x}&=\left(f(\cdot,t)-\frac{\partial u}{\partial t}(\cdot,t),w\right)_{H^{-1}(\Omega)}\\&=\int_{\Omega}|u(\boldsymbol{x},t)|^{\ell-2}u(\boldsymbol{x},t)\,w(\boldsymbol{x})\ d\boldsymbol{x}.\end{aligned}$$

Let $\varphi=J^{-1}(w)$, then $w=-\Delta\varphi$. We get

$$\begin{aligned}\int_{\Omega}\left(f(\boldsymbol{x},t)-\frac{\partial u}{\partial t}(\boldsymbol{x},t)\right)\varphi(\boldsymbol{x})\ d\boldsymbol{x}&=-\int_{\Omega}|u(\boldsymbol{x},t)|^{\ell-2}u(\boldsymbol{x},t)\,\Delta\varphi(\boldsymbol{x})\ d\boldsymbol{x}\\&=-\int_{\Omega}\Delta\left(|u(\boldsymbol{x},t)|^{\ell-2}u(\boldsymbol{x},t)\right)\varphi(\boldsymbol{x})\ d\boldsymbol{x}\end{aligned}$$

and so

$$\int_{\Omega}\left(f(\boldsymbol{x},t)-\frac{\partial u}{\partial t}(\boldsymbol{x},t)+\Delta\left(|u(\boldsymbol{x},t)|^{\ell-2}u(\boldsymbol{x},t)\right)\right)\varphi(\boldsymbol{x})\ d\boldsymbol{x}=0$$

for all $\varphi\in C_c^{\infty}(\Omega)$. This implies that

$$f(\boldsymbol{x},t)-\frac{\partial u}{\partial t}(\boldsymbol{x},t)+\Delta\left(|u(\boldsymbol{x},t)|^{\ell-2}u(\boldsymbol{x},t)\right)=0$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x}\in\Omega$.

19 Gradient Flows

Consider an energy $\Psi:D\to\mathbb{R}$ defined on an open subset of a Hilbert space H. We want to evolve u in such a way that $\Psi(u)$ decreases in time. If Ψ is smooth, a standard way to do this is to let u evolve in the direction opposite to the gradient of Ψ, namely, to consider the evolution equation

$$\frac{du}{dt}(t)=-\nabla\Psi(u(t)).\tag{164}$$

Note that by the chain rule

$$\frac{d}{dt}(\Psi\circ u)(t)=\left(\nabla\Psi(u(t)),\frac{du}{dt}(t)\right)_H=-\left\|\frac{du}{dt}(t)\right\|_H^2\leq 0.$$

This evolution equation (164) is called the *gradient flow* of Ψ over H.

When Ψ is not smooth and D is not open, one can extend the notion of gradient flow to differential inclusion and consider

$$\frac{du}{dt}(t)\in-\partial\Psi(u(t)).$$

Under the hypotheses of Theorem 58 with $f = 0$, we have shown that $\Psi \circ u$ is absolutely continuous, with

$$\frac{d}{dt}(\Psi \circ u)(t) = -\left\| \frac{du}{dt}(t) \right\|_H^2 \leq 0,$$

and so we have proved the existence of a gradient flow of Ψ over H. In applications it is important to note that by extending Ψ to ∞ outside its effective domain, we can consider Ψ defined on a possibly larger Hilbert space and study the gradient flow in this space. This gives rise to a different gradient flow.

Friday, April 18, 2014

19.1 Allen–Cahn Equation

Consider the energy

$$\Psi(v) := \frac{1}{2} \int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} + \int_\Omega W(v(\boldsymbol{x})) \, d\boldsymbol{x}, \tag{165}$$

where W is a double–well potential, for example

$$W(z) = \frac{1}{4}\left(z^2 - 1\right)^2. \tag{166}$$

Let's consider the gradient flow of this energy in $L^2(\Omega)$, so extend Ψ to $L^2(\Omega)$ by defining

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} + \int_\Omega W(v(\boldsymbol{x})) \, d\boldsymbol{x} & \text{if } v \in H^1(\Omega) \text{ and } W \circ v \in L^1(\Omega), \\ \infty & \text{otherwise in } L^2(\Omega). \end{cases}$$

Note that Ψ is not convex, so we cannot apply Theorem 58. But we can still derive the form of the gradient flow of Ψ in $L^2(\Omega)$. Let's start by computing the subdifferential of Ψ. If Ψ is subdifferentiable at some $v_0 \in H$, then $\Psi(v_0) < \infty$ and so $v_0 \in H^1(\Omega)$ and $W \circ v_0 \in L^1(\Omega)$. Let $g \in \partial\Psi(v_0)$. Then

$$\Psi(v) \geq \Psi(v_0) + \int_\Omega g(v - v_0) \, d\boldsymbol{x}$$

for all $v \in L^2(\Omega)$. Taking $v = v_0 \pm tw$, where $w \in C_c^1(\Omega)$, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega \|\nabla v_0 \pm t\nabla w\|^2 \, d\boldsymbol{x} + \int_\Omega W(v_0 \pm tw) \, d\boldsymbol{x} \\ &\quad \geq \frac{1}{2}\int_\Omega \|\nabla v_0\|^2 \, d\boldsymbol{x} + \int_\Omega W(v_0) \, d\boldsymbol{x} \pm t \int_\Omega gw \, d\boldsymbol{x} \end{aligned}$$

that is

$$\begin{aligned} &\frac{1}{2}t^2 \int_\Omega \|\nabla w\|^2 \, d\boldsymbol{x} \pm t \int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} \\ &\quad + \int_\Omega (W(v_0 \pm tw) - W(v_0)) \, d\boldsymbol{x} \geq \pm t \int_\Omega gw \, d\boldsymbol{x}. \end{aligned}$$

Dividing by $t > 0$ and letting $t \to 0^+$, we have

$$\pm \int_\Omega \nabla v_0 \cdot \nabla w \; d\boldsymbol{x} \pm \int_\Omega W'(v_0)\, w \; d\boldsymbol{x} \ge \pm \int_\Omega g w \; d\boldsymbol{x}$$

for all $w \in C_c^1(\Omega)$, that is,

$$\int_\Omega \nabla v_0 \cdot \nabla w \; d\boldsymbol{x} + \int_\Omega W'(v_0)\, w \; d\boldsymbol{x} = \int_\Omega g w \; d\boldsymbol{x} \tag{167}$$

for all $w \in C_c^1(\Omega)$. This shows that v_0 is a weak solution of $-\Delta v_0 + W'(v_0) = g$ in Ω. We claim that $\partial\Psi(v_0)$ is a singleton. To see this, let $g_1 \in \partial\Psi(v_0)$. Then by subtracting the equation (167) for g and g_1 we get

$$\int_\Omega (g - g_1)\, w \; d\boldsymbol{x} = 0$$

for all $w \in C_c^1(\Omega)$. Since $C_c^1(\Omega)$ is dense in $L^2(\Omega)$, it follows that $g = g_1$. Hence, the gradient flow

$$\frac{\partial u}{\partial t}(\cdot, t) \in -\partial\Psi(u(\cdot, t))$$

becomes the equation

$$\frac{\partial u}{\partial t} = \Delta u - W'(u).$$

In the case in which W takes the form (166), we get

$$\frac{\partial u}{\partial t} = \Delta u + u - u^3,$$

which is called the *Allen–Cahn equation.*

Theorem 115 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $u_0 \in L^\infty(\Omega)$. Then the Neumann problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u - u^3 & (\boldsymbol{x}, t) \in \Omega \times (0, \infty), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega, \\ \frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega, \quad t > 0 \end{cases} \tag{168}$$

admits a unique weak solution.

Proof. Assume first that $u_0 \in L^\infty(\Omega) \cap H^1(\Omega)$. Fix $T > 0$. We use minimizing movements. For $\ell \in \mathbb{N}$ set $\tau := \frac{T}{\ell}$ and subdivide $(0, T)$ into ℓ intervals of length τ,

$$\tau_0 := 0 < t_1 < \cdots < \tau_\ell = T,$$

where $\tau_n := n\tau$, $n = 1, \ldots, \ell$. For every $n = 1, \ldots, \ell$, let $u_n \in H^1(\Omega)$ be a solution of the minimization problem

$$\min_{v \in H^1(\Omega)} J_n(v),$$

where

$$J_n(v) := \frac{1}{2}\int_\Omega \|\nabla v\|^2 \, d\boldsymbol{x} + \int_\Omega \frac{1}{4}\left(v^2-1\right)^2 \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega \left(v-u_{n-1}\right)^2 \, d\boldsymbol{x}.$$

Note that u_n, J_n and f_n all depend on ℓ. We claim that u_n exists and that $\|u_n\|_\infty \le \max\{\|u_0\|_\infty, 1\} =: L$. The proof is by induction on n. Assume that u_{n-1} has been found and that

$$\|u_{n-1}\|_\infty \le L.$$

Note that this is certainly true for $n = 0$. Note that J_n is nonnegative and coercive in $H^1(\Omega)$. Next let

$$m_n := \inf_{v \in H^1(\Omega)} J_n(v).$$

We claim that

$$m_n = \inf_{v \in H^1(\Omega),\ \|v\|_\infty \le L} J_n(v).$$

To see this, take $v \in H^1(\Omega)$ and consider the Lipschitz function

$$f_L(s) := \begin{cases} L & \text{if } s \ge L, \\ s & \text{if } -L < s < L, \\ -L & \text{if } s \le -L. \end{cases}$$

Then $f_L \circ v \in H^1(\Omega)$, since Ω is bounded, and by the chain rule in Sobolev spaces, for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \Omega$,

$$\nabla(f_L \circ v)(\boldsymbol{x}) := \begin{cases} \mathbf{0} & \text{if } v(\boldsymbol{x}) \ge L, \\ \nabla v(\boldsymbol{x}) & \text{if } -L < v(\boldsymbol{x}) < L, \\ \mathbf{0} & \text{if } v(\boldsymbol{x}) < -L. \end{cases}$$

