Monday, August 27, 2012

1 Real Numbers

There are two ways to introduce the real numbers. The first is to give them in an axiomatic way, the second is to construct them starting from the natural numbers. We will use the first method.

The *real numbers* are a set $\mathbb{R}$ with two binary operations, *addition*

$$\begin{aligned} + : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$

and *multiplication*

$$\begin{aligned} \cdot : \mathbb{R} \times \mathbb{R} &\to \mathbb{R} \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

and a relation $\leq$ such that

(A) $(\mathbb{R}, +) \leq$ is an *commutative group*, that is,

- (A_1) for every $a, b \in \mathbb{R}$, $a + b = b + a$,
- (A_2) for every $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$,
- (A_3) there exists a unique element in $\mathbb{R}$, called *zero* and denoted 0, such that $0 + a = a + 0 = a$ for every $a \in \mathbb{R}$,
- (A_4) for every $a \in \mathbb{R}$ there exists a unique element in $\mathbb{R}$, called the *opposite* of a and denoted $-a$, such that $(-a) + a = a + (-a) = 0$,

(M)

- (M_1) for every $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$,
- (M_2) for every $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- (M_3) there exists a unique element in $\mathbb{R}$, called *one* and denoted 1, such that $1 \neq 0$ and $1 \cdot a = a \cdot 1 = a$ for every $a \in \mathbb{R}$ with $a \neq 0$,
- (M_4) for every $a \in \mathbb{R}$ with $a \neq 0$ there exists a unique element in $\mathbb{R}$, called the *inverse* of a and denoted a^{-1}, such that $a^{-1} \cdot a = a \cdot a^{-1} = 1$,

(O) $\leq$ is a *total order relation*, that is,

- (O_1) for every $a, b \in \mathbb{R}$ either $a \leq b$ or $b \leq a$,
- (O_2) for every $a, b, c \in \mathbb{R}$ if $a \leq b$ and $b \leq c$, then $a \leq c$,
- (O_3) for every $a, b \in \mathbb{R}$ if $a \leq b$ and $b \leq a$, then $a = b$,
- (O_4) for every $a \in \mathbb{R}$ we have $a \leq a$,

(AM) for every $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

(AO) for every $a, b, c \in \mathbb{R}$ if $a \leq b$, $a + c \leq b + c$,

(MO) for every $a, b \in \mathbb{R}$ if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$,

(S) **(supremum property)**

Remark 1 *Properties (A), (M), (O), (AM), (AO), (MO), and (S) completely characterize the real numbers in the sense that if $(\mathbb{R}', \oplus, \odot, \preccurlyeq)$ satisfies the same properties, then there exists a bijection $T : \mathbb{R} \to \mathbb{R}'$ such that T is an isomorphism between the two fields, that is,*

$$T(a + b) = T(a) \oplus T(b), \qquad T(a \cdot b) = T(a) \odot T(b)$$

for all $a, b \in \mathbb{R}$, and $a \leq b$ if and only if $T(a) \preccurlyeq T(b)$. Hence, for all practical purposes, we cannot distinguish $\mathbb{R}$ from $\mathbb{R}'$.

If $a \leq b$ and $a \neq b$, we write $a < b$.

Exercise 2 *Using **only** the axioms (A), (M), (O), (AO), (AM) and (MO) of $\mathbb{R}$, prove the following properties of $\mathbb{R}$:*

(i) if $a \cdot b = 0$ then either $a = 0$ or $b = 0$,

(ii) if $a \geq 0$ then $-a \leq 0$,

(iii) if $a \leq b$ and $c < 0$ then $ac \geq bc$,

(iv) for every $a \in \mathbb{R}$ we have $a^2 \geq 0$,

(v) $1 > 0$.

Definition 3 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) An element $L \in \mathbb{R}$ is called an upper bound *of E if $x \leq L$ for all $x \in E$;*

(ii) E is said to be bounded from above *if it has at least an upper bound;*

(iii) if E is bounded from above, the least of all its upper bounds, if it exists, is called the supremum *of E and is denoted* $\sup E$.

(iv) E has a maximum *if there exists $L \in E$ such that $x \leq L$ for all $x \in E$. We write $L = \max E$.*

We are now ready to state the supremum property.

(S) **(supremum property)** every nonempty set $E \subseteq \mathbb{R}$ bounded from above has a supremum in $\mathbb{R}$.

The supremum property says that in $\mathbb{R}$ the supremum of a nonempty set bounded from above always exists in $\mathbb{R}$. We will see that this is not the case for the rationals numbers.

Remark 4 *(i) Note that if a set has a maximum L, then L is also the supremum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from below, to prove that a number $L \in \mathbb{R}$ is the supremum of E, we need to show that L is an upper bound of E, that is, that $x \leq L$ for every $x \in E$, and that any number $s < L$ cannot be an upper bound of E, that is, that there exists $x \in E$ such that $s < x$.

Wednesday, August 29, 2012

Definition 5 *Let $E \subseteq \mathbb{R}$ be a nonempty set.*

(i) An element $\ell \in \mathbb{R}$ is called a lower bound *of E if $\ell \leq x$ for all $x \in E$;*

(ii) E is said to be bounded from below *if it has at least an lower bound;*

(iii) if E is bounded from above, the greatest of all its lower bounds, if it exists, is called the infimum *of E and is denoted* $\inf E$*;*

(iv) E has a minimum *if there exists $\ell \in E$ such that $\ell \leq x$ for all $x \in E$. We write $\ell = \min E$.*

Remark 6 *(i) Note that if a set has a minimum ℓ, then ℓ is also the infimum of the set.*

(ii) If $E \subseteq \mathbb{R}$ is a set bounded from above, to prove that a number $\ell \in \mathbb{R}$ is the infimum of E, we need to show that ℓ is a lower bound of E, that is, that $\ell \leq x$ for every $x \in E$, and that any number $\ell < s$ cannot be a lower bound of E, that is, that there exists $x \in E$ such that $x < s$.

Example 7 *Consider the set*

$$E = \left\{ y \in \mathbb{R} : \, y = \frac{n^2 + 2n}{n^2 + 2}, \, n \in \mathbb{N} \right\}.$$

Since $\frac{n^2+2n}{n^2+2} > 0$, the set E is bounded from below. To see if it is bounded from above, let's sketch the graph of the function

$$f(x) = \frac{x^2 + 2x}{x^2 + 2}$$

for $x \geq 1$. We have

$$\lim_{x \to \infty} \frac{x^2 + 2x}{x^2 + 2} = \lim_{x \to \infty} \frac{x^2 \left(1 + \frac{2}{x}\right)}{x^2 \left(1 + \frac{2}{x^2}\right)}$$
$$= \lim_{x \to \infty} \frac{1 + \frac{2}{x}}{1 + \frac{2}{x^2}} = \frac{1 + \frac{2}{\infty}}{1 + \frac{2}{\infty^2}} = \frac{1+0}{1+0} = 1.$$

Moreover,

$$f'(x) = \frac{d}{dx}\left(\frac{x^2+2x}{x^2+2}\right) = \frac{2\left(-x^2+2x+2\right)}{\left(x^2+2\right)^2} \geq 0$$

for all $-\sqrt{3}+1 \leq x \leq \sqrt{3}+1$. *Hence,* f *is increasing in* $\left[1, \sqrt{3}+1\right]$ *and decreasing for* $x \geq \sqrt{3}+1$. *Not that* $2 \leq \sqrt{3}+1 \leq 3$. *This implies that*

$$\sup E = \max E = \max\{f(2), f(3)\} = \max\left\{\frac{4}{3}, \frac{15}{11}\right\} = \frac{15}{11},$$

while

$$\inf E = \min\left\{f(1), \lim_{n\to\infty} f(n)\right\} = \min\{1,1\} = 1,$$

so actually $\inf E = \min E = f(1) = 1$.

Example 8 *Consider the set*

$$E = \left\{t \in \mathbb{R} : \, t = \frac{xy}{x^2+y^2}, \, x, y \in \mathbb{R}, \, x < y\right\}.$$

The set E *is bounded since* $-\frac{1}{2} \leq \frac{xy}{x^2+y^2} \leq \frac{1}{2}$ *for all* $x, y \in \mathbb{R}$, *with* $x < y$. *Moreover, by taking* $x = -1$ *and* $y = 1$, *we get* $t = \frac{-1}{2}$, *so*

$$\inf E = \min E = -\frac{1}{2}.$$

Let's prove that

$$\sup E = \frac{1}{2}.$$

We need to show that any $s < \frac{1}{2}$ *is not an upper bound for the set* E. *If* $s \leq -\frac{1}{2}$, *then we can take* $x = -1$ *and* $y = 0$, *so that* $s < t = 0$. *Thus assume that* $-\frac{1}{2} < s < \frac{1}{2}$. *Take the sequence* $x_n = 1 - \frac{1}{n} < y_n = 1$. *Then*

$$t_n = \frac{x_n y_n}{x_n^2 + y_n^2} = \frac{1-\frac{1}{n}}{\left(1-\frac{1}{n}\right)^2+1} > s,$$

which gives

$$1 - \frac{1}{n} > s\left(\left(1-\frac{1}{n}\right)^2 + 1\right) \quad \text{or} \quad 2sn^2 - 2sn + s < n^2 - n$$

$$\text{or} \quad 0 = (1-2s)\,n^2 + (2s-1)\,n - s,$$

that is,

$$n > \frac{1-2s+\sqrt{-4s^2+1}}{2(1-2s)}$$

Note that $-4s^2+1 > 0$ *and* $1-2s > 0$ *for* $-\frac{1}{2} < s < \frac{1}{2}$. *Hence, for all* n *sufficiently large,* $t_n > s$, *which shows that* s *is not an upper bound of* E. *Thus,* $\frac{1}{2}$ *is the supremum of the set. Note that* $t = \frac{xy}{x^2+y^2} = \frac{1}{2}$ *only if* $x = y$, *which is not allowed, so the set does not have a maximum.*

Friday, August 31, 2012

2 Natural Numbers

Definition 9 *A set* $E \subseteq \mathbb{R}$ *is called an* inductive set *if it has the following properties*

(i) the number 1 *belongs to* E*,*

(ii) if a number x *belongs to* E*, then* $x+1$ *also belongs to* E*.*

Example 10 *The sets* $[0,\infty) = \{x \in \mathbb{R} : 0 \leq x\}$*,* $[1,\infty) = \{x \in \mathbb{R} : 1 \leq x\}$*, and* $\mathbb{R}$ *are all inductive sets.*

Definition 11 *The set of the* natural numbers $\mathbb{N}$ *is defined as the intersection of all inductive sets of* $\mathbb{R}$*.*

Note that $\mathbb{N}$ is nonempty, since 1 belongs to every inductive set, and so also to $\mathbb{N}$. We also define

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}\,.$$

Proposition 12 *The set* $\mathbb{N}$ *is an inductive set.*

Proof. We already know that 1 belongs to $\mathbb{N}$. If x belongs to $\mathbb{N}$, then it belongs to every inductive set E but then, since E is an inductive set, it follows that $x+1$ belongs E. Hence, $x+1$ belongs to every inductive set, and so by definition of $\mathbb{N}$, we have that $x+1$ also belongs to $\mathbb{N}$. ■

The next result is very important.

Theorem 13 (Principle of mathematical induction) *Let* $\{p_n\}$*,* $n \in \mathbb{N}$*, be a family of propositions such that*

(i) p_1 *is true,*

(ii) if p_n *is true for some* $n \in \mathbb{N}$*, then* p_{n+1} *is also true.*

Then p_n *is true for* every $n \in \mathbb{N}$*.*

Proof. Let $E := \{n \in \mathbb{N}$ such that p_n is true$\}$. Note that $E \subseteq \mathbb{N}$. It follows by (i) and (ii) that E is an inductive set, and so E contains $\mathbb{N}$ (since $\mathbb{N}$ is the intersection of all inductive sets). Hence, $E = \mathbb{N}$. ■

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

If $x \neq 0$, we define $x^0 := 1$. We do not define 0^0.

The following example and exercises will be used later on.

Example 14 *Let $x \geq -1$. Let's prove that*

$$(1+x)^n \geq 1 + nx \tag{1}$$

for every $n \in \mathbb{N}$. For $n = 1$, we have $(1+x)^1 \geq 1 + 1x$, which is true. Assume that for some $n \in \mathbb{N}$, the inequality (1) holds. We want to prove that $(1+x)^{n+1} \geq 1 + (n+1)x$. To see this, observe that

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \geq (1+x)(1+nx) \\ &= 1 + (n+1)x + nx^2 \geq 1 + (n+1)x + 0 = 1 + (n+1)x,\end{aligned}$$

where in the first inequality we have used the fact that $1 + x \geq 0$. Hence, by the principle of mathematical induction, the inequality (1) holds for every $n \in \mathbb{N}$.

Remark 15 *For $x < -1$ the inequality (1) is false in general, take $x = -3$ and $n \in \mathbb{N}$. Then*

$$(1-3)^n = (-2)^n = (-1)^n 2^n \overset{??}{\geq} 1 - 3n.$$

For n odd, $(-1)^n = -1$, and so $-2^n \overset{??}{\geq} 1-3n$, or, equivalently, $2^n \overset{??}{\leq} 3n-1$,which is false for all n odd large. Take $n = 4$, you get $2^4 = 16 \leq 12-1$, which is false.

Example 16 *Prove that*

$$1 + \cdots + n = \frac{n(n+1)}{2} \tag{2}$$

for every $n \in \mathbb{N}$

Exercise 17 *Let $x \neq 1$. Prove that*

$$1 + x \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for all $n \in \mathbb{N}$.

In what follows $0! := 1$, $1! := 1$ and $n! := 1 \cdot 2 \cdot \cdots \cdot n$ for all $n \in \mathbb{N}$. The number $n!$ is called the *factorial* of n. For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}.$$

Theorem 18 (Binomial theorem) *Let $x, y \in \mathbb{R} \setminus \{0\}$ and let $n \in \mathbb{N}$. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{3}$$

Proof. Step 1: Let's prove that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

We have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!\,(n+1-k)!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{k!\,(n-k)!n! + (k-1)!\,(n+1-k)!n!}{k!\,(n-k)!\,(k-1)!\,(n+1-k)!} \\
&= (n!\,(k-1)!\,(n-k)!)\,\frac{k + (n+1-k)}{k!\,(n-k)!\,(k-1)!\,(n+1-k)!} \\
&= \frac{n!\,(n+1)}{k!\,(n+1-k)!} = \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}.
\end{aligned}$$

Step 2: Let's prove (3). The proof is by induction on n. For $n = 1$ we have

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = (x+y)^1,$$

since $\binom{1}{1} = \binom{1}{0} = 1$.

Assume that the formula (3) holds for n and let's prove it for $n+1$, we have

$$(x+y)^{n+1} = (x+y)\,(x+y)^n = x\,(x+y)^n + y\,(x+y)^n.$$

By the induction hypothesis for n, the right-hand side of the previous identity equals to

$$\begin{aligned}
x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= \sum_{l=1}^{n+1} \binom{n}{l-1} x^l y^{n+1-l} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= \binom{n}{0} x^0 y^{n+1-0} + \sum_{l=1}^{n} \binom{n}{l-1} x^l y^{n+1-l} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + \binom{n}{n} x^{n+1} y^0 \\
&= \binom{n+1}{0} x^0 y^{n+1-0} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{n+1} x^{n+1} y^0
\end{aligned}$$

where we have set $k+1 = l$ and used Step 1 and the facts that $\binom{n}{0} = \binom{n+1}{0} = 1$ and $\binom{n}{n} = \binom{n+1}{n+1} = 1$. ■

Remark 19 *If in Theorem 13 we replace property (i) with*

(i)′ if p_{n_0} is true for some $n_0 \in \mathbb{N}$,

then we can conclude that p_n is true for all $n \in \mathbb{N}$ with $n \geq n_0$. To see this, it is enough to define

$$E := \{n \in \mathbb{N} \text{ such that } p_{n+n_0-1} \text{ is true}\},$$

which is still an inductive set.

Exercise 20 *Prove that*

$$n^n > 2^n n!$$

for all $n > 6$. Hint: Use the binomial theorem.

3 The Rationals Numbers and the Supremum Property

In the previous section we have defined the natural numbers. Note that $(\mathbb{N}, +, \cdot, \leq)$ does not satisfy properties (A_3), (A_4), and (M_4). In particular, we cannot subtract two numbers $a, b \in \mathbb{N}$ unless, $a \geq b + 1$. For this reason, we define the *set of integers* $\mathbb{Z}$ as follows

$$\mathbb{Z} := \{\pm n : n \in \mathbb{N}\} \cup \{0\}.$$

Now $(\mathbb{Z}, +, \cdot, \leq)$ satisfies properties (A_3), (A_4), but not (M_4). To resolve this issue, we introduce the *set of rational numbers* $\mathbb{Q}$ defined by

$$\mathbb{Q} := \left\{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\right\},$$

where $\frac{p}{q} := p \cdot q^{-1}$. Then $(\mathbb{Q}, +, \cdot, \leq)$ satisfies properties (A), (M), (O), (AM), (AO), (MO). So, what's wrong?

Theorem 21 *There does not exist a rational number r such that $r^2 = 2$.*

Proof. The proof is by contradiction. Assume that there exists $r \in \mathbb{Q}$ such that $r^2 = 2$. Write $r = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$\left(\frac{p}{q}\right)^2 = 2.$$

By dividing by common factors, without loss of generality, we may suppose that p and q have no common integral factors other that 1. Since $p^2 = 2q^2$, it follows that p^2 is an even number. We claim that p is even. Indeed, if $p = 2k + 1$ for some $k \in \mathbb{Z}$, then

$$p^2 = (2k+1)^2 = 2\left(2k^2 + k\right) + 1,$$

which is an odd number, which is a contradiction. Hence, $p = 2k$, for some $k \in \mathbb{Z}$. But then

$$4k^2 = p^2 = 2q^2,$$

which implies that $2k^2 = q^2$. Hence, q^2 is even, and so reasoning as before, we conclude that q must be even. This contradicts the fact that p and q have no common integral factors (they are both divisible by 2). ■

Thus in the set of rational numbers the square root $\sqrt{r}$ is not defined, in general.

Monday, September 3, 2012

Memorial day, no classes

Wednesday, September 5, 2012

Theorem 22 *The rational numbers do not satisfy the supremum property.*

Proof. We need a nonempty set $E \subseteq \mathbb{Q}$ bounded from above but for which there exists no supremum in $\mathbb{Q}$. Define

$$E := \left\{x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2\right\}.$$

Then E is nonempty, since $1 \in E$. Moreover, E is bounded from above, since 2 is an upper bound.

Assume by contradiction that there exists $L \in \mathbb{Q}$ such that $L = \sup E$. It cannot be $L \leq 0$, since $1 \in E$ and $1 > 0$. Hence, $L > 0$. Let's prove that it cannot be $L^2 < 2$. Choose $n \in \mathbb{N}$ so large that $n > \frac{2L+1}{2-L^2}$ (we will see later on that this can be done). Then

$$\left(L + \frac{1}{n}\right)^2 = L^2 + \frac{1}{n^2} + \frac{2L}{n} < L^2 + \frac{1}{n} + \frac{2L}{n} = L^2 + \frac{2L+1}{n} < 2,$$

by the choice of n. Hence, $L + \frac{1}{n}$ belongs to E, which contradicts the fact that L is an upper bound of E. Similarly, taking $L - \frac{1}{n}$, for n large, we can show that $\left(L - \frac{1}{n}\right)^2 > 2$ and $L - \frac{1}{n} > 0$. Let's prove that $L - \frac{1}{n}$ is an upper bound of E. If $x \in E$, then $x > 0$ and $x^2 < 2 < \left(L - \frac{1}{n}\right)^2$. Hence,

$$0 < \left[\left(L - \frac{1}{n}\right) + x\right] \cdot \left[\left(L - \frac{1}{n}\right) - x\right].$$

Since $L - \frac{1}{n} > 0$ and $x > 0$, we have that $\left[\left(L - \frac{1}{n}\right) + x\right] > 0$. It follows from the previous inequality that $0 < \left[\left(L - \frac{1}{n}\right) - x\right]$, that is $x < \left(L - \frac{1}{n}\right)$. Hence, $L - \frac{1}{n}$ is an upper bound of E, which contradicts the fact that L is the least upper bound of E. Hence, it cannot be $L^2 > 2$, Thus, $L^2 = 2$, which is again a contradiction by Theorem 21. ■

The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of *irrational numbers.*

Theorem 23 *The set of irrational numbers is nonempty.*

Proof. Take

$$E := \left\{x \in \mathbb{R} : 0 < x \text{ and } x^2 < 2\right\}.$$

Exactly as in the previous proof, we have that E is nonempty and bounded from above. Hence, by the supremum property there exists $L \in \mathbb{R}$ such that

$L = \sup E$. It follows as in the previous proof that $L^2 = 2$, and so L belongs to $\mathbb{R} \setminus \mathbb{Q}$. ■

The number L is denoted $\sqrt{2}$ and called *square root* of 2. Similarly, for every $n \in \mathbb{N}$ with n even and every $x \in \mathbb{R}$ with $x \geq 0$, we can show that there exists a unique $y \in \mathbb{R}$ with $y \geq 0$ such that $x^n = y$. On the other hand, for every $n \in \mathbb{N}$ with n odd and every $x \in \mathbb{R}$, we can show that there exists a unique $y \in \mathbb{R}$ such that $x^n = y$.

The number y is denoted $\sqrt[n]{x}$ and called *n-th root* of x.

Exercise 24 (The n-th root of a) *Given $a > 0$ and $n \in \mathbb{N}$, we want to define the n-th root of a.*

(i) Prove that the set

$$E := \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x > 0 \text{ and } x^n < a\}$$

is nonempty and bounded from above.

(ii) Let $L = \sup E$. Prove that $L^n = a$. Hint: Use the binomial theorem.

(iii) Prove that for every $x, y \in \mathbb{R}$, $x \neq 0$, $y \neq 0$, and $n \in \mathbb{N}$,

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\right).$$

(iii) Prove that L is the only positive solution of the equation $x^n = a$.

In Theorem 22 we have used the fact that there exist arbitrarily large natural numbers. This follows from the Archimedean Property.

Proposition 25 (Archimedean Property) *If $a, b \in \mathbb{R}$ with $a > 0$, then there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. If $b \leq 0$, then $n = 1$ will do. Thus, assume that $b > 0$. Assume by contradiction that $na \leq b$ for all $n \in \mathbb{N}$ and define the set

$$E = \{na : n \in \mathbb{N}\}.$$

Then the set E is nonempty and has an upper bound, b. By the supremum property, there exists $L = \sup E$. Hence, for every $m \in \mathbb{N}$, we have that $(m + 1)\, a \leq L$, or, equivalently, $ma \leq L - a$ for all $m \in \mathbb{N}$. But this shows that $L - a$ is an upper bound of E, which contradicts the fact that L is the least upper bound. ■

Exercise 26 *Prove that every nonempty subset of the integers bounded from below has a minimum.*

The next result is left as an exercise.

Theorem 27 (The integer part) *Given a real number $x \in \mathbb{R}$, there exists an integer $k \in \mathbb{Z}$ such that $k \leq x < k + 1$.*

Definition 28 *Given a real number $x \in \mathbb{R}$, the integer k given by the previous corollary is called the* integer part *of x and is denoted $\lfloor x \rfloor$. The number $x - \lfloor x \rfloor$ is called the* fractional part *of x and is denoted $\operatorname{fr} x$ (or $\{x\}$). Note that $0 \leq \operatorname{fr} x < 1$.*

Friday, September 7, 2012

Corollary 29 (Density of the rationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. We want to find $r \in \mathbb{Q}$ such that $a < r < b$. Any $r \in \mathbb{Q}$ can be written as $r = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. So we want to find $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$a < \frac{p}{q} < b,$$

or, equivalently,

$$qa < p < qb.$$

Suppose we have chosen q. Then to find p, we apply the previous corollary to obtain an integer $p \in \mathbb{Z}$ such that

$$p - 1 \leq qa < p. \tag{4}$$

We would like $p < qb$. By (4),

$$p \leq 1 + qa \overset{?}{<} qb,$$

provided $\frac{1}{b-a} < q$. Hence, we first use the Archimedean property (applied with 1 and $\frac{1}{b-a}$ in place of a and b) to find $q \in \mathbb{N}$ such that $\frac{1}{b-a} < q$ and then find p as above, namely, $p = \lfloor qa \rfloor + 1$. ■

Corollary 30 (Density of the irrationals) *If $a, b \in \mathbb{R}$ with $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.*

Proof. Since $a < b$, we have that $\sqrt{2}a < \sqrt{2}b$. By the density of the rationals, there exists $r \in \mathbb{Q}$ such that $\sqrt{2}a < r < \sqrt{2}b$. Without loss of generality, we may assume that $r \neq 0$ (why?). Hence, $a < \frac{r}{\sqrt{2}} < b$. Since $\frac{r}{\sqrt{2}}$ is irrational (why?), the result is proved. ■

4 Topological Properties of the Real Line

Given a number $x \in \mathbb{R}$, the *absolute value* of x is the number

$$|x| := \begin{cases} +x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The absolute value satisfies the following properties, which are left as as exercise.

Theorem 31 *Let $x, y, z \in \mathbb{R}$. Then the following properties hold.*

(i) $|x| \geq 0$ *for all* $x \in \mathbb{R}$, *with* $|x| = 0$ *if and only if* $x = 0$,

(ii) $|-x| = |x|$ *for all* $x \in \mathbb{R}$,

(iii) *if* $y \geq 0$ *and* $x \in \mathbb{R}$, *then* $|x| \leq y$ *if and only if* $-y \leq x \leq y$,

(iv) $-|x| \leq x \leq |x|$ *for all* $x \in \mathbb{R}$,

(iii) $|xy| = |x|\,|y|$ *for all* $x, y \in \mathbb{R}$,

(iv) $|x + y| \leq |x| + |y|$ *for all* $x, y \in \mathbb{R}$.

Given $r > 0$ and $x_0 \in \mathbb{R}$, the *ball* of center x_0 and radius r is the set

$$B(x_0, r) := \{x \in \mathbb{R} : |x_0 - x| < r\}.$$

A subset $U \subseteq \mathbb{R}$ is *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$.

Example 32 *Some simple examples of sets that are open and of some that are not.*

(i) The set $(a, \infty) = \{x \in \mathbb{R} : x > a\}$ *is open. Indeed, if* $x > a$, *take* $r := x - a > 0$. *Then* $B(x, r) \subset (a, \infty)$. *Similarly, the set* $(-\infty, a)$ *is open.*

(ii) The set $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ *is open. Indeed, given* $a < x < b$, *take* $r := \min\{b - x, x - a\} > 0$. *Then* $B(x, r) \subseteq (a, b)$.

(iii) The set $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ *is not open, since* b *belongs to the set but there is no ball* $B(b, r)$ *contained in* $(a, b]$.

Example 33 *Consider the set*

$$E = \mathbb{R} \setminus \left(\left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right).$$

Let's prove that E *is not open. The point* $x = 0$ *belongs to* E, *but for every* $r > 0$, *by the Archimedean principle we can find* $n \in \mathbb{N}$ *such that* $n > \frac{1}{r}$, *and so* $0 < \frac{1}{n} < r$, *which shows that* $\frac{1}{n} \in (-r, r)$. *Since* $\frac{1}{n}$ *does not belong to* E, *the ball* $(-r, r)$ *is not contained in* E *for* any $r > 0$. *Hence,* E *is not open.*

Example 34 *Consider the set*

$$U = \mathbb{R} \setminus \left(\{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\}\right).$$

Let's prove that U *is open. If* $x < 0$, *take* $r = -x > 0$, *then* $B(x, r) = (-2x, 0) \subseteq U$. *If* $x > 1$, *take* $r = x - 1$, *then* $B(x, r) = (1, 2x - 1) \subseteq U$. *If* $\frac{1}{n+1} < x < \frac{1}{n}$, *take* $r = \min\left\{\frac{1}{n} - x, x - \frac{1}{n+1}\right\} = \frac{1}{n+1}$, *then* $B(x, r) \subseteq U$. *Hence,* U *is open.*

Monday, September 10, 2012

The main properties of open sets are given in the next proposition.

Proposition 35 *The following properties hold:*

(i) $\emptyset$ *and* $\mathbb{R}$ *are open.*

(ii) If $U_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$, *is a finite family of open sets of* $\mathbb{R}$, *then* $U_1 \cap \cdots \cap U_n$ *is open.*

(iii) If $\{U_i\}_{i \in I}$ *is an arbitrary collection of open sets of* $\mathbb{R}$, *then* $\bigcup_{i \in I} U_i$ *is open.*

Proof. To prove (ii), let $x \in U_1 \cap \cdots \cap U_M$. Then $x \in U_i$ for every $i = 1, \ldots, n$, and since U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$. Take $r := \min\{r_1, \ldots, r_n\} > 0$. Then

$$B(x, r) \subseteq U_1 \cap \cdots \cap U_n,$$

which shows that $U_1 \cap \cdots \cap U_n$ is open.

To prove (iii), let $x \in U := \bigcup_{i \in I} U_i$. Then there is $i \in I$ such that $x \in U_i$ and since U_i is open, there exists $r > 0$ such that $B(x, r) \subseteq U_i \subseteq U$. This shows that U is open. ■

Properties (i)–(iii) are used to define topological spaces.

Definition 36 *Let* X *be a nonempty set and let* τ *be a family of sets of* X. *The pair* (X, τ) *is called a* topological space *if the following hold.*

(i) $\emptyset, X \in \tau$.

(ii) If $U_i \in \tau$ *for* $i = 1, \ldots, M$, *then* $U_1 \cap \ldots \cap U_M \in \tau$.

(iii) If $\{U_i\}_{i \in I}$ *is an arbitrary collection of elements of* τ, *then* $\bigcup_{i \in I} U_i \in \tau$.

Remark 37 *The intersection of infinitely many open sets is not open in general. Take* $U_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ *for* $n \in \mathbb{N}$. *Then*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

but $\{0\}$ *is not open. Indeed, for every* $r > 0$, *the ball* $(-r, r)$ *is not contained in* $\{0\}$.

Given a set $E \subseteq \mathbb{R}$, a point $x \in E$ is called an *interior point* of E if there exists $r > 0$ such that $B(x, r) \subseteq E$. The *interior* E° of a set $E \subseteq \mathbb{R}$ is the union of all its interior points.

The proof of following proposition is left as an exercise.

Proposition 38 *Let $E \subseteq \mathbb{R}$. Then*

(i) E° is an open subset of E,

(ii) E° is given by the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,

(iii) E is open if and only if $E = E^\circ$,

(iv) $(E^\circ)^\circ = E^\circ$.

Example 39 *Consider the set $E = [0,1)$. Then 0 is not an in interior point of E, so $E^\circ \subseteq (0,1)$. On the other hand, since $(0,1)$ is open and contained in E, by part (ii) of the previous proposition, $E^\circ \supseteq (0,1)$, which shows that $E^\circ = (0,1)$.*

Exercise 40 *Some properties of the interior.*

(i) Prove that if E, F are subsets of $\mathbb{R}$, then

$$E^\circ \cap F^\circ = (E \cap F)^\circ ,$$
$$E^\circ \cup F^\circ \subseteq (E \cup F)^\circ .$$

(ii) Show that in general $E^\circ \cup F^\circ \neq (E \cup F)^\circ$.

(iii) Let $\{E_i\}_{i\in I}$ be an arbitrary collection of sets of R. What is the relation, if any, between $\bigcap_{i\in I} (U_i)^\circ$ and $\left(\bigcap_{i\in I} U_i\right)^\circ$? And between $\bigcup_{i\in I} (U_i)^\circ$ and $\left(\bigcup_{i\in I} U_i\right)^\circ$?

A subset $C \subseteq \mathbb{R}$ is *closed* if its complement $\mathbb{R} \setminus C$ is open.
The main properties of closed sets are given in the next proposition.

Proposition 41 *The following properties hold:*

(i) $\emptyset$ and $\mathbb{R}$ are closed.

