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1 $\mathbf{L}^p$ Spaces and Interpolation

1.1 $\mathbf{L}^p$ Spaces

Let $(X, \mathfrak{M}, \mu)$ be a a measure space and let $0 < p < \infty$. The space $L^p(X;\mu)$ is defined as the space of all measurable functions $f : X \to \mathbb{C}$ such that

$$\int_X |f|^p \, d\mu < \infty.$$

The space $L^\infty(X;\mu)$ is the space of all measurable functions $f : X \to \mathbb{C}$ for which there exists a constant $M \geq 0$ such that $|f(x)| \leq M$ for μ a.e. $x \in X$. As usual we identify functions in $L^p(X;\mu)$ that coincide up to a set of measure zero. When there is no possibility of confusion, we will write $L^p(X)$ or just L^p.

For $0 < p < \infty$ we define the quasinorm

$$\|f\|_{L^p} := \left(\int_X |f|^p \, d\mu \right)^{1/p},$$

while for $p = \infty$,

$$\|f\|_{L^\infty} := \operatorname*{esssup}_X |f|,$$

where the *essential supremum* of a measurable function $f : X \to \mathbb{C}$ is given by

$$\operatorname*{esssup}_X |f| := \inf\{M : \ |f(x)| \leq M \text{ for } \mu \text{ a.e. } x \in X\}.$$

For $1 \leq p \leq \infty$, in view of Minkowski's inequality, we have that $L^p(X)$ is a normed space and it is actually complete, so it is a Banach space. For $0 < p < 1$, $\|\cdot\|_{L^p}$ is not a norm, but one can show that

$$\|f + g\|_{L^p} \leq 2^{(1-p)/p} \|f\|_{L^p} + 2^{(1-p)/p} \|g\|_{L^p},$$

and so $L^p(X)$ is a quasi-normed space and it is actually complete, so it is a quasi-Banach space.

For $1 < p < \infty$, we will write $p' := \frac{p}{p-1}$, while $1' := \infty$ and $\infty' := 1$. Then for $1 \leq p < \infty$ the dual of $L^p(X)$ can be identified with $L^{p'}(X)$. Moreover, we can show that for all $1 \leq p \leq \infty$,

$$\|f\|_{L^p} = \sup\left\{ \left| \int_X fg \, d\mu \right| : \ \|g\|_{L^{p'}} \leq 1 \right\}. \tag{1}$$

1.2 Weak L^p Spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Given a measurable function $f : X \to \mathbb{C}$, the *distribution function* of f is the function $\varrho_f : [0, \infty) \to [0, \mu(X)]$, defined by

$$\varrho_f(s) := \mu(\{x \in X : |f(x)| > s\}), \quad s \geq 0.$$

The function f is said to *vanish at infinity* if it is measurable and $\varrho_f(s) < \infty$ for every $s > 0$. Note that for a function f vanishing at infinity the value $\varrho_f(0)$ could be infinite. For example, if X has infinite measure and f is everywhere positive, then $\varrho_f(0) = \infty$. Moreover,

$$\varrho_f(s) = 0 \quad \text{for all } s \geq \operatorname*{esssup}_X |f|. \tag{2}$$

Some important properties of ϱ_f are summarized in the next proposition.

Proposition 1 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f, g, f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, be measurable functions. Then the following properties hold:*

(i) The function $\varrho_f : [0, \infty) \to [0, \mu(X)]$ is decreasing and right continuous.

(ii) If $|f(x)| \leq |g(x)|$ for μ a.e. $x \in X$, then $\varrho_f \leq \varrho_g$. In particular, if $|f(x)| = |g(x)|$ for μ a.e. $x \in X$, then $\varrho_f = \varrho_g$.

(iii) If $|f_n(x)| \nearrow |f(x)|$ for μ a.e. $x \in X$, then $\varrho_{f_n} \nearrow \varrho_f$.

(iv) If $|f(x)| \leq \liminf_n |f_n(x)|$ for μ a.e. $x \in X$, then $\varrho_f \leq \liminf_n \varrho_{f_n}$.

Proof. In what follows, for $s \geq 0$ we set

$$X_s := \{x \in X : |f(x)| > s\}.$$

(i) If $0 \leq s \leq r$, then $X_r \subseteq X_s$, and so $\varrho_f(r) \leq \varrho_f(s)$, which shows that ϱ_f is decreasing. To prove that ϱ_f is right continuous at $s \geq 0$, consider a decreasing sequence $s_n \to s^+$. Then $X_{s_n} \subseteq X_{s_{n+1}}$ and

$$\bigcup_{n=1}^{\infty} X_{s_n} = \{x \in X : |f(x)| > s\} = X_s,$$

and so by standard properties of μ we have that

$$\lim_{n\to\infty} \varrho_f(s_n) = \lim_{n\to\infty} \mu(X_{s_n}) = \mu\left(\bigcup_{n=1}^{\infty} X_{s_n}\right) = \varrho_f(s).$$

Since ϱ_f is decreasing,

$$(\varrho_f)_+(s_0) = \lim_{n\to\infty} \varrho_f(s_n) = \varrho_f(s),$$

which shows that ϱ_f is right continuous.

(ii) For every $s \geq 0$,

$$\{x \in X : |f(x)| > s\} \subseteq \{x \in X : |g(x)| > s\} \cup F,$$

where $\mu(F) = 0$, and so $\varrho_f(s) \leq \varrho_g(s)$.

(iii) In view of part (ii), by modifying each f_n on a set of measure zero, we can assume that $|f_n(x)| \nearrow |f(x)|$ for *all* $x \in X$. For every $s \geq 0$ and $n \in \mathbb{N}$ set $X_n := \{x \in X : f_n(x) > s\}$. Since $|f_n| \leq |f_{n+1}|$, we have that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, and since $|f_n(x)| \to |f(x)|$ for all $x \in X$, it follows that

$$\bigcup_{n=1}^{\infty} X_n = \{x \in X : |f(x)| > s\}.$$

Hence, again by properties of measures,

$$\lim_{n\to\infty} \varrho_{f_n}(s) = \lim_{n\to\infty} \mu(X_n) = \mu(\{x \in X : |f(x)| > s\}) = \varrho_f(s).$$

(iv) Set $g_n := \inf_{m\geq n} |f_m|$. Then $g_n \nearrow g := \sup_n g_n = \liminf_n |f_n|$. It follows by parts (ii) and (iii) and the facts that $f \leq g$ and $g_n \leq f_n$ that

$$\varrho_f(s) \leq \varrho_g(s) = \lim_{n\to\infty} \varrho_{g_n}(s) \leq \liminf_{n\to\infty} \varrho_{f_n}(s)$$

for every $s > 0$. ■

Exercise 2 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$ be a measurable function.*

(i) Prove that if f vanishes at infinity, then

$$\lim_{s\to\infty} \varrho_f(s) = 0$$

and that for $s > 0$,

$$(\varrho_f)_-(s) = \varrho_f(s) + \mu(\{x \in X : |f(x)| = s\}).$$

(ii) Give an example of a function f not vanishing at infinity for which ϱ_f is not left continuous.

Example 3 *Let $(X, \mathfrak{M}, \mu)$ be a a measure space and let $f : X \to [0, \infty)$ be a simple function vanishing at infinity, that is,*

$$f = \sum_{i=1}^{k} c_i \chi_{E_i},$$

where $E_i \subseteq X$ are pairwise disjoint measurable sets with $\mu(E_i) < \infty$, $i = 1, \ldots, k$, and $c_1 > \cdots > c_k > 0$. If $s \geq c_1$, then $\varrho_f(s) = 0$. If $c_2 \leq s < c_1$, then

$f(x) > s$ if and only if $x \in E_1$, and in general if $c_{i+1} \leq s < c_i$, then $f(x) > s$ if and only if $x \in E_1 \cup \cdots \cup E_i$. Thus,

$$\varrho_f = \sum_{i=1}^{k} \left[\sum_{j=1}^{i} \mu\left(E_j\right) \right] \chi_{[c_{i+1}, c_i)},$$

where $c_{k+1} := 0$.

Theorem 4 *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $0 < p < \infty$ and let $f : X \to \mathbb{C}$ be a measurable function. Then*

$$\int_X |f(x)|^p \, d\mu(x) = p \int_0^\infty s^{p-1} \varrho_f(s) \, ds.$$

Proof. If $\varrho_f(s_0) = \infty$ for some $s_0 > 0$, then $\varrho_f(s) = \infty$ for all $0 \leq s < s_0$, and thus both sides of the previous equality are infinite. Thus, assume that $\varrho_f(s) < \infty$ for all $s > 0$. Restrict the measure μ to the set of σ-finite measure

$$X_0 := \{x \in X : |f(x)| > 0\}.$$

By Tonelli's theorem, which holds since $\mathcal{L}^1$ and μ restricted to X_0 are both σ-finite,

$$\begin{aligned} p \int_0^\infty s^{p-1} \varrho_f(s) \, ds &= p \int_0^\infty s^{p-1} \mu\left(\{x \in X_0 : |f(x)| > s\}\right) \, ds \\ &= p \int_0^\infty s^{p-1} \int_{X_0} \chi_{\{|f| > s\}}(x) \, d\mu(x) ds \\ &= \int_{X_0} \int_0^{|f(x)|} p s^{p-1} \, ds d\mu(x) \\ &= \int_{X_0} |f(x)|^p \, d\mu(x) = \int_X |f(x)|^p \, d\mu(x). \end{aligned}$$

■

Wednesday, January 14, 2015

We now define the weak L^p spaces.

Definition 5 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $0 < p < \infty$. The* weak space *$L^p_w(X; \mu)$ is defined as the space of all measurable functions $f : X \to \mathbb{C}$ such that*

$$\|f\|_{L^p_w} := \sup\{s \varrho_f(s)^{1/p} : \ s > 0\} < \infty. \tag{3}$$

For $p = \infty$ we define $L^\infty_w(X; \mu) := L^\infty(X; \mu)$.

We will write $L^p_w(X)$ or L^p_w for $L^p_w(X; \mu)$ when there is no possibility of confusion. In view of Proposition 1(iii), we can identify functions in $L^p_w(X)$ that coincide up to a set of measure zero. Let's prove that $\|\cdot\|_{L^p_w}$ is a seminorm.

Proposition 6 *Let* $(X, \mathfrak{M}, \mu)$ *be a a measure space and let* $0 < p < \infty$. *Then* $\|\cdot\|_{L^p_w}$ *is a seminorm.*

Proof. If $\|f\|_{L^p_w} = 0$, then $\varrho_f(s) = 0$ for all $s > 0$, which implies that $f(x) = 0$ for μ a.e. $x \in X$. Thus f coincides with the zero function.

Using the fact that for $\boldsymbol{z} \in \mathbb{C}$, $\boldsymbol{z} \neq \mathbf{0}$,

$$\varrho_{\boldsymbol{z}f}(s) = \varrho_f(s/|\boldsymbol{z}|) \tag{4}$$

for $s > 0$, it follows that

$$\begin{aligned} \|\boldsymbol{z}f\|_{L^p_w} &= \sup\{s\varrho_{\boldsymbol{z}f}(s)^{1/p} : \ s > 0\} = \sup\{s\varrho_f(s/|\boldsymbol{z}|)^{1/p} : \ s > 0\} \\ &= |\boldsymbol{z}| \sup\{s|\boldsymbol{z}|^{-1}\varrho_f(s/|\boldsymbol{z}|)^{1/p} : \ s > 0\} = |\boldsymbol{z}| \, \|f\|_{L^p_w} . \end{aligned}$$

Next observe that if $|f(x) + g(x)| > s$ then $|f(x)| > s/2$ or $|g(x)| > s/2$, and so

$$\{x \in X : |f(x) + g(x)| > s\} \subseteq \{x \in X : |f(x)| > s/2\} \cup \{x \in X : |g(x)| > s/2\} .$$

In turn,

$$\varrho_{f+g}(s) \leq \varrho_f(s/2) + \varrho_g(s/2).$$

Using the inequality $(a + b)^{1/p} \leq c_p(a^{1/p} + b^{1/p})$, where $c_p := 1$ if $p \geq 1$ and $c_p := 2^{1/p-1}$ if $p < 1$, we get

$$\begin{aligned} s\varrho_{f+g}(s)^{1/p} &\leq c_p s\varrho_f(s/2)^{1/p} + c_p\varrho_g(s/2)^{1/p} \\ &\leq c_p 2^{1/p} \|f\|_{L^p_w} + c_p 2^{1/p} \|g\|_{L^p_w} . \end{aligned}$$

By taking the supremum over all $s > 0$ we obtain that $\|\cdot\|_{L^p_w}$ is a seminorm. ■

Remark 7 *Using (4) we have that*

$$\|f\|_{L^p_w} = \inf\{M : \ \varrho_f(s) \leq M^p/s^p \text{ for all } s > 0\}.$$

Theorem 8 *Let* $(X, \mathfrak{M}, \mu)$ *be a a measure space and let* $0 < p < \infty$. *Then* $L^p_w(X)$ *is a quasi-Banach space.*

Proof. Homework. ■

In view of Theorem 4 we have the following inclusions.

Proposition 9 *Let* $(X, \mathfrak{M}, \mu)$ *be a a measure space and let* $0 < p < \infty$. *Then*

$$L^p(X) \subseteq L^p_w(X)$$

and in the general the inclusion is strict. If μ *is finite then*

$$L^p_w(X) \subseteq L^q(X)$$

for every $0 < q < p$.

Proof. Given $f \in L^p(X;\mu)$, for every $s > 0$ we have that

$$s\varrho_f(s)^{1/p} \le \left(\int_{\{|f|>s\}} |f|^p \, d\mu\right)^{1/p} \le \|f\|_{L^p} .$$

By taking the supremum over all $s > 0$ we obtain that $\|f\|_{L^p_w} \le \|f\|_{L^p}$.

Next assume that μ is finite. Let $\varepsilon := p - q > 0$. Then for $0 < s \le 1$,

$$s^{p-\varepsilon-1}\varrho_f(s) \le s^{p-\varepsilon-1}\mu(X),$$

while for $s > 1$,

$$s^{p-\varepsilon-1}\varrho_f(s) = s^{-\varepsilon-1}[s\varrho_f(s)^{1/p}]^p \le s^{-\varepsilon-1} \|f\|^p_{L^p_w} .$$

Hence, by Theorem 4,

$$\begin{aligned}\int_X |f(x)|^{p-\varepsilon} \, d\mu(x) &= (p-\varepsilon)\int_0^\infty s^{p-\varepsilon-1}\varrho_f(s) \, ds \\ &\le (p-\varepsilon)\mu(X)\int_0^1 s^{p-\varepsilon-1} \, ds + (p-\varepsilon)\|f\|^p_{L^p_w}\int_1^\infty s^{-\varepsilon-1} \, ds \\ &= \mu(X) + \frac{p-\varepsilon}{\varepsilon}\|f\|^p_{L^p_w} ,\end{aligned}$$

which concludes the proof. ■

Example 10 *Take $X = \mathbb{R}^N$ with the Lebesgue measure $\mathcal{L}^N$. Then for $p > 0$ the function $f(\boldsymbol{x}) := |\boldsymbol{x}|^{-N/p}$ does not belong to any $L^q(\mathbb{R}^N)$, $q > 0$, but*

$$\varrho_f(s) = \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ |\boldsymbol{x}|^{-N/p} > s\}) = \mathcal{L}^N(B(\mathbf{0}; s^{p/N})) = \alpha_N s^p,$$

where $\alpha_N = \mathcal{L}^N(B(\mathbf{0};1))$. Hence, $\|f\|_{L^p_w} = \alpha_N^{1/p}$.