Hence,

$$\int_\Omega \|\nabla(f_L \circ v)\|^2 \, d\boldsymbol{x} = \int_{\{|v|<L\}} \|\nabla v\|^2 \, d\boldsymbol{x} \le \int_\Omega \|\nabla v\|^2 \, d\boldsymbol{x}.$$

On the other hand, since $W'(s) = s\left(s^2-1\right) \ge 0$ for $s > 1$, if $v(\boldsymbol{x}) \ge L \ge 1$, then

$$\frac{1}{4}\left(v^2(\boldsymbol{x})-1\right)^2 \ge \frac{1}{4}\left(L^2-1\right)^2 = \frac{1}{4}\left((f_L \circ v)^2(\boldsymbol{x})-1\right)^2,$$

while if if $v(\boldsymbol{x}) \le -L \le -1$,

$$\frac{1}{4}\left(v^2(\boldsymbol{x})-1\right)^2 \ge \frac{1}{4}\left(L^2-1\right)^2 = \frac{1}{4}\left((f_L \circ v)^2(\boldsymbol{x})-1\right)^2.$$

In turn,

$$\int_\Omega \frac{1}{4}\left((f_L \circ v)^2-1\right)^2 \, d\boldsymbol{x} \le \int_\Omega \frac{1}{4}\left(v^2-1\right)^2 \, d\boldsymbol{x}.$$

Finally, if $v(\boldsymbol{x}) \geq L \geq u_{n-1}(\boldsymbol{x})$, then

$$v(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x}) \geq L - u_{n-1}(\boldsymbol{x}) = (f_L \circ v)(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x}) \geq 0,$$

while if $v(\boldsymbol{x}) \leq -L \leq -|u_{n-1}(\boldsymbol{x})|$, then

$$|v(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x})| \geq |-L - u_{n-1}(\boldsymbol{x})| = |(f_L \circ v)(\boldsymbol{x}) - u_{n-1}(\boldsymbol{x})|,$$

and so

$$\frac{1}{2\tau}\int_\Omega (f_L \circ v - u_{n-1})^2 \, d\boldsymbol{x} \leq \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x}.$$

This shows that

$$J_n(f_L \circ v) \leq J_n(v)$$

and proves the claim.

Using the definition of infimum consider a sequence $\{v_k\} \subset H^1(\Omega)$ with $\|v_n\|_\infty \leq L$ such that

$$m_n \leq J_n(v_k) \leq m_n + \frac{1}{k}.$$

Then

$$\lim_{k\to\infty} J_n(v_k) = m_n.$$

It follows from (76) and the fact that $J_n(v_k) \leq m_n + 1$ for all k, that $\{v_k\}$ is bounded in $H^1(\Omega)$ and in $L^\infty(\Omega)$. Hence, up to a subsequence, not relabelled, there exists $u_n \in H^1(\Omega)$ such that $\{v_k\}$ converges weakly to u_n in $L^2(\Omega)$ and strongly in $L^p(\Omega)$ for every $1 \leq p < \infty$ (by the Lebesgue dominated convergence theorem), and $\{\nabla v_k\}$ converges weakly to ∇u_n in $L^2(\Omega; \mathbb{R}^N)$. We claim that $J_n(u_n) = m_n$. To see this, observe that

$$\begin{aligned} m_n &= \liminf_{k\to\infty} J_n(v_k) = \liminf_{k\to\infty} \frac{1}{2}\int_\Omega \|\nabla v_k\|^2 \, d\boldsymbol{x} \\ &\quad + \lim_{k\to\infty}\int_\Omega \frac{1}{4}\left(v_k^2 - 1\right)^2 \, d\boldsymbol{x} + \lim_{k\to\infty}\frac{1}{2\tau}\int_\Omega (v_k - u_{n-1})^2 \, d\boldsymbol{x}. \\ &\geq \frac{1}{2}\int_\Omega \|\nabla v\|^2 \, d\boldsymbol{x} + \int_\Omega \frac{1}{4}\left(v^2 - 1\right)^2 \, d\boldsymbol{x} + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 \, d\boldsymbol{x} = J_n(v) \geq m_n. \end{aligned}$$

Hence, $m_n = J_n(u_n)$. We can now continue as in the proof of Theorem 60. ■

Exercise 116 *Prove that a similar proof continues to work for a potential* $W : \mathbb{R} \to [0,\infty)$ *of class* C^2 *satisfying the coercivity hypothesis*

$$\lim_{|s|\to\infty} W(s) = \infty.$$

Monday, April 21, 2014

19.2 Cahn–Hiliard Equation

The problem with the Allen–Cahn equation is that it does not preserves the average of u, which is an important property in physical applications (conservation of mass). One way around would be to work in the space $\left\{u \in L^2(\Omega) : \int_\Omega u \, d\boldsymbol{x} = 0\right\}$, but this would lead to the equation

$$\frac{\partial u}{\partial t} = \Delta u - W'(u) + \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega W'(u) \, d\boldsymbol{x},$$

which contains the nonlocal term $\frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega W'(u) \, d\boldsymbol{x}$.

We now consider the gradient flow of (165) with respect to (a variant of) $\left(H^1(\Omega)\right)'$. To be precise, let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitz boundary and let

$$V := \left\{u \in H^1(\Omega) : \int_\Omega u \, d\boldsymbol{x} = 0\right\}.$$

It follows by the Poincaré inequality that an equivalent norm in H is given by

$$\|u\|_V := \|\nabla u\|_{L^2(\Omega)}.$$

Note that V is a Hilbert space with the inner product

$$(u, v)_V := \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x}.$$

Given $L \in V'$ by the Riesz representation theorem there exists a unique $z_L \in V$ such that

$$L(w) = \int_\Omega \nabla z_L \cdot \nabla w \, d\boldsymbol{x}$$

for all $w \in V$ and

$$\|L\|_{V'} = \sup_{w \in V, \ \|w\|_V \leq 1} |\langle L, w \rangle| = \|z_L\|_V.$$

Note that z_L is a weak solution of the equation

$$\begin{cases} -\Delta z_L = L & \text{in } \Omega, \\ \frac{\partial z_L}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, if $L, M \in V'$, then

$$(L, M)_{V'} := \int_\Omega \nabla z_L \cdot \nabla z_M \, d\boldsymbol{x}$$

is an inner product in V'. Observe that this inner product produces the usual norm in V'.

We take

$$H := \left\{L \in \left(H^1(\Omega)\right)' : \ \langle L, 1 \rangle_{(H^1(\Omega))', H^1(\Omega)} = 0\right\}$$

and given $L, M \in H$ we define

$$\|L\|_H := \|L|_V\|_{V'}$$
$$(L, M)_H := \int_\Omega \nabla z_L \cdot \nabla z_M \, d\boldsymbol{x}.$$

Exercise 117 *H is closed in $\left(H^1(\Omega)\right)'$.*

Exercise 118 *$\|\cdot\|_H$ and $\|\cdot\|_{(H^1(\Omega))'}$ are equivalent norms in H.*

Consider

$$\Psi(v) := \begin{cases} \frac{1}{2}\int_\Omega \|\nabla v(\boldsymbol{x})\|^2 \, d\boldsymbol{x} + \int_\Omega W(v(\boldsymbol{x})) \, d\boldsymbol{x} & \text{if } v \in H^1(\Omega) \text{ and } W \circ v \in L^1(\Omega), \\ \infty & \text{otherwise in } H. \end{cases}$$

Given $g \in \partial\Psi(v_0)$, reasoning as in the previous subsection we have that

$$\int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} + \int_\Omega W'(v_0)\, w \, d\boldsymbol{x} = (g, w)_H$$

for all $w \in C^1\left(\overline{\Omega}\right)$. Moreover, using the fact that $C^1\left(\overline{\Omega}\right)$ is dense in H (exercise), we can prove that $\partial\Psi(v_0)$ is a singleton. Assume next that v_0 is smooth with $\frac{\partial v_0}{\partial \boldsymbol{n}} = 0$ and $\frac{\partial \Delta v_0}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. Note that $\frac{\partial}{\partial \boldsymbol{n}}\left(\Delta v_0 - W'(v_0)\right) = \frac{\partial \Delta v_0}{\partial \boldsymbol{n}} - W''(v_0)\frac{\partial v_0}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. We claim that $g = \Delta^2 v_0 - \Delta\left(W'(v_0)\right)$. To see this, let z_w be the function corresponding w, that is, $-\Delta z_w = w$ in Ω and $\frac{\partial z_w}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$. Then by the divergence theorem,

$$\begin{aligned}
&\int_\Omega \nabla v_0 \cdot \nabla w \, d\boldsymbol{x} + \int_\Omega W'(v_0)\, w \, d\boldsymbol{x} \\
&\quad = -\int_\Omega w \Delta v_0 \, d\boldsymbol{x} - \int_{\partial\Omega} w \frac{\partial v_0}{\partial \boldsymbol{n}} \, dS + \int_\Omega W'(v_0)\, w \, d\boldsymbol{x} \\
&\quad = -\int_\Omega w\left(\Delta v_0 - W'(v_0)\right) \, d\boldsymbol{x} \\
&\quad = \int_\Omega \Delta z_w \left(\Delta v_0 - W'(v_0)\right) \, d\boldsymbol{x} \\
&\quad = -\int_\Omega \nabla z_w \cdot \nabla\left(\Delta v_0 - W'(v_0)\right) \, d\boldsymbol{x} + \int_{\partial\Omega} \frac{\partial z_w}{\partial \boldsymbol{n}}\left(\Delta v_0 - W'(v_0)\right) \, dS \\
&\quad = -\int_\Omega \nabla z_w \cdot \nabla\left(\Delta v_0 - W'(v_0)\right) \, d\boldsymbol{x} = \left(\Delta\left(\Delta v_0 - W'(v_0)\right), w\right)_{V'}
\end{aligned}$$

where we have used the fact that $z_{-\Delta(\Delta v_0 - W'(v_0))} = \Delta v_0 - W'(v_0)$. This prove the claim.