(ii) If $C_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$, is a finite family of closed sets of $\mathbb{R}$, then $C_1 \cup \cdots \cup C_n$ is closed.

(iii) If $\{C_i\}_{i\in I}$ is an arbitrary collection of closed sets of $\mathbb{R}$, then $\bigcap_{i\in I} C_i$ is closed.

Wednesday, September 12, 2012

The proof follows from Proposition 35 and De Morgan's laws. If $\{E_i\}_{i\in I}$ is an arbitrary collection of subsets of a set $\mathbb{R}$, then *De Morgan's laws* are

$$\mathbb{R} \setminus \left(\bigcup_{i\in I} E_i \right) = \bigcap_{i\in I} (\mathbb{R} \setminus E_i) ,$$
$$\mathbb{R} \setminus \left(\bigcap_{i\in I} E_i \right) = \bigcup_{i\in I} (\mathbb{R} \setminus E_i) .$$

Next we present a very useful proposition for identifying open an closed sets. We will prove it later on when we discuss about continuous functions.

Proposition 42 *Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be continuous.*

(i) If D is open, then $f^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}$.

(ii) If D is closed, then $f^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}$.

Example 43 *Consider the set*

$$E = \left\{ x \in \mathbb{R} : \sin x > \frac{1}{2}, \cos x < 1 \right\}.$$

To see if this is open or closed, let's rewrite E as follows

$$E = \left\{ x \in \mathbb{R} : \sin x > \frac{1}{2} \right\} \cap \{ x \in \mathbb{R} : \cos x < 1 \}.$$

The function $f(x) = \sin x$ is continuous and defined in $\mathbb{R}$, which is open. Note that

$$\left\{ x \in \mathbb{R} : \sin x > \frac{1}{2} \right\} = f^{-1}\left(\left(\frac{1}{2}, \infty \right) \right)$$

and since $\left(\frac{1}{2}, \infty\right)$ is open, by the previous proposition $f^{-1}\left(\left(\frac{1}{2}, \infty\right)\right)$ is open. Similarly, setting $g(x) = \cos x$, we have that the set

$$\{ x \in \mathbb{R} : \cos x < 1 \} = g^{-1}((-\infty, 1))$$

is open. Thus E is open, since intersection of two open sets.

Remark 44 *Note that the majority of sets are neither open nor closed. The set $E = (0, 1]$ is neither open nor closed.*

Given a set $E \subseteq \mathbb{R}$, the closure of E, denoted $\overline{E}$, is the intersection of all closed sets that contain E; in other words, the closure of E is the smallest (with respect to inclusion) closed set that contains E. It follows by Proposition 41 that $\overline{E}$ is closed.

The proof of following proposition is left as an exercise.

Proposition 45 *Let $C \subseteq \mathbb{R}$. Then C is closed if and only if $C = \overline{C}$.*

Proposition 46 *Let $E \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Then $x \in \overline{E}$ if and only if $B(x, r) \cap E \neq \emptyset$ for every $r > 0$.*

Proof. Let $x \in \overline{E}$ and assume by contradiction that there exists $r > 0$ such that $B(x, r) \cap E = \emptyset$. Since $B(x, r)$ is open and $B(x, r) \cap E = \emptyset$, it follows that $\mathbb{R} \setminus B(x, r)$ is closed and contains E. By the definition of $\overline{E}$ we have that $\overline{E} \subseteq \mathbb{R} \setminus B(x, r)$, which contradicts the fact that $x \in \overline{E}$.

Conversely, let $x \in \mathbb{R}$ and assume that $B(x, r) \cap E \neq \emptyset$ for every $r > 0$. We claim that $x \in \overline{E}$. Indeed, if not, then $x \in \mathbb{R} \setminus \overline{E}$, which is open. Thus, there exists $B(x, r) \subseteq \mathbb{R} \setminus \overline{E}$, which contradicts the fact that $B(x, r) \cap E \neq \emptyset$. ■

The previous proposition leads us to the definition of accumulation points.

Definition 47 *Given a set $E \subseteq \mathbb{R}$, a point $x \in \mathbb{R}$ is an* accumulation point*, or* cluster point *of E if for every $r > 0$ the ball $B(x, r)$ contains at least one point of E different from x.*

Note that x does not necessarily belong to the set E.

Example 48 *Consider the set*

$$E := \{2\}.$$

Then 2 is not an accumulation points of E. Indeed, for every $r > 0$, the ball $B(2, r)$ intersects E only at the point 2, and so in $B(2, r)$ there are no point of E different from 2. If $x \neq 2$, then x is not an accumulation point of E, since taking $r = |2 - x| > 0$, we have that the ball $B(x, |2 - x|)$ does not intersect E.

Example 49 *Consider the set*

$$E := \left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \cup \left\{1 + \frac{1}{n}\right\}_{n\in\mathbb{N}}.$$

We want to prove that 0 and 1 are accumulation points of E. Note that $0 \notin E$, while $1 \in E$ (so accumulation points may or may not be in the set E). For $r > 0$, by taking a natural number $n > \frac{1}{r}$, we have that $0 < \frac{1}{n} < r$, and so $\frac{1}{n} \in B(0, r) \cap E$ (of course $\frac{1}{n} \neq 0$). This shows that 0 is an accumulation points of E.

Similarly, for $r > 0$, by taking a natural number $n > \frac{1}{r}$, we have that $0 < \frac{1}{n} < r$, and so $1 < 1 + \frac{1}{n} < 1 + r$, which shows that $1 + \frac{1}{n} \in B(1, r) \cap E$ (of course $1 + \frac{1}{n} \neq 1$). This shows that 1 is an accumulation points of E. Next we show that there are no other accumulation points of E.

Indeed, if $x < 0$, take $r = -x > 0$, then $B(x, r) = (-2x, 0)$ does not intersect E. If $x > 2$, take $r = x - 1$, then $B(x, r) = (1, 2x - 1)$ does not intersect E.

If $\frac{1}{n+1} < x < \frac{1}{n}$, take $r = \min\left\{\frac{1}{n} - x, x - \frac{1}{n+1}\right\}$, then $B(x, r)$ does not intersect E. If $x = \frac{1}{n}$, with $n > 1$, take $r = \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\}$. Then $B(x, r)$ intersects E only in $\frac{1}{n}$. Hence, U is open.

If $1 + \frac{1}{n+1} < x < 1 + \frac{1}{n}$, take $r = \min\left\{1 + \frac{1}{n} - x, x - \left(1 + \frac{1}{n+1}\right)\right\}$, then $B(x, r)$ does not intersect E. If $x = 1 + \frac{1}{n}$, take $r = \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\}$. Then $B(x, r)$ intersects E only in $1 + \frac{1}{n}$.

The set of all accumulation points of E is denoted $\operatorname{acc} E$.

Remark 50 *Note take if $x \in \mathbb{R}$ is an accumulation point of E, then by taking $r = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a sequence $\{x_n\} \subseteq E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $|x_n - x| < \frac{1}{n} \to 0$. Thus $\{x_n\}$ converges to x. Conversely, if there exists a sequence $\{x_n\} \subseteq E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $|x_n - x| \to 0$, then x is an accumulation point of E.*

Friday, September 14, 2012

It turns out that the closure of a set is given by the set and all its accumulations points.

Proposition 51 *Let* $E \subseteq \mathbb{R}$ *Then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, a set $C \subseteq \mathbb{R}$ *is closed if and only if* C *contains all its accumulation points.*

Proof. Let $x \in \overline{E}$ and assume by contradiction that $x \notin E \cup \operatorname{acc} E$. Since $x \notin \operatorname{acc} E$, then there exists a ball $B(x, r)$ that contains no other point of E other than x, but since $x \notin E$, it follows that $B(x, r) \subseteq \mathbb{R} \setminus E$. This contradicts Proposition 46.

Conversely, let $x \in E \cup \operatorname{acc} E$. If $x \in E$, then since $E \subseteq \overline{E}$, there is nothing to prove. If $x \in \operatorname{acc} E$, then the result follows from Proposition 46. ■

Exercise 52 *(i) Prove that if* $E_1, \ldots, E_n$ *are subsets of* $\mathbb{R}$*, then*

$$\overline{E_1} \cap \cdots \cap \overline{E_n} \supseteq \overline{E_1 \cap \cdots \cap E_n},$$
$$\overline{E_1} \cup \cdots \cup \overline{E_n} = \overline{E_1 \cup \cdots \cup E_n}.$$

(ii) Show that in general $\overline{E_1} \cap \cdots \cap \overline{E_n} \neq \overline{E_1 \cap \cdots \cap E_n}$.

(iii) Let $\{E_i\}_{i \in I}$ *be an arbitrary collection of sets of* $\mathbb{R}$*. What is the relation, if any, between between* $\bigcap_{i \in I} \overline{U_i}$ *and* $\overline{\bigcap_{i \in I} U_i}$*? And between* $\bigcup_{i \in I} \overline{U_i}$ *and* $\overline{\bigcup_{i \in I} U_i}$*?*

Theorem 53 (Bolzano–Weierstrass) *Every bounded set* $E \subset \mathbb{R}$ *with infinitely many elements has at least one accumulation point.*

Proof. Since E is bounded, let $b \in \mathbb{R}$ be its supremum and $a \in \mathbb{R}$ be its infimum. Then $E \subseteq [a, b]$. Divide $[a, b]$ into two closed intervals of equal length, $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Since E has infinitely many elements, at least one of these two closed intervals contains infinitely many elements of E. Let's call this closed interval $I_1 = [a_1, b_1]$ (if both interval do the job, just pick one). Then $[a_1, b_1] \subset [a, b]$, $b_1 - a_1 = \frac{b-a}{2}$ and I_1 contains infinitely many elements of E.

Divide $[a_1, b_1]$ into two closed intervals of equal length. Since E has infinitely many elements, at least one of these two closed intervals contains infinitely many elements of E. Let's call this closed interval $I_2 = [a_2, b_2]$ (if both interval do the job, just pick one). Then $[a_2, b_2] \subset [a_1, b_1]$, $b_2 - a_2 = \frac{b-a}{2^2}$ and I_2 contains infinitely many elements of E. By induction, we construct a sequence of intervals $[a_n, b_n]$, $n \in \mathbb{N}$, with

$$\cdots \subset [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a_1, b_1] \subset [a, b],$$

and for every $n \in \mathbb{N}$, $b_n - a_n = \frac{b-a}{2^n}$ and $[a_n, b_n]$ contains infinitely many elements of E. Note that

$$a \le a_1 \le \cdots \le a_n \le a_{n+1} \le \cdots , \tag{5}$$
$$b \ge b_1 \ge \cdots \ge b_n \ge b_{n+1} \ge \cdots . \tag{6}$$

Let

$$A := \{a_1, \ldots, a_n, \ldots\},$$
$$B := \{b_1, \ldots, b_n, \ldots\}.$$

We claim that $\sup A \le \inf B$. To see this, let $n, m \in \mathbb{N}$ and find $k \ge m, n$. Then by (5) and (6),

$$a_n \le a_k \le b_k \le b_m.$$

Hence, $a_n \le b_m$ for all $n, m \in \mathbb{N}$. Taking the supremum over all $n \in \mathbb{N}$, we get that

$$\sup A = \sup_{n \in \mathbb{N}} a_n \le b_m$$

for all $m \in \mathbb{N}$. Taking the infimum over all $m \in \mathbb{N}$, we get that $\sup A \le \inf B$.

Next we claim that $\sup A = \inf B$. Since $\sup A \ge a_n$ for all $n \in \mathbb{N}$ and $\inf B \le b_n$ for all $n \in \mathbb{N}$, we have that

$$0 \le t := \inf B - \sup A \le b_n - a_n = \frac{b-a}{2^n}$$

for all $n \in \mathbb{N}$. If $t > 0$, by (1), we have that

$$1 + n \le 2^n \le \frac{b-a}{t}$$

for all $n \in \mathbb{N}$, which contradicts the Archimedean property. This shows that

$$x_0 := \sup A = \inf B.$$

Finally, we prove that x_0 is an accumulation point of E. Fix $r > 0$ and consider the ball $B(x_0, r)$. Since $x_0 - r < x_0 = \sup A$, we have that $x_0 - r$ is not an upper bound of A, and so there exists a_n such that

$$x_0 - r < a_n.$$

On the other hand, since $x_0 = \inf B < x_0 + r$, we have that $x_0 + r$ is not a lower bound of B, and so there exists b_m such that

$$b_m < x_0 + r.$$

Let $k \ge m, n$. Then by (5) and (6),

$$x_0 - r < a_n \le a_k \le b_k \le b_m < x_0 + r.$$

This shows that $[a_k, b_k] \subset B(x_0, r)$. Since $[a_k, b_k]$ contains infinitely many elements of E, the same holds for $B(x_0, r)$ and so x_0 is an accumulation point of E. ■

Monday, September 17, 2012

Definition 54 *Given a set $E \subseteq \mathbb{R}$, a point $x \in \mathbb{R}$ is a* boundary point *of E if for every $r > 0$ the ball $B(x, r)$ contains at least one point of E and one point of $\mathbb{R} \setminus E$. The set of boundary points of E is denoted ∂E.*

The following theorem is left as an exercise.

Theorem 55 *Let $E \subseteq \mathbb{R}$. Then*

(i) $\overline{E} = E \cup \partial E$,

(ii) E is closed if and only if it contains all its boundary points,

(iii) $\partial E = \partial(\mathbb{R} \setminus E)$,

(iv) $\partial E = \overline{(\mathbb{R} \setminus E)} \cap \overline{E}$.

5 Sequences

Definition 56 *A* sequence *of real numbers is a function*

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{R} \\ n &\mapsto f(n) \end{aligned}$$

from the natural numbers to the real numbers.

If $f(n) = x_n$ for $n = 1, 2, \ldots$, usually we denote the sequence f by the symbol $\{x_n\}$ or $\{x_n\}_{n \in \mathbb{N}}$ or $x_1, x_2, \ldots$.

Definition 57 *We say that a sequence $\{x_n\}$ of real numbers* converges *to a number $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ such that*

$$|x_n - \ell| \leq \varepsilon$$

for all $n \geq N$. In this case we say that ℓ is the limit *of $\{x_n\}$ and we write*

$$\lim_{n\to\infty} x_n = \ell \quad \text{or} \quad x_n \to \ell.$$

Let's see some examples

Example 58 *Consider the limit $\lim_{n\to\infty} \frac{n^2+\sin n}{n^2+2n-1}$. We have*

$$\lim_{n\to\infty} \frac{n^2 + \sin n}{n^2 + 2n - 1} = \lim_{n\to\infty} \frac{n^2\left(1 + \frac{\sin n}{n^2}\right)}{n^2\left(1 + \frac{2}{n} - \frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{1 + \frac{\sin n}{n^2}}{1 + \frac{2}{n} - \frac{1}{n^2}} = \frac{1 + 0}{1 + 0 - 0} = 1.$$

Let's prove it using the previous definition. Fix $\varepsilon > 0$, we want to find $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\left|\frac{n^2 + \sin n}{n^2 + 2n - 1} - 1\right| \leq \varepsilon$$

for all $n \geq N$. *We have*

$$\left|\frac{n^2+\sin n}{n^2+2n-1}-1\right| = \left|\frac{n^2+\sin n-n^2-2n+1}{n^2+2n-1}\right|$$
$$= \frac{2n-1-\sin n}{n^2+2n-1} \leq \frac{4n}{n^2+0} = \frac{4}{n} \leq \varepsilon$$

for all $n \geq \frac{4}{\varepsilon}$, *where we have used the fact that* $2n-2 \geq 0$. *It is enough to take* $N = \left\lfloor \frac{4}{\varepsilon} \right\rfloor + 1$.

Wednesday, September 19, 2012

Next we discuss some important limits.

Example 59 *Let* $x > 0$. *We want to calculate*

$$\lim_{n\to\infty} \sqrt[n]{x}.$$

Consider first the case $x > 1$. *Then* $\sqrt[n]{x} > 1$ *and so we may write*

$$\sqrt[n]{x} = 1 + a_n,$$

where $a_n > 0$. *By inequality (1),*

$$x = (1+a_n)^n \geq 1 + na_n,$$

which implies that

$$0 < a_n \leq \frac{x-1}{n}.$$

This implies that $a_n \to 0$ *as* $n \to \infty$. *Indeed, given* $\varepsilon > 0$, *we have that* $\frac{x-1}{n} \leq \varepsilon$ *for all* $n \geq n \geq \frac{x-1}{\varepsilon}$. *It is enough to take* $N = \left\lfloor \frac{x-1}{\varepsilon} \right\rfloor + 1$. *In turn,* $\sqrt[n]{x} = 1 + a_n \to 1 + 0$, *which shows that*

$$\lim_{n\to\infty} \sqrt[n]{x} = 1$$

for $x > 1$. *If* $x = 1$, *then* $\sqrt[n]{1} = 1 \to 1$ *as* $n \to \infty$.

Consider next the case $0 < x < 1$. *Then* $\sqrt[n]{x} < 1$ *and so we may write*

$$\sqrt[n]{x} = \frac{1}{1+a_n},$$

where $a_n > 0$. *Hence,* $x = \frac{1}{(1+a_n)^n}$. *By inequality (1),*

$$(1+a_n)^n \geq 1 + na_n,$$

which implies that

$$x = \frac{1}{(1+a_n)^n} \leq \frac{1}{1+na_n},$$

and so

$$0 < a_n \le \frac{1 - \frac{1}{x}}{n}.$$

This implies that $a_n \to 0$ *as* $n \to \infty$. *Indeed, given* $\varepsilon > 0$, *we have that* $\frac{1-\frac{1}{x}}{n} \le \varepsilon$ *for all* $n \ge \frac{1-\frac{1}{x}}{\varepsilon}$. *In turn,* $\sqrt[n]{x} = \frac{1}{1+a_n} \to \frac{1}{1+0}$, *which shows that*

$$\lim_{n\to\infty} \sqrt[n]{x} = 1.$$

Definition 60 *We say that a sequence* $\{x_n\}$ *of real numbers* diverges to plus infinity *if for any real number* $M > 0$ *there exists an integer* $N = N(M)$ *such that for all* $n \ge N$ *we have*

$$x_n > M. \tag{7}$$

In this case we write

$$\lim_{n\to\infty} x_n = \infty \quad \text{or} \quad x_n \to \infty.$$

Similarly we say that a sequence $\{x_n\}$ *of real numbers* diverges to minus infinity *if for any real number* $M > 0$ *there exists an integer* $N = N(M)$ *such that for all* $n \ge N$ *we have*

$$x_n < -M. \tag{8}$$

In this case we write

$$\lim_{n\to\infty} x_n = -\infty \quad \text{or} \quad x_n \to -\infty.$$

If $\{x_n\}$ *does not converge and does not diverge to plus infinity or to minus infinity, we say that it* oscillates.

Definition 61 *Given a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *of real numbers and a strictly increasing sequence of natural numbers* $\{n_k\}_{k\in\mathbb{N}}$ *(that is* $n_1 < n_2 < \ldots$*) the sequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *is called a* subsequence *of* $\{x_n\}$.

Theorem 62 *A sequence* $\{x_n\}$ *of real numbers converges to* ℓ *if and only if* every *subsequence* $\{x_{n_k}\}$ *converges to* ℓ.

We will prove this theorem later on when we study the liminf and limsup of a sequence.

Example 63 *This theorem is very useful to show that a sequence diverges. It is enough to find two subsequences which converge to different numbers. As an example, the sequence* $\{(-1)^n\}$ *is divergent for both* $1, 1, \ldots$ *and* $-1, -1, \ldots$ *are subsequences and converge to different limits.*

Exercise 64 *Let* $0 < \theta < 1$ *be a rational number. Show that the sequence* $\{\sin(n\pi\theta)\}$ *oscillates.*

Example 65 *If $x \in \mathbb{R}$ then*

$$\lim_{n\to\infty} x^n = \begin{cases} \infty & \text{if } x > 1, \\ 1 & \text{if } x = 1, \\ 0 & \text{if } -1 < x < 1, \\ \text{oscillates} & \text{if } x \le -1. \end{cases}$$

If $x > 1$, then we can write $x = 1 + a$, where $a > 0$. By inequality (1),

$$x^n = (1+a)^n \ge 1 + na.$$

Hence, for any $M > 0$, by taking $n \ge \frac{M-1}{a}$, we have that $x^n \ge M$, which implies that $\lim_{n\to\infty} x^n = 1$.

If $x = 1$, then $x^n = 1^n = 1 \to 1$ as $n \to \infty$. If $0 < x < 1$, then we can write $x = \frac{1}{1+a}$, where $a > 0$, and so again by inequality (1),

$$0 \le x^n = \frac{1}{(1+a)^n} \le \frac{1}{1+na}.$$

Hence, given $\varepsilon > 0$, by taking $n \ge \frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$, we have that $0 \le x^n \le \varepsilon$, which implies that $\lim_{n\to\infty} x^n = 0$. The case $-1 < x < 0$ is similar and is left as an exercise.

If $x = -1$, we have already seen that the limit does not exist. If $x < -1$, then for $n = 2k$, we have that

$$x^n = (x)^{2k} = (-x)^{2k} \to \infty$$

as $k \to \infty$, since $-x > 1$. If instead we take $n = 2k+1$, then

$$x^n = (x)^{2k+1} = -(-x)^{2k+1} \to -\infty$$

as $k \to \infty$, since $-x > 1$. Thus, the limit $\lim_{n\to\infty} x^n$ does not exists.

Exercise 66 *Prove that*

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Exercise 67 *Prove that*

$$\lim_{n\to\infty} \frac{n}{x^n} = \begin{cases} 0 & \text{if } |x| > 1, \\ \infty & \text{if } 0 < x \le 1, \\ \text{does not exist} & \text{if } -1 \le x < 0. \end{cases}.$$

Friday, September 21, 2012

Definition 68 *We say that a sequence $\{x_n\}$ of real numbers is*

(i) bounded from above *if there exists $M \in \mathbb{R}$ such that $x_n \le M$ for all $n \in \mathbb{N}$;*

(ii) bounded from below *if there exists $m \in \mathbb{R}$ such that $x_n \ge m$ for all $n \in \mathbb{N}$;*

(iii) bounded *if it is bounded from above and from below.*

Theorem 69 *Let $\{x_n\} \subset \mathbb{R}$ be a sequence.*

(i) If there exists $\lim_{n\to\infty} x_n = \ell \in \mathbb{R}$, then $\{x_n\}$ is bounded.

(ii) If the limit of $\{x_n\}$ exists, it is unique.

Proof. (i) Let $\varepsilon = 1$, then there exists an integer $N = N(1)$ such that

$$|x_n - \ell| \leq 1$$

for all $n \geq N$. Thus, $|x_n| = |x_n \pm \ell| \leq |x_n - \ell| + |\ell| \leq 1 + |\ell|$. Hence, it suffices to take

$$M = \max\{|x_1|, \ldots, |x_N|, 1 + |\ell|\}.$$

(ii) Let $\lim_{n\to\infty} x_n = \ell$ and $\lim_{n\to\infty} x_n = L$, with $L \neq \ell$. If one of the two is a real number, say $\ell \in \mathbb{R}$, then the sequence is bounded by part (i), and so also L must be a real number. Let $0 < \varepsilon < \frac{1}{2}|L - \ell|$. By the definition of limit, there exist an integer $N_1 = N_1(\varepsilon)$ such that

$$|x_n - \ell| \leq \varepsilon$$

for all $n \geq N_1$ and an integer $N_2 = N_2(\varepsilon)$ such that

$$|x_n - L| \leq \varepsilon$$

for all $n \geq N_2$. Then for $n \geq \max\{N_1, N_2\}$, we have that

$$|L - \ell| = |L \pm x_n - \ell| \leq |x_n - \ell| + |x_n - L| \leq \varepsilon + \varepsilon < |L - \ell|,$$

which is a contradiction. It remains the case in which one limit is $+\infty$ and the other $-\infty$, which again gives a contradiction (take $M = 1$ in (7) and (8)). ■

Theorem 70 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that there exist*

$$\lim_{n\to\infty} x_n = \ell_1 \in \mathbb{R} \quad \text{and} \quad \lim_{n\to\infty} y_n = \ell_2 \in \mathbb{R}.$$

Then

(i) there exists $\lim_{n\to\infty} (x_n + y_n) = \ell_1 + \ell_2$,

(ii) there exists $\lim_{n\to\infty} x_n y_n = \ell_1 \ell_2$,

(iii) if $y_n \neq 0$ for all $n \in \mathbb{N}$ and $\ell_2 \neq 0$ then there exists $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\ell_1}{\ell_2}$.

Proof. (i) and (ii) are left as an exercise. (iii) Write

$$\begin{aligned}\left|\frac{x_n}{y_n}-\frac{\ell_1}{\ell_2}\right| &= \left|\frac{x_n\ell_2-y_n\ell_1}{y_n\ell_2}\right| = \left|\frac{x_n\ell_2\pm\ell_1\ell_2-y_n\ell_1}{y_n\ell_2}\right| \\ &= \frac{1}{|y_n|\,|\ell_2|}\left|\ell_2\left(x_n-\ell_1\right)+\ell_1\left(y_n-\ell_2\right)\right| \\ &\le \frac{1}{|y_n|}\left|x_n-\ell_1\right|+\frac{1}{|y_n|}\frac{|\ell_1|}{|\ell_2|}\left|y_n-\ell_2\right|.\end{aligned}$$

Thus we need to bound $\frac{1}{|y_n|}$ from above, or, equivalently, we need $|y_n|$ to stay away from zero. Since $\ell_2\neq 0$, taking $\varepsilon=\frac{|\ell_2|}{2}>0$, there exist an integer $N_1=N_1(\varepsilon)$ such that

$$|y_n-\ell_2|\le\frac{|\ell_2|}{2}$$

for all $n\ge N_1$. Hence,

$$|y_n|=|y_n\pm\ell_2|\ge|\ell_2|-|y_n-\ell_2|\ge|\ell_2|-\frac{|\ell_2|}{2}=\frac{|\ell_2|}{2}$$

for all $n\ge N_1$. It follows that

$$\begin{aligned}\left|\frac{x_n}{y_n}-\frac{\ell_1}{\ell_2}\right| &\le \frac{1}{|y_n|}\left|x_n-\ell_1\right|+\frac{1}{|y_n|}\frac{|\ell_1|}{|\ell_2|}\left|y_n-\ell_2\right| \\ &\le \frac{2}{|\ell_2|}\left|x_n-\ell_1\right|+\frac{2}{|\ell_2|}\frac{|\ell_1|}{|\ell_2|}\left|y_n-\ell_2\right|.\end{aligned}$$

Given $\varepsilon>0$ there exist an integer $N_2=N_2(\varepsilon)$ such that

$$|x_n-\ell_1|\le\frac{\varepsilon\,|\ell_2|}{4}$$

for all $n\ge N_2$ and an integer $N_3=N_3(\varepsilon)$ such that

$$|y_n-\ell_2|\le\frac{\varepsilon\,|\ell_2|^2}{4\,(1+|\ell_1|)}$$

for all $n\ge N_3$. Then for $n\ge\max\{N_1,N_2,N_3\}$, we have that

$$\begin{aligned}\left|\frac{x_n}{y_n}-\frac{\ell_1}{\ell_2}\right| &\le \frac{2}{|\ell_2|}\left|x_n-\ell_1\right|+\frac{2}{|\ell_2|}\frac{|\ell_1|}{|\ell_2|}\left|y_n-\ell_2\right| \\ &\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\frac{|\ell_1|}{1+|\ell_1|}\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}1=\varepsilon.\end{aligned}$$

This completes the proof. ■

The following theorem is left as an exercise.

Theorem 71 *Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers.*

(i) If there exists $\lim_{n\to\infty} x_n = \infty$ *and* $\{y_n\}$ *is bounded from below, then there exists* $\lim_{n\to\infty} (x_n + y_n) = \infty$.

(ii) If there exists $\lim_{n\to\infty} x_n = -\infty$ *and* $\{y_n\}$ *is bounded from above, then there exists* $\lim_{n\to\infty} (x_n + y_n) = -\infty$.

(iii) If there exist $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to\infty} y_n = \ell \in [-\infty, \infty]$ *with* $\ell \neq 0$, *then there exists* $\lim_{n\to\infty} (x_n y_n) = (\operatorname{sgn} \ell)\, \infty$.

(iv) If there exist $\lim_{n\to\infty} x_n = \infty$, *then there exists* $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

(v) If $x_n \neq 0$ *for all* $n \in \mathbb{N}$ *sufficiently large and there exist* $\lim_{n\to\infty} x_n = 0$, *then* .

$$\lim_{n\to\infty} \frac{1}{x_n} = \begin{cases} +\infty & \text{if } x_n > 0 \text{ for all } n \text{ sufficiently large,} \\ -\infty & \text{if } x_n < 0 \text{ for all } n \text{ sufficiently large,} \\ \text{does not exist} & \text{otherwise.} \end{cases}$$

Remark 72 (Important) *In any of the following cases, nothing can be said without further investigation.*

(i) If there exist $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to\infty} y_n = -\infty$, *then the limit* $\lim_{n\to\infty} (x_n + y_n)$ *needs further investigation.*

(ii) If there exist $\lim_{n\to\infty} x_n = \infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty} y_n = 0$, *then the limit* $\lim_{n\to\infty} (x_n y_n)$ *needs further investigation.*

(iii) If there exist $\lim_{n\to\infty} x_n = \infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty} y_n = 1$, *then the limit* $\lim_{n\to\infty} (y_n)^{x_n}$ *needs further investigation.*

(iv) If $y_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exist* $\lim_{n\to\infty} x_n = \infty$ *(or* $-\infty$*) and* $\lim_{n\to\infty} y_n = \infty$ *(or* $-\infty$*), then the limit* $\lim_{n\to\infty} \frac{x_n}{y_n}$ *needs further investigation.*

(v) If $y_n \neq 0$ *for all* $n \in \mathbb{N}$ *and there exist* $\lim_{n\to\infty} x_n = 0$ *and* $\lim_{n\to\infty} y_n = 0$, *then the limit* $\lim_{n\to\infty} \frac{x_n}{y_n}$ *needs further investigation.*

Example 73 *An important limit is*

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

We will prove it later (see Example **??***).*

Monday, September 24, 2012

The next theorem gives the precise speed at which $n!$ goes to infinity.