Next we prove a first interpolation theorem.

Theorem 11 *Let $(X, \mathfrak{M}, \mu)$ be a a measure space and let $0 < p < q \le \infty$. If $f \in L^p_w(X) \cap L^q_w(X)$, then $f \in L^r(X)$ for all $p < r < q$, with*

$$\|f\|_{L^r} \le C(p,q,r)\|f\|^{p(q-r)/[r(q-p)]}_{L^p_w}\|f\|^{q(r-p)/[r(q-p)]}_{L^q_w}$$

if $q < \infty$ and

$$\|f\|_{L^r} \le \left(\frac{r}{r-p} + \frac{r}{q-r}\right)^{1/r}\|f\|^{p/r}_{L^p_w}\|f\|^{(r-p)/r}_{L^\infty}$$

if $q = \infty$.

Proof. Assume first that $q < \infty$. If $f = 0$, there is nothing to prove, thus assume that $f \neq 0$ and define

$$L := \left(\frac{\|f\|_{L_w^q}^q}{\|f\|_{L_w^p}^p} \right)^{1/(q-p)} .$$

It follows from (3) that for every $s > 0$,

$$\varrho_f(s) \leq \min \left\{ \|f\|_{L_w^p}^p / s^p, \|f\|_{L_w^q}^q / s^q \right\} .$$

Hence, by Theorem 4,

$$\begin{aligned}
\int_X |f(x)|^r \, d\mu(x) &= r \int_0^\infty s^{r-1} \varrho_f(s) \, ds \\
&\leq r \int_0^\infty s^{r-1} \min \left\{ \|f\|_{L_w^p}^p / s^p, \|f\|_{L_w^q}^q / s^q \right\} \, ds \\
&= r \int_0^L s^{r-1-p} \|f\|_{L_w^p}^p \, ds + r \int_L^\infty s^{r-1-q} \|f\|_{L_w^q}^q \, ds \\
&= \frac{r}{r-p} \|f\|_{L_w^p}^p L^{r-p} + \frac{r}{q-r} \|f\|_{L_w^p}^p \frac{1}{L^{q-r}} \\
&= \left(\frac{r}{r-p} + \frac{r}{q-r} \right) \|f\|_{L_w^p}^{p(q-r)/(q-p)} \|f\|_{L_w^q}^{q(r-p)/(q-p)} .
\end{aligned}$$

On the other hand, if $q = \infty$, then $\varrho_f(s) = 0$ for all $s \geq \|f\|_{L^\infty}$ by (2) and so again by Theorem 4,

$$\begin{aligned}
\int_X |f(x)|^r \, d\mu(x) &= r \int_0^{\|f\|_{L^\infty}} s^{r-1} \varrho_f(s) \, ds \\
&\leq r \int_0^{\|f\|_{L^\infty}} s^{r-1-p} \|f\|_{L_w^p}^p \, ds \\
&= \frac{r}{r-p} \|f\|_{L_w^p}^p \|f\|_{L^\infty}^{r-p} ,
\end{aligned}$$

which concludes the proof. ■

1.3 Lorentz Spaces

Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$, be a measurable function. The *decreasing rearrangement* of f is the function $f^* : [0, \infty) \to [0, \infty]$, defined by

$$f^*(t) := \inf \{ s \in [0, \infty) : \varrho_f(s) \leq t \} , \tag{5}$$

where we set $\inf \emptyset := \infty$. Hence, if $\varrho_f(s) > t$ for all $s \in [0, \infty)$, then $f^*(t) = \infty$.

Observe that if μ is finite, then it follows from the definition of f^* that

$$f^*(t) = 0 \quad \text{for all } t \geq \mu(X). \tag{6}$$

In this case it is enough to consider the function f^* restricted to the set $[0, \mu(X))$.

Friday, January 16, 2015

In view of Proposition 1 we have the following result.

Proposition 12 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f, g, f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, be measurable functions. Then the following properties hold:*

(i) The function $f^* : [0, \infty) \to [0, \infty]$ is decreasing, right continuous and for all $s, t \geq 0$,

$$f^*(t) > s \quad \text{if and only if} \quad \varrho_f(s) > t. \tag{7}$$

(ii) If $|f(x)| \leq |g(x)|$ for μ-a.e. $x \in X$, then $f^* \leq g^*$. In particular, if $f(x) = g(x)$ for μ-a.e. $x \in X$, then $f^* = g^*$.

(iii) If $|f_n(x)| \nearrow |f(x)|$ for μ a.e. $x \in X$, then $f_n^* \nearrow f^*$.

(iv) If $|f(x)| \leq \liminf_n |f_n(x)|$ for μ a.e. $x \in X$, then $f^* \leq \liminf_n f_n^*$.

Proof. (i) If $0 \leq t_1 \leq t_2$, then

$$\{s \in [0, \infty) : \varrho_f(s) \leq t_1\} \subseteq \{s \in [0, \infty) : \varrho_f(s) \leq t_2\},$$

and so

$$\begin{aligned} f^*(t_1) &= \inf\{s \in [0, \infty) : \varrho_f(s) \leq t_1\} \\ &\geq \inf\{s \in [0, \infty) : \varrho_f(s) \leq t_2\} = f^*(t_2). \end{aligned}$$

Thus, f^* is decreasing.

To prove that f^* right continuous, fix $t_0 \geq 0$. If $f^*(t_0) = 0$, then, since f^* is decreasing, we have that $f^*(t) = 0$ for all $t \geq t_0$ and there is nothing to prove. Thus, assume that $f^*(t_0) > 0$ and fix $0 < s_0 < f^*(t_0)$. It follows from (5) that $\varrho_f(s_0) > t_0$. Hence, we can find $\delta > 0$ such that $\varrho_f(s_0) > t_0 + \delta$. Using (5) once more, we have that $f^*(t) \geq s_0$ for all $t \in [t_0, t_0 + \delta]$, and so

$$s_0 \leq (f^*)_+(t_0) \leq f^*(t_0).$$

Letting $s_0 \to f^*(t_0)$ proves that f^* is right continuous.

Next we prove (7). Assume that $f^*(t) > s$ for some $s, t \geq 0$. Then $\varrho_f(s) > t$ by (5).

Conversely, assume that $\varrho_f(s) > t$. Since ϱ_f is right continuous, we have that $\varrho_f(r) > t$ for all $r \in [s, s + \delta]$ for some $\delta > 0$. Hence $f^*(t) > s$ by (5). This proves (7). In turn, for $s \geq 0$,

$$\{t \geq 0 : f^*(t) > s\} = \{t \geq 0 : \varrho_f(s) > t\} = [0, \varrho_f(s)).$$

Hence,

$$\mathcal{L}^1(\{t \geq 0 : f^*(t) > s\}) = \varrho_f(s) = \mu(\{x \in X : f(x) > s\})$$

by the definition of ϱ_f.

(ii) By Proposition 1(ii), for every $t \geq 0$,

$$\{s \in [0, \infty) : \varrho_g(s) \leq t\} \subseteq \{s \in [0, \infty) : \varrho_f(s) \leq t\}$$

and hence $f^*(t) \leq g^*(t)$.

(iii) By part (ii) we have that $f_n^* \leq f_{n+1}^* \leq f^*$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} f_n^*(t) \leq f^*(t) \tag{8}$$

for all $t \geq 0$. Suppose that $f^*(t) > 0$ for some $t \geq 0$ and fix $s \in [0, f^*(t))$. By (7), $\varrho_f(s) > t$. Hence, by Proposition 1(iii) there exists $k \in \mathbb{N}$ such that $\varrho_{f_k}(s) > t$. Again by (7) we have that $f_k^*(t) > s$ and since $f_n^* \leq f_{n+1}^* \leq f^*$ for all $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} f_n^*(t) > s.$$

Letting $s \nearrow f^*(t)$, we conclude that $\lim\limits_{n\to\infty} f_n^*(t) \geq f^*(t)$, which, together with (8), implies that

$$\lim_{n\to\infty} f_n^*(t) = f^*(t)$$

for all $t \geq 0$.

(iv) The proof is the same as the one of Proposition 1(iv). ■

Exercise 13 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$, be a measurable function.*

(i) Prove that if f vanishes at infinity, then f^* jumps at some point $t_0 \in (0, \mu(X))$ if and only if $\varrho_f(s) \equiv t_0$ for all s in some interval $(s_1, s_2) \subset [0, \infty)$, with $s_1 < s_2$. Moreover,

$$f^*(\varrho_f(s)) \leq s \quad \textit{for every } s > 0,$$

with strict inequality holding if and only if ϱ_f is constant on some interval $[s', s] \subset [0, \infty)$, with $s' < s$, while $f^*(t) \equiv s_0$ for all t in some interval $(t_1, t_2) \subset J$, with $t_1 < t_2$, and for some $s_0 > 0$ if and only if

$$\mu(\{x \in X : |f(x)| = s_0\}) > 0$$

and $(t_1, t_2) \subset \Big(\varrho_f(s_0), (\varrho_f)_-(s_0)\Big)$.

(ii) What happens if we remove the assumption that f vanishes at infinity?

Next we prove that f and f^* have the same L^p semi-norms.

Theorem 14 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$, be a measurable function. Then the functions f and f^* are equi-measurable; that is, for all $s \geq 0$,*

$$\mu(\{x \in X : f(x) > s\}) = \mathcal{L}^1(\{t \in [0, \infty) : f^*(t) > s\}).$$

In particular, if f vanishes at infinity, then so does f^*. Moreover, for all $0 < p < \infty$,

$$\int_X |f(x)|^p \ d\mu(x) = \int_0^\infty (f^*(t))^p \ dt, \tag{9}$$

while

$$\operatorname*{esssup}_X |f| = \sup f^* = f^*(0). \tag{10}$$

Proof. We begin by showing that for all $s, t \geq 0$, It follows from Theorem 4 and (7) that

$$\begin{aligned}\int_X |f(x)|^p \ d\mu(x) &= p \int_0^\infty s^{p-1} \varrho_f(s) \ ds \\ &= p \int_0^\infty s^{p-1} \mathcal{L}^1 (\{t \geq 0 : \ f^*(t) > s\}) \ ds \\ &= \int_0^\infty (f^*(t))^p \ dt.\end{aligned}$$

Next we claim that

$$f^*(0) = \operatorname*{esssup}_X |f|. \tag{11}$$

By (5),

$$\begin{aligned}f^*(0) &= \inf \{s \in [0, \infty) : \ \varrho_f(s) = 0\} \\ &= \inf \{s \in [0, \infty) : \ \mu (\{x \in X : \ f(x) > s\}) = 0\} \\ &= \inf \{s \in [0, \infty) : \ |f(x)| \leq s \text{ for } \mu \text{ a.e. } x \in X\} \\ &= \operatorname*{esssup}_X |f|.\end{aligned}$$

■

Exercise 15 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{C}$, be a measurable function. What is the relation between $\mathcal{L}^1 (\{t \in [0, \mu(X)) : \ f^*(t) = 0\})$ and $\mu (\{x \in X : \ f(x) = 0\})$? When are they equal?*

Definition 16 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $0 < p < \infty$ and $0 < q \leq \infty$. The* Lorentz space *$L^{p,q}(X; \mu)$ is defined as the space of all measurable functions $f : X \to \mathbb{C}$ such that*

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty (t^{1/p} f^*(t))^q \ \frac{dt}{t} \right)^{1/q} < \infty \tag{12}$$

if $q < \infty$ and

$$\|f\|_{L^{p,\infty}} := \sup\{t^{1/p} f^*(t) : \ t > 0\} < \infty \tag{13}$$

if $q = \infty$.

In view of Proposition 12(iii), we can identify functions in $L^{p,q}(X; \mu)$ that coincide up to a set of measure zero.

Exercise 17 *Prove that if* $q < \infty$, $L^{\infty,q}(X;\mu) = \{0\}$.

Monday, January 19, 2015

MLK No classes.

Wednesday, January 21, 2015

Let's prove that $\|\cdot\|_{L^{p,q}}$ is a quasi-norm.

Proposition 18 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $0 < p, q \le \infty$. *Then* $\|\cdot\|_{L^{p,q}}$ *is a quasi-norm.*

Proof. Let $f, g \in L^{p,q}(X;\mu)$ and let $t > 0$. We claim that

$$(f+g)^*(t) \le f^*(t/2) + g^*(t/2). \tag{14}$$

To see this, consider the three sets

$$E_{f+g} := \{s \in [0,\infty) : \varrho_{f+g}(s) \le t\},$$
$$E_f := \{s \in [0,\infty) : \varrho_f(s) \le t/2\}, \quad E_g := \{s \in [0,\infty) : \varrho_g(s) \le t/2\}.$$

Let $s_1 \in E_f$ and $s_2 \in E_g$. Then $\varrho_f(s_1) = \mu(\{x \in X : |f(x)| > s_1\}) \le t/2$ and $\varrho_g(s_2) = \mu(\{x \in X : |g(x)| > s_2\}) \le t/2$. But since

$$\{x \in X : |f(x) + g(x)| > s_1 + s_2\} \subseteq \{x \in X : |f(x)| > s_1\} \cup \{x \in X : |g(x)| > s_2\},$$

it follows that

$$\begin{aligned} \varrho_{f+g}(s_1+s_2) &= \mu(\{x \in X : |f(x) + g(x)| > s_1 + s_2\}) \\ &\le \varrho_f(s_1) + \varrho_g(s_2) \le \frac{t}{2} + \frac{t}{2} = 2. \end{aligned}$$

Hence, $s_1 + s_2 \in E_{f+g}$, and so

$$(f+g)^*(t) = \inf E_{f+g} \le s_1 + s_2.$$

By taking first the infimum over all $s_1 \in E_f$ and then the infimum over all $s_2 \in E_g$, it follows that

$$(f+g)^*(t) = \inf E_{f+g} \le \inf E_f + \inf E_g = f^*(t/2) + g^*(t/2).$$

Multiplying both sides by $t^{1/p}$ we get

$$t^{1/p}(f+g)^*(t) \le t^{1/p} f^*(t/2) + t^{1/p} g^*(t/2).$$

If $q \ge 1$, we can apply Minkowski's inequality with respect to the measure $\frac{1}{t}\,dt$ to obtain

$$\begin{aligned} \left(\int_0^\infty (t^{1/p}(f+g)^*(t))^q \,\frac{dt}{t}\right)^{1/q} &\le \left(\int_0^\infty (t^{1/p} f^*(t/2) + t^{1/p} g^*(t/2))^q \,\frac{dt}{t}\right)^{1/q} \\ \le \left(\int_0^\infty (t^{1/p} f^*(t/2))^q \,\frac{dt}{t}\right)^{1/q} &+ \left(\int_0^\infty (t^{1/p} g^*(t/2))^q \,\frac{dt}{t}\right)^{1/q}. \end{aligned}$$

We can now consider the change of variables $s = t/2$.