Hence, the gradient flow

$$\frac{\partial u}{\partial t}(\cdot, t) \in -\partial\Psi(u(\cdot, t))$$

becomes the equation

$$\frac{\partial u}{\partial t} = -\Delta^2 u + \Delta\left(W'(u)\right),$$

which is called the *Cahn–Hilliard equation.*

20 Maximal Monotone Operators

Definition 119 *Given two nonempty sets X, Y, a* multifunction *or* correspondence *from X to Y is a map from X to the family of subsets of Y, namely,*

$$\Gamma : X \to \mathcal{P}(Y).$$

The domain *of a multifunction is the set*

$$\operatorname{dom}\Gamma := \{x \in X : \Gamma(x) \neq \emptyset\}.$$

The graph *of a multifunction Γ is the set*

$$\operatorname{graph}\Gamma := \{(x, y) \in \operatorname{dom}\Gamma \times Y : y \in \Gamma(x)\}.$$

The inverse *of a multifunction Γ is the multifunction $\Gamma^{-1} : Y \to \mathcal{P}(X)$ defined by*

$$\Gamma^{-1}(y) := \{x \in X : y \in \Gamma(x)\}. \tag{169}$$

A multifunction $\Gamma : X \to \mathcal{P}(Y)$ is *single-valued* on a set $E \subseteq X$ if $\Gamma(x)$ consists of at most one element for every $x \in E$. In this case the restriction of Γ to $E \cap \operatorname{dom}\Gamma$ may be identified with a function.

Definition 120 *Given a Hilbert space H, a multifunction $\Gamma : H \to \mathcal{P}(H)$ is called* monotone *if*

$$(y_2 - y_1, x_2 - x_1)_H \geq 0$$

for all (x_1, y_1), $(x_2, y_2) \in \operatorname{graph}\Gamma$. A monotone multifunction $\Gamma : H \to \mathcal{P}(H)$ is called maximal *if its graph is not a proper subset of the graph of a monotone multifunction.*

Remark 121 *Note that if Γ is a monotone multifunction, then Γ is maximal if and only if for all $(x_1, y_1) \in H \times H$ such that*

$$(y_2 - y_1, x_2 - x_1)_H \geq 0$$

for every $(x_2, y_2) \in \operatorname{graph}\Gamma$, we have that (x_1, y_1) belongs to $\operatorname{graph}\Gamma$. Indeed, if there exists $(x_1, y_1) \notin \operatorname{graph}\Gamma$ for which the previous inequality holds, then $y_1 \notin \Gamma(x_1)$. Define Γ_1 to be $\Gamma_1(x) := \Gamma(x)$ for all $x \neq x_1$ and $\Gamma_1(x_1) := \Gamma(x_1) \cup \{y_1\}$. Then Γ_1 is a monotone multifuction whose graph strictly contains the graph of Γ.

Remark 122 *Note that if Γ is a maximal monotone multifunction, then the set $\Gamma(x)$ is closed and convex.*

Example 123 *Given an Hilbert space H and a linear map $T : H \to H$, the multifunction $\Gamma(x) := \{T(x)\}$ is monotone and maximal provided T is positive, that is*

$$(T(x), x)_H \geq 0$$

for all $x \in H$.

Example 124 *Given an interval I and a function $f : I \to \mathbb{R}$, the multifunction $\Gamma(x) := \{f(x)\}$ if $x \in I$ and $\Gamma(x) := \emptyset$ otherwise is monotone if and only if f is increasing.*

Example 125 *Given a nonempty set $E \subseteq [0,1]$, the multifunction $\Gamma : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ defined by*

$$\Gamma(x) := \begin{cases} \{1\} & \text{if } x > 0, \\ E & \text{if } x = 0, \\ \{0\} & \text{if } x < 0, \end{cases}$$

is monotone and maximal.

Wednesday, April 23, 2014

Theorem 126 (Min-Max Theorem) *Let $C \subset \mathbb{R}^M$ and $K \subset \mathbb{R}^N$ be compact and convex sets. Let $f : C \times K \to \mathbb{R}$ be such that for every $\boldsymbol{y} \in K$ the function $\boldsymbol{x} \mapsto f(\boldsymbol{x}, \boldsymbol{y})$ is convex and lower semicontinuous in C, and for every $\boldsymbol{x} \in C$ the function $\boldsymbol{y} \mapsto f(\boldsymbol{x}, \boldsymbol{y})$ is concave and upper semicontinuous in K. Then there exists $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in C \times K$ such that*

$$f(\boldsymbol{x}_0, \boldsymbol{y}) \le f(\boldsymbol{x}_0, \boldsymbol{y}_0) \le f(\boldsymbol{x}, \boldsymbol{y}_0) \tag{170}$$

for all $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in K$. Moreover, (170) is equivalent to

$$\max_{\boldsymbol{y} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{x} \in C} \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}, \boldsymbol{y}). \tag{171}$$

Proof. Step 1: We will show that (170) and (171) are equivalent. Note that for every $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in K$,

$$\min_{\boldsymbol{w} \in C} f(\boldsymbol{w}, \boldsymbol{y}) \le f(\boldsymbol{x}, \boldsymbol{y}).$$

Hence,

$$\max_{\boldsymbol{y} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}) \le \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}, \boldsymbol{y})$$

and since this is true for all $\boldsymbol{x} \in C$, it follows that

$$\max_{\boldsymbol{y} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}) \le \min_{\boldsymbol{x} \in C} \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}, \boldsymbol{y}).$$

Hence, it is enough to show that (170) is equivalent to

$$\max_{\boldsymbol{y} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}) \ge \min_{\boldsymbol{x} \in C} \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}, \boldsymbol{y}).$$

Assume that (170) holds. Then

$$\min_{\boldsymbol{w} \in C} \max_{\boldsymbol{y} \in K} f(\boldsymbol{w}, \boldsymbol{y}) \le \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}_0, \boldsymbol{y}) \le f(\boldsymbol{x}_0, \boldsymbol{y}_0) \le \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}_0) \le \max_{\boldsymbol{z} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{z}).$$

Conversely, assume that (171) holds and let

$$\alpha := \max_{\boldsymbol{y} \in K} \min_{\boldsymbol{x} \in C} f(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{x} \in C} \max_{\boldsymbol{y} \in K} f(\boldsymbol{x}, \boldsymbol{y})$$

and let $\boldsymbol{x}_0 \in C$ and $\boldsymbol{y}_0 \in K$ be such that

$$\alpha = \max_{\boldsymbol{y}\in K} \min_{\boldsymbol{x}\in C} f(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{x}\in C} f(\boldsymbol{x}, \boldsymbol{y}_0),$$
$$\alpha = \min_{\boldsymbol{x}\in C} \max_{\boldsymbol{y}\in K} f(\boldsymbol{x}, \boldsymbol{y}) = \max_{\boldsymbol{y}\in K} f(\boldsymbol{x}_0, \boldsymbol{y}).$$

Then

$$f(\boldsymbol{x}_0, \boldsymbol{y}) \le \max_{\boldsymbol{z}\in K} f(\boldsymbol{x}_0, \boldsymbol{z}) = \alpha = \min_{\boldsymbol{w}\in C} f(\boldsymbol{w}, \boldsymbol{y}_0) \le f(\boldsymbol{x}, \boldsymbol{y}_0)$$

for all $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in K$. It follows that $\alpha = f(\boldsymbol{x}_0, \boldsymbol{y}_0)$, and so (170) holds.
Step 2: For $\varepsilon > 0$ let

$$f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) := f(\boldsymbol{x}, \boldsymbol{y}) + \varepsilon \|\boldsymbol{x}\|^2.$$

Note that for every fixed $\boldsymbol{y} \in K$ the function $\boldsymbol{x} \mapsto f_\varepsilon(\boldsymbol{x}, \boldsymbol{y})$ is strictly convex (since sum of a convex and a strictly convex functions) and lower semicontinuous in C (since sum of a lower semicontinuous and a continuous function). Hence, there exists a unique $\boldsymbol{x}_{\varepsilon,\boldsymbol{y}} \in C$ such that

$$\min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = f_\varepsilon(\boldsymbol{x}_{\varepsilon,\boldsymbol{y}}, \boldsymbol{y}).$$

Define

$$g_\varepsilon(\boldsymbol{y}) := \min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = f_\varepsilon(\boldsymbol{x}_{\varepsilon,\boldsymbol{y}}, \boldsymbol{y}).$$

We claim that g_ε is concave and upper semicontinuous. To prove concavity, let $\theta \in (0,1)$ and $\boldsymbol{y}_1$, $\boldsymbol{y}_2 \in K$. Then for every $\boldsymbol{x} \in C$,

$$\begin{aligned}\theta g_\varepsilon(\boldsymbol{y}_1) + (1-\theta) g_\varepsilon(\boldsymbol{y}_2) &\le \theta f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}_1) + (1-\theta) f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}_2) \\ &\le f_\varepsilon(\boldsymbol{x}, \theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2).\end{aligned}$$

In turn,

$$\theta g_\varepsilon(\boldsymbol{y}_1) + (1-\theta) g_\varepsilon(\boldsymbol{y}_2) \le \min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x}, \theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2) = g_\varepsilon(\theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2).$$

To prove upper semicontinuity let $\{\boldsymbol{y}_n\} \subseteq K$ be such that $\boldsymbol{y}_n \to \boldsymbol{y}_0$. Then for every $\boldsymbol{x} \in C$,

$$\limsup_{n\to\infty} g_\varepsilon(\boldsymbol{y}_n) \le \limsup_{n\to\infty} f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}_n) \le f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}_0).$$

Since this is true for every $\boldsymbol{x} \in C$, it follows that

$$\limsup_{n\to\infty} g_\varepsilon(\boldsymbol{y}_n) \le \min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}_0) = g_\varepsilon(\boldsymbol{y}_0).$$

This proves the claim. In turn, there exists $\boldsymbol{y}_\varepsilon \in K$ such that

$$g_\varepsilon(\boldsymbol{y}_\varepsilon) = \max_{\boldsymbol{y}\in K} g_\varepsilon(\boldsymbol{y}) = \max_{\boldsymbol{y}\in K} \min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x}, \boldsymbol{y}).$$