Theorem 74 (Stirling's Formula) *The following holds*

$$\lim_{n\to\infty}\frac{n!}{n^n e^{-n}\sqrt{2\pi n}}=1.$$

Second proof. Consider the function $f(x)=\log x$, for $x>0$. Then $f'(x)=\frac{1}{x}$ and $f''(x)=-\frac{1}{x^2}<0$, so the function f is convex, and the slope of f is positive and decreasing. Hence, the area of f in the interval $[n,n+1]$ is greater than the area of the trapezoid of vertices, $A=(n,0)$, $D=(n+1,0)$, $B=(n,\log n)$, $C=(n+1,\log(n+1))$, that is,

$$\frac{1}{2}\left[\log n+\log(n+1)\right]\le\int_n^{n+1}\log x\,dx. \tag{9}$$

Consider the line tangent to the curve at the point B. Its equation is

$$y-\log n=\frac{1}{n}(x-n).$$

This line intersects the vertical line $x=n+1$ at the point $G=\left(n+1,\log n+\frac{1}{n}\right)$. Since the slope of f is increasing, he area of f in the interval $[n,n+1]$ is less than the area of the trapezoid $ABGD$, that is,

$$\int_n^{n+1}\log x\,dx\le\frac{1}{2}\left[\log n+\log n+\frac{1}{n}\right]. \tag{10}$$

Similarly, consider the line tangent to the curve at the point C. Its equation is

$$y-\log(n+1)=\frac{1}{n+1}(x-n-1).$$

This line intersects the vertical line $x=n$ at the point $E=\left(n,\log(n+1)-\frac{1}{n+1}\right)$. Since the slope of f is increasing, the area of f in the interval $[n,n+1]$ is less than the area of the trapezoid $AECD$, that is,

$$\int_n^{n+1}\log x\,dx\le\frac{1}{2}\left[\log(n+1)+\log(n+1)-\frac{1}{n+1}\right]. \tag{11}$$

Summing the inequalities and dividing by 2 gives

$$\int_n^{n+1}\log x\,dx\le\frac{1}{4}\left[2\log n+2\log(n+1)+\frac{1}{n}-\frac{1}{n+1}\right].$$

Hence, also by (9),

$$\frac{1}{2}\left[\log n+\log(n+1)\right]\le\int_n^{n+1}\log x\,dx\le\frac{1}{2}\left[\log n+\log(n+1)+\frac{1}{2n}-\frac{1}{2(n+1)}\right].$$

Summing between 1 and $m-1$ gives

$$\begin{aligned}
\log m! - \frac{1}{2}\log m &= \sum_{n=1}^{m-1} \frac{1}{2}\left[\log n + \log(n+1)\right] \\
&\le \int_1^m \log x\, dx = \left[x(\log x - 1)\right]_1^m = m\log m - m + 1 \\
&\le \sum_{n=1}^{m-1} \frac{1}{2}\left[\log n + \log(n+1) + \frac{1}{2n} - \frac{1}{2(n+1)}\right] \\
&= \log m! - \frac{1}{2}\log m + \frac{1}{4} - \frac{1}{4m}.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 \le c_m &:= \int_1^{m-1} \log x\, dx - \left(\log m! - \frac{1}{2}\log m\right) \\
&= \left(m + \frac{1}{2}\right)\log m - \log m! - m + 1 \le \frac{1}{4} - \frac{1}{4m}.
\end{aligned}$$

Since c_m is given by the difference between the area of f between $x = 1$ and $x = m$ and the trapezoidal approximation to this area, we have that the sequence $\{c_m\}$ is increasing. Hence, by a theorem that we will prove later on, there exists

$$\lim_{m\to\infty} c_m = c \in \left(0, \frac{1}{4}\right].$$

Raising everything to the power e we have that

$$e^{c_m} = e^{\left(m+\frac{1}{2}\right)\log m - \log m! - m} e = \frac{m^m \sqrt{m} e^{-m}}{m!} e \to e^c.$$

That is,

$$\lim_{m\to\infty} \frac{m!}{m^m e^{-m}\sqrt{m}} = e^{1-c} =: \ell. \tag{12}$$

To find the exact value of c we use your homework. Let

$$I_n := \int_0^{\pi/2} (\sin x)^n \, dx$$

Since in $\left[0, \frac{\pi}{2}\right]$,

$$(\sin x)^{n+1} \le (\sin x)^n,$$

we have

$$I_{2k+1} < I_{2k} < I_{2k-1}.$$

Hence,

$$1 \le \frac{I_{2k}}{I_{2k+1}} \le \frac{I_{2k-1}}{I_{2k+1}} = \frac{2k+1}{2k},$$

where we have used the formula $nI_n = (n-1)\, I_{n-2}$. It follows by the squeeze theorem that

$$\lim_{k\to\infty} \frac{I_{2k}}{I_{2k+1}} = 1.$$

On the other hand, by your homework again,

$$I_{2k} = \frac{\pi}{2} \prod_{i=1}^{k} \frac{2i-1}{2i},$$
$$I_{2k+1} = \prod_{i=1}^{k} \frac{2i}{2i+1},$$

and so

$$\frac{I_{2k}}{I_{2k+1}} = \frac{\frac{\pi}{2}\prod_{i=1}^{k} \frac{2i-1}{2i}}{\prod_{i=1}^{k} \frac{2i}{2i+1}} = \frac{\pi}{2} \prod_{i=1}^{k} \frac{(2i-1)\,(2i+1)}{(2i)^2}$$
$$= \frac{\pi}{2} \frac{\left[(2k)!\right]^2 (2k+1)}{\left(2^k k!\right)^4} \to 1.$$

Using (12), we finally have

$$1 = \lim_{k\to\infty} \frac{\pi}{2} \frac{\left[(2k)!\right]^2 (2k+1)}{\left(2^k\right)^4 (k!)^4} = \lim_{k\to\infty} \frac{\pi}{2} \frac{\left(\ell\,(2k)^{2k} e^{-2k}\sqrt{2k}\right)^2 (2k+1)}{\left(2^k\right)^4 \left(\ell k^k e^{-k}\sqrt{k}\right)^4}$$
$$= \lim_{k\to\infty} \frac{\pi}{2} \frac{2k\,(2k+1)}{\ell^2 k^2} = \frac{\pi}{2}\frac{4}{\ell^2},$$

and so $\ell = \sqrt{2\pi}$. ■

Remark 75 *Stirling's formula shows that* $n!$ *goes to infinity slower than* n^n *but faster that* b^n*. Indeed,*

$$\frac{n!}{n^n} = \frac{n!}{n^n} \frac{e^{-n}\sqrt{2\pi n}}{e^{-n}\sqrt{2\pi n}} = \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \frac{\sqrt{2\pi} n^{\frac{1}{2}}}{e^n} \to 1 \cdot 0,$$

where we have used the fact that $\frac{n^{\frac{1}{2}}}{e^n} \to 0$ *(since* $e > 1$*). On the other hand, if* $b > 1$*,*

$$\frac{b^n}{n!} = \frac{b^n}{n!} \frac{n^n e^{-n}\sqrt{2\pi n}}{n^n e^{-n}\sqrt{2\pi n}} = \frac{n^n e^{-n}\sqrt{2\pi n}}{n!} \left(\frac{be}{n}\right)^n \frac{1}{n^{\frac{1}{2}}\sqrt{2\pi}} \to 1 \cdot 0,$$

where we have used the fact that $\left(\frac{be}{n}\right)^n \to 0$ *(why?).*

Wednesday, September 26, 2012

Example 76 *Let's calculate*

$$\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n.$$

We have

$$\left(\frac{n+1}{n+2}\right)^n=\left(\frac{1}{\frac{n+2}{n+1}}\right)^n=\left[\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\right]^{\frac{n}{n+1}}\to e^{-1},$$

where we have used the previous limit and the fact that $\frac{n}{n+1}=\frac{n}{n\left(1+\frac{1}{n}\right)}=\frac{1}{1+\frac{1}{n}}\to 1$.

Theorem 77 (Squeeze) *Let* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be three sequences of real numbers such that*

$$x_n\le y_n\le z_n$$

for all $n\in\mathbb{N}$.

(i) If there exist $\lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=\ell\in\mathbb{R}$, *then there exists* $\lim_{n\to\infty}y_n=\ell$.

(ii) If there exists $\lim_{n\to\infty}x_n=\infty$, *then there exists* $\lim_{n\to\infty}y_n=\infty$.

(iii) If there exists $\lim_{n\to\infty}z_n=-\infty$, *then there exists* $\lim_{n\to\infty}y_n=-\infty$.

Proof. We prove (i) and leave (ii) and (iii) as an exercise. Given $\varepsilon>0$, there exist an integer $N_1=N_1(\varepsilon)$ such that

$$\ell-\varepsilon\le x_n\le\ell+\varepsilon$$

for all $n\ge N_1$ and an integer $N_2=N_2(\varepsilon)$ such that

$$\ell-\varepsilon\le z_n\le\ell+\varepsilon$$

for all $n\ge N_2$. Then for $n\ge\max\{N_1,N_2\}$, we have that

$$\ell-\varepsilon\le x_n\le y_n\le z_n\le\ell+\varepsilon,$$

which implies that $|y_n-\ell|\le\varepsilon$. ■

Corollary 78 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of real numbers such that there exists* $\lim_{n\to\infty}x_n=0$ *and* $\{y_n\}$ *is bounded. Then there exists* $\lim_{n\to\infty}(x_ny_n)=0$.

Proof. Since $\lim_{n\to\infty}x_n=0$, we have that $\lim_{n\to\infty}|x_n|=0$. Since $\{y_n\}$ is bounded, there exists $M>0$ such that $|y_n|\le M$ for all $n\in\mathbb{N}$. Hence,

$$-M|x_n|\le x_ny_n\le M|x_n|.$$

We are in a position to apply the squeeze theorem, since $-M|x_n|\to 0$ and $M|x_n|\to 0$. ■

Example 79 *Consider*

$$\lim_{n\to\infty} \frac{\sin^4 n}{n^3}.$$

Since $0 \le \sin^4 n \le 1$ *and* $\frac{1}{n^3} \to 0$*, by the previous corollary,* $\lim_{n\to\infty} \frac{\sin^4 n}{n^3} = 0$.

Example 80 *Let's prove that* $\lim\limits_{n\to\infty} \dfrac{n}{x^n} = 0$ *if* $x > 1$*. Write* $\sqrt{x} = 1 + y$ *where* $y > 0$*. Then by Bernoulli's inequality*

$$\left(\sqrt{x}\right)^n = (1+y)^n \ge (1+ny)^2$$

and so

$$x^n \ge (1+y)^{2n} \ge (1+ny)^2 \ge n^2y^2.$$

In turn

$$0 \le \frac{n}{x^n} \le \frac{n}{n^2y^2} = \frac{1}{ny^2} \to 0.$$

Example 81 *Let* $x > 1$ *and let* $a > 0$*. Let's prove that*

$$\lim_{n\to\infty} \frac{n^a}{x^n} = 0.$$

If $a = m \in \mathbb{N}$*, then*

$$\frac{n^a}{x^n} = \left(\frac{n}{\left[(x)^{\frac{1}{m}}\right]^n}\right)^m = \frac{n}{\left[(x)^{\frac{1}{m}}\right]^n} \times \cdots \times \frac{n}{\left[(x)^{\frac{1}{m}}\right]^n}$$

and since $b = (x)^{\frac{1}{m}} > 1$*, we have that* $\frac{n}{b^n} \to 0$ *by the previous example. The result now follow from Theorem 70(ii). If* $0 < a < 1$*, then*

$$0 \le \frac{n^a}{x^n} \le \frac{n^1}{x^n}$$

and the result follows from the squeeze theorem and the previous example. Finally. if $a > 1$*, then* $\lfloor a \rfloor \le a \le \lfloor a \rfloor + 1$*, and so*

$$\frac{n^{\lfloor a \rfloor}}{x^n} \le \frac{n^a}{x^n} \le \frac{n^{\lfloor a \rfloor + 1}}{x^n}$$

and the result follows from the squeeze theorem.

Exercise 82 *Let* $\{a_n\}$ *be a sequence of real numbers such that there exists* $\lim\limits_{n\to\infty} a_n = \infty$ *and let* $x > 1$*.*

(i) Prove that

$$\lim_{n\to\infty} \frac{a_n}{x^{a_n}} = 0.$$

(ii) Prove that if $b > 0$, then

$$\lim_{n\to\infty} \frac{(a_n)^b}{x^{a_n}} = 0.$$

(iii) Use parts (i) and (ii) to prove that if $a > 0$ and $b > 0$, then

$$\lim_{n\to\infty} \frac{(\log n)^b}{n^a} = 0.$$

Example 83 *Let's calculate*

$$\lim_{n\to\infty} \frac{n^3 - n^2 + \log^2 n + 2^n - 3}{4^n - \log n + n^7}.$$

We have

$$\begin{aligned} \frac{n^3 - n^2 + \log^2 n + 2^n - 3}{4^n - \log n + n^7} &= \frac{2^n \left(\frac{n^3}{2^n} - \frac{n^2}{2^n} + \frac{\log^2 n}{2^n} + 1 - \frac{3}{2^n}\right)}{4^n \left(1 - \frac{\log n}{4^n} + \frac{n^7}{4^n}\right)} \\ &= \frac{\left(\frac{n^3}{2^n} - \frac{n^2}{2^n} + \frac{\log^2 n}{2^n} + 1 - \frac{3}{2^n}\right)}{2^n \left(1 - \frac{\log n}{4^n} + \frac{n^7}{4^n}\right)} \to \frac{(0 - 0 + 0 + 1 - 0)}{\infty (1 - 0 + 0)} = \frac{1}{\infty} = 0 \end{aligned}$$

as $n \to \infty$, where we have used the facts that, by the previous exercise,

$$\frac{\log^2 n}{2^n} = \frac{\log^2 n}{n} \frac{n}{2^n} \to 0, \qquad \frac{\log n}{4^n} = \frac{\log n}{n} \frac{n}{4^n} \to 0.$$

6 Monotone Sequences

Definition 84 *We say that a sequence $\{x_n\}$ of real numbers is*

(i) increasing *if $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$;*

(ii) decreasing *if $x_n \ge x_{n+1}$ for all $n \in \mathbb{N}$;*

(iii) strictly increasing *if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$;*

(iv) strictly decreasing *if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.*

We say that a sequence $\{x_n\}$ of real numbers is a monotone *sequence if it satisfies one of the four properties above.*

Theorem 85 *Let $\{x_n\}$ be a monotone sequence of real numbers and let*

$$E = \{x_n : n \in \mathbb{N}\}.$$

(i) If $\{x_n\}$ is increasing, then there exists $\lim_{n\to\infty} x_n = \sup E$.

(ii) If $\{x_n\}$ is decreasing, then there exists $\lim_{n\to\infty} x_n = \inf E$.

Proof. We prove (i). Let $L := \sup E \in (-\infty, \infty]$. Fix $t < L$. Since t is not an upper bound of E, there exists $N = N(t) \in \mathbb{N}$ such that $t < x_N \leq L$. Since $\{x_n\}$ is increasing, for all $n \geq N$, we have that

$$t < x_N \leq x_n \leq L.$$

We now distinguish two cases. If $L \in \mathbb{R}$ then given $\varepsilon > 0$, we may take $t = L - \varepsilon$, to obtain that

$$L - \varepsilon \leq x_n \leq L \leq L + \varepsilon$$

for all $n \geq N$, which implies that there exists $\lim_{n\to\infty} x_n = L$. On the other hand, if $L = \infty$, then we can take t to be any large number, and so $x_n \geq t$ for all $n \geq N$, which implies that there exists $\lim_{n\to\infty} x_n = \infty$. ■

Friday, September 28, 2012

The previous theorem used the supremum property (S). Actually, we can show that the opposite is also true.

Theorem 86 *Assume that every increasing sequence $\{x_n\}$ of real numbers bounded from above admits a limit in $\mathbb{R}$. Then the supremum property holds, that is, every nonempty set $E \subset \mathbb{R}$ bounded from above has a supremum.*

Proof. Exercise. ■

Using Theorem 85, we can define the number e. Consider the sequence

$$s_n := \sum_{k=0}^{n} \frac{1}{k!},$$

where we recall that $0! := 1$ and $n! := 1 \cdot 2 \cdot \cdots \cdot n$. Note that $s_{n+1} = s_n + \frac{1}{(n+1)!} > s_n$, and so the sequence $\{s_n\}$ is strictly increasing. We leave it as an exercise to prove that $\{s_n\}$ is bounded. Since $\{s_n\}$ is bounded and increasing there exists

$$\lim_{n\to\infty} s_n \in (0, \infty).$$

We call this limit e.

Next we study the behavior of sequences defined recursively.

Example 87 *Consider the sequence defined recursively as follows*

$$\begin{cases} a_1 = a \geq 0, \\ a_{n+1} = \sqrt[4]{a_n}, \end{cases}$$

where $n \in \mathbb{N}$. If $a = 0$, then by induction, we have that $a_n = 0$ for all $n \in \mathbb{N}$. Similarly, if $a = 1$, then by induction, we have that $a_n = 1$ for all $n \in \mathbb{N}$. If $0 < a < 1$, then we claim that $0 < a_n < 1$ for all $n \in \mathbb{N}$. This can be proved by *induction. Indeed, it is true for $n = 1$. Assume that $0 < a_n < 1$ for some $n \in \mathbb{N}$ and let's prove that $0 < a_{n+1} < 1$. We have that $0 < a_{n+1} = \sqrt[4]{a_n} < 1$. Hence, $0 < a_n < 1$ for all $n \in \mathbb{N}$. In turn, in this case we have that the sequence is increasing for all $n \in \mathbb{N}$. Indeed, $a_{n+1} = \sqrt[4]{a_n} \geq a_n$ for $0 < a_n < 1$. It follows from Theorem 85 that there exists*

$$\lim_{n\to\infty} a_n = \ell \in [a, 1] .$$

Hence, $\ell \leftarrow a_{n+1} = \sqrt[4]{a_n} \to \sqrt[4]{\ell}$, which implies that $\ell = \sqrt[4]{\ell}$. It follows that $\ell = 1$.

Similarly, if $a > 1$, we can show that $a_n > 1$ for all $n \in \mathbb{N}$, that the sequence is decreasing, so that

$$\lim_{n\to\infty} a_n = \ell \in [1, a] .$$

As before, we conclude that $\ell = 1$.

7 Powers with Real Exponents

If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$x^n := \underbrace{x \cdot \cdots \cdot x}_{n \text{ times}}.$$

But what does it mean $x^{\sqrt{2}}$? Or more generally, x^a if $a \in \mathbb{R}$? To define this, we will assume that $x > 0$ (this is needed to preserve the properties of powers). If a is positive and rational, say $a = \frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{\frac{n}{m}} := \left(\sqrt[m]{x}\right)^n .$$

Remark 88 *Note that $x^{\frac{n}{m}} = \sqrt[m]{x^n}$. Indeed, let $y = \sqrt[m]{x}$. Then*

$$(y^n)^m = (y^m)^n = x^n,$$

and so $y^n = \sqrt[m]{x^n}$, that is, $\left(\sqrt[m]{x}\right)^n = \sqrt[m]{x^n}$.

If a is rational and negative, say $a = -\frac{n}{m}$, where $m, n \in \mathbb{N}$, then we define

$$x^{-\frac{n}{m}} := \left(x^{-1}\right)^{\frac{n}{m}} .$$

Exercise 89 *Prove that if $x > 0$ and $r, q \in \mathbb{Q}$, then*

$$x^r x^s = x^{r+s},$$
$$(x^r)^s = (x^s)^r = x^{rs}.$$

Exercise 90 *Let $x > 1$ and $r, q \in \mathbb{Q}$.*

(i) Prove that if $r > 0$, then $x^r > 1$.

(ii) Prove that if $r < s$, then $x^r < x^s$.

(iii) Prove that if $\{t_n\} \subseteq \mathbb{Q}$ *and* $t_n \to 0$*, then* $x^{t_n} \to 1$*. Hint: Use Example 59.*

We are now ready to define x^a for a real. Assume that $x > 1$. We want to construct an increasing sequence $\{r_n\} \subset \mathbb{Q}$ such that $r_n \to a$. We proceed as follows. By the density of the rational numbers, given $a-1$ and a there exists $r_1 \in \mathbb{Q}$ such that $a - 1 < r_1 < a$. Again by the density of the rational numbers, given $\max\left\{a - \frac{1}{2}, r_1\right\}$ and a there exists $r_2 \in \mathbb{Q}$ such that

$$\max\left\{a - \frac{1}{2}, r_1\right\} < r_2 < a.$$

Inductively, assume that rational numbers $r_1 < r_2 < \cdots < r_n < a$ have been constructed in such a way that

$$a - \frac{1}{i} < r_i < a$$

for all $i = 1, \ldots, n$. We need to construct r_{n+1}. Again by the density of the rational numbers, given $\max\left\{a - \frac{1}{n+1}, r_n\right\}$ and a there exists $r_{n+1} \in \mathbb{Q}$ such that

$$\max\left\{a - \frac{1}{n+1}, r_n\right\} < r_{n+1} < a.$$

Hence, by induction we have constructed an increasing sequence $\{r_n\}$ of rational numbers with

$$a - \frac{1}{n} < r_n < a.$$

Letting $n \to \infty$ in the previous inequality, it follows by the squeeze theorem that $r_n \to a$ as $n \to \infty$.

Since $\{r_n\}$ is increasing, by part (ii) of the previous exercise, it follows that the sequence $\{x^{r_n}\}$ is also increasing, and thus by Theorem 85, there exists

$$\lim_{n\to\infty} x^{r_n} = \ell \in (-\infty, \infty].$$

Since $r_n \le a \le \lfloor a \rfloor + 1$, again by part (ii) of the previous exercise, we have that $x^{r_n} \le x^{\lfloor a \rfloor + 1}$, which implies that the sequence $\{x^{r_n}\}$ is bounded from above. Hence, $\ell \in \mathbb{R}$.

Next let $\{s_n\} \subset \mathbb{Q}$ be such that $s_n \to a$. Then by Exercise 89,

$$x^{s_n} - x^{r_n} = x^{r_n}\left(x^{s_n - r_n} - 1\right).$$

Since $x^{r_n} \to \ell \in \mathbb{R}$ and $x^{s_n - r_n} \to 1$ by part (iii) of the previous exercise, it follows that $x^{s_n} - x^{r_n} \to 0$. Thus, we have shown that there exists $\ell \in \mathbb{R}$ with the property that for *every* sequence $\{s_n\} \subset \mathbb{Q}$ such that $s_n \to a$, there exists

$$\lim_{n\to\infty} x^{s_n} = \ell.$$

Hence, we define $x^a := \ell$.

If $0 < x < 1$, we set

$$x^a := \left(x^{-1}\right)^{-a}.$$

Note that if $x > 0$ and $a, b \in \mathbb{R}$, then

$$x^a x^b = x^{a+b},$$

Indeed, let $\{r_n\} \subset \mathbb{Q}$ and $\{s_n\} \subset \mathbb{Q}$ be such that $r_n \to a$ and $s_n \to b$. Then by Exercise 89

$$x^{r_n} x^{s_n} = x^{r_n + s_n},$$

and it is enough to let $n \to \infty$.

Exercise 91 *Let $x > 1$ and $a, b \in \mathbb{R}$.*

(i) Prove that if $a > 0$, then $x^a > 1$.

(ii) Prove that if $a < b$, then $x^a < x^b$.

(iii) Prove that if $\{a_n\} \subset \mathbb{Q}$ and $a_n \to 0$, then $x^{a_n} \to 1$.

(iv) Prove that if $\{a_n\} \subset \mathbb{Q}$ and $a_n \to a$, then $x^{a_n} \to x^a$.

Exercise 92 *Let $x > 0$ and $a, b \in \mathbb{R}$. Prove that*

$$(x^a)^b = \left(x^b\right)^a = x^{ab}.$$

Hint: It is enough to show $(x^a)^b = x^{ab}$. Consider first the case in which a is real and b is rational.

Exercise 93 *Let $x > 1$ and $a \in \mathbb{R}$. Prove that*

$$|x^a - 1| \leq x^{|a|} - 1.$$

Monday, October 1, 2012

8 Limsup and Liminf

Proposition 94 *If a sequence $\{x_n\}$ of real numbers is not bounded from above, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\lim_{k \to \infty} x_{n_k} = \infty.$$

Proof. Consider the number $M_1 = 1$. Since $\{x_n\}$ is not bounded from above, there exists $n_1 \in \mathbb{N}$ such that $x_{n_1} > 1$. Consider the number

$$M_2 = \max\{2, x_1, \ldots, x_{n_1}\} < \infty.$$

Since $\{x_n\}$ is not bounded from above, there exists $n_2 \in \mathbb{N}$ such that $x_{n_2} > M_2 \geq 2$. Note that, necessarily, $n_2 > n_1$.

Inductively, assume that k positive integers $n_1 < \cdots < n_k$ have been chosen in such a way that $x_{n_i} > i$ for all $i = 1, \ldots, k$ and let's choose n_{k+1}. Consider the number

$$M_k = \max\{k+1, x_1, \ldots, x_{n_k}\} < \infty.$$

Since $\{x_n\}$ is not bounded from above, there exists $n_{k+1} \in \mathbb{N}$ such that $x_{n_{k+1}} > M_{k+1}$. Note that, necessarily, $n_{k+1} > n_k$. Thus we have constructed a subsequence $\{x_{n_k}\}$ of the original sequence $\{x_n\}$ such that $x_{n_k} > k$ for all $k \in \mathbb{N}$. Letting $k \to \infty$ and using the squeeze theorem, we get that $\lim_{k\to\infty} x_{n_k} = \infty$. ■

Similarly, we have

Proposition 95 *If a sequence $\{x_n\}$ of real numbers is not bounded from below, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = -\infty.$$

Finally, we consider the case of bounded sequences.

Proposition 96 *If a sequence $\{x_n\}$ of real numbers is bounded, then there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\ell \in \mathbb{R}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \ell.$$

Proof. Let $F := \{x_n : n \in \mathbb{N}\}$. We distinguish two cases.
Case 1: If the set F has only a finite number of different elements, then this means that there exists $\ell \in \mathbb{R}$ such that $x_n = \ell$ for infinitely many n. Let $\{n_k\}$, $k \in \mathbb{N}$, be the sequence of all natural numbers such that $x_{n_k} = \ell$. Then

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \ell = \ell$$

Thus, we have constructed a subsequence converging to ℓ.
Case 2: The sequence $\{x_n\}$ is bounded. Then the set F is bounded and it has infinitely many different elements. By the Bolzano–Weierstrass theorem, F has an accumulation point $\ell \in \mathbb{R}$. Then by Remark 50 there exists a sequence in the set F that converges to ℓ, namely there exists a subsequence $\{x_{n_k}\}$ converging to ℓ. ■

On the extended real line $[-\infty, \infty]$ we can consider a total order relation $\leq$ by setting $-\infty \leq x \leq \infty$ for all $x \in \mathbb{R}$ and by keeping the usual order relation in $\mathbb{R}$. Given a nonempty set $E \subseteq [-\infty, \infty]$, we can define the supremum of a set as for the real line. Let's prove that the supremum of a set always exist.

Theorem 97 *Every set $E \subseteq [-\infty, \infty]$ has a supremum and an infimum.*

Proof. If $\infty \in E$, then

$$\sup E = \max E = \infty.$$

If $\infty \notin E$, then there are two cases. If $E = \{-\infty\}$, then $\sup E = \max E = -\infty$. If E contains at least two elements, consider the set $F = E \setminus \{-\infty\}$. This set is contained in $\mathbb{R}$ and so it has a supremum $L \in [-\infty, \infty]$. Since $-\infty \leq x$ for all $x \in \mathbb{R}$, it follows that $\sup F = \sup E$.

The existence of the infimum can be proved in a similar way and we omit it. ■

Given a sequence $\{x_n\}$ of real numbers, consider the set

$$E := \left\{ L \in [-\infty, \infty] : \text{ there is a subsequence } \{x_{n_k}\} \text{ such that } \lim_{k \to \infty} x_{n_k} = L \right\}.$$

We define the *limit superior* of the sequence $\{x_n\}$ to be

$$\limsup_{n \to \infty} x_n := \sup E$$

and we define the *limit inferior* of the sequence $\{x_n\}$ to be

$$\liminf_{n \to \infty} x_n := \inf E.$$

Example 98 *Consider the sequence $x_n = (-1)^n$. Then $E = \{-1, 1\}$, and so*

$$\liminf_{n \to \infty} (-1)^n = \min E = -1,$$
$$\limsup_{n \to \infty} (-1)^n = \max E = +1.$$

Theorem 99 *Given a sequence $\{x_n\}$ of real numbers, the set*

$$E := \left\{ L \in [-\infty, \infty] : \textit{ there is a subsequence } \{x_{n_k}\} \textit{ such that } \lim_{k \to \infty} x_{n_k} = L \right\}$$

is nonempty.

Proof. This follows from Propositions 94, 95, and 96. ■

Wednesday, October 3, 2012

The next theorem is important for the exercises.

Theorem 100 *Let $\{x_n\}$ be a sequence bounded from above and let $\ell \in \mathbb{R}$. Then the following are equivalent:*

(a) $\ell = \limsup_{n \to \infty} x_n$;

(b) for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$x_n \leq \ell + \varepsilon \tag{13}$$

for all $n \geq n_\varepsilon$, and

$$x_n \geq \ell - \varepsilon \tag{14}$$

for infinitely many $n \in \mathbb{N}$.

Example 101 *Consider the sequence* $x_n = (-1)^n \frac{2n}{n+1}$*. To prove that*

$$2 = \limsup_{n\to\infty} (-1)^n \frac{2n}{n+1}$$

fix $\varepsilon > 0$*. We want to prove that*

$$(-1)^n \frac{2n}{n+1} \le 2 + \varepsilon$$

for all n *sufficiently large. If* n *is odd, then there is nothing to prove, since a negative number is less than a positive. If* n *is even, we have that*

$$\frac{2n}{n+1} \le 2 + \varepsilon,$$

that is, $2n \le (2+\varepsilon)(n+1)$*, which gives* $0 \le \varepsilon + n\varepsilon + 2$*. This is true for all* $n \in \mathbb{N}$*. Thus we can take* $n_\varepsilon = 1$*.*

Next we want to prove that

$$2 - \varepsilon \le (-1)^n \frac{2n}{n+1} \tag{15}$$

for infinitely many n*. Note that if* n *is odd, then the previous inequality is false. Thus assume that* n *is even. Then*

$$2 - \varepsilon \le \frac{2n}{n+1},$$

that is $(2-\varepsilon)(n+1) \le 2n$*, which gives* $2n - \varepsilon - n\varepsilon + 2 \le 2n$*, that is,* $2 - \varepsilon \le n\varepsilon$*. Hence, the inequality (15) holds for all* n ***even*** *with* $n \ge \frac{2-\varepsilon}{\varepsilon}$*. There are infinitely many such* n*.*

We now turn to the proof of the theorem.

Proof. Assume that $\limsup_{n\to\infty} x_n = \ell \in \mathbb{R}$. We need to prove (13) and (14). If (13) fails then there exist infinitely many many $n \in \mathbb{N}$ such that $x_n \ge \ell + \varepsilon$. So we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \ge \ell + \varepsilon$ for all $k \in \mathbb{N}$. Applying either Proposition 94 or Proposition 96 to the sequence $\{x_{n_k}\}$, we have that the sequence $\{x_{n_k}\}_k$ admits a further subsequence $\left\{x_{n_{k_j}}\right\}_j$ such that $x_{n_{k_j}}$ has a limit, $x_{n_{k_j}} \to L$. Note that $L \in E$. Since $x_{n_{k_j}} \ge \ell + \varepsilon$ for all $j \in \mathbb{N}$ letting $j \to \infty$ we get that $L \ge \ell + \varepsilon$, which contradicts the fact that $\ell = \sup E$. Hence (13) holds.

To prove (14) note that, since $\ell - \varepsilon$ is not an upper bound of the set E there exist $L \in E$ such that $L > \ell - \varepsilon$. By the definition of E there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L$ as $k \to \infty$. Hence, taking $\varepsilon_1 = L - (\ell - \varepsilon) > 0$ there exists $K \in \mathbb{N}$ such that

$$|x_{n_k} - L| \le \varepsilon_1 = L - (\ell - \varepsilon)$$

for all $k \geq K$, that is,

$$-L + (\ell - \varepsilon) \leq x_{n_k} - L \leq L - (\ell - \varepsilon)$$

for all $k \geq K$. In particular, $x_{n_k} \geq \ell - \varepsilon$ for all $k \geq K$. Hence (14) holds.

Conversely assume that (13) and (14) hold. Let's prove that $\ell+\varepsilon$ is an upper bound of E. Let $L \in E$. By the definition of E there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to L$ as $k \to \infty$. Since

$$x_n \leq \ell + \varepsilon$$

for all $n \geq N$ it follows that

$$x_{n_k} \leq \ell + \varepsilon$$

for all k such that $n_k \geq N$. Letting $k \to \infty$ we conclude that $L \leq \ell + \varepsilon$. But since this is true for every $\varepsilon > 0$, letting $\varepsilon \to 0^+$ we get that $L \leq \ell$. Hence $L \leq \ell$ for all $L \in E$, which shows that ℓ is an upper bound of E.