On the other hand, if $q \le 1$, then

$$\begin{aligned}\int_0^\infty (t^{1/p}(f+g)^*(t))^q \,\frac{dt}{t} &\le \int_0^\infty (t^{1/p}f^*(t/2) + t^{1/p}g^*(t/2))^q \,\frac{dt}{t} \\ &\le \int_0^\infty (t^{1/p}f^*(t/2))^q \,\frac{dt}{t} + \int_0^\infty (t^{1/p}g^*(t/2))^q \,\frac{dt}{t}.\end{aligned}$$

We can now consider the change of variables $s = t/2$ and then raise everything to power $1/q \ge 1$ and use the inequality $(a+b)^{1/q} \le 2^{1/q-1}a^{1/q} + 2^{1/q-1}b^{1/q}$. ■

Exercise 19 *Prove that the spaces $L^{p,q}(X;\mu)$ are quasi-Banach spaces.*

Theorem 20 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $0 < p, q \le \infty$. Then,*

$$L^p(X;\mu) = L^{p,p}(X;\mu), \quad L^p_w(X;\mu) = L^{p,\infty}(X;\mu).$$

Proof. If $p < \infty$, then by (9) and (12),

$$\|f\|_{L^{p,p}(X;\mu)} = \|f^*\|_{L^p([0,\infty);\mathcal{L}^1)} = \|f\|_{L^p(X;\mu)}.$$

On the other hand, if $p = \infty$, then since f^* is decreasing by Proposition 12(i), it follows from (10) and (13) that

$$\|f\|_{L^{\infty,\infty}} = \sup\{f^*(t) : \; t > 0\} = f^*(0) = \operatorname*{esssup}_X |f|.$$

This proves the first equality.

To prove that $L^p_w(X;\mu) = L^{p,\infty}(X;\mu)$, let $t > 0$. If $f^*(t) > 0$, let $0 < \varepsilon < f^*(t)$. Then $f^*(t) > f^*(t) - \varepsilon =: s_\varepsilon > 0$, and so (7) we have that $\varrho_f(s_\varepsilon) > t$. It follows from (3) that

$$\|f\|_{L^p_w} \ge (\varrho_f(s_\varepsilon))^{1/p} s_\varepsilon \ge t^{1/p}(f^*(t) - \varepsilon).$$

Letting $\varepsilon \to 0^+$ gives

$$\|f\|_{L^p_w} \ge t^{1/p} f^*(t).$$

On the other hand, if $t > 0$ is such that $f^*(t) = 0$, then this inequality continues to hold. Hence, the inequality holds for all $t > 0$. Taking the supremum over all $t > 0$ and using (13) gives $\|f\|_{L^p_w} \ge \|f\|_{L^{p,\infty}}$.

The opposite inequality follows in the same way. Given $s > 0$, if $\varrho_f(s) > 0$, let $0 < \varepsilon < \varrho_f(s)$. Since $\varrho_f(s) > \varrho_f(s) - \varepsilon =: t_\varepsilon > 0$, by (7), we have that $f^*(t_\varepsilon) > s$. It follows from (13) that

$$\|f\|_{L^{p,\infty}} \ge t_\varepsilon^{1/p} f^*(t_\varepsilon) \ge (\varrho_f(s) - \varepsilon)^{1/p} s.$$

Letting $\varepsilon \to 0^+$ gives

$$\|f\|_{L^{p,\infty}} \ge (\varrho_f(s))^{1/p} s.$$

If $s > 0$ is such that $\varrho_f(s) = 0$, then this inequality continues to hold. Taking the supremum over all $s > 0$ gives $\|f\|_{L^{p,\infty}} \ge \|f\|_{L^p_w}$. ■

Exercise 21 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $0 < p \leq \infty$ *and let* $0 < q \leq r \leq \infty$*. Prove that* $L^{p,q}(X;\mu) \subseteq L^{p,r}(X;\mu)$*, with*

$$\|f\|_{L^{p,r}} \leq C(p,q,r)\,\|f\|_{L^{p,q}}$$

for all $f \in L^{p,q}(X;\mu)$*. In particular,* $L^p(X;\mu) \subseteq L^{p,r}(X;\mu) \subseteq L^p_w(X;\mu)$ *for all* $p \leq r \leq \infty$*.*

1.4 Interpolation

We present here two important interpolation theorems. Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces, let $L^0(X) := \{f : X \to \mathbb{C} \text{ measurable}\}$, $L^0(Y) := \{g : Y \to \mathbb{C} \text{ measurable}\}$, let $V \subseteq L^0(X)$ be a subspace. We say that V is of *closed by truncation* if for every $f \in V$ and $0 \leq r_1 < r_2$, the function

$$g(x) := \begin{cases} f(x) & \text{if } r_1 \leq |f(x)| \leq r_2, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to V. The family of simple functions has this property. Let $T : V \to L^0(Y)$. We say that T is *sublinear* if

$$\begin{aligned} |T(f_1 + f_2)(y)| &\leq |T(f_1)(y)| + |T(f_2)(y)|, \\ |T(\lambda f)(y)| &= |\lambda|\,|T(f)(y)| \end{aligned}$$

for all f, f_1, $f_2 \in V$, all $\lambda \in \mathbb{C}$, and for ν-a.e. $y \in Y$.

We say that T is of *strong type* (p, q) if it is bounded from $L^p(X;\mu) \cap V$ into $L^q(Y;\nu)$, that is,

$$\|T(f)\|_{L^q(Y)} \leq C_{p,q}\,\|f\|_{L^p(X)}$$

for all $f \in L^p(X;\mu) \cap V$ and for some constant $C_{p,q} > 0$, while it is of *weak type* (p, q) if it is bounded from $L^p(X;\mu) \cap V$ into $L^q_w(Y;\nu)$, that is,

$$\|T(f)\|_{L^q_w(Y)} \leq C_{p,q,\infty}\,\|f\|_{L^p(X)} \tag{15}$$

for all $f \in L^p(X;\mu) \cap V$ and for some constant $C_{p,q,\infty} > 0$.

Theorem 22 (Marcinkiewicz) *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces, let* $0 < p < q \leq \infty$*, let* V *be a subspace of* $L^p(X;\mu) + L^q(X;\mu)$ *closed by truncation and let* $T : V \to L^0(Y)$ *be a sublinear operator of weak type* (p, p) *and* (q, q) *with*

$$\|T(f)\|_{L^p_w(Y)} \leq C_p\,\|f\|_{L^p(X)}$$

for all $f \in L^p(X;\mu) \cap V$ *and*

$$\|T(f)\|_{L^q_w(Y)} \leq C_q\,\|f\|_{L^q(X)}$$

for all $f \in L^q(X;\mu) \cap V$*. Then* T *is of strong type* (r, r) *for all* $p < r < q$*, with*

$$\|T(f)\|_{L^r(Y)} \leq C_r\,\|f\|_{L^p(X)}$$

for all $f \in L^r(X;\mu) \cap V$, *where*

$$C_r \leq \left[\frac{r\,(2C_p)^p}{r-p} + \frac{r\,(2C_q)^q}{q-r}\right]^{1/r}$$

if $q < \infty$ *and*

$$C_r \leq \left[\frac{r\,(2C_p)^p}{(r-p)(2C_\infty)^{r-p}}\right]^{1/r}$$

if $q = \infty$.

Proof. Step 1: Assume that $q < \infty$. Given $t > 0$ and $f \in L^r(X;\mu) \cap V$, write $f = f_1 + f_2$, where $f_1 := f\chi_{\{|f|>t\}}$ and $f_2 := f\chi_{\{|f|\leq t\}}$. Since V is closed by truncation, $f_2 \in V$. In turn, since V is a subspace and $f, f_2 \in V$, it follows that f_1 also belongs to V. Moreover, $f_1 \in L^p(X)$ because

$$\int_X |f_1|^p \, d\mu = \frac{t^{r-p}}{t^{r-p}} \int_{|f|>t} |f|^p \, d\mu \leq \frac{1}{t^{r-p}} \int_{|f|>t} |f|^r \, d\mu < \infty,$$

while $f_2 \in L^q(X)$ since

$$\int_X |f_2|^q \, d\mu = \int_{|f|\leq t} |f|^q \, d\mu \leq t^{q-r} \int_{|f|\leq t} |f|^r \, d\mu < \infty.$$

By the sublinearity of T,

$$\begin{aligned}\{y \in Y : \ |T(f)(y)| > t\} \subseteq &\left\{y \in Y : \ |T(f_1)(y)| > \frac{t}{2}\right\} \\ &\cup \left\{y \in Y : \ |T(f_2)(y)| > \frac{t}{2}\right\},\end{aligned} \tag{16}$$

and so, since T is of weak type (p,p) and (q,q), by (13) and (15),

$$\begin{aligned}\nu\left(\{y \in Y : \ |Tf(y)| > t\}\right) &\leq \nu\left(\left\{y \in Y : \ |Tf_1(y)| > \frac{t}{2}\right\}\right) \\ &\quad + \nu\left(\left\{y \in Y : \ |Tf_2(y)| > \frac{t}{2}\right\}\right) \\ &\leq \left(2C_p t^{-1} \|f_1\|_{L^p(X)}\right)^p + \left(2C_q t^{-1} \|f_2\|_{L^q(X)}\right)^q.\end{aligned}$$

In turn, by Theorem 4,

$$\begin{aligned}
\|Tf\|_{L^r(Y)}^r &= r\int_0^\infty t^{r-1}\nu\left(\{y\in Y:\ |Tf(y)|>t\}\right)\,dt \\
&\le r\,(2C_p)^p\int_0^\infty t^{r-1-p}\int_{\{|f|>t\}}|f(x)|^p\,d\mu(x)\,dt \\
&\quad + r\,(2C_q)^q\int_0^\infty t^{r-1-q}\int_{\{|f|\le t\}}|f(x)|^q\,d\mu(x)\,dt \\
&= r\,(2C_p)^p\int_X |f(x)|^p\int_0^{|f(x)|} t^{r-1-p}\,dtd\mu(x) \\
&\quad + r\,(2C_q)\int_X |f(x)|^q\int_{|f(x)|}^\infty t^{r-1-q}\,dtd\mu(x) \\
&= \frac{r\,(2C_p)^p}{r-p}\int_X |f(x)|^r\,d\mu(x)+\frac{r\,(2C_q)^q}{q-r}\int_X |f(x)|^r\,d\mu(x),
\end{aligned}$$

which proves that T is of strong type (r,r). ■

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Proof. Step 2: Assume that $q=\infty$. Given $t>0$ and $f\in L^r(X;\mu)\cap V$, write $f=f_1+f_2$, where $f_1:=f\chi_{\{|f|>ct\}}$ and $f_2:=f\chi_{\{|f|\le ct\}}$, where $c>0$ will be determined later on. As before, f_1 and f_2 belong to V. Moreover, $f_1\in L^p(X)$ and $f_2\in L^\infty(X)$. Since T is of weak type (p,p) and (∞,∞), we have that

$$\|T(f_2)\|_{L^\infty(Y)}\le C_\infty\|f_2\|_{L^\infty(X)}=C_\infty ct=\frac{t}{2},$$

provided $c=1/(2C_\infty)$. Hence, the set $\left\{y\in Y:\ |T(f_2)(y)|>\frac{t}{2}\right\}$ has measure ν zero, and so by (16),

$$\{y\in Y:\ |T(f)(y)|>t\}\subseteq\left\{y\in Y:\ |T(f_1)(y)|>\frac{t}{2}\right\}$$

up to a set of measure ν zero. Since T is of weak type (p,p), by (13) and (15),

$$\begin{aligned}
\nu\left(\{y\in Y:\ |Tf(y)|>t\}\right) &\le \nu\left(\left\{y\in Y:\ |Tf_1(y)|>\frac{t}{2}\right\}\right) \\
&\le \left(2C_pt^{-1}\|f_1\|_{L^p(X)}\right)^p.
\end{aligned}$$

Hence, by Theorem 4,

$$\begin{aligned}
\|Tf\|_{L^r(Y)}^r &= r\int_0^\infty t^{r-1}\nu\left(\{y\in Y:\ |Tf(y)|>t\}\right)\,dt \\
&\le r\,(2C_p)^p\int_0^\infty t^{r-1-p}\int_{\{|f|>ct\}}|f(x)|^p\,d\mu(x)\,dt \\
&= r\,(2C_p)^p\int_X |f(x)|^p\int_0^{c|f(x)|} t^{r-1-p}\,dtd\mu(x) \\
&= \frac{r\,(2C_p)^p\,c^{r-p}}{r-p}\int_X |f(x)|^r\,d\mu(x),
\end{aligned}$$

which concludes the proof. ■

Remark 23 *Note that the constant* $\frac{r(2C_p)^p}{r-p}$ *explodes as* $r \to p^-$.

Next we discuss the extension of the Marcinkiewicz interpolation theorem to Lorentz spaces. In the proof we will use the following inequalities.

Theorem 24 (Hardy's inequality) *Let* $f : (a,b) \to [0,\infty)$ *be a measurable function, where* $0 \le a < b \le \infty$, *and let* $1 \le s < \infty$ *and* $q > 0$. *Then*

$$\left(\int_a^b x^{-q-1}\left(\int_a^x f(t)\,dt\right)^s dx\right)^{\frac{1}{s}} \le \frac{s}{q}\left(\int_a^b x^{s-q-1}(f(x))^s\,dx\right)^{\frac{1}{s}}, \tag{17}$$

and

$$\left(\int_a^b x^{q-1}\left(\int_x^b f(t)\,dt\right)^s dx\right)^{\frac{1}{s}} \le \frac{s}{q}\left(\int_a^b x^{s+q-1}(f(x))^s\,dx\right)^{\frac{1}{s}}. \tag{18}$$

Proof. Extend f by zero in $(0,\infty) \setminus (a,b)$ and define the function $g : (0,\infty) \times (0,\infty) \to \mathbb{R}$ by

$$g(x,y) := \frac{1}{x^a} f(xy), \quad x, y > 0,$$

where $a := \frac{q+1}{s} - 1$. Note that by the change of variables $t = xy$,

$$\int_0^1 g(x,y)\,dy = \int_0^1 \frac{1}{x^a} f(xy)\,dy = \frac{1}{x^{a+1}}\int_0^x f(t)\,dt = \frac{1}{x^{(q+1)/s}}\int_0^x f(t)\,dt$$

By Minkowski's inequality for integrals and the change of variables $t = xy$,

$$\begin{aligned}\left(\int_0^\infty x^{-q-1}\left(\int_a^x f(t)\,dt\right)^s dx\right)^{\frac{1}{s}} &= \left(\int_0^\infty\left(\int_0^1 g(x,y)\,dy\right)^s dx\right)^{\frac{1}{s}} \\ \le \int_0^1\left(\int_0^\infty |g(x,y)|^s\,dx\right)^{\frac{1}{s}} dy &= \int_0^1 y^{a-\frac{1}{s}}\left(\int_0^\infty \frac{1}{t^{as}}(f(t))^s\,dt\right)^{\frac{1}{s}} dy \\ &= \frac{s}{q}\left(\int_0^\infty t^{s-q-1}(f(t))^s\,dt\right)^{\frac{1}{s}}.\end{aligned}$$

Recalling that $f = 0$ outside (a,b), we obtain the desired result. The other inequality is similar. ■

Theorem 25 (Marcinkiewicz in Lorentz spaces) *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces, let* $0 < p_1, q_1, p_2, q_2 \le \infty$ *and* $0 < r_1, s_1, r_2, s_2 \le \infty$, *with* $p_1 \neq p_2$ *and* $r_1 \neq r_2$, *let* V *be a subspace of* $L^{p_1,q_1}(X;\mu) + L^{p_2,q_2}(X;\mu)$ *closed by truncation and let* $T : V \to L^0(Y)$ *be a sublinear operator such that*

$$\|T(f)\|_{L^{r_i,s_i}(Y)} \le C_i \|f\|_{L^{p_i,q_i}(X)}, \quad i = 1, 2, \tag{19}$$

for all $f \in V$ and for some positive constants C_1 and $C_2 > 0$. Then for every $\theta \in [0,1]$, there exists a constant $C_\theta > 0$ such that

$$\|T(f)\|_{L^{r,s}(Y)} \le C_\theta \|f\|_{L^{p,q}(X)}$$

for all $f \in V$, where $0 < s \le q \le \infty$ and

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{r} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}. \tag{20}$$

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Proof. Since $p_1 \neq p_2$, without loss of generality we may assume that $p_1 < p_2$. We will only give the proof in the case $1 \le s < \infty$ and $p_2 < \infty$.