Given $\theta \in (0,1)$, $\boldsymbol{x} \in C$, and $\boldsymbol{y} \in K$, by the concavity of $f(\boldsymbol{x},\cdot)$ we have

$$\begin{aligned} f_\varepsilon(\boldsymbol{x}, \theta\boldsymbol{y} + (1-\theta)\boldsymbol{y}_\varepsilon) &\geq \theta f_\varepsilon(\boldsymbol{x},\boldsymbol{y}) + (1-\theta) f_\varepsilon(\boldsymbol{x},\boldsymbol{y}_\varepsilon) \\ &\geq \theta f_\varepsilon(\boldsymbol{x},\boldsymbol{y}) + (1-\theta) g_\varepsilon(\boldsymbol{y}_\varepsilon). \end{aligned}$$

In particular, taking $\boldsymbol{x} := \boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}+(1-\theta)\boldsymbol{y}_\varepsilon}$, we get

$$\begin{aligned} g_\varepsilon(\boldsymbol{y}_\varepsilon) \geq g_\varepsilon(\theta\boldsymbol{y} + (1-\theta)\boldsymbol{y}_\varepsilon) &= f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}+(1-\theta)\boldsymbol{y}_\varepsilon}, \theta\boldsymbol{y} + (1-\theta)\boldsymbol{y}_\varepsilon\right) \\ &\geq \theta f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}+(1-\theta)\boldsymbol{y}_\varepsilon}, \boldsymbol{y}\right) + (1-\theta) g_\varepsilon(\boldsymbol{y}_\varepsilon), \end{aligned}$$

and so

$$g_\varepsilon(\boldsymbol{y}_\varepsilon) \geq f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}+(1-\theta)\boldsymbol{y}_\varepsilon}, \boldsymbol{y}\right) \tag{172}$$

for all $\theta \in (0,1)$ and all $\boldsymbol{y} \in K$.

We claim that for every $\theta \in (0,1)$, and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in K$,

$$\boldsymbol{x}_\theta := \boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}_1+(1-\theta)\boldsymbol{y}_2} \to \boldsymbol{x}_{\varepsilon,\boldsymbol{y}_2} \tag{173}$$

as $\theta \to 0^+$. To see this, observe that by the concavity of $f(\boldsymbol{x}_\theta,\cdot)$ we have

$$\begin{aligned} \theta f_\varepsilon(\boldsymbol{x}_\theta,\boldsymbol{y}_1) + (1-\theta) f_\varepsilon(\boldsymbol{x}_\theta,\boldsymbol{y}_2) &\leq f_\varepsilon(\boldsymbol{x}_\theta, \theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2) \\ &= \min_{\boldsymbol{w}\in C} f_\varepsilon(\boldsymbol{w}, \theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2) \\ &\leq f_\varepsilon(\boldsymbol{x}, \theta\boldsymbol{y}_1 + (1-\theta)\boldsymbol{y}_2) \end{aligned}$$

for every $\boldsymbol{x} \in C$. Let $\theta_n \to 0^+$. Since the sequence $\{\boldsymbol{x}_{\theta_n}\} \subseteq C$ compact, up to a subsequence, $\boldsymbol{x}_{\theta_n} \to \boldsymbol{x}_* \in C$. Hence, taking $\theta = \theta_n$ and letting $n \to \infty$ in the previous inequality gives

$$\begin{aligned} f_\varepsilon(\boldsymbol{x}_*,\boldsymbol{y}_2) &\leq \liminf_{n\to\infty} \left(\theta_n f_\varepsilon(\boldsymbol{x}_{\theta_n},\boldsymbol{y}_1) + (1-\theta_n) f_\varepsilon(\boldsymbol{x}_{\theta_n},\boldsymbol{y}_2)\right) \\ &\leq \limsup_{n\to\infty} f_\varepsilon(\boldsymbol{x}, \theta_n\boldsymbol{y}_1 + (1-\theta_n)\boldsymbol{y}_2) \leq f_\varepsilon(\boldsymbol{x},\boldsymbol{y}_2), \end{aligned}$$

where we have used the semicontinuity properties of f. This shows that

$$f_\varepsilon(\boldsymbol{x}_*,\boldsymbol{y}_2) = \min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x},\boldsymbol{y}_2)$$

and so $\boldsymbol{x}_* = \boldsymbol{x}_{\varepsilon,\boldsymbol{y}_2}$. Thus, (173) holds.

In turn, letting $\theta \to 0^+$ in (172), we obtain, again by the semicontinuity properties of f,

$$g_\varepsilon(\boldsymbol{y}_\varepsilon) \geq \liminf_{\theta\to 0^+} f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\theta\boldsymbol{y}+(1-\theta)\boldsymbol{y}_\varepsilon}, \boldsymbol{y}\right) \geq f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\boldsymbol{y}_\varepsilon}, \boldsymbol{y}\right)$$

for all $\boldsymbol{y} \in K$. On the other hand, $g_\varepsilon(\boldsymbol{y}_\varepsilon) \leq f_\varepsilon(\boldsymbol{x},\boldsymbol{y}_\varepsilon)$ for all $\boldsymbol{x} \in C$, and so

$$f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\boldsymbol{y}_\varepsilon}, \boldsymbol{y}\right) \leq f_\varepsilon\left(\boldsymbol{x}_{\varepsilon,\boldsymbol{y}_\varepsilon}, \boldsymbol{y}_\varepsilon\right) = g_\varepsilon(\boldsymbol{y}_\varepsilon) \leq f_\varepsilon(\boldsymbol{x},\boldsymbol{y}_\varepsilon)$$

for all $\boldsymbol{x} \in C$ and $\boldsymbol{y} \in K$. Using the first part of the proof, we have that

$$\min_{\boldsymbol{x}\in C}\max_{\boldsymbol{y}\in K} f(\boldsymbol{x},\boldsymbol{y}) \leq \min_{\boldsymbol{x}\in C}\max_{\boldsymbol{y}\in K} f_\varepsilon(\boldsymbol{x},\boldsymbol{y}) = \max_{\boldsymbol{y}\in K}\min_{\boldsymbol{x}\in C} f_\varepsilon(\boldsymbol{x},\boldsymbol{y}) \leq \max_{\boldsymbol{y}\in K}\min_{\boldsymbol{x}\in C} f(\boldsymbol{x},\boldsymbol{y}) + \varepsilon L^2,$$

where $L := \max_{\boldsymbol{x}\in C} \|\boldsymbol{x}\|^2$. Letting $\varepsilon \to 0^+$ gives

$$\min_{\boldsymbol{x}\in C}\max_{\boldsymbol{y}\in K} f(\boldsymbol{x},\boldsymbol{y}) \le \max_{\boldsymbol{y}\in K}\min_{\boldsymbol{x}\in C} f(\boldsymbol{x},\boldsymbol{y}),$$

which proves the theorem. ■

Friday, April 25, 2014

Note that Γ is a (maximal) monotone multifunction if and only if Γ^{-1} is a (maximal) monotone multifunction.

Theorem 127 *Let H be a Hilbert space and let $\Gamma : H \to \mathcal{P}(H)$ be a multifunction. If Γ is monotone, then*

(i) for every $\varepsilon > 0$, $I + \varepsilon\Gamma$, and $(I+\varepsilon\Gamma)^{-1}$ are monotone multifunctions;

(ii) for every $\varepsilon > 0$, $(I+\varepsilon\Gamma)^{-1}$ is single-valued and the corresponding function is Lipschitz continuous with Lipschitz constant at most 1.

Conversely, if (ii) holds, then Γ is monotone.

Proof. (i) Assume that Γ is monotone. Let (x_1,z_1), $(x_2,z_2) \in \operatorname{graph}(I+\varepsilon\Gamma)$. Then $z_i = x_i + \varepsilon y_i$, where $y_i \in \Gamma(x_i)$, $i = 1,2$. By the monotonicity of Γ,

$$(y_2 - y_1, x_2 - x_1)_H \ge 0,$$

and so

$$\begin{aligned}(z_2 - z_1, x_2 - x_1)_H &= (x_2 - x_1 + \varepsilon(y_2 - y_1), x_2 - x_1)_H \\ &= \varepsilon(y_2 - y_1, x_2 - x_1)_H + \|x_2 - x_1\|_H^2 \ge \|x_2 - x_1\|_H^2 \ge 0,\end{aligned} \tag{174}$$

which shows that $I + \varepsilon\Gamma$, and, in turn, $(I+\varepsilon\Gamma)^{-1}$ are monotone multifunctions. Note that the previous inequality implies, in particular, that

$$\begin{aligned}\|x_2 - x_1\|_H^2 &\le (x_2 - x_1 + \varepsilon(y_2 - y_1), x_2 - x_1)_H \\ &\le \|(x_2 + \varepsilon y_2) - (x_1 + \varepsilon y_1)\|_H \|x_2 - x_1\|_H,\end{aligned}$$

and so

$$\|x_2 - x_1\|_H \le \|(x_2 + \varepsilon y_2) - (x_1 + \varepsilon y_1)\|_H = \|z_2 - z_1\|_H. \tag{175}$$

To prove (ii), recall that by (169), for all $z \in H$,

$$(I+\varepsilon\Gamma)^{-1}(z) := \{x \in H : z \in (I+\varepsilon\Gamma)(x)\}.$$

To prove that $(I+\varepsilon\Gamma)^{-1}$ is single-valued, fix $z \in H$ and assume that $(I+\varepsilon\Gamma)^{-1}(z)$ is nonempty. If x_1, $x_2 \in (I+\varepsilon\Gamma)^{-1}(z)$, then $z \in (I+\varepsilon\Gamma)(x_i)$, $i = 1,2$, and so we may write

$$z = \varepsilon y_2 + x_2 = \varepsilon y_1 + x_1,$$

where $y_i \in \Gamma(x_i)$, $i = 1, 2$, and from the inequality (175) we get that $x_1 = x_2$. Thus, $(I + \varepsilon\Gamma)^{-1}$ is single-valued. Hence, on the set $\operatorname{dom}(I + \varepsilon\Gamma)^{-1}$ we can define the function

$$z \in \operatorname{dom}(I + \varepsilon\Gamma)^{-1} \mapsto x,$$

where $x \in H$ is the unique element such that $z \in (I + \varepsilon\Gamma)(x)$. Inequality (175) implies that this function is Lipschitz continuous with Lipschitz constant at most 1.