Let's prove that $\ell - \varepsilon$ is not an upper bound of E. Take $\ell - \frac{\varepsilon}{2}$. By (14)

$$x_n \geq \ell - \frac{\varepsilon}{2}$$

for infinitely many $n \in \mathbb{N}$. So we can find a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \geq \ell - \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$. Applying the previous theorem to $\{x_{n_k}\}$, we have that the sequence $\{x_{n_k}\}_k$ admits a further subsequence $\left\{x_{n_{k_j}}\right\}_j$ such that $x_{n_{k_j}}$ has a limit, $x_{n_{k_j}} \to L$. Note that $L \in E$. Since $x_{n_{k_j}} \geq \ell - \frac{\varepsilon}{2}$ for all $j \in \mathbb{N}$, letting $j \to \infty$ we get that $L \geq \ell - \frac{\varepsilon}{2} > \ell - \varepsilon$, which shows $\ell - \varepsilon$ is not an upper bound of E. Hence $\ell = \sup E$. ■

Corollary 102 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\limsup_{n\to\infty} x_n = \max E,$$

that is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$.

Proof. Exercise ■

A similar theorem holds for sequences which are not bounded from above.

Theorem 103 *Given a sequence $\{x_n\}$ of real numbers, the following are equivalent:*

(a) $\limsup_{n\to\infty} x_n = \infty$;

(b) The sequence $\{x_n\}$ is not bounded from above.

(c) For every $M > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \geq M.$$

Proof. We have already proved that (a) and (b) are equivalent. On the other hand, (b) and (c) are clearly equivalent, and so we are done. ■

We have similar theorems for the limit inferior.

Theorem 104 *Given a sequence $\{x_n\}$ of real numbers bounded from below and a number $\ell \in \mathbb{R}$, the following are equivalent:*

(a) $\liminf_{n\to\infty} x_n = \ell$;

(b) for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\ell - \varepsilon \leq x_n \quad \text{for all } n \geq N$$

and there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \leq \ell + \varepsilon.$$

Corollary 105 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\liminf_{n\to\infty} x_n = \min E,$$

that is, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \liminf_{n\to\infty} x_n$.

Theorem 106 *Given a sequence $\{x_n\}$ of real numbers, the following are equivalent:*

(a) $\liminf_{n\to\infty} x_n = \infty$;

(b) The sequence $\{x_n\}$ is not bounded from below.

(c) For every $M > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n \leq -M.$$

Friday, October 5, 2012

First midterm.

Monday, October 8, 2012

The relation between limit, limit superior and limit inferior is given by the following theorem.

Theorem 107 *Given a sequence $\{x_n\}$ of real numbers, then*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n. \tag{16}$$

Moreover there exists $\lim_{n\to\infty} x_n$ if and only if equality holds in (16), and in this case the limit coincides with the common value in (16).

Proof. We have

$$\liminf_{n\to\infty} x_n = \inf E \le \sup E = \limsup_{n\to\infty} x_n.$$

To prove the second part of the theorem assume that there exists $\lim_{n\to\infty} x_n = \ell$. I will consider only the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. By definition of limit, for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\ell - \varepsilon \le x_n \le \ell + \varepsilon$$

for all $n \ge N$. Since properties (13) and (14) are satisfied, it follows from Theorem 100 that $\ell = \limsup_{n\to\infty} x_n$. Similarly, by Theorem 104, we have that $\ell = \liminf_{n\to\infty} x_n$.

Conversely, assume that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = L$$

for some $L \in [-\infty, \infty]$. Again we consider the case $L \in \mathbb{R}$ and leave the cases $L = \infty$ and $L = -\infty$ as an exercise. Fix $\varepsilon > 0$. By Theorems 100 there exists an integer $N_1 = N_1(\varepsilon)$ such that

$$x_n \le L + \varepsilon$$

for all $n \ge N_1$ while by and Theorem 104 there exists an integer $N_2 = N_2(\varepsilon)$ such that

$$L - \varepsilon \le x_n$$

for all $n \ge N_2$. Then for $n \ge \max\{N_1, N_2\}$, we have that

$$L - \varepsilon \le x_n \le L + \varepsilon,$$

which implies that there exists $\lim_{n\to\infty} x_n = L$. ■

As a corollary of this theorem, we can finally prove Theorem 62.

Proof of Theorem 62. Let $\{x_n\}$ be a sequence of real numbers. Assume that there exists

$$\lim_{n\to\infty} x_n = \ell \in [-\infty, \infty].$$

We want to prove that *every* subsequence $\{x_{n_k}\}$ converges to ℓ. Consider the set

$$E := \{L \in [-\infty, \infty] : \text{there is a subsequence of } \{x_n\} \text{ converging to } L\}.$$

Since $x_n \to \ell$, by the previous theorem,

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell.$$

Hence,

$$\inf E = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \sup E,$$

and so $\inf E = \sup E = \ell$. But this implies that $E = \{\ell\}$ (if E had two or more elements the infimum and the supremum would not coincide). By the definition of E and the previous theorem, we have that every subsequence must convergence to ℓ.

Conversely, if every subsequence of $\{x_n\}$ goes to $\ell \in [-\infty, \infty]$, then $E = \{\ell\}$, and so $\inf E = \sup E = \ell$, which implies that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell.$$

Again by the previous theorem, it follows that

$$\lim_{n\to\infty} x_n = \ell.$$

■

Exercise 108 *Consider a sequence $\{x_n\}$ of real numbers. Prove that*

$$\liminf_{n\to\infty} (-x_n) = -\limsup_{n\to\infty} x_n,$$
$$\limsup_{n\to\infty} (-x_n) = -\liminf_{n\to\infty} x_n.$$

The next two theorems are very important to calculate liminf and limsup in exercises.

Theorem 109 *Consider two sequences $\{x_n\}$ and $\{y_n\}$ of real numbers. Then*

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$$
$$\le \limsup_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

provided we exclude the case $\infty - \infty$. Moreover, all inequalities may be strict in general.

Finally, if one of the two sequences has a limit, then

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = \liminf_{n\to\infty} (x_n + y_n)$$

and

$$\limsup_{n\to\infty} (x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n,$$

Wednesday, October 10, 2012

Proof. We only consider the case in which the sequences are bounded and leave the other cases as an exercise. We prove that

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} (x_n + y_n). \tag{17}$$

Let

$$\liminf_{n\to\infty} x_n = \ell_1, \quad \liminf_{n\to\infty} y_n = \ell_2,$$
$$\liminf_{n\to\infty} (x_n + y_n) = \ell_3.$$

For every $\varepsilon > 0$ there exist two positive integer N_1, $N_2 \in \mathbb{N}$ such that

$$\ell_1 - \varepsilon \le x_n \quad \text{for all } n \ge N_1, \quad \ell_2 - \varepsilon \le y_n \quad \text{for all } n \ge N_2.$$

Summing the two inequalities we obtain that

$$\ell_1 + \ell_2 - 2\varepsilon \le x_n + y_n \quad \text{for all } n \ge \max\{N_1, N_2\}.$$

On the other hand, there exist infinitely many $n \in \mathbb{N}$ such that

$$x_n + y_n \le \ell_3 + \varepsilon.$$

By combining the last two inequalities, it follows that for infinitely many $n \ge \max\{N_1, N_2\}$,

$$\ell_1 + \ell_2 - 2\varepsilon \le x_n + y_n \le \ell_3 + \varepsilon.$$

Hence,

$$\ell_1 + \ell_2 - 2\varepsilon \le \ell_3 + \varepsilon$$

but this is true for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we conclude that $\ell_1 + \ell_2 \le \ell_3$. Hence (17) is true.

Now we prove that

$$\liminf_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \liminf_{n\to\infty} y_n. \tag{18}$$

Let

$$\limsup_{n\to\infty} x_n = L_1.$$

For every $\varepsilon > 0$ there exists a positive integer $N_3 \in \mathbb{N}$ such that

$$x_n \le L_1 + \varepsilon \quad \text{for all } n \ge N_3$$

and there exist infinitely many $n \in \mathbb{N}$ for which

$$y_n \le \ell_2 + \varepsilon.$$

Summing the two inequalities we obtain that

$$x_n + y_n \le L_1 + \ell_2 + 2\varepsilon \quad \text{for infinitely many } n \ge N_3.$$

On the other hand, there exists a positive integer $N_4 \in \mathbb{N}$ such that

$$\ell_3 - \varepsilon \le x_n + y_n \quad \text{for all } n \ge N_4.$$

By combining the last two inequalities, it follows that for infinitely many $n \ge \max\{N_3, N_4\}$,

$$\ell_3 - \varepsilon \le x_n + y_n \le L_1 + \ell_2 + 2\varepsilon.$$

and so

$$\ell_3 - \varepsilon \le L_1 + \ell_2 + 2\varepsilon.$$

but this is true for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain that (18) holds. The other inequalities are very similar and so we omit them. ■

Example 110 *By taking* $x_n = (-1)^n$ *and* $y_n = (-1)^{n-1}$*, we have that*

$$\liminf_{n\to\infty} x_n = -1, \quad \liminf_{n\to\infty} y_n = -1,$$
$$\lim_{n\to\infty} (x_n + y_n) = 0, \quad \limsup_{n\to\infty} x_n = 1, \quad \limsup_{n\to\infty} y_n = 1,$$

which shows that the inequalities in the previous theorem may be strict.

Theorem 111 *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of nonnegative numbers. Then*

$$\begin{aligned}\liminf_{n\to\infty} x_n \cdot \liminf_{n\to\infty} y_n &\le \liminf_{n\to\infty} (x_n \cdot y_n) \\ &\le \liminf_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} (x_n \cdot y_n) \\ &\le \limsup_{n\to\infty} x_n \cdot \limsup_{n\to\infty} y_n,\end{aligned}$$

provided we exclude the case $0 \cdot \infty$ *and* $\infty \cdot 0$*. Moreover, all inequalities can be strict.*

Example 112 *By taking* $x_n = 3 + (-1)^n$ *and* $y_n = 4 + (-1)^{n-1}$*, we have that*

$$\liminf_{n\to\infty} x_n = 2, \quad \liminf_{n\to\infty} y_n = 3, \quad \liminf_{n\to\infty} (x_n \cdot y_n) = 10,$$
$$\limsup_{n\to\infty} (x_n \cdot y_n) = 12, \quad \limsup_{n\to\infty} x_n = 4, \quad \limsup_{n\to\infty} y_n = 5,$$

so all inequalities are strict in the previous theorem.

Example 113 *By taking* $x_n = (-1)^n$ *and* $y_n = (-1)^{n-1}$*, we have that*

$$\liminf_{n\to\infty} x_n = -1, \quad \liminf_{n\to\infty} y_n = -1, \quad \lim_{n\to\infty} (x_n \cdot y_n) = -1,$$
$$\limsup_{n\to\infty} x_n = 1, \quad \limsup_{n\to\infty} y_n = 1,$$

so Theorem 111 fails is the sequences can take both positive and negative values.

Theorem 114 *Consider two sequences* $\{x_n\}$ *and* $\{y_n\}$ *of real numbers. Prove that if there exists* $\lim\limits_{n\to\infty} x_n = \ell \in (0, \infty)$*, then*

$$\lim_{n\to\infty} x_n \liminf_{n\to\infty} y_n = \liminf_{n\to\infty} (x_n y_n),$$
$$\limsup_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

Remark 115 *Note that Theorem 114 does not follow from Theorem 111 since in Theorem 114 we are not assuming that* $y_n \ge 0$*.*

Example 116 *Consider the sequence* $x_n = (-1)^n \frac{2n}{n+1}$*. Since there exists* $\lim_{n\to\infty} \frac{2n}{n+1} = 2$*, by the previous theorem,*

$$\liminf_{n\to\infty} (-1)^n \frac{2n}{n+1} = \liminf_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{2n}{n+1} = -1 \cdot 2 = -2,$$
$$\limsup_{n\to\infty} (-1)^n \frac{2n}{n+1} = \limsup_{n\to\infty} (-1)^n \lim_{n\to\infty} \frac{2n}{n+1} = 1 \cdot 2 = 2.$$

Exercise 117 *Let $\{x_n\}$ be a sequence of real numbers, with $x_n > 0$ for all $n \in \mathbb{N}$. Prove that*

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Example 118 *The inequalities in the previous exercise can be strict. Take*

$$x_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is even,} \\ \frac{1}{3^n} & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\frac{x_{n+1}}{x_n} = \begin{cases} \frac{2^n}{3^{n+1}} & \text{if } n \text{ is even,} \\ \frac{3^n}{2^{n+1}} & \text{if } n \text{ is odd,} \end{cases}$$

and so

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = \liminf_{k\to\infty} \frac{2^{2k}}{3^{2k+1}} = 0, \quad \limsup_{n\to\infty} \frac{3^{2k+1}}{2^{2k+2}} = \infty,$$

while

$$\sqrt[n]{x_n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even,} \\ \frac{1}{3} & \text{if } n \text{ is odd,} \end{cases}$$

and so

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} = 0 < \liminf_{n\to\infty} \sqrt[n]{x_n} = \frac{1}{3}$$
$$< \frac{1}{2} = \limsup_{n\to\infty} \sqrt[n]{x_n} < \limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = \infty.$$

Next we prove some important limits.

Theorem 119 $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$

Proof. Let

$$s_n := \sum_{k=0}^{n} \frac{1}{k!}, \qquad t_n := \left(1 + \frac{1}{n}\right)^n.$$

By the binomial theorem,

$$t_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} 1^{n-k}.$$

Note that

$$\binom{n}{k} \frac{1}{n^k} = \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} = \frac{(n-k)!}{k!\,(n-k)!} \frac{n\,(n-1)\cdots(n-k+1)}{n\cdot\cdots\cdot n}$$
$$= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k+1}{n}\right) \le \frac{1}{k!}.$$

Hence, $t_n \leq s_n$, and so

$$\limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = e.$$

On the other hand, if $n \geq m$,

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \geq \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k+1}{n}\right). \end{aligned}$$

Fixing m and letting $n \to \infty$ in the previous inequality gives

$$\begin{aligned} \liminf_{n\to\infty} t_n &\geq \liminf_{n\to\infty} \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k+1}{n}\right) \\ &\geq \sum_{k=0}^{m} \liminf_{n\to\infty} \left(\frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k+1}{n}\right)\right) \\ &= \sum_{k=0}^{m} \frac{1}{k!}. \end{aligned}$$

Note that in the last inequality it was important to have a finite sum $\sum_{k=0}^{m}$ rather than $\sum_{k=0}^{n}$. Finally, letting $m \to \infty$ gives

$$\liminf_{n\to\infty} t_n \geq \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = e.$$

Thus, we have shown that

$$e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e,$$

and so by Theorem 107, there exists $\lim_{n\to\infty} t_n = e$. ■

Friday, October 12, 2012

9 Metric Spaces and Completeness

Definition 120 *A* metric *on a set X is a map $d : X \times X \to [0, \infty)$ such that*

(i) $d(x, y) \leq d(x, z) + d(z, y)$ *for all $x, y, z \in X$ (triangle inequality),*

(ii) $d(x, y) = d(y, x)$ *for all $x, y \in X$ (symmetry),*

(iii) $d(x, y) = 0$ *if and only if $x = y$.*

A *metric space* (X, d) is a set X endowed with a metric d. When there is no possibility of confusion, we abbreviate by saying that X is a metric space.

Example 121 *Here are some of the most important examples of metric spaces.*

(i) In $\mathbb{R}$ *we have that* $d(x,y) := |x-y|$ *is a metric.*

(ii) In $\mathbb{R}^N$, $N \geq 1$, *for* $x = (x_1, \ldots, x_N)$ *and* $y = (y_1, \ldots, y_N)$,

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + \ldots + (x_N - y_N)^2}$$

is a metric.

(iii) Taking X *to be the set of all bounded functions* $f : [a,b] \to \mathbb{R}$, *it can be shown that*

$$d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$$

is a metric.

(iv) In $\mathbb{R}$ *we have that*

$$d_1(x,y) := \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \tag{19}$$

is a metric.

Definition 122 *Given a metric space* (X,d) *and a sequence* $\{x_n\} \subseteq X$, *we say that*

(i) $\{x_n\}$ *is a* Cauchy sequence *if for every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \mathbb{N}$ *such that*

$$d(x_n, x_m) \leq \varepsilon$$

for all $n, m \geq N_\varepsilon$,

(ii) $\{x_n\}$ converges to $x \in X$ *if for every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \mathbb{N}$ *such that*

$$d(x_n, x) \leq \varepsilon$$

for all $n \geq N_\varepsilon$.

Remark 123 *The previous definition does not change if we replace* $\leq$ *with* $<$. *Thus, we can say that* $\{x_n\}$ converges to $x \in X$ *if for every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \mathbb{N}$ *such that*

$$x_n \in B(x, \varepsilon)$$

for all $n \geq N_\varepsilon$, *where* $B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$ *is the ball centered at* x *and radius* $\varepsilon > 0$.

Similarly, if (X, τ) *is a topological space, we can say that* $\{x_n\}$ converges to $x \in X$ *if for every open set* $U \in \tau$ *containing* x *there exists* $N_U \in \mathbb{N}$ *such that*

$$x_n \in U$$

for all $n \geq N_U$.

Proposition 124 *Given a metric space* (X,d) *and a sequence* $\{x_n\} \subseteq X$*, if* $\{x_n\}$ *converges to* $x \in X$*, then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. Since $\{x_n\}$ converges to $x \in X$, given $\varepsilon > 0$, consider $\frac{\varepsilon}{2}$ in the definition of convergence. Then there exists $N_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x) \leq \frac{\varepsilon}{2}$$

for all $n \geq N_\varepsilon$. Hence, by the triangle inequality and symmetry of d, if $n, m \geq N_\varepsilon$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = d(x_n, x) + d(x_m, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

■

The opposite is not true, that is, there are Cauchy sequences that do not have a limit.

Example 125 *Consider* $X = (0,1]$ *with the metric* $d(x,y) = |x-y|$ *and consider the sequence* $x_n = \frac{1}{n}$*. Then* $x_n \to 0$ *which does not belong to* $X = (0,1]$*, but* $\{x_n\}$ *is a Cauchy (just applied the previous proposition in the metric space* $\mathbb{R}$*).*

Exercise 126 *Let* $\{x_n\}$ *be a sequence of real numbers.*

(i) Prove that if $\{x_n\}$ *is a Cauchy sequence and if a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *converges to some* $x \in \mathbb{R}$*, then* $\{x_n\}$ *converges to* x*.*

(ii) Prove that if there exists $x \in \mathbb{R}$ *such that for* every *subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *there exists a further subsequence* $\left\{x_{n_{k_j}}\right\}$ *that converges to* x*, then* $\{x_n\}$ *converges to* x*.*

A metric space (X,d) is said to be *complete* if every Cauchy sequence is convergent.

Example 127 *Let* $X = (0,1)$ *with the metric* $d(x,y) = |x-y|$*. The sequence* $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ *converges to* 0 *in* $\mathbb{R}$*, and so it is a Cauchy sequence in* $\mathbb{R}$*. In particular, it is a Cauchy sequence in* X*. However, it does not converge to an element of* X*, since* $0 \notin X$*.*

Theorem 128 $(\mathbb{R}, d)$ *is a complete metric space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence.
Step 1: We claim that $\{x_n\}$ is bounded. Fix $\varepsilon = 1$. By the definition of Cauchy sequence, there is exists $N_1 \in \mathbb{N}$ such that

$$|x_n - x_m| \leq 1$$

for all $n, m \geq N_1$. In particular, taking $m = N_1$, we have that

$$|x_n - x_{N_1}| \leq 1,$$

for all $n \geq N_1$, and so

$$|x_n| = |x_n \pm x_{N_1}| \leq |x_n - x_{N_1}| + |x_{N_1}| \leq 1 + |x_{N_1}| = M_1$$

for all $n \geq N_1$. On the other hand, if $n < N_1$, then

$$|x_n| \leq |x_1| + |x_2| + \cdots + |x_{N_1}| = M_2.$$

Hence, for every $n \in \mathbb{N}$, we have that

$$|x_n| \leq \max\{M_1, M_2\}.$$

This shows that $\{x_n\}$ is bounded.

Step 2: Since $\{x_n\}$ is bounded, by Proposition 96, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\ell \in \mathbb{R}$ such that $\lim_{k\to\infty} x_{n_k} = \ell$. Let's prove that the entire sequence converges to ℓ.

Given $\varepsilon > 0$, by the definition of Cauchy sequence, there is exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - x_m| \leq \frac{\varepsilon}{2}$$

for all $n, m \geq N_\varepsilon$. On the other hand, since $\lim_{k\to\infty} x_{n_k} = \ell$, there is exists $K_\varepsilon \in \mathbb{N}$ such that

$$|x_{n_k} - \ell| \leq \frac{\varepsilon}{2}$$

for all $k \geq K_\varepsilon$. Since $\{n_k\}$ is strictly increasing, $n_k \to \infty$ as $k \to \infty$ and so $n_k \geq N_\varepsilon$ for all k large, say $k \geq K_1$. Fix $k \geq \max\{K_\varepsilon, K_1\}$. Then for all $n \geq N_\varepsilon$,

$$|x_n - \ell| = |x_n - \ell \pm x_{n_k}| \leq |x_n - x_{n_k}| + |x_{n_k} - \ell| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

where we have used the facts that $n, n_k \geq N_\varepsilon$ and that $k \geq K_\varepsilon$.

This implies that $\{x_n\}$ converges to ℓ. ■

Remark 129 (Important) *The previous theorem relies on Proposition 96, which in turn is based on the Bolzano–Weierstrass theorem. Note that in infinite dimensional spaces (such as the space of bounded functions), the Bolzano–Weierstrass theorem fails in general.*

Exercise 130 *Prove that the function d_1 defined in* (19) *is a metric and that* $(\mathbb{R}, d_1)$ *is not complete.*

The completeness of the real line together with the Archimedean property is equivalent to Dedekind axiom.

Theorem 131 *The following are equivalent.*

- (D) *Dedekind axiom.*
- (S) *The supremum property.*
- (M) *Every increasing sequence bounded from above has a limit in $\mathbb{R}$.*
- (C) *Every Cauchy sequence has a limit and the Archimedean property holds.*

Monday, October 15, 2012

10 Sequences and Topology

A set $C \subseteq \mathbb{R}$ is *sequentially closed* if for every sequence $\{x_n\} \subseteq C$ such that $\{x_n\}$ converges to some $x \in \mathbb{R}$, then x belongs to C.

Proposition 132 *Let $C \subseteq \mathbb{R}$. Then C is closed if and only if C is sequentially closed.*

Proof. Step 1: Assume that C is closed and let $\{x_n\} \subseteq C$ be such that $\{x_n\}$ converges to some $x \in \mathbb{R}$. We need to show that x belongs to C. If not, then $x \in \mathbb{R}\backslash C$. Since $\mathbb{R}\backslash C$ is open, there exists $r > 0$ such that $B(x, r) \subseteq \mathbb{R}\backslash C$. But then, taking $\varepsilon = r$ there exists $n_r \in \mathbb{N}$ such that $|x_n - x| < r$ for all $n \geq n_r$, which implies that $x_n \in B(x, r) \subseteq \mathbb{R} \setminus C$ for all $n \geq n_r$. This contradicts the fact that $\{x_n\} \subseteq C$.
Step 2: Assume that C is sequentially closed. We need to show that $\mathbb{R} \setminus C$ is open. Let $x \in \mathbb{R}\backslash C$. We claim that there exists $r > 0$ such that $B(x, r) \subseteq \mathbb{R}\backslash C$. If not, then for every $r > 0$ we can find $y \in C$ such that $y \in B(x, r)$. Taking $r = \frac{1}{n}$ we can find $x_n \in C$ such that $|x_n - x| < \frac{1}{n} \to 0$, which shows that $\{x_n\}$ converges to x. Since C is sequentially closed, it follows that $x \in C$, which is a contradiction. ■

Definition 133 *A set $K \subseteq \mathbb{R}$ is* sequentially compact *if for every sequence $\{x_n\} \subseteq K$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.*

Example 134 *A finite set $K \subseteq \mathbb{R}$ is sequentially compact. The set $E = [0, 1)$ is not sequentially compact, since the sequence $\left\{1 - \frac{1}{n}\right\}$ converges to 1, which does not belong to E. The problem here is that E is not closed.*

The set $F = [0, \infty)$ is closed but not sequentially compact, since the sequence $\{n\}$ converges to ∞, which does not belong to F. The problem here is that F is not bounded.

The following theorem is one of the main results of section.

Theorem 135 *Let $K \subseteq \mathbb{R}$. Then the following are equivalent.*

(i) K is sequentially compact.

(ii) K is closed and bounded.

Proof. Assume that K is sequentially compact. We claim that K is sequentially closed. To see this, let $\{x_n\} \subseteq K$ be a sequence such that $x_n \to x$ for some $x \in \mathbb{R}$. Since K is sequentially compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $y \in K$ such that $x_{n_k} \to y$ as $k \to \infty$. By the uniqueness of limits, it follows that $x = y \in K$, which shows that K is sequentially closed.

Next we prove that K is bounded from above. Indeed, if not, then for every $n \in \mathbb{N}$ we would find $x_n \in K$ such that $x_n \geq n$, but then $x_n \to \infty$, and so every subsequence of $\{x_n\}$ converges to ∞, which contradicts the fact that K is sequentially compact.

Conversely, assume that K is closed and bounded and let $\{x_n\} \subseteq K$. Since K is bounded, so is $\{x_n\}$. Hence,by Proposition 96 there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{k\to\infty} x_{n_k} = \ell$. Since K is closed, it is sequentially closed, and so $\ell \in K$. This shows that K is sequentially compact. ■

Definition 136 *A set $K \subseteq \mathbb{R}$ is* compact *if for every open cover of K, i.e., for every collection $\{U_i\}$ of open sets such that $\bigcup_i U_i \supset K$, there exists a finite subcover (i.e., a finite subcollection of $\{U_i\}$ whose union still contains K).*

Example 137 *The set $E = \mathbb{N}$ is not compact. To see this, consider the family of balls $B\left(n, \frac{1}{2}\right)$. These balls cover E but since they are disjoint, if we remove even one of them, we cannot cover E anymore. The problem here is that E is unbounded.*

Example 138 *The set $E = [0, 1)$ is not compact. Consider the family of open sets $U_n = \left(-\infty, 1 - \frac{1}{n}\right)$. Then*

$$\bigcup_{n=1}^{\infty} \left(-\infty, 1 - \frac{1}{n}\right) = (-\infty, 1),$$

but if we consider a finite number of the sets U_n, say, $U_{n_1}, \ldots, U_{n_\ell}$, letting $N = \max\{n_1, \ldots, n_\ell\}$, we have that

$$\bigcup_{k=1}^{\ell} \left(-\infty, 1 - \frac{1}{n_k}\right) = \left(-\infty, 1 - \frac{1}{N}\right),$$

which does not cover E. The problem is that E is not bounded.

Exercise 139 *Prove that if K is closed and bounded, then K is compact. Hint: Assume that there exists a collection $\{U_i\}$ of open sets such that $\bigcup_i U_i \supset K$ for which there is no finite subcover. Repeat the proof of the Bolzano–Weierstrass theorem.*

Exercise 140 *Prove that if K is compact, then K is closed and bounded.*

Example 141 *In view of the previous theorem, the interval $[a, b] \subset \mathbb{R}$ is sequentially compact.*

Wednesday, October 17, 2012

11 Functions

Consider a function $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. The set E is called the *domain* of f. If E is not specified, then E should be taken to be the largest set of x for which $f(x)$ makes sense. This means that:

If there are even roots, their arguments should be nonnegative. If there are logarithms, their arguments should be strictly positive. Denominators should be different from zero. If a function is raised to an irrational number, then the function should be nonnegative.

Given a set $F \subseteq E$, the set $f(F) = \{y \in \mathbb{R} : y = f(x) \text{ for some } x \in F\}$ is called the *image* of F through f. The function f is said to be bounded from above in F, bounded from below in F, bounded in F if the set $f(F)$ is bounded from above, bounded from below, bounded, respectively.

Given a set $G \subseteq \mathbb{R}$, the set $f^{-1}(G) = \{x \in E : f(x) \in G\}$ is called the *inverse image* of F through f. It has NOTHING to do with the inverse function. It is just one of those unfortunate cases in which we use the same symbol for two different objects.

The *graph* of a function is the set of $\mathbb{R}^2$ defined by

$$\operatorname{gr} f = \{(x, f(x)) : x \in E\}.$$

A function f is said to be

- *increasing* if $f(x) \leq f(y)$ for all $x, y \in E$ with $x < y$,
- *strictly increasing* if $f(x) < f(y)$ for all $x, y \in E$ with $x < y$,
- *decreasing* if $f(x) \geq f(y)$ for all $x, y \in E$ with $x < y$,
- *strictly decreasing* if $f(x) > f(y)$ for all $x, y \in E$ with $x < y$,
- *monotone* if one of the four property above holds,
- *one-to-one* or *injective* if $f(x) \neq f(y)$ for all $x, y \in E$ with $x \neq y$.

If $f : E \to F$, where $E, F \subseteq \mathbb{R}$, then f is said to be

- *onto* or *surjective* if $f(E) = F$,
- *bijective* or *invertible* if it is one-to-one and onto. The function $f^{-1} : F \to E$, which assigns to each $y \in F = f(E)$ the unique $x \in E$ such that $f(x) = y$, is called the *inverse* function of f.

12 Limits of Functions

Definition 142 *Let $E \subseteq \mathbb{R}$, let $x_0 \in \mathbb{R}$ be an accumulation point of E, and let $f : E \to \mathbb{R}$. We say that*

- *a number $\ell \in \mathbb{R}$ is the* limit *of $f(x)$ as x approaches x_0 if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ with the property that*

$$|f(x) - \ell| \leq \varepsilon$$

for all $x \in E$ with $0 < |x - x_0| \leq \delta$. We write

$$\lim_{x \to x_0} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to x_0.$$

- ∞ *is the* limit *of* $f(x)$ *as* x *approaches* x_0 *if for every* $M > 0$ *there exists a real number* $\delta = \delta(M, x_0) > 0$ *with the property that*

$$f(x) \geq M$$

for all $x \in E$ *with* $0 < |x - x_0| \leq \delta$*. We write*

$$\lim_{x \to x_0} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to x_0.$$

- $-\infty$ *is the* limit *of* $f(x)$ *as* x *approaches* x_0 *if for every* $M > 0$ *there exists a real number* $\delta = \delta(M, x_0) > 0$ *with the property that*

$$f(x) \leq -M$$

for all $x \in E$ *with* $0 < |x - x_0| \leq \delta$*. We write*

$$\lim_{x \to x_0} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to x_0.$$

Note that even when $x_0 \in E$, we cannot take $x = x_0$ since in the definition we require $0 < |x - x_0|$.

Theorem 143 *Let* $E \subseteq \mathbb{R}$ *and let* $x_0 \in \mathbb{R}$ *be an accumulation point of* E*. Given a function* $f : E \to \mathbb{R}$ *there exists*

$$\lim_{x \to x_0} f(x) = \ell \in [-\infty, \infty]$$

if and only if

$$\lim_{n \to \infty} f(x_n) = \ell$$

for ***every*** *sequence* $\{x_n\} \subseteq E \setminus \{x_0\}$ *converging to* x_0*. In particular, if the limit exists, it is unique.*

Proof. We consider only the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. Assume that there exists $\ell = \lim_{x \to x_0} f(x)$ and let $\{x_n\} \subseteq E \setminus \{x_0\}$ converge to x_0. Fix $\varepsilon > 0$ and find $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$|f(x) - \ell| \leq \varepsilon$$

for all $x \in E$ with $0 < |x - x_0| \leq \delta$. Since $x_n \to x_0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_n - x_0| \leq \delta$ for all $n \geq n_\varepsilon$ and so

$$|f(x_n) - \ell| \leq \varepsilon$$

for all $n \geq n_\varepsilon$, which shows that $f(x_n) \to \ell$.