It follows from Exercise 21 that, by possibly changing the constants $C_i > 0$, we can assume that $s_i = \infty$ and $q_i = 1$, so that (19) becomes

$$\begin{aligned} \|T(f)\|_{L^{r_i,\infty}(Y)} &= \sup_{t>0} t^{1/r_i} (T(f))^*(t) \le C_i \|f\|_{L^{p_i,1}(X)} \\ &= C_i \int_0^\infty t^{1/p_i - 1} f^*(t)\, dt \end{aligned} \tag{21}$$

for all $f \in V$ and all $i = 1, 2$.

By (20),

$$\gamma := \frac{\frac{1}{r_1} - \frac{1}{r}}{\frac{1}{p_1} - \frac{1}{p}} = \frac{\frac{1}{r} - \frac{1}{r_1}}{\frac{1}{p} - \frac{1}{p_2}}. \tag{22}$$

For $\tau > 0$ define the functions

$$f_\tau(x) := \begin{cases} f(x) & \text{if } |f(x)| \le f^*(\tau^\gamma), \\ 0 & \text{otherwise,} \end{cases} \qquad g_\tau(x) := \begin{cases} f(x) & \text{if } |f(x)| > f^*(\tau^\gamma), \\ 0 & \text{otherwise,} \end{cases} .$$

Since V is closed by truncation, $f_\tau \in V$. In turn, since V is a subspace and $f, f_\tau \in V$, it follows that g_τ also belongs to V. Since $|f_\tau| \le |f|$ and $|g_\tau| \le |f|$, by Proposition 12(ii) $(f_\tau)^* \le f^*$ and $(g_\tau)^* \le f^*$. Moreover, (exercise)

$$(f_\tau)^*(t) \le \begin{cases} f^*(\tau^\gamma) & \text{if } 0 < t < \tau^\gamma, \\ f^*(t) & \text{otherwise,} \end{cases} \qquad (g_\tau)^*(t) \le \begin{cases} f^*(t) & \text{if } 0 < t < \tau^\gamma, \\ 0 & \text{otherwise,} \end{cases} \tag{23}$$

Since T is sublinear, by Proposition 12(ii) and (14)

$$\begin{aligned} (T(f))^*(t) &= (T(f_\tau + g_\tau))^*(t) \le (T(f_\tau) + T(g_\tau))^*(t) \\ &\le (T(f_\tau))^*(t/2) + (T(g_\tau))^*(t/2) \end{aligned}$$

for all $t > 0$. In particular, taking $t = \tau$ we get

$$\begin{aligned} (T(f))^*(\tau) &\le (T(f_\tau) + T(g_\tau))^*(\tau) \\ &\le (T(f_\tau))^*(\tau/2) + (T(g_\tau))^*(\tau/2). \end{aligned}$$

By (12), the previous inequality, Minkowski's inequality and a change of variables,

$$\begin{aligned}\|T(f)\|_{L^{r,s}(Y)} &= \left(\int_0^\infty (\tau^{1/r}(T\,(f))^*(\tau))^s \,\frac{d\tau}{\tau}\right)^{1/s} \\ &\leq \left(\int_0^\infty [\tau^{1/r}(T(f_\tau))^*(\tau/2) + \tau^{1/r}(T(g_\tau))^*(\tau/2)]^s \,\frac{d\tau}{\tau}\right)^{1/s} \\ &\leq 2^{1/r}\left(\int_0^\infty [\tau^{1/r}(T(f_\tau))^*(\tau)]^s \,\frac{d\tau}{\tau}\right)^{1/s} \\ &\quad + 2^{1/r}\left(\int_0^\infty [\tau^{1/r}(T(g_\tau))^*(\tau)]^s \,\frac{d\tau}{\tau}\right)^{1/s}.\end{aligned}$$

Using (21) we can bound the right-hand of the previous inequality from above by

$$\begin{aligned}&2^{1/r}\left(\int_0^\infty \left[C_1\tau^{1/r-1/r_1}\int_0^\infty y^{1/p_1-1}(f_\tau)^*(y)\;dy\right]^s \,\frac{dt}{\tau}\right)^{1/s} \\ &\quad + 2^{1/r}\left(\int_0^\infty \left[C_2\tau^{1/r-1/r_2}\int_0^\infty y^{1/p_2-1}(g_\tau)^*(y)\;dy\right]^s \,\frac{dt}{\tau}\right)^{1/s}.\end{aligned}$$

We now use (23) and Minkowski's inequality to dominate this sum by

$$\begin{aligned}&2^{1/r}C_1\left(\int_0^\infty \left[\tau^{1/r-1/r_1}\int_0^{\tau^\gamma} y^{1/p_1-1}f^*(y)\;dy\right]^s \,\frac{d\tau}{\tau}\right)^{1/s} \\ &\quad + 2^{1/r}C_2\left(\int_0^\infty \left[\tau^{1/r-1/r_2}f^*(\tau^\gamma)\int_0^{\tau^\gamma} y^{1/p_2-1}\;dy\right]^s \,\frac{d\tau}{\tau}\right)^{1/s} \\ &\quad + 2^{1/r}C_2\left(\int_0^\infty \left[\tau^{1/r-1/r_2}\int_{\tau^\gamma}^\infty y^{1/p_2-1}f^*(y)\;dy\right]^s \,\frac{d\tau}{\tau}\right)^{1/s}.\end{aligned}$$

Using the change of variables $t = \tau^\gamma$ and (22) we can rewrite the previous expression as

$$\begin{aligned}&2^{1/r}\frac{C_1}{\gamma^{1/s}}\left(\int_0^\infty \left[t^{1/p-1/p_1}\int_0^t y^{1/p_1-1}f^*(y)\;dy\right]^s \,\frac{dt}{t}\right)^{1/s} \\ &\quad + 2^{1/r}\frac{C_2p_2}{\gamma^{1/s}}\left(\int_0^\infty \left[t^{1/p-1/p_2}f^*(t)t^{1/p_2}\right]^s \,\frac{dt}{t}\right)^{1/s} \\ &\quad + 2^{1/r}\frac{C_2}{\gamma^{1/s}}\left(\int_0^\infty \left[t^{1/p-1/p_2}\int_t^\infty y^{1/p_2-1}f^*(y)\;dy\right]^s \,\frac{dt}{t}\right)^{1/s}.\end{aligned}$$

where we used the facts that

$$\frac{d\tau}{\tau} = \frac{1}{\gamma t}dt, \quad \frac{1}{\gamma}\left(\frac{1}{r_1} - \frac{1}{r}\right) = \frac{1}{p_1} - \frac{1}{p},$$

$$\frac{1}{\gamma}\left(\frac{1}{r} - \frac{1}{r_2}\right) = \frac{1}{p} - \frac{1}{p_2}.$$

We now use Hardy's inequalities to dominate the previous expression by

$$\frac{2^{1/r}}{\gamma^{1/s}}\left(\frac{C_1}{\frac{1}{p} - \frac{1}{p_1}} + C_2 p_2 + \frac{C_2}{\frac{1}{p_2} - \frac{1}{p}}\right)\left(\int_0^\infty \left[t^{1/p} f^*(t)\right]^s \frac{dt}{t}\right)^{1/s}.$$

Hence we have shown that

$$\|T(f)\|_{L^{r,s}(Y)} \le \frac{2^{1/r}}{\gamma^{1/s}}\left(\frac{C_1}{\frac{1}{p} - \frac{1}{p_1}} + C_2 p_2 + \frac{C_2}{\frac{1}{p_2} - \frac{1}{p}}\right)\|f\|_{L^{p,s}(X)}.$$

Since $s \le q$ it follows from Exercise 21 that $\|f\|_{L^{p,s}(X)} \le C\|f\|_{L^{p,q}(X)}$, and so the proof is complete. ■

The Marcinkiewicz interpolation theorem has been generalized to an abstract setting, called the real interpolation method.

Wednesday, January 28, 2015

Under stronger assumptions on T we can obtain better constants that do not explode as we approach p and q.

Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces, and let $\mathcal{S}$ be the family of simple functions vanishing at infinity.

Theorem 26 (Riesz–Thorin) *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces, let $1 \le p_1, q_1, p_2, q_2 \le \infty$, let $T : \mathcal{S} \to L^0(Y)$ be a linear operator of strong type (p_1, q_1) and (p_2, q_2). Then for every $\theta \in [0, 1]$, T is of strong type (p, q), with*

$$\|T(f)\|_{L^q(Y)} \le c_{p_1,q_1}^{1-\theta} c_{p_2,q_2}^{\theta} \|f\|_{L^p(X)} \tag{24}$$

for all $f \in \mathcal{S}$, with

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}. \tag{25}$$

In particular, T can be uniquely extended to $L^p(X)$ as a linear operator of strong type (p, q).

The proof makes use of the following result.

Lemma 27 (Hadamard's three lines lemma) *Let $R := \{\boldsymbol{z} \in \mathbb{C} : 0 \le \operatorname{Re} \boldsymbol{z} \le 1\}$ and let $h : R \to \mathbb{C}$ be such that h is holomorphic in R° and continuous and bounded in R, with*

$$|h(0 + \boldsymbol{i}y)| \le M_0 \quad \text{and} \quad |h(1 + \boldsymbol{i}y)| \le M_1 \quad \text{for all } y \in \mathbb{R}$$

and for some positive constants M_0 and M_1. Then for every $0 < \theta < 1$,

$$|h(\theta + \boldsymbol{i}y)| \le M_0^{1-\theta} M_1^{\theta} \quad \text{for all } y \in \mathbb{R}.$$

Proof. Define

$$g(\boldsymbol{z}) := h(\boldsymbol{z})(M_0^{1-\boldsymbol{z}} M_1^{\boldsymbol{z}})^{-1}, \quad g_n(\boldsymbol{z}) := g(\boldsymbol{z}) e^{(\boldsymbol{z}^2-1)/n}.$$

Since h is bounded in R and $M_0^{1-\boldsymbol{z}} M_1^{\boldsymbol{z}}$ is bounded from below, there exists a constant M such that $|g(\boldsymbol{z})| \le M$ for all $\boldsymbol{z} \in R$. Moreover, $|g(\boldsymbol{z})| \le 1$ on ∂R. In turn, writing $\boldsymbol{z} = x + \boldsymbol{i}y$, we have that

$$\boldsymbol{z}^2 - 1 = (x + \boldsymbol{i}y)(x + \boldsymbol{i}y) - 1 = x^2 - 1 - y^2 + 2xy\boldsymbol{i}$$

and so

$$|g_n(\boldsymbol{z})| = |g(\boldsymbol{z})||e^{(\boldsymbol{z}^2-1)/n}| \le M e^{-y^2/n} e^{(x^2-1)/n}.$$

It follows that $g_n \to 0$ as $|y| \to \infty$ uniformly for $x \in [0,1]$. Let $y_n > 0$ be such that $|g_n(\boldsymbol{z})| \le 1$ for all $x \in [0,1]$ and $|y| \ge y_n$. It follows by the maximum principle that $|g_n(\boldsymbol{z})| \le 1$ in $[0,1] \times [-y_n, y_n]$, since $|g_n(\boldsymbol{z})| \le 1$ on the boundary of this rectangle. This shows that $|g_n(\boldsymbol{z})| \le 1$ for all $\boldsymbol{z} \in R$. Letting $n \to \infty$ gives $|g(\boldsymbol{z})| \le 1$ for all $\boldsymbol{z} \in R$, that is,

$$|h(\boldsymbol{z})| \le |M_0^{1-\boldsymbol{z}} M_1^{\boldsymbol{z}}|$$

for all $\boldsymbol{z} \in R$, which gives the desired result. ■

We turn to the proof of Theorem 26.

Proof of Theorem 26. In view of (1) for every $f \in \mathcal{S}$ we have

$$\|T(f)\|_{L^q(Y)} = \sup\left\{\left|\int_Y T(f) g \, d\nu\right| : \ \|g\|_{L^{q'}} = 1\right\}.$$

By the density of simple functions in $L^{q'}(Y)$ and the linearity of T to prove (24) it is enough to show that

$$\left|\int_Y T(f) g \, d\nu\right| \le c_{p_1,q_1}^{1-\theta} c_{p_2,q_2}^{\theta}. \tag{26}$$

for all simple functions $f \in L^p(X)$ and $g \in L^{q'}(Y)$ with

$$\|f\|_{L^p(X)} = \|g\|_{L^{q'}(Y)} = 1. \tag{27}$$

Assume first that $p < \infty$ and $q' < \infty$. Then in view of (27) we can write

$$f = \sum_{j=1}^{m} c_j \chi_{E_j}, \quad g = \sum_{k=1}^{n} d_k \chi_{F_k},$$

where $c_j \in \mathbb{C}$, $c_j \neq 0$, $E_i \subseteq X$ are pairwise disjoint measurable sets with $\mu(E_j) < \infty$, $j = 1, \dots, m$, $d_k \in \mathbb{C}$, $d_k \neq 0$, and $F_k \subseteq Y$ are pairwise disjoint measurable sets with $\nu(F_k) < \infty$, $k = 1, \dots, n$, and

$$\sum_{j=1}^{m} |c_j|^p \mu(E_j) = \sum_{k=1}^{n} |d_k|^p \nu(F_k) = 1.$$

Using polar coordinates, write $c_j = r_j e^{\boldsymbol{i}\theta_j}$ and $d_k = \rho_k e^{\boldsymbol{i}\omega_k}$, where where $r_j > 0$, $\theta_j \in [0, 2\pi)$, $\rho_k > 0$, $\omega_k \in [0, 2\pi)$. For $\boldsymbol{z} \in R$ define

$$p(\boldsymbol{z}) := \frac{p}{p_1}(1 - \boldsymbol{z}) + \frac{p}{p_2}\boldsymbol{z}, \quad q(\boldsymbol{z}) := \frac{q'}{q'_1}(1 - \boldsymbol{z}) + \frac{q'}{q'_2}\boldsymbol{z} \tag{28}$$

and

$$f_{\boldsymbol{z}} = \sum_{j=1}^{m} r_j^{p(\boldsymbol{z})} e^{\boldsymbol{i}\theta_j} \chi_{E_j}, \quad g_{\boldsymbol{z}} = \sum_{k=1}^{n} \rho_k^{q(\boldsymbol{z})} e^{\boldsymbol{i}\omega_k} \chi_{F_k}. \tag{29}$$

Consider the function

$$\begin{aligned} h(\boldsymbol{z}) &:= \int_X T(f_{\boldsymbol{z}}) g_{\boldsymbol{z}} \, d\mu \\ &= \sum_{j=1}^{m} \sum_{k=1}^{n} r_j^{p(\boldsymbol{z})} \rho_k^{q(\boldsymbol{z})} e^{\boldsymbol{i}\theta_j} e^{\boldsymbol{i}\omega_k} \int_X T(\chi_{E_j}) \chi_{F_k} \, d\mu. \end{aligned}$$

Since $r_j > 0$ and $\rho_k > 0$, the function h is holomorphic in R°.