Conversely, assume that (ii) holds. Then by squaring both sides of (175), we obtain

$$\begin{aligned} \|x_2 - x_1\|_H^2 &\leq \|(x_2 + \varepsilon y_2) - (x_1 + \varepsilon y_1)\|_H^2 = \|x_2 - x_1 + \varepsilon(y_2 - y_1)\|_H^2 \\ &= \|x_2 - x_1\|_H^2 + \varepsilon^2 \|y_2 - y_1\|_H^2 + 2\varepsilon(y_2 - y_1, x_2 - x_1)_H . \end{aligned}$$

Hence,

$$0 \leq \varepsilon \|y_2 - y_1\|_H^2 + 2(y_2 - y_1, x_2 - x_1)_H .$$

By sending $\varepsilon \to 0^+$, we obtain that Γ is monotone. ■

The following characterization of monotone operators is due to Minty.

Theorem 128 (Minty) *Let H be a Hilbert space and let $\Gamma : H \to \mathcal{P}(H)$ be a monotone multifunction. Then Γ is maximal if and only if the domain of $(I + \varepsilon\Gamma)^{-1}$ is H for every $\varepsilon > 0$.*

We begin with some preliminary results.

Lemma 129 *Let H be a Hilbert space and let $\Gamma : H \to \mathcal{P}(H)$ be a monotone multifunction. Then for every $y_0 \in H$, there exists $x_0 \in H$ such that*

$$(y - (y_0 - x_0), x - x_0)_H \geq 0$$

for all $(x, y) \in \operatorname{graph}\Gamma$.

Proof. Given $(x, y) \in \operatorname{graph}\Gamma$ define

$$K_{(x,y)} := \{z \in X : \ (y - (y_0 - z), x - z)_H \geq 0\} .$$

To prove the lemma, it is enough to show that

$$\bigcap_{(x,y) \in \operatorname{graph}\Gamma} K_{(x,y)} \neq \emptyset. \tag{176}$$

The set $K_{(x,y)}$ is closed, nonempty, since $x \in K_{(x,y)}$. Moreover, writing

$$(y - y_0 + z, x - z)_H = (y - y_0, x)_H - (y - y_0, z)_H + (z, x)_H - \|z\|_H^2 ,$$

and noting that the function $z \mapsto (y - y_0, x)_H - (y - y_0, z)_H + (z, x)_H - \|z\|_H^2$ is concave, it follows that the set $K_{(x,y)}$ is convex and bounded. We now use the facts that a closed and convex set is weakly closed, and a weakly closed and bounded set is weakly compact to conclude that $K_{(x,y)}$ is weakly compact. Hence, to prove (176), it is enough to prove that the family $\{K_{(x,y)}\}_{(x,y)\in \operatorname{graph}\Gamma}$ of weakly compact sets satisfies the finite intersection property, that is, that if $(x_i, y_i) \in \operatorname{graph}\Gamma$, $i = 1, \ldots, n$, then

$$\bigcap_{i=1}^{n} K_{(x_i,y_i)} \neq \emptyset. \tag{177}$$

To prove (177), let

$$C := \left\{ \boldsymbol{t} \in \mathbb{R}^N : \ t_i \geq 0 \text{ for all } i = 1, \ldots, n, \ \sum_{i=1}^{n} t_i = 1 \right\}.$$

The set C is convex and compact. Define the function $f : C \times C \to \mathbb{R}$ by

$$f(\boldsymbol{s}, \boldsymbol{t}) := \sum_{i=1}^{n} t_i \left(y_0 - y_i - z(\boldsymbol{s}), x_i - z(\boldsymbol{s})\right)_H,$$

where

$$z(\boldsymbol{s}) := \sum_{i=1}^{n} s_i x_i.$$

We claim that $f(\boldsymbol{s}, \boldsymbol{s}) \leq 0$ for all $\boldsymbol{s} \in C$. To see this, observe that using the fact that $\sum_{i=1}^{n} s_i = 1$, we have

$$\begin{aligned}
f(\boldsymbol{s}, \boldsymbol{s}) &= \sum_{i=1}^{n} s_i \left(y_0 - y_i - z(\boldsymbol{s}), x_i - z(\boldsymbol{s})\right)_H \\
&= -\sum_{i=1}^{n} s_i \left(y_i, x_i - z(\boldsymbol{s})\right)_H + \left(y_0 - z(\boldsymbol{s}), z(\boldsymbol{s}) - z(\boldsymbol{s})\right)_H \\
&= -\sum_{i=1}^{n} \sum_{k=1}^{n} s_k s_i \left(y_i, x_i - x_k\right)_H + 0 \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} s_k s_i \left(y_i, x_i - x_k\right)_H - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} s_k s_i \left(y_k, x_k - x_i\right)_H \\
&= -\frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} s_k s_i \left(y_i - y_k, x_i - x_k\right)_H \leq 0,
\end{aligned}$$

where in the last inequality we used the facts that $(x_i, y_i) \in \operatorname{graph}\Gamma$ for all $i = 1, \ldots, n$, and that Γ is a monotone multifunction. This proves the claim.

Next, observe that the function f is continuous, linear (and so concave) as a function of $\boldsymbol{t}$ and convex as a function of $\boldsymbol{s}$. Hence, by the minimax theorem, there exist $\boldsymbol{s}_0 \in C$ and $\boldsymbol{t}_0 \in C$ such that

$$f(\boldsymbol{s}_0, \boldsymbol{t}) \leq f(\boldsymbol{s}_0, \boldsymbol{t}_0) \leq f(\boldsymbol{s}, \boldsymbol{t}_0)$$

for all $\boldsymbol{s} \in C$ and $\boldsymbol{t} \in C$. In particular, taking $\boldsymbol{s} = \boldsymbol{t}_0$ and using the fact that $f(\boldsymbol{t}_0, \boldsymbol{t}_0) \leq 0$, it follows that $f(\boldsymbol{s}_0, \boldsymbol{t}) \leq 0$ for all $\boldsymbol{s} \in C$. Taking $\boldsymbol{t} = \boldsymbol{e}_j$ we get

$$(y_0 - y_j - z(\boldsymbol{s}_0), x_j - z(\boldsymbol{s}_0))_H = f(\boldsymbol{s}_0, \boldsymbol{e}_j) \leq 0,$$

which shows that $z(\boldsymbol{s}_0) \in K_{(x_j, y_j)}$ for all $j = 1, \ldots, n$. ■

Monday, April 28, 2014

We turn to the proof of Theorem 128.

Proof of Theorem 128.. Assume that the domain of $(I + \varepsilon\Gamma)^{-1}$ is H. Let $(x_1, y_1) \in H \times H$ be such that

$$(y_2 - y_1, x_2 - x_1)_H \geq 0$$

for all $(x_2, y_2) \in \operatorname{graph}\Gamma$. We claim that (x_1, y_1) belongs to $\operatorname{graph}\Gamma$. Since the domain of $(I + \varepsilon\Gamma)^{-1}$ is H there exists a unique $x \in H$ such that $x_1 + \varepsilon y_1 \in (I + \varepsilon\Gamma)(x)$. Hence $x_1 + \varepsilon y_1 = x + \varepsilon w_1$, where $w_1 \in \Gamma(x)$. Since $(x, w_1) \in \operatorname{graph}\Gamma$, we have that

$$0 \leq (w_1 - y_1, x - x_1)_H = -(x - x_1, x - x_1)_H = -\|x - x_1\|_H^2,$$

which implies that $x = x_1$ and, in turn, that $y_1 = w_1$. Thus $(x_1, y_1) = (x, w_1) \in \operatorname{graph}\Gamma$. By Remark 121, Γ is maximal.

Conversely, assume that Γ is maximal. Given $y_0 \in H$, by the previous lemma there exists $x_0 \in H$ such that

$$(y - (y_0 - x_0), x - x_0)_H \geq 0$$

for all $(x, y) \in \operatorname{graph}\Gamma$. By Remark 121, this implies that $(x_0, y_0 - x_0) \in \operatorname{graph}\Gamma$, that is, $x_0 \in \operatorname{dom}\Gamma$ and $y_0 - x_0 \in \Gamma(x_0)$. In turn, $y_0 \in (I + \Gamma)(x_0)$, which shows that $I + \Gamma$ is surjective, or, equivalently, that $\operatorname{dom}(I + \Gamma)^{-1} = H$. This proves the result for $\varepsilon = 1$.

If Γ is maximal, then so is $\varepsilon\Gamma$, and so by what we just proved applied to $\varepsilon\Gamma$ in place of Γ, we have that $I + \varepsilon\Gamma$ is surjective. This concludes the proof. ■

Remark 130 *Note that to prove that Γ is maximal we only used the fact that $(I + \varepsilon\Gamma)^{-1}$ is H for some $\varepsilon > 0$ and not for every $\varepsilon > 0$.*

Next we study evolution equations of the form

$$\begin{cases} f(t) - \frac{du}{dt}(t) \in \Gamma(u(t)), \\ u(0) = u_0, \end{cases} \tag{178}$$

where $u : [0, T] \to H$, with H a Hilbert space, $\Gamma : H \to \mathcal{P}(H)$ be a maximal monotone multifunction, $u_0 \in \operatorname{dom}\Gamma$, and $f : [0, T] \to H$.

Definition 131 *Let H be a Hilbert space, let $\Gamma : H \to \mathcal{P}(H)$ be a multifunction, let $f : [0, T] \to H$, and let $u_0 \in H$.* A strong solution *of the Cauchy problem (178) is a function $u \in C([0, T]; H)$ such that u is absolutely continuous on compact sets of $(0, T)$, $u(t) \in \operatorname{dom}\Gamma$ for $\mathcal{L}^1$ a.e. $t \in (0, T)$ with $f(t) - \frac{du}{dt}(t) \in \Gamma(u(t))$, and $u(0) = u_0$.*

Definition 132 *Let H be a Hilbert space, let $\Gamma : H \to \mathcal{P}(H)$ be a multifunction, let $f \in L^1((0,T);H)$, and let $u_0 \in H$. A function $u \in C([0,T];H)$ is a* weak solution *of the Cauchy problem* (178) *if there exist $\{f_n\} \subseteq L^1((0,T);H)$ converging to f in $L^1((0,T);H)$ and $\{u_n\} \subseteq C((0,T);H)$ converging to u in $C((0,T);H)$ such that u_n is a strong solution of* (178) *with f_n in place of f.*

We begin with the case $f = 0$.