Conversely, assume that $\lim_{n \to \infty} f(x_n) = \ell$ for every sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ converging to x_0. If either the limit $\lim_{x \to x_0} f(x)$ does not exist or it exists but it it different from ℓ, then there exists $\varepsilon > 0$ with the property that for every δ there exists $x \in E \setminus \{x_0\}$ with $0 < |x - x_0| < \delta$ such that

$$|f(x) - \ell| > \varepsilon.$$

Take $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$ and find $x_n \in E \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$ such that

$$|f(x_n) - \ell| > \varepsilon. \tag{20}$$

Then $x_0 - \frac{1}{n} \le x_n \le x_0 + \frac{1}{n}$ for every n, and so by the squeeze theorem, $x_n \to x_0$. But then, by hypothesis $f(x_n) \to \ell$ as $n \to \infty$, which contradicts (20). ■

Remark 144 *In view of the previous theorem, in order to show that the limit* $\lim_{x \to x_0} f(x)$ *does not exist, it is enough to find two sequences* $\{x_n\} \subseteq E \setminus \{x_0\}$ *and* $\{y_n\} \subseteq E \setminus \{x_0\}$ *both converging to* x_0 *and such that*

$$\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n).$$

Example 145 *Consider the function* $f(x) = \cos \frac{1}{x}$ *defined in* $E = \mathbb{R} \setminus \{0\}$. *Note that* 0 *is an accumulation point of* E. *To prove that the limit*

$$\lim_{x \to 0} \cos \frac{1}{x}$$

does not exist, consider the sequences $x_n = \frac{1}{2n\pi} \to 0$ *and* $x_n = \frac{1}{n\pi} \to 0$. *Then*

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \cos(2n\pi) = 1 \neq \lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} \cos(n\pi) = -1.$$

We now list some important operations for limits.

Theorem 146 *Let* $E \subseteq \mathbb{R}$ *and let* $x_0 \in \mathbb{R}$ *be an accumulation point of* E. *Given three functions* $f, g, h : E \to \mathbb{R}$, *assume that there exist*

$$\lim_{x \to x_0} f(x) = \ell_1 \in \mathbb{R}, \qquad \lim_{x \to x_0} g(x) = \ell_2 \in \mathbb{R}.$$

Then

(i) there exists $\lim_{x \to x_0} (f + g)(x) = \ell_1 + \ell_2$,

(ii) there exists $\lim_{x \to x_0} (fg)(x) = \ell_1 \ell_2$,

(iii) if $\ell_2 \neq 0$ *and* $F := \{x \in E : g(x) \neq 0\}$, *then* x_0 *is an accumulation point of* F *and there exists* $\lim_{x \to x_0} \left(\frac{f}{g}\Big|_F \right)(x) = \frac{\ell_1}{\ell_2}$,

(iv) ***(Squeeze Theorem)*** *if* $\ell_1 = \ell_2$ *and* $f(x) \le h(x) \le g(x)$ *for every* $x \in E$, *then there exists* $\lim_{x \to x_0} h(x) = \ell_1$.

Proof. Parts (i) and (ii) follow from Theorems 70 and 143, part (iv) from Theorems 77 and 143. The fact that x_0 is an accumulation point of F is left as an exercise. ■

Remark 147 *As in the case of sequences, the previous theorem continues to hold if* $\ell_1, \ell_1 \in [-\infty, \infty]$*, provided we avoid the cases* $\infty - \infty$*,* 0∞*,* $\frac{0}{0}$*,* $\frac{\infty}{\infty}$*.*

Theorem 148 *Let* $E \subseteq \mathbb{R}$ *and let* $x_0 \in \mathbb{R}$ *be an accumulation point of* E*. Given two functions* $f, g : E \to \mathbb{R}$*, assume that there exists*

$$\lim_{x \to x_0} f(x) = 0,$$

and that g *is bounded. Then there exists* $\lim_{x \to x_0} (fg)(x) = 0$*.*

Proof. This follows from Corollary 78 and Theorem 143. ■

Example 149 *The previous theorem can be used for example to show that for* $a > 0$

$$\lim_{x \to 0} x^a \sin \frac{1}{x} = 0.$$

Example 150 *We list below some important limits.*

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}, \qquad \lim_{x \to 0} \frac{\log(1 + x)}{x} = 1,$$
$$\lim_{x \to 0} \frac{(1 + x)^a - 1}{x} = a \quad \text{for } a \in \mathbb{R}, \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Friday, October 19, 2012

Midsemester break.

Monday, October 22, 2012

Theorem 151 $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

Proof. Let $0 < x < \frac{\pi}{2}$ and consider the triangle of vertices $O = (0, 0)$, $A = (1, 0)$, and $B = (\cos x, \sin x)$. Consider the point $D = (1, \tan x)$. Then the area of the triangle OAB is less than the area of the circular sector[1] that contains the triangle, and in turn the area of the circular sector is less than the area of the triangle OAD. Note that the base of the triangle OAB is one and the height is $\sin x$. Thus, we have

$$0 < \frac{1}{2} \sin x < \frac{x}{2} < \frac{1}{2} \tan x.$$

[1] A circular sector is the portion of a circle enclosed by two radii and an arc. It θ is the angle in radians formed by the two radii and r is the radius of the circle, the area of the circular sector is given by

$$A = \frac{1}{2} r^2 \theta.$$

Dividing everything by $\frac{1}{2}\sin x$ and inverting the inequalities, we get

$$\cos x < \frac{\sin x}{x} < 1,$$

and so

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} \leq 2\left(\frac{x}{2}\right)^2.$$

By the squeeze theorem we get that

$$\lim_{x\to 0^+}\frac{\sin x}{x} = 1.$$

Since $\sin(-x) = -\sin x$, considering the change of variables $y = -x$, we have that

$$\lim_{y\to 0^+}\frac{\sin y}{y} = \lim_{x\to 0^+}\frac{\sin(-x)}{-x} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1,$$

and so there exists $\lim_{x\to 0}\frac{\sin x}{x} = 1$. ■

Remark 152 *Note that we have also shown that*

$$|\sin x| \leq |x|$$

for every $0 < |x| < \frac{\pi}{2}$*. On the other hand, if* $|x| \geq \frac{\pi}{2}$*, then*

$$|\sin x| \leq 1 < \frac{\pi}{2} \leq |x|$$

and so the inequality $|\sin x| \leq |x|$ *holds for every* x*. In particular, by the squeeze theorem,*

$$\lim_{x\to 0}\sin x = 0.$$

Theorem 153 $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

Proof. We have

$$1 - \cos x = 2\sin^2\frac{x}{2}$$

and so

$$\frac{1-\cos x}{x^2} = 2\frac{\sin^2\frac{x}{2}}{x^2} = \frac{2}{4}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \to \frac{1}{2}1$$

as $x \to 0$. ■

Exercise 154 *Prove that for every* $x_0 \in \mathbb{R}$*,*

$$\lim_{x\to x_0}\cos x = \cos x_0.$$

We next study the limit of composite functions.

Theorem 155 *Let $E, F \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of E. Given two functions $f : E \to F$ and $g : F \to \mathbb{R}$, assume that there exist*

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R},$$

that ℓ is an accumulation point of F, and that there exists

$$\lim_{y \to \ell} g(y) = L \in [-\infty, \infty].$$

Assume that either there exists $\delta_1 > 0$ such that $f(x) \neq \ell$ for all $x \in E$ with $0 < |x - x_0| \leq \delta_1$, or that $\ell \in F$, $L \in \mathbb{R}$ and $g(\ell) = L$. Then there exists $\lim\limits_{x \to x_0} g(f(x)) = L$.

Proof. We consider only the case $L \in \mathbb{R}$ and leave the cases $L = \infty$ and $L = -\infty$ as an exercise. Fix $\varepsilon > 0$ and find $\eta = \eta(\varepsilon, \ell) > 0$ such that

$$|g(y) - L| \leq \varepsilon \tag{21}$$

for all $y \in F$ with $0 < |y - \ell| \leq \eta$.

Since $\lim_{x \to x_0} f(x) = \ell$, there exists $\delta_2 = \delta_2(x_0, \eta) > 0$ such that

$$|f(x) - \ell| \leq \eta$$

for all $x \in E$ with $0 < |x - x_0| \leq \delta_1$.

We now distinguish two cases.

Case 1: Assume that $f(x) \neq \ell$ for all $x \in E$ with $0 < |x - x_0| \leq \delta_1$. Then taking $\delta = \min\{\delta_1, \delta_2\}$, we have that for all $x \in E$ with $0 < |x - x_0| \leq \delta$,

$$0 < |f(x) - \ell| \leq \eta.$$

Hence, taking $y = f(x)$, by (21), it follows that

$$|g(f(x)) - L| \leq \varepsilon$$

for all $x \in E$ with $0 < |x - x_0| \leq \delta$. This shows that there exists $\lim\limits_{x \to x_0} g(f(x)) = L$.

Case 2: Assume that $\ell \in F$ and $g(\ell) = L$. Let $x \in E$ with $0 < |x - x_0| \leq \delta_1$. If $f(x) = \ell$, then $g(f(x)) = L$, and so

$$|g(f(x)) - L| = 0 \leq \varepsilon,$$

while if $f(x) \neq \ell$, then taking $y = f(x)$, by (21), it follows that

$$|g(f(x)) - L| \leq \varepsilon.$$

■

Example 156 *Let's prove that the previous theorem fails without the hypotheses that either* $f(x) \neq \ell$ *for all* $x \in E$ *near* x_0 *or* $\ell \in F$, $L \in \mathbb{R}$ *and* $g(\ell) = L$. *Consider the function*

$$g(y) := \begin{cases} 1 & \text{if } y \neq 0, \\ 2 & \text{if } y = 0. \end{cases}$$

Then there exists

$$\lim_{y \to 0} g(y) = 1.$$

So $L = 1$. *Consider the function* $f(x) := 0$ *for all* $x \in \mathbb{R}$. *Then for every* $x_0 \in \mathbb{R}$, *we have that*

$$\lim_{x \to x_0} f(x) = 0.$$

So $\ell = 0$. *However,* $g(f(x)) = g(0) = 2$ *for all* $x \in \mathbb{R}$. *Hence,*

$$\lim_{x \to x_0} g(f(x)) = \lim_{x \to x_0} 2 = 2 \neq 1,$$

which shows that the conclusion of the theorem is violated.

Note that the previous theorem can be used to change variables in limits.

Example 157 *Let's try to calculate*

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{x}.$$

For $\sin x \neq 0$, *we have*

$$\frac{\log(1 + \sin x)}{x} = \frac{\log(1 + \sin x)}{x} \frac{\sin x}{\sin x} = \frac{\log(1 + \sin x)}{\sin x} \frac{\sin x}{x}.$$

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, *it remains to study*

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{\sin x}.$$

Consider the function $g(y) = \frac{\log(1+y)}{y}$ *and the function* $f(x) = \sin x$. *As* $x \to 0$, *we have that* $\sin x \to 0 = \ell$, *while*

$$\lim_{y \to 0} \frac{\log(1 + y)}{y} = 1.$$

Moreover $\sin x \neq 0$ *for all* $x \in E := \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$. *Hence, we can apply the previous theorem to conclude that*

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{\sin x} = 1.$$

In turn,

$$\lim_{x \to 0} \frac{\log(1 + \sin x)}{x} = 1.$$

Definition 158 *Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$ be a function. Given a subset $F \subseteq E$ we denote by $f|_F$ the restriction of the function f to the set F, that is the function $f : F \to \mathbb{R}$.*

Theorem 159 *Let $E \subseteq \mathbb{R}$ be such that $E = F \cup G$ and let $x_0 \in \mathbb{R}$ be an accumulation point of both F and G. Consider a function $f : E \to \mathbb{R}$. Then there exists $\lim_{x\to x_0} f(x)$ if and only if there exist $\lim_{x\to x_0} f|_F(x)$ and $\lim_{x\to x_0} f|_G(x)$ and they are equal.*

Proof. Assume that there exist $\lim_{x\to x_0} f|_F(x)$ and $\lim_{x\to x_0} f|_G(x)$ and that their equal. Let ℓ be the common value. Assume that $\ell \in \mathbb{R}$ (the cases $\ell = \infty$ and $\ell = -\infty$ are left as an exercise). Fix $\varepsilon > 0$. Since $\lim_{x\to x_0} f|_F(x) = \ell$, there exists $\delta_1 = \delta_1(\varepsilon, x_0) > 0$ with the property that

$$|f(x) - \ell| \le \varepsilon$$

for all $x \in F$ with $0 < |x - x_0| \le \delta_1$. Since $\lim_{x\to x_0} f|_G(x) = \ell$, there exists $\delta_2 = \delta_2(\varepsilon, x_0) > 0$ with the property that

$$|f(x) - \ell| \le \varepsilon$$

for all $x \in G$ with $0 < |x - x_0| \le \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\} > 0$.

If $x \in E$, with $0 < |x - x_0| \le \delta$, then, since $E = F \cup G$, we have that either $x \in F$ or $x \in G$ (or both), and so

$$|f(x) - \ell| \le \varepsilon.$$

Conversely, if the limit $\lim_{x\to x_0} f(x) = \ell$ exists, assume that $\ell \in \mathbb{R}$ (the cases $\ell = \infty$ and $\ell = -\infty$ are left as an exercise). Fix $\varepsilon > 0$. Then there exists $\delta = \delta(\varepsilon, x_0) > 0$ with the property that

$$|f(x) - \ell| \le \varepsilon$$

for all $x \in E$ with $0 < |x - x_0| \le \delta$. By restricting x to F, we get that there exists $\lim_{x\to x_0} f|_F(x) = \ell$, while by restricting x to G, we get that there exists $\lim_{x\to x_0} f|_F(x) = \ell$. ■

An important special case is obtained by taking as sets F and G,

$$E^- := E \cap (-\infty, x_0], \quad E^+ := E \cap (x_0, \infty).$$

Whenever they exist, the limits $\lim_{x\to x_0} f|_{E^-}(x)$ and $\lim_{x\to x_0} f|_{E^+}(x)$ are called the *left* and *right limit* of f as $x \to x_0$ and they are denoted, respectively, by

$$\lim_{x\to x_0^-} f(x) \quad \text{and} \quad \lim_{x\to x_0^+} f(x).$$

Exercise 160 *Prove that there exists*

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}.$$

Hint: Calculate the left and right limits separately.

Theorem 161 *Let $E \subseteq \mathbb{R}$, let $x_0 \in \mathbb{R}$ be an accumulation point of E, and let $f : E \to \mathbb{R}$ be a monotone function. If x_0 is an accumulation point of $E^- := E \cap (-\infty, x_0]$, then there exists*

$$\lim_{x\to x_0^-} f(x) = \ell \in [-\infty, \infty],$$

while if x_0 is an accumulation point of $E^+ := E \cap (x_0, \infty)$, then there exists

$$\lim_{x\to x_0^+} f(x) = L \in [-\infty, \infty].$$

Moreover, if f is increasing and x_0 is an accumulation point of E^- and E^+, then $\ell \leq L$, while if f is decreasing and x_0 is an accumulation point of E^- and E^+, then $\ell \geq L$.

Proof. Assume that f is increasing and let

$$\ell := \sup\{f(x) : x \in E,\ x < x_0\}.$$

Assume that $\ell \in \mathbb{R}$ (the case $\ell = \infty$ is left as an exercise). Fix $\varepsilon > 0$. We need to find $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$\ell - \varepsilon \leq f(x) \leq \ell + \varepsilon$$

for all $x \in E$ with $x_0 - \delta \leq x < x_0$. By the definition of supremumn, we have that $f(x) \leq \ell$ for all $x \in E$ with $x < x_0$. On the other hand, since $\ell - \varepsilon$ is not an upper bound, there exists $x_1 \in E$, with $x_1 < x_0$ such that $\ell - \varepsilon < f(x_1)$. Take $\delta := x_0 - x_1$. If $x \in E$ with $x_0 - \delta = x_1 \leq x < x_0$, then, since f is increasing,

$$\ell - \varepsilon < f(x_1) \leq f(x),$$

which gives the other inequality. ■

Exercise 162 *Prove that*

$$\lim_{x\to 0^+} x^a \log^b x = 0$$

for $a > 0$ and $b \in \mathbb{R}$. Hint: Use Exercise 82.

Another important theorem is the following.

Exercise 163 *Let $E \subseteq \mathbb{R}$, let $x_0 \in \mathbb{R}$ be an accumulation point of E, and let $f : E \to \mathbb{R}$ be a function. Assume that there exist*

$$\lim_{x\to x_0} f(x) = \ell \in (0, \infty].$$

Prove that there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that for all $x \in E$ with $|x - x_0| < \delta$ we have

$$f(x) > 0.$$

Remark 164 *A similar result continues to hold if $\ell < 0$.*

13 Limits at Infinity

Definition 165 *Let $E \subseteq \mathbb{R}$ be unbounded from above and let $f : E \to \mathbb{R}$. We say that*

- *a number $\ell \in \mathbb{R}$ is the* limit *of $f(x)$ as x diverges to ∞ if for every $\varepsilon > 0$ there exists a real number $M = M(\varepsilon) > 0$ with the property that*

$$|f(x) - \ell| \le \varepsilon$$

  for all $x \in E$ with $x \ge M$. We write

$$\lim_{x \to \infty} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to \infty.$$

- *∞ is the* limit *of $f(x)$ as x diverges to ∞ if for every $L > 0$ there exists a real number $M = M(L) > 0$ with the property that*

$$f(x) \ge L$$

  for all $x \in E$ with $x \ge M$. We write

$$\lim_{x \to \infty} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \text{ as } x \to \infty.$$

- *$-\infty$ is the* limit *of $f(x)$ as x diverges to ∞ if for every $L > 0$ there exists a real number $M = M(L) > 0$ with the property that*

$$f(x) \le -L$$

  for all $x \in E$ with $x \ge M$. We write

$$\lim_{x \to \infty} f(x) = -\infty \quad \text{or} \quad f(x) \to -\infty \text{ as } x \to \infty.$$

Note that if $E \subseteq \mathbb{R}$ be unbounded from above, then there exists a sequence $\{x_n\} \subseteq E$ such that $x_n \to \infty$, so the previous definition makes sense, since for every $M > 0$ there will always be infinitely many $x \in E$ with $x \ge M$.

Similarly, we have

Definition 166 *Let $E \subseteq \mathbb{R}$ be unbounded from below and let $f : E \to \mathbb{R}$. We say that*

- *a number $\ell \in \mathbb{R}$ is the* limit *of $f(x)$ as x diverges to $-\infty$ if for every $\varepsilon > 0$ there exists a real number $M = M(\varepsilon) > 0$ with the property that*

$$|f(x) - \ell| \le \varepsilon$$

  for all $x \in E$ with $x \le -M$. We write

$$\lim_{x \to -\infty} f(x) = \ell \quad \text{or} \quad f(x) \to \ell \text{ as } x \to -\infty.$$

- ∞ *is the* limit *of* $f(x)$ *as* x *diverges to* $-\infty$ *if for every* $L > 0$ *there exists a real number* $M = M(L) > 0$ *with the property that*

$$f(x) \geq L$$

for all $x \in E$ *with* $x \leq -M$. *We write*

$$\lim_{x \to -\infty} f(x) = \infty \quad \textit{or} \quad f(x) \to \infty \textit{ as } x \to -\infty.$$

- $-\infty$ *is the* limit *of* $f(x)$ *as* x *diverges to* $-\infty$ *if for every* $L > 0$ *there exists a real number* $M = M(L) > 0$ *with the property that*

$$f(x) \leq -L$$

for all $x \in E$ *with* $x \leq -M$. *We write*

$$\lim_{x \to -\infty} f(x) = -\infty \quad \textit{or} \quad f(x) \to -\infty \textit{ as } x \to -\infty.$$

Theorems 143, 146, 148, 155 continue to hold if we replace $x_0 \in \mathbb{R}$ with ∞ or $-\infty$. We omit the details.

Exercise 167 *Calculate the following limits*

1. $\displaystyle\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$,
2. $\displaystyle\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x$,
3. $\displaystyle\lim_{x \to 0^+} (1 + x)^{\frac{1}{x}}$.

Exercise 168 *Using the previous exercise, prove that*

$$\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$$

and that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Friday, October 26, 2012

14 Continuity

Definition 169 *Let* $E \subseteq \mathbb{R}$. *A point* $x_0 \in E$ *is called an* isolated point *of* E *if there exists* $\delta > 0$ *such that*

$$(x_0 - \delta, x_0 + \delta) \cap E = \{x_0\}.$$

It is clear that if a point of the set E is not an isolated point of E then it is an accumulation point of E.

Definition 170 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$. Given a function $f : E \to \mathbb{R}$ we say that f is* continuous *at x_0 if for every $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon, x_0) > 0$ such that for all $x \in E$ with $|x - x_0| < \delta$ we have*

$$|f(x) - f(x_0)| < \varepsilon.$$

If f is continuous at every point of E we say that f is continuous *on E and we write $f \in C(E)$ or $f \in C^0(E)$.*

Theorem 171 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$. Given a function $f : E \to \mathbb{R}$,*

(i) if x_0 is an isolated point of E then f is continuous at x_0;

(ii) if x_0 is an accumulation point of E then f is continuous at x_0 if and only if there exists $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Proof of part (i). If x_0 is an isolated point of E then there exists $\delta_0 > 0$ such that

$$(x_0 - \delta_0, x_0 + \delta_0) \cap E = \{x_0\}.$$

Fix $\varepsilon > 0$ and take $\delta := \delta_0$ in the definition of continuity. Clearly if $x \in E$ and $|x - x_0| < \delta$ then necessarily $x = x_0$ so that we have $|f(x) - f(x_0)| = 0$. ■

Exercise 172 *Prove that the functions $\sin x$, $\cos x$, x^n, where $n \in \mathbb{N}$, are continuous.*

The following theorems follows from the analogous results for limits.

Theorem 173 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$. Given two functions f, $g : E \to \mathbb{R}$ assume that f and g are continuous at x_0. Then*

(i) $f + g$ and fg are continuous at x_0;

(ii) if $g(x_0) \neq 0$ then $\dfrac{f}{g}$ restricted to the set $F := \{x \in E : g(x) \neq 0\}$ is continuous at x_0.

Example 174 *In view of Exercise 172 and the previous theorem, the functions $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$ are continuous in their domain of definition.*

Theorem 175 *Let E, $F \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an accumulation point of E. Given two functions $f : E \to F$ and $g : F \to \mathbb{R}$ assume that f is continuous at x_0 and that g is continuous at $f(x_0)$. Then $g \circ f : E \to \mathbb{R}$ is continuous at x_0.v*

15 Discontinuities

Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$. Given $x_0 \in E$, what happens when f is discontinuous at x_0? Then x_0 is an accumulation point of E. The following situations can arise. It can happen that there exists

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$$

but $\ell \neq x_0$. In this case, we say that x_0 is a *removable discontinuity*. Indeed, consider the function $g : E \to \mathbb{R}$ defined by

$$g(x) := \begin{cases} f(x) & \text{if } x \neq x_0, \\ \ell & \text{if } x = x_0. \end{cases}$$

Then there exists

$$\lim_{x \to x_0} g(x) = \ell = g(x_0),$$

and so the new function g is continuous at x_0.

Another type of discontinuity is when x_0 is an accumulation point of $E^- := E \cap (-\infty, x_0]$ and of $E^+ := E \cap (x_0, \infty)$ and there exist

$$\lim_{x \to x_0^-} f(x) = \ell \in \mathbb{R}, \quad \lim_{x \to x_0^+} f(x) = L \in \mathbb{R}$$

but $\ell \neq L$. In this case the point x_0 is called a *jump discontinuity* of f.

Example 176 *The integer and fractional part of x have jump discontinuity at every integer.*

An important class of functions that exhibit only jump discontinuities are monotone functions.

Definition 177 *A set $E \subseteq \mathbb{R}$ is* countable *if there exists an injective function $f : E \to \mathbb{N}$.*

Example 178 *The following sets are countable.*

1. *A finite set E is countable. Let $E = \{x_1, x_2, \ldots, x_k\}$ and define the function $f(x_i) := i$. Then f is injective.*
2. *The set of integers $\mathbb{Z}$ is countable. Define*

$$f_1(k) := \begin{cases} 2^k & \textit{if } k \geq 1, \\ 3^{-k} & \textit{if } k < 0. \end{cases}$$

Then $f_2 : \mathbb{Z} \to \mathbb{N}$ is injective (exercise).

3. *The set of integers $\mathbb{N} \times \mathbb{N}$ is countable. Define*

$$f_2(n, m) := 2^n 3^m$$

Then $f_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is injective (exercise).

4. *The set of integers* $\mathbb{Z} \times \mathbb{N}$ *is countable. Define*

$$f_3(k,m) := \begin{cases} 2^k 5^m & \text{if } k \geq 1, \\ 3^{-k} 5^m & \text{if } k < 0. \end{cases}$$

   Then $f_3 : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$ *is injective (exercise).*

5. *The set of rationals* $\mathbb{Q}$ *is countable. Given* $r \in \mathbb{Q}$ *write* $r = \frac{k}{m}$ *where* $m \in \mathbb{N}$ *and* $k \in \mathbb{Z}$*, where* $r = 0$ *take* $k = 0$ *and* $m = 1$*, while if* $r \neq 0$*, take* m *and* n *with no common divisors. The function* $f_4 : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ *defined by*

$$f_4(k,m) := (k,m)$$

   Since composition of injective functions is injective the composition of f_4 *and* f_3 *is injective (exercise) and so* $\mathbb{Q}$ *is countable.*

Exercise 179 *Prove that if* $E_n \subseteq \mathbb{R}$*,* $n \in \mathbb{N}$*, is countable, then*

$$E = \bigcup_{n=1}^{\infty} E_n$$

is countable.

Remark 180 *It can be shown that* $\mathbb{R}$ *is NOT countable.*

Example 181 *The set of irrationals* $\mathbb{R} \setminus \mathbb{Q}$ *is not countable. Indeed, if* $\mathbb{R} \setminus \mathbb{Q}$ *were countable, then since* $\mathbb{Q}$ *is countable, it would follows that* $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ *and so* $\mathbb{R}$ *would be countable, since union of two countable sets.*

Example 182 *The set* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is not countable. Indeed,*

Definition 183 *A set* $I \subseteq \mathbb{R}$ *is an* interval *if for every* $x, y \in I$*, with* $x < y$*, we have that the interval* $[x,y]$ *is contained in* I*.*

Theorem 184 *Let* $I \subseteq \mathbb{R}$ *be an interval and let* $f : I \to \mathbb{R}$ *be a monotone function. Then* f *has at most countably many discontinuity points.*

Proof. Step 1: Assume that $I = [a,b]$ and, without loss of generality, that f is increasing. For every $x \in (a,b)$ there exist

$$\lim_{y \to x^+} f(y) =: f_+(x), \quad \lim_{y \to x^-} f(y) =: f_-(x).$$

Let

$$E := \{x \in (a,b) : f \text{ is discontinuous at } x\}.$$

Let $S(x) := f_+(x) - f_-(x) \geq 0$ be the jump of f at x. Then f is continuous at x if and only if $S(x) = 0$. Hence,

$$E = \{x \in (a,b) : S(x) > 0\}.$$

For each $n \in \mathbb{N}$ define

$$E_n := \left\{ x \in (a, b) : S(x) \geq \frac{1}{n} \right\}.$$

Fix $n \in \mathbb{N}$. We claim that E_n has at most $\ell := \lfloor n(f(b) - f(a)) \rfloor$ elements. To see this, assume by contradiction that E_n has more than ℓ elements. Let $x_1, \dots, x_{\ell+1} \in E$. By reordering the lements, we can assume that

$$x_1 < x_2 < \cdots < x_{\ell+1}.$$

Since f is increasing, we have that

$$\begin{aligned} f(a) &\leq f_-(x_1) \leq f_+(x_1) \leq f_-(x_2) \leq f_+(x_2) \\ &\leq \cdots \leq f_-(x_{\ell+1}) \leq f_+(x_{\ell+1}) \leq f(b), \end{aligned}$$

and so, also using the definition of E_n,

$$\frac{\ell+1}{n} \leq \sum_{i=1}^{\ell+1} S(x_i) = \sum_{i=1}^{\ell+1} (f_+(x_i) - f_-(x_i)) \leq f(b) - f(a),$$

which implies that $\ell + 1 \leq n(f(b) - f(a))$. This contradicts the fact that $\ell := \lfloor n(f(b) - f(a)) \rfloor$. Hence, E_n has finitely elements, and so it is countable.

Next we claim that

$$E = \bigcup_{n=1}^{\infty} E_n. \tag{22}$$

Since $E_n \subseteq E$ for every $n \in \mathbb{N}$, we have that $\bigcup_{n=1}^{\infty} E_n \subseteq E$. To prove the opposite inclusion, let $x \in E$. Then $S(x) > 0$. By the Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{S(x)}$ and so $S(x) > \frac{1}{n_0}$. This means that $x \in E_{n_0}$, and, in turn, that $x \in \bigcup_{n=1}^{\infty} E_n$. Thus, we have proved that $E \subseteq \bigcup_{n=1}^{\infty} E_n$.

Since (22) holds and each E_n is countable, it follows by Exercise 179 that E is countable.

Step 2: If I is an arbitrary interval, construct an increasing sequence of intervals $[a_n, b_n]$ such that

$$a_n \searrow \inf I, \quad b_n \nearrow \sup I.$$

Since the union of countable sets is countable by Exercise 179 and on each interval $[a_n, b_n]$ the set of discontinuity points of f is at most countable, by the previous step it follows that the set of discontinuity points of f in I is at most countable. ■

Finally, the last type of discontinuity is when at least one of the limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ is not finite or does not exist. In this case, the point x_0 is called an *essential discontinuity* of f.

Example 185 *The function*

$$f(x) := \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

and

$$g(x) := \begin{cases} \log x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \end{cases}$$

have an essential discontinuity at $x = 0$.

Monday, October 29, 2012

16 Important Theorems on Continuity

In this section we study some important consequences of continuity. The next theorem shows that continuity preserves the sign of a function.

Theorem 186 *Let* $E \subseteq \mathbb{R}$ *,and let* $f : E \to \mathbb{R}$ *be continuous at some* $x_0 \in E$ *with* $f(x_0) > 0$. *Then there exists* $\delta = \delta(\varepsilon, x_0) > 0$ *such that for all* $x \in E$ *with* $|x - x_0| < \delta$ *we have*

$$f(x) > 0.$$

Proof. This follows from 186. ■

Remark 187 *A similar result continues to hold if* $f(x_0) < 0$.

Example 188 *The previous theorem implies in particular that sets of the form*

$$\{x \in \mathbb{R} : 4 \sin x - \log(1 + |x|) > 0\}$$

are open. We used this in the exercises.

More generally, we have the following theorem.

Definition 189 *Given a set* $E \subseteq \mathbb{R}$, *a set* $F \subseteq E$ *is said to be* relatively open *in* E *if there exists an open set* $U \subseteq \mathbb{R}$ *such that* $F = U \cap E$. *A set* $G \subseteq E$ *is said to be* relatively closed *in* E *if there exists a closed set* $C \subseteq \mathbb{R}$ *such that* $G = C \cap E$.

Theorem 190 *Let* $E \subseteq \mathbb{R}$ *and let* $f : E \to \mathbb{R}$.

(i) The function f *is continuous if and only if* $f^{-1}(U)$ *is relatively open for every open set* $U \subseteq \mathbb{R}$.

(ii) The function ,f *is continuous if and only if* $f^{-1}(C)$ *is relatively closed for every closed set* $C \subseteq \mathbb{R}$.

Proof. Exercise. ■

Remark 191 *The previous characterization of continuous functions is useful to define continuity in a topological space.*

Another important theorem on continuity is the following.

Theorem 192 (Zeros of a continuous function) *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. If there exist $x_1, x_2 \in I$ such that $f(x_1) < 0$ and $f(x_2) > 0$, then there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = 0$.*

Proof. Without loss of generality, we may assume that $x_1 < x_2$, the case $x_1 > x_2$ is similar. Then $[x_1, x_2] \subseteq I$, and so we may define the set

$$E := \{x \in [x_1, x_2] : f(x) < 0\}.$$

Let $x_0 := \sup E$. Then $x_0 \in [x_1, x_2]$ and $f(x) \geq 0$ for all $x \in (x_0, x_2]$.