Next we claim that if $\boldsymbol{z} = y\boldsymbol{i}$, $y \in \mathbb{R}$, then

$$|h(\boldsymbol{z})| \leq c_{p_1, q_1}.$$

Indeed, if $p_1 = \infty$, then $p(\boldsymbol{z}) = \frac{p}{p_2}\boldsymbol{z}$ and so $\operatorname{Re}(p(y\boldsymbol{i})) = 0$. In turn, by (29),

$$f_{\boldsymbol{z}} = \sum_{j=1}^{m} e^{\boldsymbol{i}\theta_j} \chi_{E_j}$$

and so, since at least one of the sets E_j must be nonempty by (27), it follows that $\|f_{\boldsymbol{z}}\|_{L^\infty(X)} = 1$. On the other hand, if $p_1 < \infty$, then by (25), $\operatorname{Re}(p(y\boldsymbol{i})) = p/p_1$ and so again by (27),

$$\|f_{\boldsymbol{z}}\|_{L^{p_1}(X)}^{p_1} = \sum_{j=1}^{m} |r_j^{\operatorname{Re}(p(y\boldsymbol{i}))}|^{p_1} \mu(E_j) = \sum_{j=1}^{m} r_j^p \mu(E_j) = 1.$$

In a similar way we can prove that $\|g_{\boldsymbol{z}}\|_{L^{q'_1}(Y)} = 1$.

Hence, by Hölder's inequality and the fact that T is of strong type (p_1, q_1) it follows that

$$|h(\boldsymbol{z})| \leq \|T(f_{\boldsymbol{z}})\|_{L^{q_1}(Y)} \|g_{\boldsymbol{z}}\|_{L^{q'_1}(Y)} \leq c_{p_1,q_1} \|f_{\boldsymbol{z}}\|_{L^{p_1}(X)} \|g_{\boldsymbol{z}}\|_{L^{q'_1}(Y)} = c_{p_1,q_1}.$$

Similarly, for $\boldsymbol{z} = 1 + y\boldsymbol{i}$, $y \in \mathbb{R}$, we have that

$$\|f_{\boldsymbol{z}}\|_{L^{p_2}(X)} = \|g_{\boldsymbol{z}}\|_{L^{q'_2}(Y)} = 1,$$

which gives

$$|h(\boldsymbol{z})| \leq c_{p_2,q_2}.$$

We are now in a position to apply the Hadamard's three points lemma to conclude that for all $0 < \theta < 1$,

$$|h(\theta + \boldsymbol{i}y)| \leq c_{p_1,q_1}^{1-\theta} c_{p_2,q_2}^{\theta}.$$

Taking $y = 0$ from (25) we get that $p(\theta) = q(\theta) = 1$, and so $f_\theta = f$ and $g_\theta = g$ and $h(\theta) = \int_Y T(f)g \, d\nu$. It follows that

$$\left| \int_Y T(f)g \, d\nu \right| \leq c_{p_1,q_1}^{1-\theta} c_{p_2,q_2}^{\theta},$$

which shows (26) in the case $p < \infty$ and $q' < \infty$.

If $\theta = 1$ then $p = p_2$ and $q = q_2$, while if $\theta = 0$, then $p = p_1$ and $q = q_1$ and so in both cases there is nothing to prove. Thus, assume that $0 < \theta < 1$. In this case, if $q = 1$, then necessarily, $q_1 = q_2 = 1$. Indeed, if $q_1 \neq q_2$, say $q_1 > q_2$, by (25) and the fact that $0 < \theta < 1$, we have that

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2} < \frac{1-\theta}{q_2} + \frac{\theta}{q_2} = \frac{1}{q_2},$$

and so $q > q_2 \geq 1$. The case $q_1 < q_2$ is similar. Thus $q_1 = q_2$. In turn, again by (25), $q = q_1 = q_2 = 1$. This proves the claim.

We now consider various cases. If $p = \infty$ and $q = 1$, then by (25), $p_1 = p_2 = \infty$ and so there is nothing to prove, since $(p, q) = (p_1, q_1) = (\infty, 1)$.

If $p = \infty$ and $q > 1$, so that $q' < \infty$, define $g_{\boldsymbol{z}}$ as before but take $f_{\boldsymbol{z}} :\equiv f$ for all $\boldsymbol{z}$. The previous proof continues to hold. The case $p < \infty$ and $q = 1$ is similar. ■

Remark 28 *If the functions f and g are real-valued instead of complex-valued, we extend T by setting*

$$T_1(f + \boldsymbol{i}h) := T(f) + \boldsymbol{i}T(h).$$

Then

$$\begin{aligned} \|T_1(f + \boldsymbol{i}h)\|_{L^{q_k}(Y;\mathbb{C})} &\leq \|T(f)\|_{L^{q_k}(Y;\mathbb{R})} + \|T(h)\|_{L^{q_k}(Y;\mathbb{R})} \\ &\leq C_{p_k,q_k}(\|f\|_{L^{p_k}(X;\mathbb{R})} + \|h\|_{L^{p_k}(X;\mathbb{R})}) \\ &\leq 2C_{p_k,q_k} \|f + \boldsymbol{i}h\|_{L^{p_k}(X;\mathbb{C})}. \end{aligned}$$

Hence, we still obtain (24) with $2C_{p_1,q_1}^{1-\theta} C_{p_2,q_2}^{\theta}$ in place of $C_{p_1,q_1}^{1-\theta} C_{p_2,q_2}^{\theta}$ on the right-hand side.

Friday, January 30, 2015

In the next sections we are going to apply these interpolation theorems to maximal functions, Fourier transforms, and convolutions.

2 Maximal Functions

We begin with some covering theorems.

Theorem 29 (Besicovitch covering theorem) *There exists a constant ℓ, depending only on the dimension N of $\mathbb{R}^N$, such that for any collection $\mathcal{F}$ of (nondegenerate) closed balls with*

$$\sup\left\{\operatorname{diam}\overline{B}:\ \overline{B}\in\mathcal{F}\right\}<\infty \tag{30}$$

there exist $\mathcal{F}_1,\ldots,\mathcal{F}_\ell\subseteq\mathcal{F}$ such that each $\mathcal{F}_n$, $n=1,\ldots,\ell$, is a countable family of disjoint balls in $\mathcal{F}$ and

$$E\subseteq\bigcup_{n=1}^{\ell}\bigcup_{\overline{B}\in\mathcal{F}_n}\overline{B},$$

where E is the set of centers of balls in $\mathcal{F}$.

Corollary 30 *Let $\mathcal{F}$ be a collection of (nondegenerate) closed balls. Assume that the set E of centers of balls in $\mathcal{F}$ is bounded. Then there exist $\mathcal{F}_1,\ldots,\mathcal{F}_\ell\subseteq\mathcal{F}$ such that each $\mathcal{F}_n$, $n=1,\ldots,\ell$, is a countable family of disjoint balls in $\mathcal{F}$ and*

$$E\subseteq\bigcup_{n=1}^{\ell}\bigcup_{\overline{B}\in\mathcal{F}_n}\overline{B},$$

where ℓ is the number given in the previous theorem.

In what follows we only consider cubes Q with sides parallel to the axes, that is, of the form $Q(\boldsymbol{x}_0,r):=\boldsymbol{x}_0+\left(-\frac{r}{2},\frac{r}{2}\right)^N$ for some $\boldsymbol{x}_0$ and $r>0$.

Remark 31 *Similar theorems also hold if instead of balls we use cubes Q with sides parallel to the axes, that is, of the form $Q(\boldsymbol{x}_0,r):=\boldsymbol{x}_0+\left(-\frac{r}{2},\frac{r}{2}\right)^N$ for some $\boldsymbol{x}_0$ and $r>0$. More generally, one could use Morse sets. See our book.*

Let $\mathfrak{M}$ be a σ-algebra containing the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ and let $\mu:\mathfrak{M}\to[0,\infty]$ be a complete measure, finite on compact sets. Assume that $\mu\neq 0$. If $f\in L^1_{\text{loc}}\left(\mathbb{R}^N;\mu\right)$ we define the *centered maximal Hardy-Littlewood function*

$$\operatorname{M}_\mu(f)(\boldsymbol{x}):=\sup\frac{1}{\mu(\overline{B(\boldsymbol{x},r)})}\int_{B(\boldsymbol{x},r)}|f|\,d\mu,\quad \boldsymbol{x}\in\mathbb{R}^N, \tag{31}$$

where the supremum is taken over all $r>0$ such that $\mu(\overline{B(\boldsymbol{x},r)})>0$. Note that operator M_μ is sublinear.

Exercise 32 $\operatorname{M}_\mu(f)$ *is Lebesgue measurable. If the function $(\boldsymbol{x},r)\mapsto\mu(\overline{B(\boldsymbol{x},r)})$ is continuous, or if $\mu(\partial B(\boldsymbol{x},r))=0$ for all $\boldsymbol{x}\in\mathbb{R}^N$ and $r>0$, then $\operatorname{M}_\mu(f)$ is lower semicontinuous.*

In the special case in which μ is the Lebesgue measure $\mathcal{L}^N$ we denote $\mathrm{M}_{\mathcal{L}^N}$ simply by M.

Theorem 33 *Let $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ be a measure finite on compact sets. Then the operator M_μ is of weak type $(1, 1)$ and of strong type (p, p) for $1 < p \leq \infty$.*

Proof. If $f \in L^\infty(\mathbb{R}^N; \mu)$, then

$$\frac{1}{\mu(\overline{B(\boldsymbol{x}, r)})} \int_{\overline{B(\boldsymbol{x}, r)}} |f| \, d\mu \leq \|f\|_{L^\infty} ,$$

and so

$$0 \leq \mathrm{M}_\mu(f)(\boldsymbol{x}) \leq \|f\|_{L^\infty}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$, which shows that M_μ is of strong type (∞, ∞). Next we prove that M_μ is of weak type $(1, 1)$.

Let $f \in L^1(\mathbb{R}^N; \mu)$, fix $R > 0$ and $s > 0$ and consider the set

$$E_{R,s} := \{\boldsymbol{x} \in B(\mathbf{0}, R) : \ \mathrm{M}_\mu(f)(\boldsymbol{x}) > s\} .$$

By (31) for every $\boldsymbol{x} \in E_{R,s}$ there exists $r_{\boldsymbol{x}} > 0$ such that

$$\int_{\overline{B(\boldsymbol{x}, r_{\boldsymbol{x}})}} |f| \, d\mu > s\mu(\overline{B(\boldsymbol{x}, r_{\boldsymbol{x}})}). \tag{32}$$

Consider the family of closed balls $\mathcal{F} := \{\overline{B(\boldsymbol{x}, r_{\boldsymbol{x}})} : \ \boldsymbol{x} \in E_{R,s}\}$. By Corollary 30 there exist $\mathcal{F}_1, \ldots, \mathcal{F}_\ell \subseteq \mathcal{F}$ such that each $\mathcal{F}_n$, $n = 1, \ldots, \ell$, is a countable family of disjoint balls in $\mathcal{F}$ and

$$E_{R,s} \subseteq \bigcup_{n=1}^{\ell} \bigcup_{\overline{B} \in \mathcal{F}_n} \overline{B}.$$

Then, also by (32) and the fact that each $\mathcal{F}_n$ is a countable family of disjoint balls,

$$\begin{aligned} \mu(E_{R,s}) &\leq \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \mu(\overline{B}) \leq \frac{1}{s} \sum_{n=1}^{\ell} \sum_{\overline{B} \in \mathcal{F}_n} \int_{\overline{B}} |f| \, d\mu \\ &= \frac{1}{s} \sum_{n=1}^{\ell} \int_{\cup_{\overline{B} \in \mathcal{F}_n} \overline{B}} |f| \, d\mu \leq \frac{\ell}{s} \int_{\mathbb{R}^N} |f| \, d\mu. \end{aligned}$$

Letting $R \to \infty$ in the previous inequality gives

$$\mu(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}_\mu(f)(\boldsymbol{x}) > s\}) \leq \frac{\ell}{s} \int_{\mathbb{R}^N} |f| \, d\mu,$$

which shows that

$$\|\mathrm{M}_\mu(f)\|_{L^{1,\infty}} \leq \ell \|f\|_{L^1} .$$

It now follows by the Marcinkiewicz interpolation theorem (see Theorem 22) that M_μ is of strong type (p, p) for all $1 < p \leq \infty$. ■

Remark 34 *In view of Remark 31 setting*

$$\mathrm{N}_\mu(f)(\boldsymbol{x}) := \sup \frac{1}{\mu(\overline{Q(\boldsymbol{x},r)})} \int_{\overline{Q(\boldsymbol{x},r)}} |f| \, d\mu, \quad \boldsymbol{x} \in \mathbb{R}^N, \tag{33}$$

where the supremum is taken over all $r > 0$ such that $\mu(\overline{Q(\boldsymbol{x},r)}) > 0$, we have that N_μ is of weak type $(1,1)$ and strong type (p,p) for all $1 < p \leq \infty$.

Next we show that in general M_μ is not of strong type $(1,1)$.

Example 35 *Let $\mu = \mathcal{L}^N$ and let $f \in L^1(\mathbb{R}^N)$ be such that $f \neq 0$. Then there exists $R > 0$ such that*

$$\int_{B(\mathbf{0},R)} |f| \, d\boldsymbol{y} > 0.$$

Then if $|\boldsymbol{x}| \geq R$ it follows that $B(\mathbf{0},R) \subset B(x, 2|\boldsymbol{x}|)$, and thus

$$\mathrm{M}(f)(\boldsymbol{x}) \geq \frac{1}{|B(\boldsymbol{x}, 2|\boldsymbol{x}|)|} \int_{B(\boldsymbol{x},2|\boldsymbol{x}|)} |f| \, d\boldsymbol{y} \geq \frac{1}{\alpha_N 2^N |\boldsymbol{x}|^N} \int_{B(\mathbf{0},R)} |f| \, d\boldsymbol{y},$$

with $|\boldsymbol{x}|^{-N} \notin L^1(\mathbb{R}^N \setminus B(\mathbf{0},R))$. Hence, $\mathrm{M}(f) \notin L^1(\mathbb{R}^N)$ for any *$f \in L^1(\mathbb{R}^N)$ be such that $f \neq 0$.*

There are several variations of the maximal functions that are important in applications.