Theorem 133 *Let H be a Hilbert space, $\Gamma : H \to \mathcal{P}(H)$ be a maximal monotone multifunction, and let $u_0 \in \operatorname{dom}\Gamma$. Then the Cauchy problem*

$$\begin{cases} -\frac{du}{dt}(t) \in \Gamma(u(t)), \\ u(0) = u_0, \end{cases} \tag{179}$$

admits a unique strong solution $u : [0,\infty) \to H$. Moreover u is Lipschitz continuous, with

$$\left\| \frac{du}{dt}(t) \right\|_H \le \|\gamma_0(u_0)\|_H$$

for $\mathcal{L}^1$ a.e. $t > 0$, the function u admits a right derivative $\frac{d^+u}{dt}(t)$ for every $t \ge 0$ with

$$-\frac{d^+u}{dt}(t) = \gamma_0(u(t)),$$

the function $t \mapsto \|\gamma_0(u(t))\|_H$ is decreasing, and the function $t \mapsto \gamma_0(u(t))$ is continuous from the right. Here

$$\gamma_0(x) := \arg\min\{\|y\|_H : \ y \in \Gamma(x)\}.$$

Let $\tau > 0$. We construct a sequence a sequence $\{u_n\} \subset H$ inductively as follows. Since Γ is maximal, by Theorem 128 the domain of $(I + \varepsilon\Gamma)^{-1}$ is H for every $\varepsilon > 0$. Hence given $u_{n-1} \in H$, there exists $u_n \in H$ such that

$$(I + \tau\Gamma)^{-1}(u_{n-1}) = \{u_n\} \tag{180}$$

or, equivalently,

$$-\frac{u_n - u_{n-1}}{\tau} \in \Gamma(u_n).$$

Note that u_n is unique, since $(I + \tau\Gamma)^{-1}$ is single-valued (see Theorem 127).

Lemma 134 *Let H be a Hilbert space, $\Gamma : H \to \mathcal{P}(H)$ be a maximal monotone multifunction, let $u_0 \in \operatorname{dom}\Gamma$, and let $\{u_n\}$ be the sequence defined in* (180). *Then*

$$\frac{1}{2}\|u_n - x\|_H^2 - \frac{1}{2}\|u_m - x\|_H^2 \le \tau \sum_{k=m}^{n} (y, x - u_k)_H$$

for all $(x,y) \in \operatorname{graph}\Gamma$ and for all $m \le n$.

Proof. Since Γ is monotone, and $\left(u_k, -\frac{u_k-u_{k-1}}{\tau}\right)$, $(x,y) \in \operatorname{graph}\Gamma$, we have

$$0 \leq \left(-\frac{u_k - u_{k-1}}{\tau} - y, u_k - x\right)_H = \frac{1}{\tau}\left(u_{k-1} - u_k - \tau y, u_k - x\right)_H,$$

that is,

$$\left(u_{k-1} - u_k, x - u_k\right)_H \leq \tau\left(y, x - u_k\right)_H.$$

On the other hand,

$$\begin{aligned}
\left(u_{k-1} - u_k, x - u_k\right)_H &= \left(u_{k-1} - x + x - u_k, x - u_k\right)_H \\
&= \left(u_{k-1} - x, x - u_k\right)_H + \|u_k - x\|_H^2 \\
&\geq -\frac{1}{2}\|u_k - x\|_H^2 - \frac{1}{2}\|u_{k-1} - x\|_H^2 + \|u_k - x\|_H^2 \\
&= \frac{1}{2}\|u_k - x\|_H^2 - \frac{1}{2}\|u_{k-1} - x\|_H^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{2}\|u_k - x\|_H^2 - \frac{1}{2}\|u_{k-1} - x\|_H^2 &\leq \left(u_{k-1} - u_k, x - u_k\right)_H \\
&\leq \tau\left(y, x - u_k\right)_H.
\end{aligned}$$

Summing for $k = m, \ldots, n$, we get the desired result. ■

Proposition 135 (Kobayashi's inequality) *Let H be a Hilbert space, $\Gamma : H \to \mathcal{P}(H)$ be a maximal monotone multifunction, and let $u_0, \hat{u}_0 \in \operatorname{dom}\Gamma$. Let $\tau > 0$ and $\hat{\tau} > 0$ and let $\{u_n\}$ and $\{\hat{u}_n\}$ be defined inductively by*

$$(I + \tau\Gamma)^{-1}(u_{n-1}) = \{u_n\}, \quad (I + \hat{\tau}\Gamma)^{-1}(\hat{u}_{n-1}) = \{\hat{u}_n\}.$$

Then for every $x \in \operatorname{dom}\Gamma$ and for all $n, m \in \mathbb{N}_0$,

$$\|u_n - \hat{u}_m\|_H \leq \|u_0 - x\|_H + \|\hat{u}_0 - x\|_H + \|\gamma_0(x)\| \sqrt{(n\tau - m\hat{\tau})^2 + n\tau^2 + m\hat{\tau}^2}. \tag{181}$$

Proof. Step 1. We claim that if (x_1, y_1), $(x_2, y_2) \in \operatorname{graph}\Gamma$ and $\lambda, \mu > 0$, then

$$(\lambda + \mu)\|x_1 - x_2\|_H \leq \lambda\|x_2 + \mu y_2 - x_1\|_H + \mu\|x_1 + \lambda y_1 - x_2\|_H.$$

To see this, we compute

$$\begin{aligned}
(\lambda + \mu)\|x_1 - x_2\|_H^2 &= \lambda\left(x_2 - x_1, x_2 - x_1\right)_H + \mu\left(x_1 - x_2, x_1 - x_2\right)_H \\
&= \lambda\left(x_2 + \mu y_2 - x_1, x_2 - x_1\right)_H + \mu\left(x_1 + \lambda y_1 - x_2, x_1 - x_2\right)_H \\
&\quad + \lambda\mu\left(y_1 - y_2, x_2 - x_1\right)_H \\
&\leq \left[\lambda\|x_2 + \mu y_2 - x_1\|_H + \mu\|x_1 + \lambda y_1 - x_2\|_H\right]\|x_1 - x_2\|_H,
\end{aligned}$$

where we have used the monotonicity of Γ.

Step 2: Set

$$c_{n,m} := \sqrt{(n\tau - m\hat{\tau})^2 + n\tau^2 + m\hat{\tau}^2}. \tag{182}$$

The proof of (181) is by induction on n and m. We begin by proving that (181) holds for all $(n,0)$, $n \in \mathbb{N}_0$. By (175), for all (x_1,y_1), $(x_2,y_2) \in \operatorname{graph}\Gamma$ and all $\varepsilon > 0$ we have

$$\|x_2 - x_1\|_H \le \|x_2 - x_1 + \varepsilon (y_2 - y_2)\|_H .$$

Hence, since $\left(u_1, -\frac{u_1 - u_0}{\tau}\right) \in \operatorname{graph}\Gamma$ and $(x, \gamma_0(x)) \in \operatorname{graph}\Gamma$, we have

$$\begin{aligned}\|u_1 - x\|_H &\le \left\| u_1 - x + \tau \left(-\frac{u_1 - u_0}{\tau} - \gamma_0(x) \right) \right\|_H \\ &= \|u_0 - x - \tau\gamma_0(x)\|_H \le \|u_0 - x\|_H + \|\gamma_0(x)\| \tau_1,\end{aligned}$$

where we used (**??**). Inductively, we obtain

$$\begin{aligned}\|u_n - x\|_H &\le \|u_0 - x\|_H + \|\gamma_0(x)\| n\tau, \\ &= \|u_0 - x\|_H + \|\gamma_0(x)\| n\tau,\end{aligned}$$

and so

$$\begin{aligned}\|u_n - \hat{u}_0\|_H &\le \|u_n - x\|_H + \|\hat{u}_0 - x\|_H \\ &\le \|u_0 - x\|_H + \|\hat{u}_0 - x\|_H + \|\gamma_0(x)\| n\tau,\end{aligned}$$

which implies (181) for $(n,0)$. Similarly, we have that (181) holds for $(0,m)$. ■

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Proof. Next assume that (181) holds for $(n-1,m)$ and $(n,m-1)$. We claim it holds for (n,m). To see this, we apply Step 1 to obtain

$$\begin{aligned}(\tau + \hat{\tau}) \|u_n - \hat{u}_m\|_H &\le \tau \left\| \hat{u}_m + \hat{\tau} \left(-\frac{\hat{u}_m - \hat{u}_{m-1}}{\tau} \right) - u_n \right\|_H + \hat{\tau} \left\| u_n + \tau \left(-\frac{u_n - u_{n-1}}{\tau} \right) - \hat{u}_m \right\|_H \\ &= \tau \|\hat{u}_{m-1} - u_n\|_H + \hat{\tau} \|u_{n-1} - \hat{u}_m\|_H\end{aligned}$$

by (**??**). Hence by the induction hypothesis and (182),

$$\begin{aligned}\|u_n - \hat{u}_m\|_H &\le \frac{\tau}{\tau + \hat{\tau}} \|\hat{u}_{m-1} - u_n\|_H + \frac{\hat{\tau}}{\tau + \hat{\tau}} \|u_{n-1} - \hat{u}_m\|_H \\ &\le \frac{\tau}{\tau + \hat{\tau}} \left(\|u_0 - x\|_H + \|\hat{u}_0 - x\|_H + \|\gamma_0(x)\| c_{n,m-1} \right) \\ &\quad + \frac{\hat{\tau}}{\tau + \hat{\tau}} \left(\|u_0 - x\|_H + \|\hat{u}_0 - x\|_H + \|\gamma_0(x)\| c_{n-1,m} \right) \\ &= \|u_0 - x\|_H + \|\hat{u}_0 - x\|_H + \left(\frac{\tau}{\tau + \hat{\tau}} c_{n,m-1} + \frac{\hat{\tau}}{\tau + \hat{\tau}} c_{n-1,m} \right) \|\gamma_0(x)\| .\end{aligned}$$