We claim that $x_1 < x_0 < x_2$. Indeed, since $f(x_1) < 0$, by Theorem 186, we can find $\delta_1 > 0$ such that $f(x) < 0$ for all $x \in I \cap (x_1 - \delta_1, x_1 + \delta_1)$. Note that this imply that $\delta_1 \leq x_2 - x_1$. It follows that $x_0 \geq x_1 + \delta > x_1$. Similarly, since $f(x_2) > 0$, by Theorem 186, we can find $\delta_2 > 0$ such that $f(x) > 0$ for all $x \in I \cap (x_2 - \delta_2, x_2 + \delta_2)$, so that $\delta_2 \leq x_2 - x_1$ and $x_0 \leq x_2 - \delta_2 < x_2$.

Next, we claim that $f(x_0) = 0$. Indeed, if $f(x_0) < 0$, then by Theorem 186, we can find $\delta > 0$ such that $f(x) < 0$ for all $x \in I \cap (x_0 - \delta_0, x_0 + \delta_0)$. Since $x_0 < x_2$, taking $\delta_0 < x_2 - x_0$, this implies that $f(x) < 0$ for all $x \in [x_0, x_0 + \delta_0)$, which contradicts the fact that x_0 is the supremum of E.

On the other hand, if $f(x_0) > 0$, then again by Theorem 186, we can find $\delta > 0$ such that $f(x) > 0$ for all $x \in I \cap (x_0 - \delta_0, x_0 + \delta_0)$. Since $x_0 < x_2$, taking $\delta_0 < x_0 - x_1$, this implies that $f(x) > 0$ for all $x \in (x_0 - \delta_0, x_0]$. Together with the fact that $f(x) \geq 0$ for all $x \in (x_0, x_2]$, it follows that $E \subseteq [x_1, x_0 - \delta_0]$, and so its supremum cannot be x_0.This shows that $f(x_0) = 0$. ■

Remark 193 *By applying the previous theorem to the function $g(x) = f(x) - t$, we can show that if $I \subseteq \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ a continuous function such that $f(x_1) < t$ and $f(x_2) > t$ for some $x_1, x_2 \in I$, then there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = t$.*

Corollary 194 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Then f takes all the values between $\inf_I f$ and $\sup_I f$. Moreover, $f(I)$ is an interval (possibly degenerate).*

Proof. If $\inf_I f = \sup_I f$, then f is constant. In this case $f(I)$ is a singleton and there is nothing to prove.

Thus, in what follows we assume that $\inf_I f < \sup_I f$ and let $\inf_I f < t < \sup_I f$. By the definition of infimum and of supremum, there exist $x_1, x_2 \in I$ such that $f(x_1) < t$ and $f(x_2) > t$. By the previous remark, there exists $x_0 \in I$ in the interval of endpoints x_1, x_2 such that $f(x_0) = t$. This shows that $f(I) \supseteq (\inf_I f, \sup_I f)$. It remains to show that $f(I)$ is an interval. Let $y_1, y_2 \in f(I)$ with $y_1 < y_2$ and let $y_1 < y < y_2$. Since $\inf_I f \leq y_1 < y_2 \leq \sup_I f$, by what we just proved, there exists $x \in I$ such that $f(y) = x$. ■

Corollary 195 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a monotone function. Then f is continuous if and only if $f(I)$ is an interval.*

Proof. If f is continuous, then the result follows from the previous corollary. Assume that $f(I)$ is an interval. We claim that f is continuous. Let $x_0 \in I$ and assume that x_0 is an interior point (the case in which x_0 is an endpoint of I is similar). Assume that f is not continuous at x_0. Then by Theorem 161 there exist

$$\lim_{x \to x_0^-} f(x) = \ell \in [-\infty, \infty] < \lim_{x \to x_0^+} f(x) = L \in [-\infty, \infty].$$

Since f is increasing, we have that

$$\ell = \sup_{x < x_0} f(x), \quad L = \inf_{x > x_0} f(x),$$

and so $f(x) \leq \ell$ for all $x < x_0$, while $f(x) \geq L$ for all x_0. This implies that $f(I)$ does not contain the set $(\ell, L) \setminus \{f(x_0)\}$, which contradicts the fact that $f(I)$ is an interval because f takes values less than or equal to ℓ and values greater than or equal to L. ■

Wednesday, October 31, 2012

Next we show that continuous functions preserve compactness.

Proposition 196 *Consider a continuous function $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. Then $f(K)$ is sequentially compact for every sequentially compact set $K \subseteq E$.*

Proof. Let $K \subseteq E$ be sequentially compact. By Theorem 135, it is enough to show that $f(K)$ is sequentially compact. Let $\{y_n\} \subseteq f(K)$. Then for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that $f(x_n) = y_n$. Then there exist a subsequence $\{x_{n_k}\}$ and $x_0 \in K$ such that $x_{n_k} \to x_0$. By the continuity of f,

$$y_{n_k} = f(x_{n_k}) \to f(x_0) \in f(K).$$

■

The following theorem is important.

Theorem 197 (Weierstrass) *Let $K \subset \mathbb{R}$ be sequentially compact and let $f : K \to \mathbb{R}$ be a continuous function. Then there exists $x_0, x_1 \in K$ such that*

$$f(x_0) = \min_{x \in K} f(x), \quad f(x_1) = \max_{x \in K} f(x).$$

Proof. Let

$$t := \inf_{x \in K} f(x).$$

By the previous proposition, t is a real number. For every $n \in \mathbb{N}$, $t + \frac{1}{n}$ is not a lower bound, and so by the definition of infimum, we may find $x_n \in K$ such that

$$t < f(x_n) < t + \frac{1}{n}.$$

Letting $n \to \infty$, by the squeeze theorem, we get

$$\lim_{n\to\infty} f(x_n) = t. \tag{23}$$

Since $\{x_n\} \subseteq K$, and K is sequentially compact (see Theorem 135), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Using the continuity of f and (23), we get

$$t = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f(x),$$

which shows that the infimum is a minimum. ■

The sequence $\{x_n\}$ constructed in the previous proof is called a *minimizing sequence*.

Remark 198 *Note that to prove the existence of a minimum, we only used a weaker form of continuity, namely that the set*

$$\liminf_{j\to\infty} f(x_j) \geq f(x)$$

for all sequences $\{x_j\}$ converging to $x \in \mathbb{R}$. A function satisfying this property is called sequentially lower semicontinuous*.*

Remark 199 *A typical application of the Weierstrass theorem is the following. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f is bounded from below, so that*

$$\ell = \inf_{x\in\mathbb{R}} f(x) > -\infty$$

and that

$$\lim_{|x|\to\infty} f(x) = \infty.$$

By the definition of limit, we can find $R > 0$ such that $f(x) > \ell$ for all $x \in \mathbb{R}$ such that $|x| \geq R$. Thus,

$$\ell = \inf_{x\in\mathbb{R}} f(x) = \inf_{x\in[-R,R]} f(x).$$

By the Weierstrass theorem, f has a minimum in $[-R, R]$ and we are done.

We now discuss the continuity of inverse functions and of composite functions. If a continuous function f is invertible its inverse function f^{-1} may not be continuous.

Example 200 *Let*

$$f(x) := \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ x-1 & \text{if } 2 < x \leq 3. \end{cases}$$

Then $f^{-1} : [0,2] \to R$ is given by

$$f^{-1}(x) := \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ x+1 & \text{if } 1 < x \leq 2, \end{cases}$$

which is not continuous at $x = 1$.

We will see that this cannot happen if E is an interval or a sequentially compact set.

Theorem 201 *Let $K \subset \mathbb{R}$ be a sequentially compact set and let $f : K \to \mathbb{R}$ be one-to-one and continuous. Then the inverse function $f^{-1} : f(K) \to \mathbb{R}$ is continuous.*

Proof. Let $y_0 \in f(K)$ and let $y_n \in f(K)$ be such that $y_n \to y_0$. To prove that f^{-1} is continuous at y_0, we need to show that there exists

$$\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0).$$

Assume by contradiction that this does not happen. Then there exists $\varepsilon > 0$ such that

$$\left|f^{-1}(y_n) - f^{-1}(y_0)\right| > \varepsilon$$

for infinitely many n, so we can find a subsequence $\left\{f^{-1}(y_{n_k})\right\}_k$ for which the previous inequality holds. Let $x_k := f^{-1}(y_{n_k}) \in K$. Since K is sequentia;;y compact, there exist a subsequence $\{x_{k_i}\}$ and $x \in K$ such that $x_{k_i} \to x$. Note that, since

$$\left|x_{k_i} - f^{-1}(y_0)\right| = \left|f^{-1}\left(y_{n_{k_i}}\right) - f^{-1}(y_0)\right| > \varepsilon,$$

letting $i \to \infty$ gives

$$\left|x - f^{-1}(y_0)\right| \geq \varepsilon,$$

and so $x \neq f^{-1}(y_0)$. On the other hand, since f is continuous, it follows that

$$\lim_{i\to\infty} f(x_{k_i}) = f(x).$$

But $f(x_{k_i}) = f\left(f^{-1}\left(y_{n_{k_i}}\right)\right) = y_{n_{k_i}} \to y_0$, and so $y_0 = f(x)$, which is a contradiction since f is injective. ■

Friday, November 02, 2012

Theorem 202 *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be one-to-one and continuous. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. Step 1: Let's prove that if $x_0 \in I^\circ$ then for every $x \in I$ with $x > x_0$, $f(x) - f(x_0)$ always has the same sign. Fix $x_1 \in I$ with $x_1 > x_0$ and say that $f(x_1) - f(x_0) > 0$ (the case $f(x_1) - f(x_0) < 0$ is similar). We claim that $f(x) - f(x_0) > 0$ for all $x \in I$ with $x > x_0$. If not, then there exists $x_2 \in I$ with $x_2 > x_0$ such that $f(x_2) - f(x_0) < 0$. But then by Remark 193 (with $t = f(x_0)$) we would find $x_3 \in I$ in the interval of endpoints x_1 and x_2 such that $f(x_3) = f(x_0)$, which contradicts the fact that f is one-to-one.

In a similar way we can show that for every $x \in I$ with $x < x_0$, $f(x) - f(x_0)$ always has the same sign.

Step 2: Let's prove that f is monotone. Fix $a, b \in I^\circ$ with $a < b$ and assume that $f(b) > f(a)$ (the case $f(b) < f(a)$ is similar). We claim that f is strictly

increasing. Let $x_1, x_2 \in I$ with $x_1 < x_2$ and let $b_1 = \max\{x_2, b\}$. Since $f(b) > f(a)$, by Step 1 (with $x_0 = a$), we have that $f(x) > f(a)$ for all $x \in I$ with $x > a$. In particular, $f(b_1) > f(a)$. Again by Step 1 (with $x_0 = b_1$), we have that $f(b_1) > f(x)$ for all $x \in I$ with $x < b_1$. In particular, $f(b_1) > f(x_1)$. By Step 1, once more, (with $x_0 = x_1$), we have that $f(x) > f(x_1)$ for all $x \in I$ with $x > x_1$. In particular, $f(x_2) > f(x_1)$, which proves that f is strictly increasing.
Step 3: Since f is continuous and I is an interval, by Corollary 194, $f(I)$ is an interval. Moreover, since f is monotone, it follows (exercise) that $f^{-1} : f(I) \to \mathbb{R}$ is also monotone. By Corollary 195 (applied to the function f^{-1}), it follows that f^{-1} is continuous. ■

Example 203 *In view of the previous theorem and of Exercise 172, the functions* $\arccos x$, $\arcsin x$, $\arctan x$ *are continuous.*

Given $a > 0$, the function $\log_a x$ is continuous for $x > 0$, since it is the inverse of a^x.

Given $n \in \mathbb{N}$, the function $\sqrt[2n+1]{x}$, $x \in \mathbb{R}$, is continuous, since it is the inverse of x^{2n+1}. The function $\sqrt[2n]{x}$, $x \in [0, \infty)$, is continuous, since it is the inverse of x^{2n}.

Given $a > 0$, since e^x and $\log x$ are continuous in $(0, \infty)$, by writing

$$x^a = e^{\log x^a} = e^{a \log x},$$
$$x^x = e^{\log x^x} = e^{x \log x},$$

it follows from Theorems 173 and 175, that x^a and x^x are continuous in $(0, \infty)$.

17 Uniform Continuity

Next we introduce the notion of uniform continuity.

Definition 204 *Consider a function $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. The function f is said to be* uniformly continuous *if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$|f(x) - f(y)| \le \varepsilon$$

for all $x, y \in E$ with $|x - y| \le \delta$.

Remark 205 *To negate uniform continuity it is enough to fins two sequences $\{x_n\}, \{y_n\} \subseteq E$ such that*

$$\lim_{n \to \infty} |x_n - y_n| = 0$$

and $|f(x_n) - f(y_n)| \nrightarrow 0$ (so either the limit does not exist or it exists but it is not zero).

Example 206 *The function* $f(x) = x$*,* $x \in \mathbb{R}$*, is uniformly continuous, while the function* $g(x) = x^2$*,* $x \in \mathbb{R}$*, is not. To see this, take* $\varepsilon = \delta$ *for the function* f*. To prove that* g *is not uniformly continuous, consider the two sequences* $x_n = n + \frac{1}{n}$ *and* $y_n = n$*. Then* $x_n - y_n = \frac{1}{n} \to 0$*, while*

$$f(x_n) - f(y_n) = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n} \to 2 \neq 0,$$

which implies that g *is not uniformly continuous, by the previous remark.*

Example 207 *The continuous function* $f(x) = \sin \frac{1}{x}$ *is not uniformly continuous, by the previous remark. Consider the two sequences* $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ *and* $y_n = \frac{1}{\frac{3\pi}{2} + 2n\pi}$*. Then* $x_n \to 0$*,* $y_n \to 0$*, so* $x_n - y_n \to 0$ *while*

$$f(x_n) - f(y_n) = 1 - (-1) \to 2 \neq 0,$$

Example 208 *The continuous function* $f(x) = \frac{1}{x}$ *is not uniformly continuous, by the previous remark. Consider the two sequences* $x_n = \frac{1}{n}$ *and* $y_n = \frac{1}{n+1}$*. Then* $x_n \to 0$*,* $y_n \to 0$*, so* $x_n - y_n \to 0$ *while*

$$f(x_n) - f(y_n) = n - (n+1) \to 1 \neq 0,$$

Monday, November 05, 2012

Solutions of Exercises 3 and 4 in Homework #6

Wednesday, November 07, 2012

Simple examples of uniformly continuous functions are Lipschitz continuous functions.

Definition 209 *Consider a function* $f : E \to \mathbb{R}$*, where* $E \subseteq \mathbb{R}$*. The function* f *is said to be* Lipschitz continuous *if there exists* $L > 0$ *such that*

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in E$*.*

Remark 210 *To negate Lipschitz continuity it is enough to fins two sequences* $\{x_n\}, \{y_n\} \subseteq E$ *with* $x_n \neq y_n$ *such that*

$$\left|\frac{f(x_n) - f(y_n)}{x_n - y_n}\right| \to \infty.$$

Proposition 211 *Let* $E \subseteq \mathbb{R}$ *and let* $f : E \to \mathbb{R}$ *be Lipschitz continuous. Then* f *is uniformly continuous.*

Proof. Since f is Lipschitz continuous, there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in E$.

Fix $\varepsilon > 0$, and take $\delta = \frac{\varepsilon}{L}$. If $x, y \in E$ with $|x - y| \leq \delta$, then

$$|f(x) - f(y)| \leq L|x - y| \leq L\delta = L\frac{\varepsilon}{L} = \varepsilon.$$

■

Example 212 *The function* $f(x) = \sqrt{x}$*,* $x \in [0,1]$*, is uniformly continuous (we will see this later), but not Lipschitz. Indeed, let* $x_n = 0$ *and* $y_n = \frac{1}{n}$*. Then*

$$\left| \frac{f(x_n) - f(y_n)}{x_n - y_n} \right| = \left| \frac{0 - \sqrt{\frac{1}{n}}}{0 - \frac{1}{n}} \right| = \sqrt{n} \to \infty,$$

which shows that f *is not Lipschitz continuous.*

Proposition 213 *If* $I \subseteq \mathbb{R}$ *is an interval and* $f : I \to \mathbb{R}$ *is differentiable in* I *and the derivative* f' *is bounded, then* f *is Lipschitz continuous.*

Proof. By hypothesis, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Let $x, y \in I$, with, say $y < x$. By the mean value theorem (we will prove this later), there exists $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y).$$

Hence,

$$|f(x) - f(y)| = |f'(z)| |x - y| \leq M |x - y|,$$

which shows that f is Lipschitz continuous. ■

Example 214 *The function* $f(x) = |x|$ *is Lipschitz continuous, since*

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|,$$

but it is not differentiable in $x = 0$*.*

Remark 215 *Thus we have shown that if* $I \subseteq \mathbb{R}$ *is an interval and* $f : I \to \mathbb{R}$*, then*

$$\begin{aligned} f \text{ differentiable with bounded derivative } &\Longrightarrow f \text{ Lipschitz continuous} \\ &\Longrightarrow f \text{ uniformly continuous } \Longrightarrow f \text{ continuous} \end{aligned}$$

but none of the opposite implications is true.

Example 216 *The functions* $f(x) = \cos x$*,* $f(x) = \sin x$*,* $f(x) = \arctan x$ *all have bounded derivatives, and so they are Lipschitz continuous, and, in turn, uniformly continuous.*

To prove that the function $f(x) = \sqrt{x}$, $x \in [0,1]$, is uniformly continuous in $[0,1]$, we apply the following theorem:

Theorem 217 *Let* $K \subseteq \mathbb{R}$ *be sequentially compact and let* $f : K \to \mathbb{R}$ *be a continuous function. Then* f *is uniformly continuous.*

Proof. Assume by contradiction that f is not uniformly continuous. Then there exist $\varepsilon_0 > 0$ and two sequences $\{x_n\}, \{y_n\} \subseteq K$ such that

$$\lim_{n\to\infty} |x_n - y_n| = 0$$

and $|f(x_n) - f(y_n)| > \varepsilon_0$. Since K is sequentially compact, there exist a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and $x \in K$ such that $x_{n_k} \to x_0$ as $k \to \infty$. It follows that

$$|y_{n_k} - x_0| = |y_{n_k} \pm x_{n_k} - x_0| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - x_0| \to 0$$

and so by the squeeze theorem $y_{n_k} \to x_0$. By continuity of f at x_0, we have that $f(x_{n_k}) \to f(x_0)$ and $f(y_{n_k}) \to f(x_0)$, which implies that $f(x_{n_k}) - f(y_{n_k}) \to f(x_0) - f(x_0) = 0$. Hence, for all k sufficiently large

$$|f(x_{n_k}) - f(y_{n_k}) - 0| \leq \frac{\varepsilon_0}{2},$$

which contradicts the fact that $|f(x_{n_k}) - f(y_{n_k})| > \varepsilon_0$ for all k. ■

Example 218 *Since* $[0,1]$ *is sequentially compact and the function* $f(x) = \sqrt{x}$, $x \in [0,1]$, *is continuous, it follows by the previous theorem that it is uniformly continuous.*

Example 219 *We have seen that the continuous function* $f(x) = x^2$, $x \in \mathbb{R}$, *is not uniformly continuous in* $\mathbb{R}$. *However, by the previous theorem it is uniformly continuous in every set* $[a,b]$.

Example 220 *Next we study continuous functions of the type* $f(x) = \sin \frac{1}{x}$. *Consider the set* $E = (0,1]$. *This set is not sequentially compact (it is bounded but not closed), thus we cannot apply the previous theorem.*

Example 221 *Consider the function* $f(x) = \frac{\sin x}{x}$ *in* $(0,1]$. *Let's prove that* f *is uniformly continuous. Consider the function*

$$g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x = 0. \end{cases}$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, *we have that* g *is continuous in* $[0,1]$ *and since* $[0,1]$ *is sequentially compact,* g *is uniformly continuous. If we restrict* g *to* $(0,1]$, *it remains uniformly continuous, and so* f *is uniformly continuous.*

The previous example can be generalized.

Proposition 222 *Let* $f : E \to \mathbb{R}$ *be uniformly continuous, where* $E \subseteq \mathbb{R}$. *Then for every* $x \in \overline{E} \setminus E$ *there exists the limit*

$$\lim_{y\to x} f(y) \in \mathbb{R}$$

and the function $g : \overline{E} \to \mathbb{R}$, *defined by*

$$g(x) := \begin{cases} f(x) & \text{if } x \in E, \\ \lim_{y\to x} f(y) & \text{if } x \in \overline{E} \setminus E, \end{cases}$$

is uniformly continuous in $\overline{E}$.

Proof. Exercise. ■

Exercise 223 *Given a function $f : [0, \infty) \to \mathbb{R}$, prove that if f is uniformly continuous, then there exist A, $B \in \mathbb{R}$ such that*

$$|f(x)| \leq A + Bx$$

for all $x \geq 0$.

Exercise 224 *Let $f : [a, \infty) \to \mathbb{R}$ be a continuous function.*

1. *Prove that if $g : [a, \infty) \to \mathbb{R}$ is uniformly continuous and*

$$\lim_{x \to \infty} (f(x) - g(x)) = 0$$

   then f is uniformly continuous.

2. *Prove that if there exist A, $B \in \mathbb{R}$ such that*

$$\lim_{x \to \infty} (f(x) - A - Bx) = 0$$

   then f is uniformly continuous.

Friday, November 09, 2012

Second midterm.

Monday, November 12, 2012

18 Differentiation

Definition 225 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$ we say that f is* differentiable *at x_0 if there exists*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$$

In this case the limit is called derivative *of f at x_0 and is denoted $f'(x_0)$ or $\frac{df}{dx}(x_0)$. If f is differentiable at every point of $E \cap \operatorname{acc} E$, we say that f is* differentiable *on E.*

Definition 226 *Given two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ and a point $x_0 \in \operatorname{acc} E$, we say that the function f is a* little o *of g as $x \to x_0$, and we write $f = \mathrm{o}(g)$, if $g \neq 0$ in E and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Hence, a little o of g is simply a function that goes to zero faster than g as $x \to x_0$.

If f is differentiable at x_0, then by setting $x := x_0 + h$ we may write

$$f(x_0 + h) - f(x_0) = f'(x_0)h + o(h). \tag{24}$$

Note that (24) expresses $f(x_0 + h) - f(x_0)$ as the sum of the linear functional

$$\begin{aligned} L : \mathbb{R} &\to \mathbb{R} \\ h &\mapsto f'(x_0)h \end{aligned}$$

plus a small reminder. We can therefore regard the derivatives of f at x_0, not as a real number, but as a linear operator on $\mathbb{R}$ that takes h to $f'(x_0)h$. So an alternative definition of differentiability is the following

Definition 227 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$ we say that f is* differentiable *at x_0 if there exists a linear functional $L_{x_0} : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L_{x_0}(x - x_0)}{x - x_0} = 0.$$

This alternative form will be useful when we define differentiability for functions of several variables.

Theorem 228 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. If a function $f : E \to \mathbb{R}$ is differentiable at x_0, then it is continuous at x_0.*

Proof. For $x \in E$, $x \neq x_0$, write

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0).$$

Then by the product of limits

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \to f'(x_0) \cdot 0$$

as $x \to x_0$. ∎

The converse of this theorem is not true.

Example 229 *(a) The functions $f(x) := |x|$ and $g(x) := \sqrt{x}$ are continuous but not differentiable at $x = 0$.*

(b) By direct application of the definitions one easily proves that the derivative of any constant is clearly zero and that if $f(x) := x$ then $f'(x) = 1$. The basic elementary functions $\sin x$, $\cos x$, e^x are known to satisfy

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (e^x)' = e^x.$$

To prove that $(e^x)' = e^x$ we need a few preliminary results.

Lemma 230
$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

Proof. If $x \to 0^+$, then $y = \frac{1}{x} \to \infty$, and so changing variables,

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{y\to\infty} \left(1+\frac{1}{y}\right)^y = e,$$

while if $x \to 0^-$, then $y = \frac{1}{x} \to -\infty$, and so changing variables,

$$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{y\to-\infty} \left(1+\frac{1}{y}\right)^y = e.$$

Thus, by the continuity of $\log x$,

$$\frac{\log(1+x)}{x} = \log(1+x)^{\frac{1}{x}} \to \log e = 1.$$

■

Lemma 231 $\lim_{x\to 0} \frac{e^x-1}{x} = 1$.

Proof. Put $y = e^x - 1$. Then $y + 1 = e^x$ and so $\log(y+1) = x$. If $x \to 0$, then $y \to 0$, and so changing variables,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log(y+1)} = 1$$

■

Theorem 232 $(e^x)' = e^x$.

Proof. Let $x_0 \in \mathbb{R}$. Then for $x \neq x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{e^x - e^{x_0}}{x - x_0} = e^{x_0}\frac{e^{x-x_0} - 1}{x - x_0}.$$

Let $t = x - x_0$. When $x \to x_0$, we have that $t \to 0$, and so

$$\lim_{x\to x_0} \frac{e^x - e^{x_0}}{x - x_0} = \lim_{t\to 0} e^{x_0}\frac{e^t - 1}{t} = e^{x_0}1,$$

where we have used the previous lemma. ■

Theorem 233 $(\sin x)' = \cos x$.

Proof. Let $x_0 \in \mathbb{R}$. Then for $x \neq x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sin x - \sin x_0}{x - x_0} = 2\frac{\sin\left(\frac{x-x_0}{2}\right)}{x - x_0}\cos\left(\frac{x+x_0}{2}\right).$$

Let $t = \frac{x-x_0}{2}$. When $x \to x_0$, we have that $t \to 0$, and so

$$\lim_{x\to x_0}\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$$

Hence, as $x \to x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}\cos\left(\frac{x+x_0}{2}\right) \to 1\cos\left(\frac{x_0+x_0}{2}\right) = \cos x_0,$$

where we have used Theorem 151 and Exercise 154. ■

Theorem 234 (Weierstrass) *Let $0 < a < 1$ and let $b \in \mathbb{N}$ be an odd integer such that $ab > 1 + \frac{3}{2}\pi$. Then the function*

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad x \in \mathbb{R},$$

is continuous, but nowhere differentiable.

We now list some elementary operations for derivatives.

Theorem 235 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given two functions $f, g : E \to \mathbb{R}$ assume that f and g are differentiable at x_0. Then*

(a) $f + g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;

(b) fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$;

(c) if $g(x_0) \neq 0$ then $\dfrac{f}{g}$ restricted to the set $F := \{x \in E : g(x) \neq 0\}$ differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Example 236 *Repeated application of (b) and (c) shows that if $k \in \mathbb{Z}$ then the function $f(x) = x^k$ is differentiable with $f'(x) = kx^{k-1}$ and that $(\tan x)' = \tan^2 x + 1$.*

Of course, if $k < 0$ we have to restrict ourselves to $x \neq 0$

We now study the differentiation of inverse and composite functions. We begin with composite functions.

Theorem 237 (Chain rule) *Let E, $F \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given two functions $f : E \to F$ and $g : F \to \mathbb{R}$ assume that f is differentiable at x_0, that $f(x_0)$ is an accumulation point of F and that g is differentiable at $f(x_0)$. Then $g \circ f : E \to \mathbb{R}$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0).$$

Proof. Consider the function

$$h(y) := \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & \text{if } y \neq f(x_0), \\ g'(f(x_0)) & \text{if } y = f(x_0). \end{cases}$$

Since g is differentiable at x_0, we have that $\lim_{y \to f(x_0)} h(y) = h(f(x_0)) = g'(f(x_0))$. For $x \in E$, $x \neq x_0$, write

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x)) \frac{f(x) - f(x_0)}{x - x_0}.$$

Since f is differentiable at x_0, it is continuous at x_0. But h is continuous at $f(x_0)$, thus the composition $h \circ f$ is continuous at x_0. It follows that $\lim_{x \to x_0} h(f(x)) = h(f(x_0)) = g'(f(x_0))$. Hence, by the product of limits

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \to g'(f(x_0)) f'(x_0)$$

as $x \to x_0$. ■

Wednesday, November 14, 2012

Example 238 *The function*

$$f(x) := \begin{cases} x^2 \sin \dfrac{1}{x} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0 \end{cases}$$

is continuous in $\mathbb{R}$ since by Example 149

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 = f(0).$$

Using Theorem 233 and Theorems 235 and 237 for $x \neq 0$ we have

$$\frac{d}{dx}\left(x^2 \sin \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

For $x = 0$ we cannot apply the previous theorems and so we have to use the definition. We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x} \to 0$$

as $x \to 0$ again by Example 149 and so $f'(0) = 0$. Note that

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$ since the limit

$$\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist (exercise).

Remark 239 (Important) *In the exercises, when checking the differentiability of a function f, one has to distinguish between "good" points in the domain, where one can apply Theorems 235, 237, and* **??**, *and "bad" points in the domain, where one has to use the definition. Examples of bad points are points where an absolute vaue is zero (for example, for $f(x) = |\sin x|$ all points where $\sin x = 0$, that is, $x = 2k\pi$, are bad points), or where the argument of an n-root is zero (for example, for $f(x) = \sqrt[3]{\sin^2 x}$, all points where $\sin^2 x = 0$, that is, $x = 2k\pi$, are bad points), or points where the law of the function changes as $x = 0$ in the previous example.*

Theorem 240 *Let $E \subseteq \mathbb{R}$ and let $x_0 \in E$ be an accumulation point of E. Given a function $f : E \to \mathbb{R}$ assume that f is one-to-one and differentiable at x_0. Suppose also that the inverse function $f^{-1} : f(E) \to \mathbb{R}$ is continuous at $f(x_0)$. Then $f(x_0)$ is an accumulation point of $f(E)$ and f^{-1} is differentiable at $f(x_0)$ if and only if $f'(x_0) \neq 0$ and in this case*

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Proof. We first prove that $f(x_0)$ is an accumulation point of the set $f(E)$. Since x_0 is an accumulation point of E by Remark 50 there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ which converges to x_0. Define $y_n := f(x_n)$. Since f is one-to-one and $\{x_n\} \subseteq E \setminus \{x_0\}$ we have that $\{y_n\} \subseteq f(E) \setminus \{f(x_0)\}$. By the continuity of f in x_0 (which follows from Theorem 228) and Theorems 171 and 143 we have that $y_n = f(x_n) \to f(x_0)$. Hence $f(x_0)$ is an accumulation point of the set $f(E)$.

Next we show that $f^{-1} : f(E) \to \mathbb{R}$ is differentiable at $f(x_0)$ if and only if $f'(x_0) \neq 0$. Consider the quotient

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)}$$

for $y \in f(E)$. For every $y \in f(E)$ there exists a unique $x \in E$ such that $f(x) = y$ and so we may write

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{x - x_0}{f(x) - f(x_0)}.$$

By assumption the function f^{-1} is continuous at $f(x_0)$ and so $x = f^{-1}(y) \to x_0 = f^{-1}(f(x_0))$ as $y \to f(x_0)$. Hence since f is differentiable at x_0 we have that

$$\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \to \frac{1}{f'(x_0)}$$

as $y \to f(x_0)$. ■

Remark 241 *Note that by writing $y_0 = f(x_0)$, the previous theorem says that*

$$\frac{df^{-1}}{dy}(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Exercise 242 *Using the previous theorem prove that*

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \quad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}},$$
$$(\arctan x)' = \frac{1}{x^2+1}, \quad (\log x)' = \frac{1}{x}.$$

Next we study local minima and maxima.