Let $\mathfrak{M}$ be a σ-algebra containing the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ and let $\mu : \mathfrak{M} \to [0,\infty]$ be a complete measure, finite on compact sets. Assume that $\mu \neq 0$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N; \mu)$ we define the *uncentered maximal Hardy-Littlewood function*

$$\mathrm{M}^{nc}_\mu(f)(\boldsymbol{x}) := \sup \frac{1}{\mu(\overline{B})} \int_{\overline{B}} |f| \, d\mu, \quad \boldsymbol{x} \in \mathbb{R}^N, \tag{34}$$

where the supremum is taken over all closed balls such that $\boldsymbol{x} \in \overline{B}$ and $\mu(\overline{B}) > 0$.

Exercise 36 *Prove that $\mathrm{M}^{nc}_\mu(f)$ is lower semicontinuous.*

If the measure μ is doubling, then Theorem 33 continues to hold for M^{nc}_μ.

Definition 37 *Let $\mathfrak{M}$ be a σ-algebra containing the Borel σ-algebra $\mathcal{B}(\mathbb{R}^N)$ and let $\mu : \mathfrak{M} \to [0,\infty]$ be a measure. We say that μ is a* doubling measure *if there exists a constant $c_\mu > 0$ such that*

$$\mu(\overline{B(\boldsymbol{x},2r)}) \leq c_\mu \mu(\overline{B(\boldsymbol{x},r)})$$

for all $\boldsymbol{x} \in \mathbb{R}^N$ and all $r > 0$.

Remark 38 *Using the fact that*

$$\overline{B(\boldsymbol{x},r/2)} \subset \overline{Q(\boldsymbol{x},r)} \subset \overline{B(\boldsymbol{x},\sqrt{N}r)},$$

we could have defined doubling measures using cubes instead of balls.

Example 39 *Consider the measure* μ *defined by*

$$\mu(E) := \mathcal{L}^2(E) + \mathcal{H}^1(E \cap ([-1,1] \times \{0\}))$$

for every $E \subseteq \mathbb{R}^2$ *Borel measurable. Consider the sequence of balls* $B((0,r_n), 3r_n/4)$ *where* $r_n \to 0$*. Then* $B((0,r_n), 3r_n/4)$ *does not intersect* $[-1,1] \times \{0\}$ *and so*

$$\mu(\overline{B((0,r_n), 3r_n/4)}) = \alpha_1 r_n^2 \frac{9}{16}.$$

On the other hand, $B((0,r_n), 3r_n/2)$ *intersects* $[-1,1] \times \{0\}$ *and*

$$\mu(\overline{B((0,r_n), 3r_n/2)}) = \alpha_1 r_n^2 \frac{9}{4} + c_1 r_n.$$

Hence, the doubling condition does not hold.

Theorem 40 *Let* $\mathfrak{M}$ *be a* σ*-algebra containing the Borel* σ*-algebra* $\mathcal{B}(\mathbb{R}^N)$ *and let* $\mu : \mathfrak{M} \to [0,\infty]$ *be a complete measure, finite on compact sets, and doubling. Then*

$$\mathrm{M}_\mu(f) \le \mathrm{M}_\mu^{nc}(f) \le c_\mu^2 \, \mathrm{M}_\mu(f).$$

In particular, M_μ^{nc} *is of weak type* $(1,1)$ *and of strong type* (p,p) *for* $1 < p \le \infty$*.*

Proof. Given $\boldsymbol{x} \in \mathbb{R}^N$, then for any closed ball $\overline{B} = \overline{B(\boldsymbol{y}, r)}$ such that $\boldsymbol{x} \in \overline{B}$ and $\mu(\overline{B}) > 0$, we have that $\overline{B(\boldsymbol{y}, r)} \subset \overline{B(\boldsymbol{x}, 2r)} \subset \overline{B(\boldsymbol{y}, 4r)}$, and so by the doubling condition

$$\mu(\overline{B(\boldsymbol{x}, 2r)}) \le \mu(\overline{B(\boldsymbol{y}, 4r)}) \le c_\mu^2 \mu(\overline{B(\boldsymbol{y}, r)}),$$

which implies that

$$\frac{1}{\mu(\overline{B})} \int_{\overline{B}} |f| \, d\mu \le c_\mu^2 \frac{1}{\mu(\overline{B(\boldsymbol{x}, 2r)})} \int_{\overline{B(\boldsymbol{x}, 2r)}} |f| \, d\mu \le c_\mu^2 \, \mathrm{M}_\mu(f)(\boldsymbol{x}).$$

Taking the supremum over all such balls $\overline{B}$ shows that

$$\mathrm{M}_\mu(f)(\boldsymbol{x}) \le \mathrm{M}_\mu^{nc}(f)(\boldsymbol{x}) \le c_\mu^2 \, \mathrm{M}_\mu(f)(\boldsymbol{x}).$$

Hence, in view of Theorem 33 we have that M_μ^{nc} is of weak type $(1,1)$ and of strong type (p,p) for $1 < p \le \infty$. ■

Remark 41 *With a similar proof we can show that for doubling measures the uncentered* uncentered maximal Hardy-Littlewood function

$$\mathrm{N}_\mu^{nc}(f)(\boldsymbol{x}) := \sup \frac{1}{\mu(\overline{Q})} \int_{\overline{Q}} |f| \, d\mu, \quad \boldsymbol{x} \in \mathbb{R}^N, \tag{35}$$

where the supremum is taken over all closed cubes such that $\boldsymbol{x} \in \overline{Q}$ *and* $\mu(\overline{Q}) > 0$*, is of weak type* $(1,1)$ *and strong type* (p,p) *for all* $1 < p \le \infty$*.*

It can be shown that the previous theorem fails for non doubling measures.

Monday, February 02, 2015

3 Fourier Transforms

3.1 Rapidly Decreasing Functions and Tempered Distributions

Definition 42 *The* space of rapidly decreasing functions $\mathcal{S}\left(\mathbb{R}^N\right)$ *is the space of all functions* $f : \mathbb{R}^N \to \mathbb{C}$ *of class* C^∞ *such that for all multi-indeces* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$*,*

$$\|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\boldsymbol{x}\in\mathbb{R}^N} |\boldsymbol{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} f(\boldsymbol{x})| < \infty.$$

Thus $\mathcal{S}\left(\mathbb{R}^N\right)$ consists of all functions that, together with all their derivatives, decay to zero faster than any polynomial.

Remark 43 *The space* $C_c^\infty(\mathbb{R}^N)$ *of all* C^∞ *functions* $f : \mathbb{R}^N \to \mathbb{C}$ *with compact support is contained in* $\mathcal{S}\left(\mathbb{R}^N\right)$*. The function* $f(\boldsymbol{x}) := e^{-|\boldsymbol{x}|^2}$ *is an example of a function in* $\mathcal{S}\left(\mathbb{R}^N\right)$ *without compact support.*

Theorem 44 *The space* $\mathcal{S}\left(\mathbb{R}^N\right)$ *with the the topology induced by the family of seminorms* $\|\cdot\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ *is a Fréchet space.*

Proof. Since there are countably many seminorms $\|\cdot\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, the space $\mathcal{S}\left(\mathbb{R}^N\right)$ is metrizable, with metric

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}}{1+\|f-g\|_{\boldsymbol{\alpha}_n,\boldsymbol{\beta}_n}}.$$

It remains to show that it is complete. Let $\{f_n\} \subset \mathcal{S}\left(\mathbb{R}^N\right)$ be a Cauchy sequence. Then $\left\{\boldsymbol{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} f_n\right\}$ is a Cauchy sequence in $C_b\left(\mathbb{R}^N;\mathbb{C}\right)$ for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$ and thus it converges uniformly to a function $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Let $f := g_{\boldsymbol{0},\boldsymbol{0}}$. By the fundamental theorem of calculus

$$f_n(\boldsymbol{x}+t\boldsymbol{e}_i) = f_n(\boldsymbol{x}) + \int_0^t \frac{\partial f_n}{\partial x_i}(\boldsymbol{x}+s\boldsymbol{e}_i)\, ds.$$

Letting $n \to \infty$ it follows by uniform convergence that

$$f(\boldsymbol{x}+t\boldsymbol{e}_i) = f(\boldsymbol{x}) + \int_0^t g_{\boldsymbol{0},\boldsymbol{e}_i}(\boldsymbol{x}+s\boldsymbol{e}_i)\, ds.$$

Hence, there exists $\frac{\partial f}{\partial x_i} = g_{\boldsymbol{0},\boldsymbol{e}_i}$. This proves that f is of class C^1. In a similar way we can show that f is of class C^∞ with $g_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \boldsymbol{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} f$. Thus $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and since $\|f_n - f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} \to 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$ it follows that $\mathcal{S}\left(\mathbb{R}^N\right)$ is complete. ■

Definition 45 *The dual of* $\mathcal{S}\left(\mathbb{R}^N\right)$ *is called the* space of tempered distributions *and is denoted* $\mathcal{S}'\left(\mathbb{R}^N\right)$*.*

The following theorem is important for applications. For $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and $m, n \in \mathbb{N}_0$ we define

$$\|f\|_{m,n} := \sum_{|\boldsymbol{\alpha}|\leq n} \sum_{|\boldsymbol{\beta}|\leq n} \|f\|_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Theorem 46 *A linear functional $T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ is continuous if and only if there exist a constant $C > 0$ and some $m, n \in \mathbb{N}_0$ such that*

$$|T(f)| \leq C \|f\|_{m,n}. \tag{36}$$

for every $f \in \mathcal{S}\left(\mathbb{R}^N\right)$.

Proof. Exercise. ■

Next we show that $\mathcal{S}\left(\mathbb{R}^N\right)$ is embedded in L^p for every p.

Theorem 47 *The space $\mathcal{S}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^p\left(\mathbb{R}^N\right)$ for all $1 \leq p \leq \infty$, while $L^p(\mathbb{R}^N)$ is continuously embedded in $\mathcal{S}'\left(\mathbb{R}^N\right)$ for all $1 \leq p \leq \infty$.*

Proof. We only need to consider the case $1 \leq p < \infty$. Write

$$\begin{aligned}\int_{\mathbb{R}^N} |f| \, d\boldsymbol{x} &= \int_{\mathbb{R}^N} \frac{1+|\boldsymbol{x}|^{N+1}}{1+|\boldsymbol{x}|^{N+1}} |f| \, d\boldsymbol{x} \\ &\leq C \|f\|_{N+1,0} \int_{\mathbb{R}^N} \frac{1}{1+|\boldsymbol{x}|^{N+1}} \, d\boldsymbol{x}.\end{aligned}$$

For $1 < p < \infty$ it is enough to observe that

$$\int_{\mathbb{R}^N} |f|^p \, d\boldsymbol{x} \leq \|f\|_\infty^{p-1} \int_{\mathbb{R}^N} |f| \, d\boldsymbol{x} \leq C \|f\|_{N+1,0}^p.$$

This shows that $\mathcal{S}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^p\left(\mathbb{R}^N\right)$. Given $g \in L^p(\mathbb{R}^N)$, consider the linear functional $T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ defined by

$$T_g(f) := \int_{\mathbb{R}^N} fg \, d\boldsymbol{x}. \tag{37}$$

Then by Hölder's inequality

$$|T_g(f)| \leq \|f\|_{L^{p'}} \|g\|_{L^p} \leq C \|f\|_{N+1,0} \|g\|_{L^p}.$$

Hence, by (36) the functional T_g belongs to $\mathcal{S}'\left(\mathbb{R}^N\right)$ and the linear mapping $g \in L^p(\mathbb{R}^N) \mapsto T_g$ is a continuous embedding. ■

Remark 48 *In what follows we identify g with T_g. Hence, $L^p(\mathbb{R}^N)$, and in particular $\mathcal{S}\left(\mathbb{R}^N\right)$, can be thought as contained in $\mathcal{S}'\left(\mathbb{R}^N\right)$.*

Example 49 *Given a measure* $\mu : \mathcal{B}(\mathbb{R}^N) \to [0,\infty]$ *with the property that*

$$\mu(\overline{B(\mathbf{0},r)}) \le C_0(1+r)^k$$

for some $C_0 > 0$*, some* $k \in \mathbb{N}$*, and for all* $r > 0$*, the linear functional* $T_\mu : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ *defined by*

$$T_\mu(f) := \int_{\mathbb{R}^N} f \; d\mu$$

is well-defined and continuous. Indeed, write

$$\begin{aligned}\int_{\mathbb{R}^N} |f| \; d\mu &= \int_{B(\mathbf{0},1)} |f| \; d\mu + \sum_{n=2}^{\infty} \int_{\overline{B(\mathbf{0},n)} \setminus B(\mathbf{0},n-1)} |f| \; d\mu \\ &\le \|f\|_\infty 2C_0 + \sum_{n=2}^{\infty} \int_{\overline{B(\mathbf{0},n)}} \frac{(1+|\boldsymbol{x}|)^{2k}}{(1+|\boldsymbol{x}|)^{2k}} |f| \; d\mu \\ &\le \|f\|_\infty 2C_0 + CC_0 \|f\|_{2k,0} \sum_{n=1}^{\infty} \frac{(1+n)^k}{(1+n)^{2k}} < \infty.\end{aligned}$$

Hence by (36), $T_\mu \in \mathcal{S}'\left(\mathbb{R}^N\right)$*.*

Example 50 (Principal value of $1/x$**)** *Let's prove that the linear mapping*

$$T(f) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} \; dx, \quad f \in \mathcal{S}(\mathbb{R}),$$

is well-defined and belongs to $\mathcal{S}'(\mathbb{R})$*. The functional* T *is called the* principal value *of* $\frac{1}{x}$ *and is denoted* pv $\frac{1}{x}$*. Write*

$$\begin{aligned}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} \; dx &= \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} \; dx + \int_{\mathbb{R}\setminus[-1,1]} \frac{f(x)}{x} \; dx \\ &=: I_1 + I_2.\end{aligned}$$

The term I_2 *does not give any troubles, since*

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]} \left| \frac{f(x)}{x} \right| \; dx &\le \int_{\mathbb{R}\setminus[-1,1]} |f(x)| \; dx \\ &\le 2 \|f\|_{1,0} \int_1^{\infty} \frac{1}{x^2} \; dx = 2 \|f\|_{0,1}.\end{aligned}$$

Let's study I_1*. Since* $1/x$ *is an odd function,*

$$\int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{1}{x} \; dx = 0, \tag{38}$$

we can write

$$I_1 = \int_{[-1,1]\setminus[-\varepsilon,\varepsilon]} \frac{f(x) - f(0)}{x - 0} \; dx.$$

Since $f \in \mathcal{S}(\mathbb{R})$, by the mean value theorem

$$\left|\frac{f(x)-f(0)}{x-0}\right| = |f'(\theta)| \le \|f\|_{0,1}$$

for all $x \in [-1,1]$, with $x \neq 0$, and so by the Lebesgue dominated convergence theorem, there exists

$$\lim_{\varepsilon \to 0+} I_1 = \int_{-1}^{1} \frac{f(x)-f(0)}{x-0}\, dx.$$

Moreover, since $|I_1| \le 2\|f\|_{0,1}$, it follows that $|\lim_{\varepsilon\to 0+} I_1| \le 2\|f\|_{0,1}$. Thus, we have shown that $T(f)$ is well-defined and

$$|T(f)| \le 2\|f\|_{0,1} + 2\|f\|_{1,0},$$

which, by Theorem 46, implies that $\operatorname{pv}\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$.