It remains to show that

$$\frac{\tau}{\tau + \hat{\tau}} c_{n,m-1} + \frac{\hat{\tau}}{\tau + \hat{\tau}} c_{n-1,m} \le c_{n,m}. \tag{183}$$

By the convexity of t^2 we have

$$\left(\frac{\tau}{\tau+\hat{\tau}}c_{n,m-1}+\frac{\hat{\tau}}{\tau+\hat{\tau}}c_{n-1,m}\right)^2\leq\frac{\tau}{\tau+\hat{\tau}}c_{n,m-1}^2+\frac{\hat{\tau}}{\tau+\hat{\tau}}c_{n-1,m}^2.$$

On the other hand, by (182),

$$\begin{aligned}c_{n,m}^2&=(n\tau-m\hat{\tau})^2+n\tau^2+m\hat{\tau}^2\\&=(\tau+(n-1)\tau-m\hat{\tau})^2+\tau^2+(n-1)\tau^2+m\hat{\tau}^2\\&=((n-1)\tau-m\hat{\tau})^2+2\tau^2+2\tau((n-1)\tau-m\hat{\tau})+(n-1)\tau^2+m\hat{\tau}^2\\&=c_{n-1,m}^2+2\tau(n\tau-m\hat{\tau}),\end{aligned}$$

while

$$c_{n,m}^2=c_{n,m-1}^2+2\hat{\tau}(m\hat{\tau}-n\tau)=c_{n,m-1}^2-2\hat{\tau}(n\tau-m\hat{\tau}).$$

Combining the last three inequalities gives

$$\begin{aligned}&\left(\frac{\tau}{\tau+\hat{\tau}}c_{n,m-1}+\frac{\hat{\tau}}{\tau+\hat{\tau}}c_{n-1,m}\right)^2\leq\frac{\tau}{\tau+\hat{\tau}}c_{n,m-1}^2+\frac{\hat{\tau}}{\tau+\hat{\tau}}c_{n-1,m}^2\\&=\frac{\tau}{\tau+\hat{\tau}}\left(c_{n,m}^2+2\hat{\tau}(n\tau-m\hat{\tau})\right)+\frac{\hat{\tau}}{\tau+\hat{\tau}}\left(c_{n,m}^2-2\tau(n\tau-m\hat{\tau})\right)\\&=c_{n,m}^2,\end{aligned}$$

which is (183). ■

We now turn to the second proof of Theorem 133.

Second proof. Step 1: For $\tau>0$ and subdivide $(0,\infty)$ into intervals of length τ,

$$0<\tau<\cdots<n\tau<\cdots$$

and define

$$u_\tau(t):=\begin{cases}u_0 & \text{if } t=0,\\ u_n^{(\tau)} & \text{for } t\in((n-1)\tau,n\tau],\ n\in\mathbb{N},\end{cases}$$

where $\left\{u_n^{(\tau)}\right\}$ is the sequence given in (180). Given $\eta>0$, and $t,s\geq 0$, let $n:=\left\lceil\frac{t}{\tau}\right\rceil$ and $m:=\left\lceil\frac{s}{\eta}\right\rceil$. By Kobayashi's inequality with $x=u_0$ and $\hat{\tau}:=\eta$, we have

$$\|u_\tau(t)-u_\eta(s)\|_H\leq\|\gamma_0(u_0)\|\sqrt{(n\tau-m\eta)^2+n\tau^2+m\eta^2}.\tag{184}$$

Note that

$$t-s-\eta\leq n\tau-m\eta\leq t-s+\tau,\quad n\tau^2+m\eta^2\leq(t+\tau)\tau+(s+\eta)\eta.\tag{185}$$

Hence, taking $s=t$ and letting $\tau,\eta\to 0^+$, we have that the right-hand side of (184) tends to zero, and so the sequence $\{u_\tau(t)\}$ is Cauchy and so it converges in H to some $u(t)$ as $\tau\to 0^+$.

On the other hand, letting $\tau, \eta \to 0^+$ in (184), again by (185), gives

$$\|u(t) - u(s)\|_H \le \|\gamma_0(u_0)\| (t - s). \tag{186}$$

Hence, u is Lipschitz continuous with Lipschitz constant less than or equal to $\|\gamma_0(u_0)\|$. In particular, it is absolutely continuous, and so by Komura's theorem it is differentiable for $\mathcal{L}^1$ a.e. $t \ge 0$. Finally, taking $s = t$ and letting $\eta \to 0^+$ in (184), again by (185), implies that

$$\|u_\tau(t) - u(t)\|_H \le \|\gamma_0(u_0)\| \sqrt{\tau^2 + (t+\tau)\tau},$$

which shows that sequence of functions $\{u_\tau\}$ converges to u uniformly on each interval $[0, T]$, $T > 0$.

Step 2: We claim that

$$\frac{1}{2}\|u(t) - x\|^2 - \frac{1}{2}\|u(s) - x\|^2 \le \int_s^t (y, x - u(r))\ dr \tag{187}$$

for all $(x, y) \in \operatorname{graph}\Gamma$ and for all $0 \le s < t$. Let $n := \left\lceil \frac{t}{\tau} \right\rceil$ and $m := \left\lceil \frac{s}{\tau} \right\rceil$. Then by Lemma 134,

$$\begin{aligned}
\frac{1}{2}\|u_\tau(t) - x\|_H^2 - \frac{1}{2}\|u_\tau(s) - x\|_H^2 &\le \tau \sum_{k=m}^{n} (y, x - u_\tau(k\tau))_H \\
&= \int_{\tau\lceil s/\tau\rceil}^{\tau\lceil t/\tau\rceil} (y, x - u_\tau(r))_H\ dr.
\end{aligned}$$

Letting $\tau \to 0^+$ and using the fact that $\{u_\tau\}$ converges to u uniformly on each interval $[0, T]$, $T > 0$, gives (187). In particular, if $t_0 \ge 0$ is such that u is differentiable at t_0, taking $s = t_0$ and $t = t_0 + h$ in (187) and dividing by $h > 0$, we obtain

$$\frac{\frac{1}{2}\|u(t_0 + h) - x\|^2 - \frac{1}{2}\|u(t_0) - x\|^2}{h} \le \frac{1}{h}\int_{t_0}^{t_0+h} (y, x - u(r))\ dr.$$

Letting $h \to 0^+$ yields

$$\left(\frac{du}{dt}(t_0), u(t_0) - x\right)_H \le (y, x - u(t_0)),$$

that is,

$$0 \le \left(y + \frac{du}{dt}(t_0), x - u(t_0)\right)$$

for all $(x, y) \in \operatorname{graph}\Gamma$. By the maximality of Γ (see Remark 121), this implies that $-\frac{du}{dt}(t_0) \in \Gamma(u(t_0))$, which shows that u is a strong solution of (179). ■

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Proof. Step 3: Uniqueness. Let u_1, u_2 be strong solutions of

$$\begin{cases} -\frac{du_i}{dt}(t) \in \Gamma(u_i(t)), \\ u_i(0) = u_{0,i}, \end{cases}$$

$i = 1, 2$, where $u_{0,i} \in \operatorname{dom}\Gamma$. By the monotonicity of Γ we have

$$\frac{1}{2}\frac{d}{dt}\left\|u_2(t) - u_1(t)\right\|_H^2 = \left(\frac{du_2}{dt}(t) - \frac{du_1}{dt}(t), u_2(t) - u_1(t)\right)_H \leq 0.$$

Hence,

$$\left\|u_2(t) - u_1(t)\right\|_H \leq \left\|u_{0,2} - u_{0,1}\right\|_H$$

for all $t \geq 0$. In particular, if $u_{0,1} = u_{0,2}$, it follows that $u_1 = u_2$.
Step 4: Additional properties. Since u is Lipschitz with Lipschitz constant at most $\left\|\gamma_0(u_0)\right\|_H$, we have that

$$\left\|\frac{du}{dt}(t)\right\|_H \leq \left\|\gamma_0(u_0)\right\|_H,$$

for $\mathcal{L}^1$ a.e. $t > 0$. Moreover, $-\frac{du}{dt}(t) \in \Gamma(u(t))$ for for $\mathcal{L}^1$ a.e. $t > 0$. Hence, by the definition of γ_0, we get

$$\left\|\gamma_0(u(t))\right\|_H \leq \left\|\gamma_0(u_0)\right\|_H \tag{188}$$

for $\mathcal{L}^1$ a.e. $t > 0$. Let's prove that this inequality holds for every $t > 0$. Fix $t_0 > 0$ and let $t_n \to t_0^+$ be such that the previous inequality holds at t_n. Since the sequence $\{\gamma_0(u(t_n))\}$ is bounded in H, there exist a subequence $\{\gamma_0(u(t_{n_k}))\}$ and $\xi \in H$ such that $\gamma_0(u(t_{n_k})) \rightharpoonup \xi$. Hence, given $(x, y) \in \operatorname{graph}\Gamma$, by the monotonicity of Γ and we get

$$(\gamma_0(u(t_{n_k})) - y, u(t_{n_k}) - x)_H \geq 0.$$

Letting $n \to \infty$ and using the facts that $\gamma_0(u(t_{n_k})) \rightharpoonup \xi$ and $u(t_{n_k}) \to u(t_0)$, we obtain

$$(\xi - y, u(t_0) - x)_H \geq 0$$

for all $(x, y) \in \operatorname{graph}\Gamma$. Since Γ is maximal, it follows by Remark 121 that $(u(t_0), \xi)$ belongs to $\operatorname{graph}\Gamma$. On the other hand, by the lower semicontinuity of the norm with respect to weak convergence, we have

$$\left\|\gamma_0(u(t_0))\right\|_H \leq \left\|\xi\right\|_H \leq \liminf_{kn \to \infty} \left\|\gamma_0(u(t_{n_k}))\right\|_H \leq \left\|\gamma_0(u_0)\right\|_H,$$

which proves the claim.