Definition 243 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$, and let $x_0 \in E$. We say that*

(i) f attains a local minimum *at x_0 if there exists $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in E \cap B(x_0, r)$,*

(ii) f attains a global minimum *at x_0 if $f(x) \geq f(x_0)$ for all $x \in E$,*

(iii) f attains a local maximum *at x_0 if there exists $r > 0$ such that $f(x) \leq f(x_0)$ for all $x \in E \cap B(x_0, r)$,*

(iv) f attains a global maximum *at x_0 if $f(x) \leq f(x_0)$ for all $x \in E$.*

Theorem 244 *Let $f : E \to \mathbb{R}$, where $E \subseteq \mathbb{R}$. If f attains a local minimum (or maximum) at some interior point $x_0 \in E$ and if f is differentiable at x_0, then $f'(x_0) = 0$.*

Remark 245 *In view of the previous theorem, when looking for minima and maxima, we have to search among the following:*

- *Interior points at which f is differentiable and $f'(x) = 0$, these are called* critical points. *Note that if $f'(x_0) = 0$, the function f may not attain a local minimum or maximum at x_0. Indeed, consider the function $f(x) = x^3$. Then $f'(0) = 0$, but f is strictly increasing, and so f does not attain a local minimum or maximum at 0.*

- *Interior points at which f is not differentiable. The function $f(x) = |x|$ attains a global minimum at $x = 0$, but f is not differentiable at $x = 0$.*

- *Boundary points.*

Friday, November 16, 2012

Proof. Assume that f attains a local minimum (the case of a local maximum is similar). Then there exists $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in E \cap B(x_0, r)$. Since $x_0 \in E$ is an interior point, by taking r smaller, we can assume that $B(x_0, r) \subseteq E$ so that $f(x) \geq f(x_0)$ for all $x \in B(x_0, r)$. If $x > x_0$, then $f(x) - f(x_0) \geq 0$ and so

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Letting $x \to x_0^+$ and using the fact that f is differentiable at x_0, we get that $f'(x_0) \geq 0$.

If $x < x_0$, then $f(x) - f(x_0) \geq 0$ and so

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Letting $x \to x_0^-$ and using the fact that f is differentiable at x_0, we get that $f'(x_0) \leq 0$. This shows that $f'(x_0) = 0$. ■

Remark 246 *If $x_0 \in E$ is not an interior point, then it is on the boundary. In this case it is either an isolated point or an accumulation point. In the first case it makes not sense to talk about $f'(x_0)$, since we cannot take limits. On the other end, it $x_0 \in \operatorname{acc} E$ and f is differentiable at x_0, then If f attains a local minimum at x_0 we can adapt the previous proof to show that $f'(x_0) \geq 0$ if there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ such that $x_n \to x_0^+$, while $f'(x_0) \leq 0$ if there exists a sequence $\{x_n\} \subseteq E \setminus \{x_0\}$ such that $x_n \to x_0^-$. Similar conclusions can be reached for local maxima (reversing the inequalities).*

An important application of the previous theorem is given by the following result.

Theorem 247 (Rolle) *Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since $[a, b]$ is sequentially compact, by the Weierstrass theorem, f has a global maximum and a global minimum in $[a, b]$. If

$$\max_{[a,b]} f = \min_{[a,b]} f,$$

then f is constant, and so $f'(x) = 0$ for all $x \in (a, b)$. If $\max_{[a,b]} f > \min_{[a,b]} f$, then since $f(a) = f(b)$, there f admits one of them at some interior point $c \in (a, b)$. By the previous theorem, $f'(c) = 0$. ■

Theorem 248 (Lagrange or Mean Value Theorem) *Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. The function

$$g(x) = f(x) - (x-a)\frac{f(b) - f(a)}{b-a}$$

is continuous in $[a,b]$, differentiable in (a,b), and $g(a) = g(b) = f(a)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a,b)$ such that $g'(c) = 0$, or, equivalently,

$$0 = f'(x) - 1\frac{f(b) - f(a)}{b-a}.$$

■

Corollary 249 *Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and differentiable in (a,b) with f' bounded. Then f is Lipschitz continuous.*

Proof. Let $L > 0$ be such that $|f'(x)| \leq L$ for all $x \in (a,b)$. By the mean value theorem, for all $x, y \in [a,b]$ with $x < y$, there exists $z \in (x,y)$ such that

$$f(y) - f(x) = f'(z)(y-x).$$

Hence,

$$|f(y) - f(x)| \leq L(y-x),$$

which shows that f is Lipschitz continuous. ■

Corollary 250 *Let $f : (a,b) \to \mathbb{R}$ be differentiable in (a,b). If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.*

Proof. By the mean value theorem, for all $x, y \in [a,b]$ with $x < y$, there exists $z \in (x,y)$ such that

$$f(y) - f(x) = f'(z)(y-x) = 0.$$

Hence, $f(y) - f(x) = 0$, which shows that f is constant. ■

Remark 251 *If $A \subseteq \mathbb{R}$ is open and $f : (a,b) \to \mathbb{R}$ be differentiable in A with $f'(x) = 0$ for all $x \in A$, then we can decompose A into a countable number of disjoint open intervals. By applying the previous result in each interval, we conclude that f is constant in every interval (the constant may change from interval to interval).*

Corollary 252 *Let $f : (a,b) \to \mathbb{R}$ be differentiable in (a,b). Then f is increasing if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.*

Proof. If f is increasing, then $f(x) \geq f(x_0)$ for $x > x_0$, and so

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Letting $x \to x_0^+$, we get $f'(x_0) \geq 0$.

Conversely, if $f' \geq 0$, by the mean value theorem, for all $x, y \in [a, b]$ with $x < y$, there exists $z \in (x, y)$ such that

$$f(y) - f(x) = f'(z)(y - x) \geq 0.$$

Hence, $f(y) \geq f(x)$. ■

Remark 253 *If $f' > 0$ is (a, b), then with the same proof we can show that f is strictly increasing, but the opposite is not true. Indeed, the function $f(x) = x^3$ is strictly increasing but $f'(x) = 3x^2$, which is zero for $x = 0$.*

Remark 254 *If $f'(x_0) > 0$, we cannot conclude that f is increasing near x_0 but only that $f(x) < f(x_0)$ for $x_0 - \delta < x < x_0$ and that $f(x) > f(x_0)$ for $x_0 < x < x_0 + \delta$ for some small $\delta > 0$. Indeed, consider the function*

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Let's look at the differentiability at $x = 0$. We have

$$\frac{f(x) - f(0)}{x - 0} = \frac{x + 2x^2 \sin \frac{1}{x} - 0}{x - 0} = 1 + 2x \sin \frac{1}{x} \to 1$$

as $x \to 0$, since $0 \leq \left|2x \sin \frac{1}{x}\right| \leq |2x| \to 0$. Hence,

$$f'(x) = \begin{cases} 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Hence $f'(0) = 1$ but f is not increasing near $x = 0$. Indeed, if it were, then $f' \geq 0$ in $(-\delta, \delta)$ for some small $\delta > 0$. However, since $4x \sin \frac{1}{x} \to 0$ as $x \to 0$, we have that

$$f'(x) = 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x^2}$$

oscillates between -1 and 1 as $x \to 0$. Note that this is also an example of a differentiable function, whose derivative is discontinuous.

Theorem 255 (Cauchy) *Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in (a, b) with $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. The function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

is continuous in $[a, b]$, differentiable in (a, b), and $h(a) = h(b) = f(a) g(b) - g(a) f(b)$. Hence, we are in a position to apply Rolle's theorem to find $c \in (a, b)$ such that $h'(c) = 0$, or, equivalently,

$$0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)).$$

■

Remark 256 *Note that by Rolle's theorem, $g(b) \neq g(a)$.*

Monday, November 19, 2012

Theorem 257 (De l'Hôpital) *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and assume that $f(x_0) = g(x_0) = 0$ for some $x_0 \in [a,b]$. Assume that f and g are differentiable in $(a,b) \setminus \{x_0\}$ with $g(x), g'(x) \neq 0$ for all $x \in (a,b) \setminus \{x_0\}$. If there exists*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \ell \in [-\infty, \infty],$$

then there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

Proof. Let $\{x_n\} \subseteq (a,b) \setminus \{x_0\}$ be such that $x_n \to x_0$. Apply Cauchy's theorem in the interval of endpoints x_n and x_0 to find y_n between x_n and x_0 such that

$$\frac{f(x_n) - 0}{g(x_n) - 0} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(y_n)}{g'(y_n)}.$$

As $x_n \to x_0$, we have that $y_n \to x_0$, and so by hypothesis

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)} \to \ell.$$

Since this is true for every sequence $\{x_n\}$ converging to x_0, it follows that there exists

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \ell.$$

■

Example 258 *Let's calculate*

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

Consider the functions $f(x) = \sin x - x$ and $g(x) = x^3$. Then $f(0) = g(0) = 0$ and both functions are differentiable. We have that $f'(x) = \cos x - 1$ and $g'(x) = 3x^2$. Let's calculate

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2}. \tag{25}$$

We get $\frac{0}{0}$. Consider the functions $f_1(x) = \cos x - 1$ and $g_1(x) = 3x^2$. Then $f_1(0) = g_1(0) = 0$ and both functions are differentiable. We have that $f_1'(x) = -\sin x$ and $g_1'(x) = 6x$. Let's calculate

$$\lim_{x \to 0} \frac{f_1'(x)}{g_1'(x)} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6}. \tag{26}$$

By (26) and De l'Hôpital's theorem, there exists

$$\lim_{x\to 0}\frac{f_1(x)}{g_1(x)}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=-\frac{1}{6}.$$

Finally, by (25) and De l'Hôpital's theorem, there exists

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{\sin x-x}{x^3}=-\frac{1}{6}.$$

Remark 259 *The converse of De l'Hôpital's theorem does not hold, that is, if there exists* $\lim_{x\to x_0}\frac{f(x)}{g(x)}$*, we cannot conclude that there exists the limit* $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$*. To see this, take*

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

and $g(x)=x$*. Then* f *and* g *are continuous and for* $x\neq 0$*, have that* $f'(x)=2x\sin\frac{1}{x}-x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}$ *and* $g'(x)=1$*. Then*

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{x}=\lim_{x\to 0}x\sin\frac{1}{x}=0,$$

since $0\leq\left|x\sin\frac{1}{x}\right|\leq|x|\to 0$*, but the limit*

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{2x\sin\frac{1}{x}+\cos\frac{1}{x}}{1}$$

does not exist (it oscillates between -1 *and* 1*).*

Another version of De l'Hôpital's theorem is the following:

Theorem 260 (De l'Hôpital) *Let* $f:[a,b]\setminus\{x_0\}\to\mathbb{R}$ *and* $g:[a,b]\setminus\{x_0\}\to\mathbb{R}$ *be continuous in* $[a,b]\setminus\{x_0\}$ *and assume that*

$$\lim_{x\to x_0}|f(x)|=\lim_{x\to x_0}|g(x)|=\infty$$

and that f *and* g *are differentiable in* $(a,b)\setminus\{x_0\}$ *with* $g(x),g'(x)\neq 0$ *for all* $x\in(a,b)\setminus\{x_0\}$*. If there exists*

$$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=\ell\in[-\infty,\infty],$$

then there exists

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\ell.$$

The proof is left as an exercise.

Exercise 261 *Calculate*

$$\lim_{x\to 0^+} x^a \log x,$$

where $a > 0$.

Exercise 262 *Calculate*

$$\lim_{x\to 0^+} x^{\sin x}.$$

Another version of the previous theorem is the following.

Theorem 263 (De l'Hôpital) *Let* $f : [a,\infty) \to \mathbb{R}$ *and* $g : [a,\infty) \to \mathbb{R}$ *be continuous in* $[a,\infty)$ *and differentiable in* (a,∞). *Assume that*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$$

and that $g(x), g'(x) \neq 0$ *for all* $x \in (a,\infty)$. *If there exists*

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \ell \in [-\infty,\infty],$$

then there exists

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \ell.$$

The proof is similar and we omit it.

Remark 264 *A similar result holds if* $\lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |g(x)| = \infty$.

Exercise 265 *Prove the following:*

$$\lim_{x\to\infty} \frac{x^a}{\log^b x} = 0,$$

where $a > 0$ *and* $b \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{x^a}{b^x} = 0,$$

where $a \in \mathbb{R}$ *and* $b > 1$.

Exercise 266 *Let's calculate*

$$\lim_{x\to\infty} \frac{\log x - x}{\log(1+e^x)}.$$

Next we study Taylor's formula.

Definition 267 *Given an open set* $U \subseteq \mathbb{R}$ *and an integer* $n \in \mathbb{N}$, *a function* $f : U \to \mathbb{R}$ *is said to be of class* C^n *if* f *is differentiable up to order* n *with continuous derivatives* f', f'', f''', $f^{(4)}, \cdots, f^{(n)}$. *The space of all functions of class* C^n *is denoted by* $C^n(U)$. *A function of class* C^n *for every* $n \in \mathbb{N}$ *is said to be of class* C^∞ *and the space of all functions of class* C^∞ *is denoted by* $C^\infty(U)$.

Theorem 268 (Taylor's Formula) *Let $f \in C^{(n)}((a,b))$ and let $x_0 \in (a,b)$. Then for every $x \in (a,b)$,*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x, x_0),$$

where the remainder $R_n(x, x_0)$ satisfies

$$\lim_{x \to x_0} \frac{R_n(x, x_0)}{(x - x_0)^n} = 0.$$

Lemma 269 *Let $g \in C^{(n)}((a,b))$ and let $x_0 \in (a,b)$. Then*

$$\lim_{x \to x_0} \frac{g(x)}{(x - x_0)^n} = 0 \tag{27}$$

if and only if

$$g(x_0) = g'(x_0) = \cdots = g^{(n)}(x_0) = 0. \tag{28}$$

Proof. Assume that (28) holds. By applying De l'Hôpital's theorem several times we get

$$\lim_{x \to x_0} \frac{g(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{g'(x)}{n(x - x_0)^{n-1}} = \lim_{x \to x_0} \frac{g^{(2)}(x)}{n(n-1)(x - x_0)^{n-2}}$$
$$= \cdots = \lim_{x \to x_0} \frac{g^{(n-1)}(x)}{n!(x - x_0)} = \lim_{x \to x_0} \frac{g^{(n)}(x)}{n!1} = \frac{g^{(n)}(x_0)}{n!} = 0.$$

Conversely, assume (27). If $g^{(k)}(x_0) \neq 0$ for some $0 \leq k < n$, then by what we just proved (with k in place of n)

$$\lim_{x \to x_0} \frac{g(x)}{(x - x_0)^k} = \frac{g^{(k)}(x_0)}{k!} \neq 0.$$

On the other hand,

$$\frac{g(x)}{(x - x_0)^k} = \frac{g(x)}{(x - x_0)^k} \frac{(x - x_0)^{n-k}}{(x - x_0)^{n-k}} = \frac{g(x)}{(x - x_0)^n}(x - x_0)^{n-k} \to 0$$

as $x \to x_0$, which is a contradiction. ■

We now turn to the proof of Theorem 268.

Proof of Theorem 268. Note that given a polynomial of degree n,

$$p(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n = \sum_{i=0}^{n} a_i (x - x_0)^i,$$

we have that

$$p^{(k)}(x) = \sum_{i=k}^{n} i(i-1)\cdots(i-k+1)\,a_i\,(x-x_0)^{i-k},$$

so that

$$p^{(k)}(x_0) = k!a_k.$$

We apply the lemma to the function

$$g(x) := f(x) - p(x)$$

to conclude that

$$\lim_{x\to x_0} \frac{f(x)-p(x)}{(x-x_0)^n} = 0$$

if and only if for all $k = 0, \ldots, n$,

$$0 = g^{(k)}(x_0) = f^{(k)}(x_0) - p^{(k)}(x_0) = f^{(k)}(x_0) - k!a_k,$$

that is,

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Thus

$$g(x) = R_n(x, x_0) = f(x) - \left[f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \right].$$

■

Definition 270 *Given two functions $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ and a point $x_0 \in \operatorname{acc} E$, we say that the function f is a* little o *of g as $x \to x_0$, and we write $f = \mathrm{o}(g)$, if $g \neq 0$ in E and*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

Hence, a little o of g is simply a function that goes to zero faster than g as $x \to x_0$. Hence, Taylor's formula can be written as

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + o((x-x_0)^n)$$

as $x \to x_0$.

Wednesday, November 21, 2012

Thanksgiving, no classes

Friday, November 23, 2012

Thanksgiving, no classes

Monday, November 24, 2012

Corollary 271 *Let $f:(a,b)\to\mathbb{R}$ be a function of class $C^n((a,b))$, $n\geq 2$, and let $x_0\in(a,b)$ be such that*

$$f'(x_0)=\cdots=f^{(m-1)}(x_0)=0 \tag{29}$$

for some $2\leq m\leq n$. Then the following hold:

(i) if m is even and $f^{(m)}(x_0)>0$, then f has a local minimum at x_0,

(ii) if m is even and $f^{(m)}(x_0)<0$, then f has a local maximum at x_0,

(iii) if m is odd and $f^{(m)}(x_0)\neq 0$, then f has neither a local minimum nor a local maximum at x_0,

(iv) if $f'(x_0)=\cdots=f^{(n)}(x_0)=0$, then anything can happen.

Proof. (i) For x close to x_0 by Taylor's formula of order m and center x_0

$$\begin{aligned}f(x)&=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2+\frac{1}{3!}f'''(x_0)(x-x_0)^3\\&\quad+\cdots+\frac{1}{m!}f^{(m)}(x_0)(x-x_0)^m+R_m(x),\end{aligned}$$

where the remainder $R_m(x,x_0)$ satisfies

$$\lim_{x\to x_0}\frac{R_m(x)}{(x-x_0)^m}=0. \tag{30}$$

By (29),

$$\begin{aligned}f(x)&=f(x_0)+\frac{1}{m!}f^{(m)}(x_0)(x-x_0)^m+R_m(x)\\&=f(x_0)+(x-x_0)^m\left[\frac{1}{m!}f^{(m)}(x_0)+\frac{R_m(x)}{(x-x_0)^m}\right].\end{aligned}$$

Take $\varepsilon=\frac{1}{2}\frac{1}{m!}f^{(m)}(x_0)>0$ then by (30) there exists $\delta>0$ such that

$$\left|\frac{R_m(x)}{(x-x_0)^m}-0\right|\leq\varepsilon,$$

that is

$$-\varepsilon\leq\frac{R_m(x)}{(x-x_0)^m}-0\leq\varepsilon,$$

for all $x\in(a,b)$ with $|x-x_0|\leq\delta$. Hence

$$\begin{aligned}\frac{1}{m!}f^{(m)}(x_0)+\frac{R_m(x)}{(x-x_0)^m}&\geq\frac{1}{m!}f^{(m)}(x_0)-\varepsilon\\&=\frac{1}{m!}f^{(m)}(x_0)-\frac{1}{2}\frac{1}{m!}f^{(m)}(x_0)=\frac{1}{2}\frac{1}{m!}f^{(m)}(x_0)>0\end{aligned}$$

for all $x \in (a,b)$ with $|x - x_0| \leq \delta$. Since m is even, we have that $(x - x_0)^m > 0$ for $x \neq x_0$ so

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)^m \left[\frac{1}{m!} f^{(m)}(x_0) + \frac{R_m(x)}{(x - x_0)^m} \right] \\ &> f(x_0) + 0 \end{aligned}$$

for all $x \in (a,b)$ with $0 < |x - x_0| \leq \delta$, which shows that f has a local minimum at x_0;

(iii) If m is odd then $(x - x_0)^m > 0$ if $x > x_0$ and so we proceed exactly as in the previous part to conclude that

$$f(x) > f(x_0)$$

for all $x \in (a,b)$ with if $x_0 < x < x_0 + \delta$. On the other hand, $(x - x_0)^m < 0$ if $x < x_0$ and so

$$f(x) < f(x_0)$$

for all $x \in (a,b)$ with if $x_0 - \delta < x < x_0$. Thus f has neither a local minimum nor a local maximum at x_0. ■

Important Taylor's formulas with center $x = 0$

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots + \frac{1}{n!}x^n + o(x^n)$, hence the first order formula is

  $$e^x = 1 + x + o(x),$$

  while the second order formula is

  $$e^x = 1 + x + \frac{1}{2!}x^2 + o\left(x^2\right).$$

- $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1} \frac{1}{n}x^n + o(x^n)$, hence the first order formula is

  $$\log(1 + x) = x + o(x),$$

  while the second order formula is

  $$\log(1 + x) = x - \frac{1}{2}x^2 + o\left(x^2\right).$$

- $(1 + x)^a = 1 + ax + \frac{1}{2}a(a-1)x^2 + \cdots + \frac{1}{n!}a(a-1)(a-2)(a-3)\cdots(a-n+1)x^n + o(x^n)$, hence the first order formula is

  $$(1 + x)^a = 1 + ax + o(x),$$

  while the second order formula is

  $$(1 + x)^a = 1 + ax + \frac{1}{2}a(a-1)x^2 + o\left(x^2\right).$$

- $\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^{n+1} x^n + o(x^n)$ hence the first order formula is

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + o(x),$$

while the second order formula is

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 + o\left(x^2\right).$$

- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \cdots + (-1)^k \frac{1}{2k!}x^{2k} + o\left(x^{2k+1}\right)$, hence the third order formula is

$$\cos x = 1 - \frac{1}{2!}x^2 + o\left(x^3\right),$$

while the fifth order formula is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right),$$

- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + o\left(x^{2k+2}\right)$, hence the second order formula is

$$\sin x = x + o\left(x^2\right),$$

while the fourth order formula is

$$\sin x = x - \frac{1}{3!}x^3 + o\left(x^4\right),$$

- $\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \cdots + (-1)^k \frac{1}{(2k+1)}x^{2k+1} + o\left(x^{2k+2}\right)$, hence the second order formula is

$$\arctan x = x + o\left(x^2\right),$$

while the fourth order formula is

$$\arctan x = x - \frac{1}{3}x^3 + o\left(x^4\right).$$

Example 272 *Note that as $x \to 0$,*

$$x^4 = o(x), \quad x^4 = o\left(x^2\right), \quad x^4 = o\left(x^3\right), \quad x^4 = o(x^n) \text{ for every } n < 4.$$

but

$$x^4 \text{ is not } o\left(x^4\right), \quad x^4 \text{ is not } o\left(x^5\right), \quad x^4 \text{ is not } o(x^m) \text{ for every } m \geq 5.$$

Example 273 *Note that as* $x \to 0$,

$$3x + x^2 + o(x^3) - x^4 + 6x^7 = 3x + x^2 + o(x^3)$$

since $-x^4 = o(x^3)$ *and* $6x^7 = o(x^3)$.

Example 274 *Note that*

$$\begin{aligned}\left(x - \frac{1}{6}x^3 + o\left(x^4\right)\right)^2 &= x^2 + \frac{1}{36}x^6 + \left(o\left(x^4\right)\right)^2 \\ &\quad + 2x\left(-\frac{1}{6}x^3\right) + 2xo\left(x^4\right) - \frac{2}{6}x^3 o\left(x^4\right) \\ &= x^2 + \frac{1}{36}x^6 + o\left(x^8\right) - \frac{1}{3}x^4 + 2o\left(x^5\right) - \frac{2}{6}o\left(x^7\right) \\ &= x^2 + \frac{1}{36}x^6 + o\left(x^8\right) - \frac{1}{3}x^4 + o\left(x^5\right) + o\left(x^7\right) \\ &= x^2 - \frac{1}{3}x^4 + o\left(x^5\right).\end{aligned}$$

Example 275 *Let's calculate*

$$\lim_{x \to 0^+} \frac{\sin^{100} x - x^{100}}{x^a},$$

where $a > 0$. *We have*

$$\sin x = x - \frac{1}{3!}x^3 + o\left(x^4\right).$$

Hence,

$$\sin^{100} x = \left(x - \frac{1}{3!}x^3 + o\left(x^4\right)\right)^{100} = x^{100}\left(1 - \frac{1}{3!}x^2 + o\left(x^3\right)\right)^{100}.$$

On the other hand, as $t \to 0$,

$$(1+t)^{100} = 1 + 100t + o(t),$$

and so taking $t = -\frac{1}{3!}x^2 + o\left(x^3\right) \to 0$ *as* $x \to 0$, *we get*

$$\begin{aligned}\left(1 - \frac{1}{3!}x^2 + o\left(x^3\right)\right)^{100} &= 1 + 100\left(-\frac{1}{3!}x^2 + o\left(x^3\right)\right) + o\left(-\frac{1}{3!}x^2 + o\left(x^3\right)\right) \\ &= 1 - \frac{100}{3!}x^2 + o\left(x^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sin^{100} x &= \left(x - \frac{1}{3!}x^3 + o\left(x^4\right)\right)^{100} = x^{100}\left(1 - \frac{1}{3!}x^2 + o\left(x^3\right)\right)^{100} \\ &= x^{100}\left(1 - \frac{100}{3!}x^2 + o\left(x^2\right)\right) = x^{100} - \frac{100}{3!}x^{102} + o\left(x^{102}\right).\end{aligned}$$

It follows that

$$\sin^{100} x - x^{100} = x^{100} - \frac{100}{3!}x^{102} + o\left(x^{102}\right) - x^{100} = -\frac{100}{3!}x^{102} + o\left(x^{102}\right),$$

and so

$$\frac{\sin^{100} x - x^{100}}{x^a} = \frac{-\frac{100}{3!}x^{102} + o\left(x^{102}\right)}{x^a} = \frac{x^{102}}{x^a}\left(-\frac{100}{3!} + \frac{o\left(x^{102}\right)}{x^{102}}\right)$$

$$\to \begin{cases} 0\left(-\frac{100}{3!}+0\right) & \text{if } a < 102 \\ \infty\left(-\frac{100}{3!}+0\right) & \text{if } a > 102 \\ 1\left(-\frac{100}{3!}+0\right) & \text{if } a = 102 \end{cases}$$

$$= \begin{cases} 0 & \text{if } a < 102 \\ -\infty & \text{if } a > 102 \\ -\frac{100}{3!} & \text{if } a = 102 \end{cases}$$

Wednesday, November 26, 2012

Example 276 *Let's calculate*

$$\lim_{x\to 0} \frac{\frac{1}{2}x^2 + \log\cos x}{x^4}.$$

We have

$$\lim_{x\to 0} \frac{\frac{1}{2}x^2 + \log\cos x}{x^4} = \frac{0 + \log\cos 0}{0} = \frac{0 - \log 1}{0} = \frac{0}{0}.$$

Taylor's formula of $\cos x$ *of order five is given by*

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right).$$

Hence,

$$\log\cos x = \log\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)\right).$$

Let's use now Taylor's formula

$$\log(1+t) = t - \frac{1}{2}t^2 + o\left(t^2\right),$$

where for us $t = -\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)$. *We get*

$$\begin{aligned}
&\log\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)\right) \\
&= \left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)\right) - \frac{1}{2}\left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)\right)^2 + o\left(\left(-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right)\right)^2\right) \\
&= -\frac{1}{2}x^2 + \frac{1}{4!}x^4 + o\left(x^5\right) - \frac{1}{2}\left(\frac{1}{4}x^4\right) + o\left(x^4\right) \\
&= -\frac{1}{2}x^2 + \left(-\frac{1}{8} + \frac{1}{4!}\right)x^4 + o\left(x^4\right) \\
&= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + o\left(x^4\right),
\end{aligned}$$

where all the powers of order bigger than 4 have been absorbed by $o\left(x^4\right)$.

Hence,

$$\begin{aligned}
\frac{\frac{1}{2}x^2 + \log\cos x}{x^4} &= \frac{\frac{1}{2}x^2 - \frac{1}{2}x^2 - \frac{1}{12}x^4 + o\left(x^4\right)}{x^4} = \frac{x^4}{x^4}\left(-\frac{1}{12} + \frac{o\left(x^4\right)}{x^4}\right) \\
&= -\frac{1}{12} + \frac{o\left(x^4\right)}{x^4} \to -\frac{1}{12} + 0
\end{aligned}$$

as $x \to 0$.

Remark 277 (Important) *Note that if in the exercises we end up with*

$$\lim_{x \to x_0} \frac{o\left((x - x_0)^n\right)}{(x - x_0)^m}$$

where $n < m$, *then we cannot conclude anything. This means that we have go back and use more terms in our Taylor's formula.*

In the previous example, if we had used

$$\cos x = 1 - \frac{1}{2!}x^2 + o\left(x^3\right).$$

Then

$$\log\cos x = \log\left(1 - \frac{1}{2!}x^2 + o\left(x^3\right)\right).$$

Let's use now Taylor's formula

$$\log(1 + t) = t - \frac{1}{2}t^2 + o\left(t^2\right),$$

where for us $t = 1 - \frac{1}{2!}x^2 + o\left(x^3\right)$ *. We get*

$$\begin{aligned}
&\log\left(1 - \frac{1}{2!}x^2 + o\left(x^3\right)\right) \\
&= \left(-\frac{1}{2!}x^2 + o\left(x^3\right)\right) - \frac{1}{2}\left(-\frac{1}{2!}x^2 + o\left(x^3\right)\right)^2 + o\left(\left(-\frac{1}{2!}x^2 + o\left(x^3\right)\right)^2\right) \\
&= -\frac{1}{2}x^2 + o\left(x^3\right)
\end{aligned}$$

Then

$$\frac{\frac{1}{2}x^2 + \log\cos x}{x^4} = \frac{\frac{1}{2}x^2 - \frac{1}{2}x^2 + o\left(x^3\right)}{x^4} = \frac{o\left(x^3\right)}{x^4}$$

and we are stuck.

19 Integration

Given an interval $[a, b]$ with $a < b$, by a *partition* P of $[a, b]$ we mean a finite set of points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Consider a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P. We define the *lower* and *upper sums* of f for the partition P respectively by

$$\begin{aligned}
L(f, P) &:= \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} f(x), \\
U(f, P) &:= \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x_{i-1} \le x \le x_i} f(x).
\end{aligned}$$

Since f is bounded, we have that

$$(b - a) \inf_{a \le x \le b} f(x) \le L(f, P) \le U(f, P) \le (b - a) \sup_{a \le x \le b} f(x). \tag{31}$$

The *lower* and *upper integrals* of f over $[a, b]$ are defined respectively by

$$\begin{aligned}
\underline{\int_a^b} f(x)\, dx &:= \sup\{L(f, P) : \ P \text{ partition of } [a, b]\}, \\
\overline{\int_a^b} f(x)\, dx &:= \inf\{U(f, P) : \ P \text{ partition of } [a, b]\}.
\end{aligned}$$

If $a = b$ we set

$$\underline{\int_a^a} f(x)\, dx = \overline{\int_a^a} f(x)\, dx := 0.$$

We study some properties of the lower and upper integrals of f.