Similarly, for $x_0 \in \mathbb{R}$ we can define the tempered distribution

$$\left(\operatorname{pv}\frac{1}{x-x_0}\right)(f) := \lim_{\varepsilon\to 0+} \int_{\mathbb{R}\setminus[x_0-\varepsilon, x_0+\varepsilon]} \frac{f(x)}{x-x_0}\, dx, \quad f \in \mathcal{S}(\mathbb{R}).$$

Note that $\operatorname{pv}\frac{1}{x}$ is not of the form 49.

Remark 51 *The cancellation property (38) will turn out to play a crucial role in the theory of singular operators.*

Exercise 52 *Let $g : \mathbb{R}^N \to \mathbb{C}$ be a function of class C^∞ such that for every multi-index $\boldsymbol{a}$ there exist $C_{\boldsymbol{a}}$ and $n_{\boldsymbol{a}} \in \mathbb{N}$ such that*

$$|\partial^{\boldsymbol{\alpha}} g(\boldsymbol{x})| \le C_{\boldsymbol{a}}(1+|\boldsymbol{x}|^2)^{n_{\boldsymbol{a}}} \tag{39}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$.

(i) Prove that if $f \in \mathcal{S}(\mathbb{R}^N)$ then $fg \in \mathcal{S}(\mathbb{R}^N)$.

(ii) Prove that if $h : \mathbb{R}^N \to \mathbb{C}$ is a measurable function such that $hf \in \mathcal{S}(\mathbb{R}^N)$ for all $f \in \mathcal{S}(\mathbb{R}^N)$ and the mapping $f \in \mathcal{S}(\mathbb{R}^N) \mapsto hf$ is continuous, then h must be of class C^∞ and satisfy (39).

(iii) Given $T \in \mathcal{S}'(\mathbb{R}^N)$ prove that the linear functional $gT : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ defined by

$$(gT)(f) := T(fg), \quad f \in \mathcal{S}(\mathbb{R}^N),$$

belongs to $\mathcal{S}'(\mathbb{R}^N)$.

We now define the notion of a derivative of a tempered distribution.

Definition 53 *Given $T \in \mathcal{S}'(\mathbb{R}^N)$ and a multi-index $\boldsymbol{\alpha}$, we define the $\boldsymbol{\alpha}$-th* derivative *of T as the linear functional $\partial^{\boldsymbol{\alpha}} T : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C}$ defined by*

$$(\partial^{\boldsymbol{\alpha}} T)(f) := (-1)^{|\boldsymbol{\alpha}|} T(\partial^{\boldsymbol{\alpha}} f), \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Theorem 54 *For every $T \in \mathcal{S}'(\mathbb{R}^N)$ and every multi-index $\boldsymbol{\alpha}$, the functional $\partial^{\boldsymbol{\alpha}} T$ belongs to $\mathcal{S}'(\mathbb{R}^N)$.*

Proof. Since $T \in \mathcal{S}'(\mathbb{R}^N)$, by Theorem 46 there exist a constant $C > 0$ and some $m, n \in \mathbb{N}_0$ such that

$$|T(f)| \le C \|f\|_{m,n}.$$

for every $f \in \mathcal{S}(\mathbb{R}^N)$. In turn, since for $f \in \mathcal{S}(\mathbb{R}^N)$, $\partial^{\boldsymbol{\alpha}} f$ still belongs to $\mathcal{S}(\mathbb{R}^N)$,

$$|(\partial^{\boldsymbol{\alpha}} T)(f)| = |T(\partial^{\boldsymbol{\alpha}} f)| \le C \|\partial^{\boldsymbol{\alpha}} f\|_{m,n} \le C \|f\|_{m,n+|\boldsymbol{\alpha}|},$$

and so, again by Theorem 46 it follows that $\partial^{\boldsymbol{\alpha}} T$ belongs to $\mathcal{S}'(\mathbb{R}^N)$. ■

Exercise 55 *The derivative of $\log|x|$ is the principal value.*

Exercise 56 *Prove that if $T \in \mathcal{D}'(\mathbb{R}^N)$ has compact support then $T \in \mathcal{S}'(\mathbb{R}^N)$.*

Exercise 57 *Prove that $\mathcal{D}(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$ and that the inclusion*

$$i : \mathcal{D}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$$

is continuous.

Exercise 58 *Prove that if P is a polynomial, $f \in \mathcal{S}(\mathbb{R}^N)$, and $T \in \mathcal{S}'(\mathbb{R}^N)$, then PT and $fT \in \mathcal{S}'(\mathbb{R}^N)$.*

3.2 Fourier Transforms

Given $f \in \mathcal{S}(\mathbb{R}^N)$, the *Fourier transform of f* is the function

$$\widehat{f}(\boldsymbol{x}) = \mathcal{F}(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} \tag{40}$$

while the inverse *Fourier transform of f* is the function

$$f^{\vee}(\boldsymbol{x}) := \widehat{f}(-\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y}. \tag{41}$$

Since $\mathcal{S}(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$, the functions $\widehat{f}$ and $f^{\vee}$ are well-defined.

Wednesday, February 04, 2015

Theorem 59 *The Fourier transform $\mathcal{F}$ maps $\mathcal{S}\left(\mathbb{R}^N\right)$ into $\mathcal{S}\left(\mathbb{R}^N\right)$. Moreover, for every $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and for every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^+$,*

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} \widehat{f}(\boldsymbol{x}) = \widehat{g_{\boldsymbol{\alpha},\boldsymbol{\beta}}}(\boldsymbol{x}) \tag{42}$$

where $g_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\boldsymbol{x}) := \frac{1}{(2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \partial^{\boldsymbol{\alpha}}((-2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\beta}} f(\boldsymbol{x}))$.

Proof. By differentiating under the integral sign we have that

$$\begin{aligned} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} \frac{\partial^{\boldsymbol{\beta}}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}} (e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}}) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y}. \end{aligned}$$

Hence,

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) &= \frac{1}{(-2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} (-2\pi \boldsymbol{i}\boldsymbol{x})^{\boldsymbol{\alpha}} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \frac{1}{(-2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} (-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y}) \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} (e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}})\, d\boldsymbol{y}. \end{aligned}$$

By integrating by parts and using the fact that f and its derivatives decay to zero at infinity we get

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \frac{\partial^{\boldsymbol{\beta}} \widehat{f}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) = \frac{1}{(2\pi \boldsymbol{i})^{\boldsymbol{\alpha}}} \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}} \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-2\pi \boldsymbol{i}\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right)\, d\boldsymbol{y},$$

which proves (42). It follows from Leibnitz rule that

$$\begin{aligned} \left\|\widehat{f}\right\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} &\le (2\pi)^{|\boldsymbol{\beta}|} \int_{\mathbb{R}^N} \left| \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right) \right|\, d\boldsymbol{y} \\ &= (2\pi)^{|\boldsymbol{\beta}|} \int_{\mathbb{R}^N} \frac{1 + |\boldsymbol{y}|^{N+1}}{1 + |\boldsymbol{y}|^{N+1}} \left| \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{y}^{\boldsymbol{\alpha}}} \left((-\boldsymbol{y})^{\boldsymbol{\beta}} f(\boldsymbol{y})\right) \right|\, d\boldsymbol{y} \\ &\le C(N, \boldsymbol{\alpha}, \boldsymbol{\beta}) \|f\|_{N+1+|\boldsymbol{\beta}|,|\boldsymbol{\alpha}|}, \end{aligned}$$

which shows that $\widehat{f} \in \mathcal{S}\left(\mathbb{R}^N\right)$ and that the linear operator $\mathcal{F} : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathcal{S}\left(\mathbb{R}^N\right)$ is continuous. ■

Example 60 *We compute the Fourier transform of the function $f(\boldsymbol{x}) = e^{-\pi|\boldsymbol{x}|^2}$. By Fubini's theorem and by completing the square we have*

$$\begin{aligned} \widehat{f}(\boldsymbol{x}) &= \prod_{k=1}^{N} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i} x_k y_k - \pi y_k^2}\, dy_k \\ &= \prod_{k=1}^{N} e^{\pi(\boldsymbol{i} x_k)^2} \int_{\mathbb{R}} e^{-\pi(\boldsymbol{i} x_k + y_k)^2}\, dy_k. \end{aligned}$$

Next observe that the function

$$g(x) := \int_{\mathbb{R}} e^{-\pi(\boldsymbol{i}x+y)^2}\ dy$$

is constant since

$$\begin{aligned} g'(x) &= \int_{\mathbb{R}} -2\pi\boldsymbol{i}(\boldsymbol{i}x+y)e^{-\pi(\boldsymbol{i}x+y)^2}\ dy \\ &= \int_{\mathbb{R}} \boldsymbol{i}\frac{d}{dy}(e^{-\pi(\boldsymbol{i}x+y)^2})\ dy = 0. \end{aligned}$$

Hence,

$$g(x) = g(0) = \int_{\mathbb{R}} e^{-\pi y^2}\ dy = 1.$$

If follows that $\widehat{f}(\boldsymbol{x}) = \prod_{k=1}^{N} e^{\pi(\boldsymbol{i}x_k)^2} = f(\boldsymbol{x})$.

Similarly, by taking

$$f_\varepsilon(\boldsymbol{x}) = e^{2\pi\boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{x}_0}e^{-\pi\varepsilon^2|\boldsymbol{x}|^2},$$

where $\varepsilon > 0$ *and* $\boldsymbol{x}_0 \in \mathbb{R}^N$ *we get*

$$\begin{aligned} \widehat{f_\varepsilon}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi\boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}}e^{2\pi\boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}_0}e^{-\pi\varepsilon^2|\boldsymbol{y}|^2}\ d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} e^{-2\pi\boldsymbol{i}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{y}}e^{-\pi\varepsilon^2|\boldsymbol{y}|^2}\ d\boldsymbol{y} \\ &= \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} e^{-2\pi\boldsymbol{i}\varepsilon^{-1}(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol{z}}e^{-\pi\varepsilon^2|\boldsymbol{z}|^2}\ d\boldsymbol{z} \\ &= \frac{1}{\varepsilon^N}\widehat{f}((\boldsymbol{x}-\boldsymbol{x}_0)/\varepsilon) = \frac{1}{\varepsilon^N}e^{-\pi|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}|^2} \end{aligned}$$

where we have made the change of variables $\boldsymbol{z} := \varepsilon\boldsymbol{y}$.

Next we prove that $\mathcal{F}$ is invertible with inverse given by $\mathcal{F}^{-1}(f) = f^\vee$.

Proposition 61 *For every* $f,\ g \in \mathcal{S}\left(\mathbb{R}^N\right)$*, we have*

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\ d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})g(\boldsymbol{x})\ d\boldsymbol{x}. \tag{43}$$

Proof. By Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\ d\boldsymbol{x} &= \int_{\mathbb{R}^N} f(\boldsymbol{x})\int_{\mathbb{R}^N} e^{-2\pi\boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}}g(\boldsymbol{y})\ d\boldsymbol{y}d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} g(\boldsymbol{y})\int_{\mathbb{R}^N} e^{-2\pi\boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{y}}f(\boldsymbol{x})\ d\boldsymbol{x}d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} g(\boldsymbol{y})\widehat{f}(\boldsymbol{y})\ d\boldsymbol{y}, \end{aligned}$$

which shows (43). ■

Theorem 62 (Fourier inversion theorem) *For every* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$(\widehat{f})^\vee = \widehat{(f^\vee)} = f.$$

In particular, the Fourier transform $\mathcal{F}$ *is an isomorphism from* $\mathcal{S}\left(\mathbb{R}^N\right)$ *to* $\mathcal{S}\left(\mathbb{R}^N\right)$ *with inverse* $\mathcal{F}^{-1}$ *given by* $\mathcal{F}^{-1}(f) = f^\vee$ *for every* $f \in \mathcal{S}\left(\mathbb{R}^N\right)$.

Proof. Fix $\boldsymbol{x}_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ and define $g_\varepsilon(\boldsymbol{x}) := e^{2\pi \boldsymbol{i}\boldsymbol{x}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2|\boldsymbol{x}|^2}$. By Example 60 we have that $\widehat{g_\varepsilon}(\boldsymbol{x}) = \frac{1}{\varepsilon^N} e^{-\pi|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}|^2}$ and so, taking $g = g_\varepsilon$ in (43), we get

$$\int_{\mathbb{R}^N} f(\boldsymbol{x}) \frac{1}{\varepsilon^N} e^{-\pi|(\boldsymbol{x}-\boldsymbol{x}_0)/\boldsymbol{\varepsilon}|^2}\, d\boldsymbol{x} = \int_{\mathbb{R}^N} e^{2\pi \boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}_0} e^{-\pi\varepsilon^2|\boldsymbol{y}|^2} \widehat{f}(\boldsymbol{y})\, d\boldsymbol{y}.$$

Note that $\widehat{g_\varepsilon}$ is a mollifier. Hence, the left-hand side converges to $f(\boldsymbol{x}_0)$. On the other hand, by the Lebesgue dominated convergence theorem the right-hand side converges to $(\widehat{f})^\vee(\boldsymbol{x}_0)$. Hence,

$$f(\boldsymbol{x}_0) = (\widehat{f})^\vee(\boldsymbol{x}_0)$$

which shows that $(\widehat{f})^\vee = f$. Similarly we can show that, $\widehat{(f^\vee)} = f$.

Next observe that if $\widehat{f} = 0$, then $f = (\widehat{f})^\vee = 0^\vee = 0$, and so $\mathcal{F}$ is one-to-one. Since $\widehat{(f^\vee)} = f$, it follows that $\mathcal{F}$ is onto and that the inverse of $\mathcal{F}$ is $\mathcal{F}^{-1}(f) = f^\vee$. ■

We recall that for a complex number $z = \operatorname{Re} z + \boldsymbol{i} \operatorname{Im} z$, the complex conjugate of z is the number $\overline{z} := \operatorname{Re} z - \boldsymbol{i} \operatorname{Im} z$.