With a similar proof we can conclude that

$$\lim_{t \to 0^+} \gamma_0(u(t)) = \gamma_0(u_0). \tag{189}$$

Indeed, if $t_n \to t_0^+$, then since (188) holds for t_n we can repeat the same proof to obtain

$$\left\|\gamma_0(u_0)\right\|_H \leq \left\|\xi\right\|_H \leq \liminf_{k \to \infty} \left\|\gamma_0(u(t_{n_k}))\right\|_H \leq \left\|\gamma_0(u_0)\right\|_H.$$

Hence, $\|\gamma_0(u_0)\|_H = \|\xi\|_H$, which, by the definition of γ_0, implies that $\gamma_0(u_0) = \xi$. This shows that $\gamma_0(u(t_{n_k})) \rightharpoonup \gamma_0(u_0)$. On the other hand,

$$\liminf_{k\to\infty} \|\gamma_0(u(t_{n_k}))\|_H = \|\gamma_0(u_0)\|_H ,$$

and so the convergence is actually strong, that is, $\gamma_0(u(t_{n_k})) \to \gamma_0(u_0)$. By the arbitrariness of the sequence $\{t_n\}$, it follows that (189) holds.

Finally, let's show that there exists

$$\frac{d^+u}{dt}(0) = -\gamma_0(u_0). \tag{190}$$

Let

$$E := \left\{ t \in (0,\infty) : \ u \text{ is differentiable at } t \text{ and } \left(u(t), -\frac{du}{dt}(t)\right) \in \operatorname{graph}\Gamma \right\}.$$

By Step 3, the complement of E in $(0,\infty)$ has measure zero. Given $t_0 \geq 0$, the function $t \mapsto u(t+t_0)$ is a solution of (179) with initial datum $u(t_0)$. Hence, this function is Lipschitz continuous with Lipschitz constant less than or equal to $\|\gamma_0(u(t_0))\|_H$. It follows that

$$\left\| \frac{u(t_0+h) - u(t_0)}{h} \right\|_H \leq \|\gamma_0(u(t_0))\|_H .$$

In particular, if $t_0 \in E$, letting $h \to 0^+$ gives

$$\left\| \frac{du}{dt}(t_0) \right\|_H \leq \|\gamma_0(u(t_0))\|_H .$$

On the other hand, since $\left(u(t_0), -\frac{du}{dt}(t_0)\right) \in \operatorname{graph}\Gamma$ by the definition of E, it follows that $-\frac{du}{dt}(t_0) = \gamma_0(u(t_0))$. Integrating this identity in $(0,t) \cap E$ gives

$$-\frac{u(t) - u(0)}{t} = \frac{1}{t}\int_0^t \gamma_0(u(s)) \, ds.$$

Letting $t \to 0^+$ and using (189) proves (190).

To conclude the proof, it remains to show that (189) and (190) hold with t_0 in place of 0. This follows again by the fact that the function $t \mapsto u(t+t_0)$ is a solution of (179) with initial datum $u(t_0)$. ■

Theorem 136 *Let H be a Hilbert space, $\Gamma : H \to \mathcal{P}(H)$ be a maximal monotone multifunction, let $f \in L^1((0,T);H)$ and let $u_0 \in \overline{\operatorname{dom}\Gamma}$. Then the Cauchy problem* (178) *admits a unique weak solution.*

Proof. Step 1: Given $y_0 \in H$, prove that the multifunction

$$\Gamma_1(x) := -y_0 + \Gamma(x), \quad x \in H,$$

is monotone and maximal. Indeed, to check maximality, observe that given $z_0 \in H$, since $I+\Gamma$ is surjective, we can find $x_0 \in \operatorname{dom}\Gamma$ and $z_0 + y_0 \in (I+\Gamma)(x_0)$. In turn, $z_0 \in (I+\Gamma_1)(x_0)$, which shows that $I+\Gamma_1$ is surjective.

Step 2: Let $u_0 \in \operatorname{dom}\Gamma$ and let $s:(0,T)\to H$ be a simple function of the form

$$s(x) = \sum_{k=1}^{n} \chi_{I_k} y_k,$$

where $y_k \in H$ and the intervals $I_1,\dots,I_n$ form a partition of $[0,T]$, namely,

$$I_1 = (0,t_1], \quad I_2 = (t_2,t_1], \dots, I_n = (t_{n-1},T),$$

with $0 < t_1 < \cdots < t_{n-1} < T$. We claim that the Cauchy problem

$$\begin{cases} s(t) - \frac{du}{dt}(t) \in \Gamma(u(t)), \\ u(0) = u_0, \end{cases} \tag{191}$$

admits a solution. To see this, consider the Cauchy problem

$$\begin{cases} -\frac{dv}{dt}(t) \in -y_1 + \Gamma(v(t)), \\ v(0) = u_0 \end{cases}$$

in $[0,t_1]$. By the previous theorem there exists a unique strong solution u_1. Note that u_1 solves (191) in $[0,t_1]$. Next, consider the Cauchy problem

$$\begin{cases} -\frac{dv}{dt}(t) \in -y_2 + \Gamma(v(t)), \\ v(t_1) = u_1(t_1) \end{cases}$$

in $[t_1,t_2]$. By the previous problem there exists a unique strong solution u_2. Inductively, define u_k to be the solution of the Cauchy problem

$$\begin{cases} -\frac{dv}{dt}(t) \in -y_k + \Gamma(v(t)), \\ v(t_{k-1}) = u_{k-1}(t_{k-1}) \end{cases}$$

in $[t_{k-1},t_k]$. The function

$$u(t) := \begin{cases} u_0 & \text{if } t=0, \\ u_k(t) & \text{for } t \in (t_{k-1},t_k],\ k=1,\dots,n, \end{cases}$$

is Lipschitz continuous and solves (191) in $[0,T]$.

Step 3: Let $u_{0,1}, u_{0,2} \in \operatorname{dom}\Gamma$ and let s_1, s_2 be two simple functions as in Step 2 and let u_1, u_2 be strong solutions of the Cauchy problem

$$\begin{cases} s_1(t) - \frac{du_1}{dt}(t) \in \Gamma(u_1(t)), \\ u(0) = u_{0,1}, \end{cases} \qquad \begin{cases} s_2(t) - \frac{du_2}{dt}(t) \in \Gamma(u_2(t)), \\ u(0) = u_{0,2}. \end{cases}$$

By the monotonicity of Γ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_2(t)-u_1(t)\|_H^2 &= \left(\frac{du_2}{dt}(t) - \frac{du_1}{dt}(t), u_2(t)-u_1(t)\right)_H \\ &= \left(-s_2(t) + \frac{du_2}{dt}(t) + s_1(t) - \frac{du_1}{dt}(t), u_2(t)-u_1(t)\right)_H \\ &\quad + (s_2(t)-s_1(t), u_2(t)-u_1(t))_H \le (s_2(t)-s_1(t), u_2(t)-u_1(t))_H . \end{aligned}$$

Hence, upon integration over $[s,t] \subseteq [0,T]$,

$$\begin{aligned}\frac{1}{2}\left\|u_1(t)-u_2(t)\right\|_H^2 &\leq \frac{1}{2}\left\|u_1(s)-u_2(s)\right\|_H^2+\int_s^t\left(s_1(r)-s_2(r), u_1(r)-u_2(r)\right)_H \,dr \\ &\leq \frac{1}{2}\left\|u_1(s)-u_2(s)\right\|_H^2+\int_0^t\left\|s_1(r)-s_2(r)\right\|_H\left\|u_1(r)-u_2(r)\right\|_H \,dr\end{aligned}$$

for all $0 \leq s \leq t \leq T$. By Lemma 67,

$$\left\|u_1(t)-u_2(t)\right\|_H \leq \left\|u_1(s)-u_2(s)\right\|_H+\int_s^t\left\|s_1(r)-s_2(r)\right\|_H \,dr \qquad (192)$$

for all $0 \leq s \leq t \leq T$.

Step 4: Finally, given $f \in L^1((0,T);H)$ and let $u_0 \in \overline{\operatorname{dom}\Gamma}$, by density we can find a sequence of steps functions $\{s_n\}$ as in Step 2 and $u_{0,n} \in \operatorname{dom}\Gamma$ such that $s_n \to f$ in $f \in L^1((0,T);H)$ and $u_{0,n} \to u_0$. Let u_n be the solutions of the Cauchy problem

$$\begin{cases} s_n(t)-\frac{dv}{dt}(t) \in \Gamma(v(t)), \\ v(0)=u_{0,n}. \end{cases}$$

Then by Step 3,

$$\left\|u_n(t)-u_m(t)\right\|_H \leq \left\|u_{0,n}-u_{0,m}\right\|_H+\int_0^t\left\|s_n(r)-s_m(r)\right\|_H \,dr$$

for every $t \in [0,T]$ and for all $n,m \in \mathbb{N}$. This shows that $\{u_n\}$ is a Cauchy sequence in $C([0,T];H)$ and so it converges uniformly to a weak solution of (178). ■

References

[DieU77] Diestel, J., Uhl, J.J., Jr.: Vector Measures. With a foreword by B. J. Pettis. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I. (1977).

[DuSc88] Dunford, N., Schwartz, J.T.: Linear Operators. Part I. General Theory. With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1958 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York (1988).

[Ed95] Edwards, R.E.: Functional Analysis. Theory and Applications. Corrected reprint of the 1965 original. Dover Publications, Inc., New York (1995).

[FL07] Fonseca, Irene; Leoni, Giovanni: Modern methods in the calculus of variations: Lp spaces. Springer Monographs in Mathematics. Springer, New York, 2007.

[SY05] Schwabik, S., Ye, G.: Topics in Banach Space Integration. Series in Real Analysis, 10. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2005).

[Yo95] Yosida, K.: Functional Analysis. Reprint of the sixth (1980) edition. Classics in Mathematics. Springer-Verlag, Berlin (1995).