Proposition 278 *Consider a bounded function $f : [a,b] \to \mathbb{R}$. Then*

$$(b-a) \inf_{a \le x \le b} f(x) \le \underline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} f(x)\, dx \le (b-a) \sup_{a \le x \le b} f(x) . \quad (32)$$

Proof. Step 1: Let $P = \{x_0, \ldots, x_n\}$ be a partition and consider a point $y \in [a,b]$ not in P. Then there exists $i_0 \in \{1, \ldots, n\}$ such that $x_{i_0 - 1} < y < x_{i_0}$. Consider the new partition P' given by $P \cup \{y\}$. Then

$$\begin{aligned} &(x_{i_0} - x_{i_0-1}) \inf_{x_{i_0-i} \le x \le x_{i_0}} f(x) \\ &\qquad = (x_{i_0} - y) \inf_{x_{i_0-1} \le x \le x_{i_0}} f(x) + (y - x_{i_0-1}) \inf_{x_{i_0} \le x \le x_{i_0}} f(x) \\ &\qquad \le (x_{i_0} - y) \inf_{y \le x \le x_{i_0}} f(x) + (x_{i_0} - y) \inf_{y \le x \le x_{i_0}} f(x) . \end{aligned}$$

Since $L(f,P)$ and $L(f,P')$ only differ in the interval $[x_{i_0-i}, x_{i_0}]$, this shows that

$$L(f,P) \le L(f,P') . \quad (33)$$

A similar argument shows that

$$U(f,P') \le U(f,P) .$$

Using an induction argument, we have proved that by adding a finite number of points to a partition, we increase the lower sum and decrease the upper sum.
Step 2: Let $P = \{x_0, \ldots, x_n\}$ and $Q = \{y_0, \ldots, y_m\}$ be two partitions of $[a,b]$ and consider the new partition $P' = P \cup Q$. By what we just proved in Step 1,

$$L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q) . \quad (34)$$

Hence, $L(f,P) \le U(f,Q)$ for all partitions P and Q of $[a,b]$. Taking the supremum over all partitions P of $[a,b]$, we get

$$\underline{\int_a^b} f(x)\, dx = \sup_{P \text{ partition of } [a,b]} L(f,P) \le U(f,Q)$$

for all partitions Q of $[a,b]$. Taking the infimum over all partitions Q of $[a,b]$, we get

$$\underline{\int_a^b} f(x)\, dx \le \inf_{Q \text{ partition of } [a,b]} U(f,Q) = \overline{\int_a^b} f(x)\, dx.$$

The remaining inequalities in (32) follow from (31). ■

Friday, November 28, 2012

Remark 279 *If $f : [a,b] \to \mathbb{R}$ is bounded, then there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$. In turn,*

$$-M \le f(x) \le M$$

for all $x \in [a,b]$*, and so*

$$-M \leq \inf_{a\leq x\leq b} f(x) \leq \sup_{a\leq x\leq b} f(x) \leq M.$$

It follows by the previous proposition

$$-(b-a)M \leq (b-a)\inf_{a\leq x\leq b} f(x) \leq \underline{\int_a^b} f(x)\ dx$$
$$\leq \overline{\int_a^b} f(x)\ dx \leq (b-a)\sup_{a\leq x\leq b} f(x) \leq (b-a)M,$$

so that

$$\left|\underline{\int_a^b} f(x)\ dx\right| \leq (b-a)M,$$
$$\left|\overline{\int_a^b} f(x)\ dx\right| \leq (b-a)M.$$

We will use this property later on.

Exercise 280 *Consider a bounded function* $f:[a,b]\to\mathbb{R}$*. Prove that for every constant* $C\in\mathbb{R}$*,*

$$\underline{\int_a^b} (f(x)+C)\ dx = \underline{\int_a^b} f(x)\ dx + C(b-a),$$
$$\overline{\int_a^b} (f(x)+C)\ dx = \overline{\int_a^b} f(x)\ dx + C(b-a).$$

Exercise 281 *Consider a bounded function* $f:[a,b]\to\mathbb{R}$ *and let* $c\in(a,b)$*. Prove that*

$$\underline{\int_a^b} f(x)\ dx = \underline{\int_a^c} f(x)\ dx + \underline{\int_c^b} f(x)\ dx, \qquad (35)$$
$$\overline{\int_a^b} f(x)\ dx = \overline{\int_a^c} f(x)\ dx + \overline{\int_c^b} f(x)\ dx. \qquad (36)$$

The next theorem will be used to prove the fundamental theorem of calculus.

Theorem 282 *Consider a bounded function* $f:[a,b]\to\mathbb{R}$ *and let*

$$F(x) := \underline{\int_a^x} f(y)\ dy \qquad G(x) := \overline{\int_a^x} f(y)\ dy$$

for $x\in[a,b]$*. Then* F *and* G *are Lipschitz continuous and*

$$F'(x_0) = G'(x_0) = f(x_0)$$

at every point $x_0\in[a,b]$ *at which* f *is continuous.*

Proof. Step 1: We only study the function F. To prove that F is Lipschitz continuous, let $M > 0$ be such that $|f(x)| \le M$ for all $x \in [a,b]$. Fix $x, y \in [a,b]$. Without loss of generality, we may assume $x < y$. By (35) with x in place of c and $[a,y]$ in place of $[a,b]$, we have

$$F(y) = \underline{\int_a^x} f(y)\, dy + \underline{\int_x^y} f(y)\, dy = F(x) + \underline{\int_x^y} f(y)\, dy.$$

Hence,

$$|F(y) - F(x)| \le \left| \underline{\int_x^y} f(y)\, dy \right| \le M(y - x)$$

by Remark 279, which shows that F is Lipschitz.
Step 2: Assume that f is continuous at $x_0 \in [a,b]$. We consider the case $x_0 \in (a,b)$ (the cases $x_0 = a$ and $x_0 = b$) are simpler. We want to prove that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

For $h \ne 0$ consider the different quotient

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \begin{cases} \frac{1}{x - x_0} \underline{\int_{x_0}^x} (f(y) - f(x_0))\, dy & \text{if } x > x_0, \\ -\frac{1}{x - x_0} \underline{\int_x^{x_0}} (f(y) - f(x_0))\, dy & \text{if } x < x_0, \end{cases}$$

where we have used (35) and Exercise 280. Since f is continuous at x_0, given $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$|f(x) - f(x_0)| \le \varepsilon$$

for all $x \in [a,b]$ with $|x - x_0| \le \delta$. Take $|x - x_0| \le \delta$. Then for $x > x_0$ (the case $x < x_0$ is similar), by Remark 279, we have

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \frac{1}{x - x_0} \left| \underline{\int_{x_0}^x} (f(y) - f(x_0))\, dy \right| \\ &\le \frac{1}{x - x_0} \varepsilon (x - x_0), \end{aligned}$$

which shows that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \le \varepsilon.$$

This concludes the proof. ■

As a corollary of the previous theorem, we have the mean value theorem for integrals.

Corollary 283 (Mean Value Theorem for Integrals) *Let $f : [a,b] \to \mathbb{R}$ be bounded. Assume that f is continuous in (a,b). Then there exists $c \in (a,b)$ such that*

$$\frac{1}{b - a} \underline{\int_a^b} f(x)\, dx = f(c).$$

A similar result holds for the upper integral.

Proof. Consider the function F defined in the previous theorem. Since f is continuous in (a,b), by the previous theorem, F is differentiable in (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$. Moreover, F is Lipschitz continuous in $[a,b]$ and so it is continuous in $[a,b]$. By the mean value theorem applied to the function F there exists $c \in (a,b)$ such that

$$\underline{\int_a^b} f(x)\, dx - 0 = F(b) - F(a) = F'(c)(b-a) = f(c)(b-a).$$

■

Given a bounded function $f : [a,b] \to \mathbb{R}$, we say that f is *Riemann integrable* over $[a,b]$ if the lower and upper integral coincide. We call the common value the *Riemann integral* of f over $[a,b]$ and we denote it by $\int_a^b f(x)\, dx$. Thus, for a Riemann integrable function,

$$\int_a^b f(x)\, dx := \underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx.$$

The family of all Riemann integrable functions over $[a,b]$ is denoted $\mathcal{R}([a,b])$.

Example 284 *The function*

$$f(x) := \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}, \end{cases}$$

is called the Dirichlet function. By the density of the rationals and of the irrationals we have that

$$\inf_{x_{i-1} \le x \le x_i} f(x) = 0, \qquad \sup_{x_{i-1} \le x \le x_i} f(x) = 1,$$

and so

$$L(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} f(x) = 0,$$
$$U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1})\, 1 = 1 - 0 = 1.$$

Thus

$$\underline{\int_0^1} f(x)\, dx = \sup 0 = 0, \qquad \overline{\int_0^1} f(x)\, dx = \inf 1 = 1.$$

To determine when a function is Riemann integrable we need to introduce the notion of sets of Lebesgue measure zero.

Definition 285 *A set $E \subseteq \mathbb{R}$ has* Lebesgue measure zero *if for every $\varepsilon > 0$ there exists a countable family of open intervals (a_n, b_n) such that*

$$E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} (b_n - a_n) \le \varepsilon.$$

Here

$$\sum_{n=1}^{\infty}(b_n - a_n) := \lim_{\ell\to\infty}\sum_{n=1}^{\ell}(b_n - a_n)$$

and this limit exists since the sequence $\left\{\sum_{n=1}^{\ell}(b_n - a_n)\right\}_\ell$ *is increasing. Indeed,*

$$\sum_{n=1}^{\ell+1}(b_n - a_n) = \sum_{n=1}^{\ell}(b_n - a_n) + (b_{\ell+1} - a_{\ell+1}) \geq \sum_{n=1}^{\ell}(b_n - a_n) + 0.$$

Monday, December 03, 2012

Example 286 *Let* $x \neq 1$*. By Exercise 17,*

$$1 + x \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for all $n \in \mathbb{N}$*. Hence, if* $x > 0$*,*

$$\sum_{n=1}^{\infty} x^n = \lim_{\ell\to\infty}\sum_{n=1}^{\ell} x^n = \lim_{\ell\to\infty}\frac{x^{\ell+1} - x}{x - 1} = \begin{cases} \frac{0-x}{x-1} & \text{if } 0 < x < 1, \\ \infty & \text{if } x > 1. \end{cases}$$

Example 287 *Let's see some examples.*

(i) A singleton $E = \{c\}$ *has Lebesgue measure zero. Given* $\varepsilon > 0$*, take* $I_1 = \left(c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right)$*.*

(ii) If a set E *contains an open interval* (a, b)*, then it cannot have Lebesgue measure zero. Indeed, for any countable family of open intervals* (a_n, b_n) *we have*

$$(a, b) \subseteq E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n),$$

and so

$$b - a \leq \sum_{n=1}^{\infty}(b_n - a_n).$$

Taking $\varepsilon < b - a$*, we obtain a contradiction.*

(iii) A countable set $E = \{x_n\}_n$ *has Lebesgue measure zero. Given* $\varepsilon > 0$*, take* $I_n = \left(x_n - \frac{\varepsilon}{2^{n-1}}, x_n + \frac{\varepsilon}{2^{n-1}}\right)$*. Then*

$$\sum_{n=1}^{\infty}\operatorname{length} I_n = \varepsilon\sum_{n=1}^{\infty}\frac{1}{2^n} \leq \varepsilon.$$

In particular, $\mathbb{N}$*,* $\mathbb{Q}$*, and* $\mathbb{Z}$ *all have Lebesgue measure zero.*

(iv) If $\{E_k\}_k$ is a countable family of sets, each with Lebesgue measure zero, then their union

$$E := \bigcup_{k=1}^{\infty} E_k$$

has Lebesgue measure zero. Indeed, given $\varepsilon > 0$ fix k. Since E_k has Lebesgue measure zero, there exists a countable family of open intervals $I_n^{(k)}$ such that

$$E_k \subseteq \bigcup_{n=1}^{\infty} I_n^{(k)} \quad \text{and} \quad \sum_{n=1}^{\infty} \text{length}\, I_n^{(k)} \le \frac{\varepsilon}{2^k}.$$

Consider the family $\left\{I_n^{(k)}\right\}_{n,k}$. It is still countable,

$$E = \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_n^{(k)}$$

and

$$\sum_{n,k} \text{length}\, I_n^{(k)} = \sum_k \sum_n \text{length}\, I_n^{(k)} \le \sum_k \frac{\varepsilon}{2^k} \le \varepsilon.$$

(v) There are uncountable sets that have Lebesgue measure zero. One such example is given by the Cantor set.

The following theorem characterizes Riemann integrable functions.

Theorem 288 *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of its discontinuity points has Lebesgue measure zero.*

We will not prove this theorem.

In view of the previous theorem, we have the following.

Corollary 289 *Given $f : [a, b] \to \mathbb{R}$,*

(i) if f is continuous, then f is Riemann integrable,

(ii) if f is monotone, then f is Riemann integrable.

Proof. If f is continuous, then by the Weierstrass theorem it is bounded, and thus by the previous theorem it is Riemann integrable.

If f is monotone, then it is bounded from below by $\min\{f(a), f(b)\}$ and from above by $\max\{f(a), f(b)\}$. Moreover, by Theorem 184 its set of discontinuity points is at most countable. Since a countable set has Lebesgue measure zero, it follows from the previous theorem that f is Riemann integrable. ∎

Example 290 *Some examples of functions that are Riemann integrable and others that are not.*

(i) The function

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

is Riemann integrable over $[0,1]$*, since it is bounded and discontinuous only at* $x = 0$*.*

(ii) The function

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

is Riemann integrable over $[0,1]$*, since it is bounded and its set of discontinuity points is*

$$E = \left\{ \frac{1}{n} \right\}_n \cup \{0\},$$

which is countable.

(iii) The function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

is bounded but not Riemann integrable over $[0,1]$*, since it is bounded and its set of discontinuity points is* $[0,1]$*.*

Exercise 291 *Consider the function* $g : [0,1] \to [0,1]$ *defined by*

$$g(x) := \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{p} & \text{if } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N} \text{ relatively prime, } 0 < p < q, \\ 1 & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

(i) Prove that g *is discontinuous at every rational point of* $[0,1]$*.*

(ii) Prove that g *is continuous at every irrational point of* $[0,1]$*.*

(iii) Prove that g *is Riemann integrable.*

Next we discuss some properties of Riemann integration.

Theorem 292 *Let* $f, g : [a,b] \to \mathbb{R}$ *be bounded. Then*

$$\begin{aligned} \underline{\int_a^b} f(x)\, dx + \underline{\int_a^b} g(x)\, dx &\leq \underline{\int_a^b} (f(x) + g(x))\, dx \\ &\leq \overline{\int_a^b} (f(x) + g(x))\, dx \leq \overline{\int_a^b} f(x)\, dx + \overline{\int_a^b} g(x)\, dx. \end{aligned}$$

Proof. Let P and Q be two partitions of $[a,b]$ and let $P\cup Q=\{x_0,\ldots,x_n\}$. Then

$$\inf_{x_{i-1}\le x\le x_i}(f(x)+g(x))\ge\inf_{x_{i-1}\le x\le x_i}f(x)+\inf_{x_{i-1}\le x\le x_i}g(x),$$

and so

$$\begin{aligned}L(f+g,P\cup Q)&=\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x_{i-1}\le x\le x_i}(f(x)+g(x))\\&\ge\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x_{i-1}\le x\le x_i}f(x)+\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x_{i-1}\le x\le x_i}g(x)\\&=L(f,P\cup Q)+L(g,P\cup Q).\end{aligned}$$

Hence,

$$\begin{aligned}\underline{\int_a^b}(f(x)+g(x))\,dx&=\sup_R L(f+g,R)\ge L(f+g,P\cup Q)\\&\ge L(f,P\cup Q)+L(g,P\cup Q)\\&\ge L(f,P)+L(g,Q),\end{aligned}$$

where in the last inequality we have used (34). Hence,

$$\underline{\int_a^b}(f(x)+g(x))\,dx\ge L(f,P)+L(g,Q)$$

for all partitions P and Q of $[a,b]$. Now we fix Q. Then

$$\underline{\int_a^b}(f(x)+g(x))\,dx-L(g,Q)\ge L(f,P)$$

for all partitions P of $[a,b]$. Hence, the number $\underline{\int_a^b}(f(x)+g(x))\,dx-L(g,Q)$ is an upper bound for the set

$$E:=\{L(f,P):\ P\text{ partition of }[a,b]\}.$$

It follows that

$$\underline{\int_a^b}(f(x)+g(x))\,dx-L(g,Q)\ge\sup E=\underline{\int_a^b}f(x)\,dx.$$

Thus, we have proved that

$$\underline{\int_a^b}(f(x)+g(x))\,dx-\underline{\int_a^b}f(x)\,dx\ge L(g,Q)$$

for all all partitions Q of $[a,b]$. Hence, the number $\underline{\int}_a^b (f(x)+g(x))\,dx - \underline{\int}_a^b f(x)\,dx$ is an upper bound for the set

$$F := \{L(g,Q) : Q \text{ partition of } [a,b]\}.$$

It follows that

$$\underline{\int}_a^b (f(x)+g(x))\,dx - \underline{\int}_a^b f(x)\,dx \ge \sup F = \underline{\int}_a^b g(x)\,dx.$$

This proves that

$$\underline{\int}_a^b (f(x)+g(x))\,dx \ge \underline{\int}_a^b f(x)\,dx + \underline{\int}_a^b g(x)\,dx.$$

The proof with the upper integral is similar and relies on the fact that

$$\sup_{x_{i-1}\le x\le x_i} (f(x)+g(x)) \le \sup_{x_{i-1}\le x\le x_i} f(x) + \sup_{x_{i-1}\le x\le x_i} g(x).$$

We omit the details. ■

Wednesday, December 05, 2012

Important properties of lower and upper integrals are the following.

Theorem 293 *Let $f:[a,b]\to\mathbb{R}$ be bounded and let $\lambda\in\mathbb{R}$. If $\lambda\ge 0$, then*

$$\underline{\int}_a^b (\lambda f)(x)\,dx = \lambda \underline{\int}_a^b f(x)\,dx, \quad \overline{\int}_a^b (\lambda f)(x)\,dx = \lambda \overline{\int}_a^b f(x)\,dx, \tag{37}$$

while if $\lambda<0$,

$$\underline{\int}_a^b (\lambda f)(x)\,dx = \lambda \overline{\int}_a^b f(x)\,dx, \quad \overline{\int}_a^b (\lambda f)(x)\,dx = \lambda \underline{\int}_a^b f(x)\,dx. \tag{38}$$

Proof. If $\lambda=0$, then $\lambda f=0$ and there is nothing to prove since we get $0=0$ in (37) and (38). If $\lambda>0$, then

$$\begin{aligned} L(\lambda f,P) &= \sum_{i=1}^n (x_i-x_{i-1}) \inf_{x_{i-1}\le x\le x_i} \lambda f(x) \\ &= \lambda \sum_{i=1}^n (x_i-x_{i-1}) \inf_{x_{i-1}\le x\le x_i} f(x) = \lambda L(f,P), \end{aligned}$$

so that

$$\begin{aligned} \underline{\int}_a^b \lambda f(x)\,dx &= \sup\{L(\lambda f,P) : P \text{ partition of } [a,b]\} \\ &= \sup\{\lambda L(f,P) : P \text{ partition of } [a,b]\} \\ &= \lambda \sup\{L(f,P) : P \text{ partition of } [a,b]\} \\ &= \lambda \underline{\int}_a^b f(x)\,dx = \lambda \int_a^b f(x)\,dx, \end{aligned}$$

while if $\lambda < 0$, then

$$\begin{aligned} L(\lambda f, P) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} \lambda f(x) \\ &= \lambda \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x_{i-1} \le x \le x_i} f(x) = \lambda U(f, P), \end{aligned}$$

so that

$$\begin{aligned} \underline{\int_a^b} \lambda f(x)\, dx &= \sup\{L(\lambda f, P) : P \text{ partition of } [a,b]\} \\ &= \sup\{\lambda U(f, P) : P \text{ partition of } [a,b]\} \\ &= \lambda \inf\{U(f, P) : P \text{ partition of } [a,b]\} \\ &= \lambda \overline{\int_a^b} f(x)\, dx = \lambda \int_a^b f(x)\, dx. \end{aligned}$$

The other cases are similar. ■

Theorem 294 *Let $f, g : [a,b] \to \mathbb{R}$ be bounded with $f(x) \le g(x)$ for all $x \in [a,b]$. Then*

$$\underline{\int_a^b} f(x)\, dx \le \underline{\int_a^b} g(x)\, dx, \quad \overline{\int_a^b} f(x)\, dx \le \overline{\int_a^b} g(x)\, dx.$$

Proof. Assume $f \le g$. Then for every partition P,

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} f(x) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x_{i-1} \le x \le x_i} g(x) = L(g, P), \end{aligned}$$

so that

$$\begin{aligned} \underline{\int_a^b} f(x)\, dx &= \sup\{L(f, P) : P \text{ partition of } [a,b]\} \\ &\le \sup\{L(g, P) : P \text{ partition of } [a,b]\} = \underline{\int_a^b} g(x)\, dx. \end{aligned}$$

The proof for the upper integral is the same. ■

Proposition 295 *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable.*

(i) If $c \in (a,b)$, then f is Riemann integrable over $[a,c]$ and $[c,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{39}$$

(ii) If $\lambda \in \mathbb{R}$, then λf is Riemann integrable and

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx. \tag{40}$$

(iii) The functions $f+g$ and fg are Riemann integrable and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{41}$$

(iv) If $f \le g$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

(v) The function $|f|$ is Riemann integrable and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. (i) Since f is Riemann integrable, the set of discontinuity points in $[a,b]$ has Lebesgue measure zero. Hence, the set of discontinuity points in $[a,c]$ and $[c,b]$ has Lebesgue measure zero. It follows that f is Riemann integrable over $[a,c]$ and $[c,b]$. Property (39) follows from (35).

(ii) If $\lambda = 0$, then $\lambda f = 0$ and both sides of (40) are zero. If $\lambda \neq 0$, then the set of discontinuities points of λf is the same of f. Hence, λf is Riemann integrable. Property (40) follows from Theorem 293.

(iii) Since the set of discontinuities points of $f+g$ and fg is contained in the union of the sets of discontinuities points of f and g, using the fact that the finite union of sets of Lebesgue measure zero still has measure zero, it follows that the functions $f+g$ and fg are Riemann integrable.

To prove (41), we apply Theorem 292.

(iv) Apply Theorem 294.

(v) Since the set of discontinuities points of $|f|$ is contained in the set of discontinuities points of f, it follows that $|f|$ is Riemann integrable. Using parts (ii) and (iv) and the facts that $f \le |f|$ and that $-f \le |f|$, we get

$$\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

$$-\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx,$$

which give (v). ■

Exercise 296 *Give an example of a bounded function $f : [a, b] \to \mathbb{R}$ such that $|f|$ is Riemann integrable over $[a, b]$, but f is not.*

Remark 297 *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, recall that*

$$\int_b^a f(x)\, dx := -\int_a^b f(x)\, dx.$$

Then property (v) should be replaced by

$$\left|\int_b^a f(x)\, dx\right| \le \left|\int_b^a |f(x)|\, dx\right|.$$

The next theorem will be used to prove the fundamental theorem of calculus.

Theorem 298 *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable then for every $\varepsilon > 0$ there exists a partition P^ε of $[a, b]$ such that for every partition $P = \{x_0, \ldots, x_n\}$ that contains P^ε and for every $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,*

$$\left|\int_a^b f(x)\, dx - \sum_{i=1}^n (x_i - x_{i-1}) f(y_i)\right| \le \varepsilon. \tag{42}$$

Proof. Using the definition of supremum and of infimum, we may find a partition P of $[a, b]$ and a partition Q of $[a, b]$ such that

$$L(f, P) \ge \underline{\int_a^b} f(x)\, dx - \frac{\varepsilon}{2},$$
$$U(f, Q) \le \overline{\int_a^b} f(x)\, dx + \frac{\varepsilon}{2}.$$

Then $P^\varepsilon := P \cup Q$ is a partition of $[a, b]$. Then by (34), $L(f, P^\varepsilon) \ge L(f, P)$ and $U(f, Q) \le U(f, P^\varepsilon)$. Hence,

$$\begin{aligned} 0 &\le U(f, P^\varepsilon) - L(f, P^\varepsilon) \le U(f, P) - L(f, Q) \\ &\le \overline{\int_a^b} f(x)\, dx + \frac{\varepsilon}{2} - \left(\underline{\int_a^b} f(x)\, dx - \frac{\varepsilon}{2}\right) = 0 + \varepsilon, \end{aligned}$$

where we have used $\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx$. Write $P^\varepsilon = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ that contains P^ε.

For every $x \in [x_{i-1}, x_i]$, we have that

$$|f(x) - f(y_i)| \le \sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f,$$

and so by Remark 279 and Proposition 295,

$$\begin{aligned}\left|\int_a^b f(x)\,dx - \sum_{i=1}^n (x_i - x_{i-1}) f(y_i)\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(y_i)\,dx\right| \\ &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f(x) - f(y_i))\,dx\right| \\ &\le \sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f(x) - f(y_i)\,dx\right| \\ &\le \sum_{i=1}^n (x_i - x_{i-1}) \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f\right) \\ &= U(f,P) - L(f,P) \le \varepsilon.\end{aligned}$$

This concludes the proof. ■

Theorem 299 (Fundamental Theorem of Calculus) *Let $F : [a,b] \to \mathbb{R}$ be a differentiable function. Assume that F' is Riemann integrable over $[a,b]$. Then*

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{43}$$

Proof. Fix $\varepsilon > 0$ and apply Theorem 298 to find a partition $P^\varepsilon = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that for *every* $y_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$,

$$\left|\int_a^b F'(x)\,dx - \sum_{i=1}^n (x_i - x_{i-1}) F'(y_i)\right| \le \varepsilon. \tag{44}$$

By the mean value theorem, for every $i = 1, \ldots, n$, there exists $y_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = (x_i - x_{i-1}) F'(y_i).$$

Hence,

$$F(b) - F(a) = F(x_n) - F(x_0) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = \sum_{i=1}^n (x_i - x_{i-1}) F'(y_i).$$

By substituting this expression in (44), we find that

$$\left|\int_a^b F'(x)\,dx - (F(b) - F(a))\right| \le \varepsilon.$$

By letting $\varepsilon \to 0^+$, we get (43). ■

As a corollary of the fundamental theorem of calculus, we have the formula for integration by parts.

Corollary 300 (Integration by Parts) *Let $F, G : [a,b] \to \mathbb{R}$ be a differentiable functions. Assume that F' and G' are Riemann integrable over $[a,b]$. Then*

$$\int_a^b F(x)\, G'(x)\; dx = F(b)\, G(b) - F(a)\, G(a) = \int_a^b F'(x)\, G(x)\; dx.$$

Proof. G and F are differentiable, so they are continuous. Hence, they are Riemann integrable. By Proposition 295, the functions FG' and $F'G$ are also Riemann integrable. Consider the function $H = FG$. Then $H' = F'G + FG'$. By the fundamental theorem of calculus applied to H, we have

$$\begin{aligned} F(b)\, G(b) - F(a)\, G(a) = H(b) - H(a) &= \int_a^b H'(x)\; dx \\ &= \int_a^b \left(F'(x)\, G(x) + F(x)\, G'(x)\right)\; dx, \end{aligned}$$

which concludes the proof. ■

Example 301 *Let's calculate*

$$\int_1^2 x \log^2 x\, dx.$$

Take $F(x) = \log^2 x$ and $G'(x) = x$. Then $F'(x) = \frac{2}{x} \log x$ while $G(x) = \frac{x^2}{2}$. Hence,

$$\begin{aligned} \int_1^2 x \log^2 x\, dx &= \left[\frac{x^2}{2} \log^2 x\right]_{x=1}^{x=2} - \int_1^2 \frac{x^2}{2} \frac{2}{x} \log x\, dx \\ &= 2 \log^2 2 - 0 - \int_1^2 x \log x\, dx. \end{aligned}$$

We integrate by parts once more, taking $F(x) = \log x$ and $G'(x) = x$. Then $F'(x) = \frac{1}{x}$ while $G(x) = \frac{x^2}{2}$. Hence,

$$\begin{aligned} \int_1^2 x \log x\, dx &= \left[\frac{x^2}{2} \log x\right]_{x=1}^{x=2} - \int_1^2 \frac{x^2}{2} \frac{1}{x}\, dx \\ &= 2 \log 2 - 0 - \frac{1}{2} \int_1^2 x\, dx = 2 \log 2 - \frac{3}{4}. \end{aligned}$$

In conclusion,

$$\int_1^2 x \log^2 x\, dx = 2 \log^2 2 - 2 \log 2 + \frac{3}{4}.$$

Friday, December 07, 2012

Next we discuss integration by substitution. The classical formula that you see in calculus is given by

$$\int_a^b f(G(x))\, G'(x)\ dx = \int_{G(a)}^{G(b)} f(y)\ dy.$$

We will prove a change of variables formula that requires weaker hypotheses.

Theorem 302 (Change of Variables) *Let $g : [a,b] \to \mathbb{R}$ be Riemann integrable and let*

$$G(x) := G(a) + \int_a^x g(t)\ dt, \quad x \in [a,b].$$

Assume that $f : G([a,b]) \to \mathbb{R}$ is Riemann integrable. Then $(f \circ G)\, g$ is Riemann integrable over $[a,b]$ and the following change of variables formula holds

$$\int_a^b f(G(x))\, g(x)\ dx = \int_{G(a)}^{G(b)} f(y)\ dy,$$

where, if $G(a) \geq G(b)$, we are using the notation (**??**) *and* (**??**).

Note that we are not assuming that G is differentiable in $[a,b]$.

Proof. We will give the proof in the very special case in which both f and g are continuous.

Case 1: Assume first that $G(a) \leq G(b)$ and define the function

$$F(t) := \int_{G(a)}^{t} f(y)\ dy, \quad t \in [G(a), G(b)].$$

Since f and g are continuous, by Theorem **??** and Exercise , F and G are differentiable, with $F' = f$ and $G' = g$. Hence, by Theorem 237, the composite function $H := F \circ G$ is differentiable, with

$$H'(x) = F'(G(x))\, G'(x) = f(G(x))\, g(x)$$

for all $x \in [a,b]$. By the fundamental theorem of calculus applied to the function $H := F \circ G$, we have that

$$\begin{aligned} \int_{G(a)}^{G(b)} f(y)\ dy - 0 = H(b) - H(a) &= \int_a^b H'(x)\ dx \\ &= \int_a^b f(G(x))\, g(x)\ dx. \end{aligned}$$

Case 2: If $G(a) > G(b)$, we define the function

$$F(t) := \int_t^{G(a)} f(y)\ dy, \quad t \in [G(b), G(a)].$$

By Exercise **??**, $F' = -f$. Hence, by Theorem 237, the composite function $H := F \circ G$ is differentiable, with

$$H'(x) = F'(G(x))\,G'(x) = -f(G(x))\,g(x)$$

for all $x \in [a, b]$. By the fundamental theorem of calculus applied to the function $H := F \circ G$, we have that

$$\begin{aligned}\int_{G(b)}^{G(a)} f(y)\ dy - 0 = H(b) - H(a) &= \int_a^b H'(x)\ dx \\ &= -\int_a^b f(G(x))\,g(x)\ dx.\end{aligned}$$

Using (**??**), we obtain

$$\int_{G(a)}^{G(b)} f(y)\ dy = -\int_{G(b)}^{G(a)} f(y)\ dy = \int_a^b f(G(x))\,g(x)\ dx.$$

■

Remark 303 *Note that there are examples of functions f and g for which $f(G(x))\,g(x)$ is integrable, but not $f(G(x))$.*

Next we discuss the composition of Riemann integrable functions. The next example shows that the composition of Riemann integrable functions may not be Riemann integrable.

Example 304 *Let f be the function defined in Exercise 291 and let $g : [0,1] \to [0,1]$ be the function*

$$g(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then f and g are both Riemann integrable, but their composition $g \circ f$ is not. Indeed, of $x \in [0,1]$

$$g(f(x)) = \begin{cases} 0 & \text{if } x \text{ is irrational}, \\ 1 & \text{if } x \text{ is rational}, \end{cases}$$

and we have seen that this function is not Riemann integrable.

The next proposition shows that if f is continuous, then the composition is Riemann integrable.

Proposition 305 *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable, let $g : [c,d] \to \mathbb{R}$ be continuous, and let $f([a,b]) \subseteq [c,d]$. Then $g \circ f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof. We begin by observing that if f is continuous at some point $x \in [a, b]$, then since g is continuous, the function g will be continuous at $f(x)$ and so $g \circ f$ will be continuous at x. This shows that set of discontinuities points of $g \circ f$ is contained in the set of discontinuities points of g, and since g is Riemann integrable, this set has Lebesgue measure zero. In turn, $g \circ f$ is Riemann integrable. ■

If f is continuous and g is Riemann integrable, then $g \circ f$ may not be Riemann integrable.

Exercise 306 *Since the rationals are countable, we can write* $[0, 1] \cap \mathbb{Q}$ *as a sequence* $\{r_n\}_n$. *Consider the open set*

$$U := \bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{4^n}, r_n + \frac{1}{4^n} \right)$$

and let $C := [0, 1] \setminus U$. *Note that* C *is closed.*

1. *Prove that* C *has empty interior.*

2. *Prove that,* C *does not have Lebesgue measure zero.*

3. *Let* $f(x) = \operatorname{dist}(x, C)$ *and let*

$$g(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

   Prove that $g \circ f$ *is not Riemann integrable.*