Corollary 63 *For every* $f,\, h \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\overline{h(\boldsymbol{x})}\, d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})\overline{\widehat{h}(\boldsymbol{x})}\, d\boldsymbol{x} \qquad \text{Parseval identity}$$

and

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2\, d\boldsymbol{x} = \int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{x})|^2\, d\boldsymbol{x} = \int_{\mathbb{R}^N} |f^\vee(\boldsymbol{x})|^2\, d\boldsymbol{x}. \qquad \text{Plancherel identity}$$

In particular, $\mathcal{F}$ *extends uniquely to an isomorphism of* $L^2\left(\mathbb{R}^N\right)$ *onto itself.*

Proof. Let $g := \overline{\widehat{h}}$. Then, using the facts that cos is even and sin is odd, we have

$$\begin{aligned} \widehat{g}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi \boldsymbol{i}\boldsymbol{y}\cdot\boldsymbol{x}} \overline{\widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} = \int_{\mathbb{R}^N} \overline{e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}}}\,\overline{\widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} \overline{e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}} \widehat{h}(\boldsymbol{y})}\, d\boldsymbol{y} \\ &= \overline{\int_{\mathbb{R}^N} e^{2\pi \boldsymbol{i}\boldsymbol{z}\cdot\boldsymbol{x}} \widehat{h}(\boldsymbol{y})\, d\boldsymbol{y}} = \overline{(\widehat{h})^\vee(\boldsymbol{x})} = \overline{h(\boldsymbol{x})}, \end{aligned}$$

where in the last equality we have used the inversion theorem. Hence, Parseval's identity follows by (43). Taking $h = f$ and using the fact that $f(\boldsymbol{x})\overline{f(\boldsymbol{x})} = |f(\boldsymbol{x})|^2$ gives the first equality Plancherel's identity. The second equality follows by replacing f with $f^\vee$ and using the inversion theorem.

Since $\mathcal{S}\left(\mathbb{R}^N\right)$ is dense in $L^2\left(\mathbb{R}^N\right)$, if $\{f_n\} \subset \mathcal{S}\left(\mathbb{R}^N\right)$ converges to f in $L^2\left(\mathbb{R}^N\right)$, then by Plancherel's identity the sequence $\{\widehat{f_n}\}$ is a Cauchy sequence in $L^2\left(\mathbb{R}^N\right)$ and so it converges to a function $g \in L^2\left(\mathbb{R}^N\right)$. Again by Plancherel's identity, the function g does not depend on the particular sequence $\{f_n\}$. We define $\widehat{f} := g$. Similar we can extend uniquely the inverse Fourier transform to $L^2\left(\mathbb{R}^N\right)$ and reasoning as in the last part of the proof of the inversion theorem we have that the Fourier transform $\mathcal{F} : L^2\left(\mathbb{R}^N\right) \to L^2\left(\mathbb{R}^N\right)$ is an isomorphism with inverse given by the extension of $\mathcal{F}^{-1}$ to $L^2\left(\mathbb{R}^N\right)$. ■

Remark 64 (Important) *Note that the Fourier transform of a function f in $L^2\left(\mathbb{R}^N\right)$ is obtained as a limit in $L^2\left(\mathbb{R}^N\right)$ of functions of the type (40), but in general we cannot say that $\widehat{f}$ has the form (40), since the integral in (40) is well-defined for functions in $L^1\left(\mathbb{R}^N\right)$ but not for functions in $L^2\left(\mathbb{R}^N\right)$. On the other hand, if $f \in L^1(\mathbb{R}^N)$, then (40) is well-defined. Hence, the Fourier transform of a function in $L^1(\mathbb{R}^N)$ is defined pointwise by (40), while the Fourier transform of a function in $L^2(\mathbb{R}^N)$ is defined as a limit in $L^2(\mathbb{R}^N)$.*

Friday, February 06, 2015

Given an open set $\Omega \subseteq \mathbb{R}^N$, the space $C_0(\Omega)$ is defined as the space of all continuous functions f such that for every $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|f(\boldsymbol{x})| < \varepsilon$ for all $\boldsymbol{x} \in \Omega \setminus K$.

Theorem 65 (Riemann–Lebesgue lemma) $\mathcal{F} : L^1(\mathbb{R}^N) \to C_0(\mathbb{R}^N)$ *with*

$$\sup |\mathcal{F}(f)| \leq \|f\|_{L^1} \tag{44}$$

In particular,

$$\lim_{|\boldsymbol{x}|\to\infty} |\widehat{f}(\boldsymbol{x})| = 0.$$

Proof. By (40), for every $f \in L^1(\mathbb{R}^N)$,

$$|\widehat{f}(\boldsymbol{x})| \leq \|f\|_{L^1}$$

for every $\boldsymbol{x} \in \mathbb{R}^N$. Since $\mathcal{S}\left(\mathbb{R}^N\right)$ is dense in $L^1\left(\mathbb{R}^N\right)$ let $\{f_n\} \subset \mathcal{S}\left(\mathbb{R}^N\right)$ converge to f in $L^1\left(\mathbb{R}^N\right)$. By the previous inequality

$$\sup |\widehat{f_n} - \widehat{f}| \leq \|f_n - f\|_{L^1}.$$

Hence, the sequence $\{\widehat{f_n}\}$ converges uniformly to $\widehat{f}$. On the other hand, by Theorem 59 we have that $\widehat{f_n} \in \mathcal{S}\left(\mathbb{R}^N\right) \subset C_0(\mathbb{R}^N)$ and hence, since $C_0(\mathbb{R}^N)$ is a closed under uniform convergence, it follows that $\widehat{f} \in C_0(\mathbb{R}^N)$. ■

Exercise not onto.

Corollary 66 (Hausdorff–Young inequality) *Let $1 < p < 2$. Then $\mathcal{F}$ can be extended uniquely to $L^p\left(\mathbb{R}^N\right)$ and*

$$\|\mathcal{F}(f)\|_{L^{p'}} \le \|f\|_{L^p} \tag{45}$$

for every $f \in L^p\left(\mathbb{R}^N\right)$.

Proof. By Plancherel's identity and (44),

$$\sup|\mathcal{F}(f)| \le \|f\|_{L^1}, \quad \|\mathcal{F}(f)\|_{L^2} = \|f\|_{L^2}$$

for every simple function f vanishing at infinity. Hence, $\mathcal{F}$ is of strong type $(1,\infty)$ and $(2,2)$ and so we are in a position to apply the Riesz–Thorin theorem to obtain that $\mathcal{F}$ is of strong type (p,q) for every $\theta \in (0,1)$, where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

Hence, $p = \frac{2}{2-\theta} \in (0,1)$ and $q = p' = \frac{2}{2-\theta}/(\frac{2}{2-\theta} - 1) = 2/\theta$. Moreover, (45) holds. ■

Corollary 67 (Weak Hausdorff–Young inequality) *Let $1 < p < 2$. Then $\mathcal{F}$ can be extended uniquely to $L^p_w\left(\mathbb{R}^N\right)$ and*

$$\|\mathcal{F}(f)\|_{L^{p'}_w} \le \|f\|_{L^p_w} \tag{46}$$

for every $f \in L^p_w\left(\mathbb{R}^N\right)$.

Proof. We proceed as in the previous corollary but we apply instead Theorem 25 with $p_1 = q_1 = 1$, $p_2 = q_2 = 2$, $r_1 = s_1 = \infty$ and $r_2 = s_2 = 2$. For every $\theta \in (0,1)$ take $s = q = \infty$, $p = \frac{2}{2-\theta} \in (0,1)$ and $r = p' = \frac{2}{\theta}$. Then the identities (20) hold and so

$$\|\mathcal{F}(f)\|_{L^{p'}_w} \le C_\theta \|f\|_{L^p_w}$$

for all $f \in L^1\left(\mathbb{R}^N\right) + L^2\left(\mathbb{R}^N\right)$. ■

We can also define the Fourier transform for tempered distributions $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ by setting

$$\widehat{T}(f) := T(\widehat{f}).$$

Given $g \in \mathcal{S}\left(\mathbb{R}^N\right)$, consider the linear functional $T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ defined by

$$T_g(f) := \int_{\mathbb{R}^N} fg \, d\boldsymbol{x}.$$

By (43), for every $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, we have

$$T_{\widehat{g}}(f) = \int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})g(\boldsymbol{x}) \, d\boldsymbol{x} = T_g(\widehat{f}) = \widehat{T_g}(f).$$

Since we are identifying g with T_g in $\mathcal{S}'\left(\mathbb{R}^N\right)$, this shows that the Fourier transform defined on $\mathcal{S}'\left(\mathbb{R}^N\right)$ extends the Fourier transform defined in $\mathcal{S}\left(\mathbb{R}^N\right)$.

Exercise 68 *Let $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$.*

(i) Prove that $\widehat{T} \in \mathcal{S}'\left(\mathbb{R}^N\right)$.

(ii) Prove that if $\{T_n\} \subset \mathcal{S}'\left(\mathbb{R}^N\right)$ is such that $T_n \overset{*}{\rightharpoonup} T$ in $\mathcal{S}'\left(\mathbb{R}^N\right)$, then $\widehat{T_n} \overset{*}{\rightharpoonup} \widehat{T}$.

(iii) Prove that $\mathcal{F} : \mathcal{S}'\left(\mathbb{R}^N\right) \to \mathcal{S}'\left(\mathbb{R}^N\right)$ is a bijection.

3.3 Convolutions

Given two measurable functions $f : \mathbb{R}^N \to \mathbb{C}$ and $g : \mathbb{R}^N \to \mathbb{C}$, the *convolution* of f and g is the function $f * g$ defined by

$$(f * g)(\boldsymbol{x}) := \int_{\mathbb{R}^N} f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y} \tag{47}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$ for which the right-hand side is well-defined.

Theorem 69 *Given $f, g \in \mathcal{S}\left(\mathbb{R}^N\right)$, the function $f * g$ belongs to $\mathcal{S}\left(\mathbb{R}^N\right)$.*

Proof. Fix $\boldsymbol{x} \in \mathbb{R}^N$. For $m \in \mathbb{N}$ with $m > N$, we can write

$$\begin{aligned} |(f * g)(\boldsymbol{x})| &\le \int_{\mathbb{R}^N} |f(\boldsymbol{x} - \boldsymbol{y})||g(\boldsymbol{y})|\, d\boldsymbol{y} \\ &\le C \|g\|_{0,m} \|f\|_{0,m} \int_{\mathbb{R}^N} \frac{1}{(1 + |\boldsymbol{y}|)^m} \frac{1}{(1 + |\boldsymbol{x} - \boldsymbol{y}|)^m}\, d\boldsymbol{y}. \end{aligned}$$

We now split $\mathbb{R}^N$ in the sets $E := \{\boldsymbol{y} \in \mathbb{R}^N : \frac{1}{2}|\boldsymbol{x}| \le |\boldsymbol{x} - \boldsymbol{y}|\}$ and $\mathbb{R}^N \setminus E$. Then we have

$$\begin{aligned} &\int_E \frac{1}{(1 + |\boldsymbol{y}|)^m} \frac{1}{(1 + |\boldsymbol{x} - \boldsymbol{y}|)^m}\, d\boldsymbol{y} \\ &\le \frac{2^m}{(2 + |\boldsymbol{x}|)^m} \int_{\mathbb{R}^N} \frac{1}{(1 + |\boldsymbol{y}|)^m}\, d\boldsymbol{y} \le \frac{C(m, N)}{(2 + |\boldsymbol{x}|)^m}, \end{aligned}$$

while in $\mathbb{R}^N \setminus E$, $|\boldsymbol{y}| \ge |\boldsymbol{x}| - |\boldsymbol{x} - \boldsymbol{y}| \ge |\boldsymbol{x}| - \frac{1}{2}|\boldsymbol{x}| = \frac{1}{2}|\boldsymbol{x}|$, and so

$$\begin{aligned} &\int_{\mathbb{R}^N \setminus E} \frac{1}{(1 + |\boldsymbol{y}|)^m} \frac{1}{(1 + |\boldsymbol{x} - \boldsymbol{y}|)^m}\, d\boldsymbol{y} \\ &\le \frac{2^m}{(2 + |\boldsymbol{x}|)^m} \int_{\mathbb{R}^N} \frac{1}{(1 + |\boldsymbol{x} - \boldsymbol{y}|)^m}\, d\boldsymbol{y} \le \frac{C(m, N)}{(2 + |\boldsymbol{x}|)^m}. \end{aligned}$$

Hence,

$$(2 + |\boldsymbol{x}|)^m |(f * g)(\boldsymbol{x})| \le C \|g\|_{0,m} \|f\|_{0,m}.$$

This shows that f decays to zero faster than any power of $|\boldsymbol{x}|$.

On the other hand, by differentiating under the integral sign, for every multi-index $\boldsymbol{\alpha}$,

$$\begin{aligned}\frac{\partial^{\boldsymbol{\alpha}}(f*g)}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} \frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}-\boldsymbol{y})\, g(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \left(\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} * g\right)(\boldsymbol{x}),\end{aligned}$$

and so by repeating the same calculations above with f replaced by $\frac{\partial^{\boldsymbol{\alpha}} f}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}$, we get that all derivatives of $f*g$ decay to zero faster than any power of $|\boldsymbol{x}|$, which shows that $f*g \in \mathcal{S}(\mathbb{R}^N)$. ■

Exercise 70 *Prove that for every* $f,\ g,\ h \in \mathcal{S}(\mathbb{R}^N)$,

$$(f*g)*h = f*(g*h).$$

Theorem 71 *For every* $f,\ g \in \mathcal{S}(\mathbb{R}^N)$,

$$\widehat{f*g} = \widehat{f}\widehat{g}.$$

Proof. For $\boldsymbol{x} \in \mathbb{R}^N$ by Fubini's theorem we have

$$\begin{aligned}\widehat{(f*g)}(\boldsymbol{x}) &= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} (f*g)(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y}-\boldsymbol{\xi})\, g(\boldsymbol{\xi})\, d\boldsymbol{\xi} d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} g(\boldsymbol{\xi}) \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} f(\boldsymbol{y}-\boldsymbol{\xi})\, d\boldsymbol{y} d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} g(\boldsymbol{\xi}) \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot(\boldsymbol{y}-\boldsymbol{\xi})} f(\boldsymbol{y}-\boldsymbol{\xi})\, d\boldsymbol{y} d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} g(\boldsymbol{\xi}) \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\eta}} f(\boldsymbol{\eta})\, d\boldsymbol{\eta} d\boldsymbol{\xi} \\ &= \widehat{g}(\boldsymbol{x})\widehat{f}(\boldsymbol{x}),\end{aligned}$$

where we have made the change of variables $\boldsymbol{\eta} := \boldsymbol{y} - \boldsymbol{\xi}$. ■

Remark 72 *The previous theorem continues to hold for* $f \in L^1(\mathbb{R}^N)$ *and* $g \in \mathcal{S}(\mathbb{R}^N)$.

Given two measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$,

Theorem 73 *Let* $f \in L^p(\mathbb{R}^N)$, $1 \le p \le \infty$, *and* $g \in L^1(\mathbb{R}^N)$. *Then* $(f*g)(\boldsymbol{x})$ *exists for* $\mathcal{L}^N$*-a.e.* $\boldsymbol{x} \in \mathbb{R}^N$ *and*

$$\|f*g\|_{L^p} \le \|f\|_{L^p} \|g\|_{L^1}.$$

Proof. Consider two Borel functions f_0 and g_0 such that $f_0(\boldsymbol{x}) = f(\boldsymbol{x})$ and $g_0(\boldsymbol{x}) = g(\boldsymbol{x})$ for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$. Since the integral in (47) is unchanged if we replace f and g with f_0 and g_0, respectively, in what follows, without loss of generality we may assume that f and g are Borel functions.

Let $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be the function defined by

$$h(\boldsymbol{x}, \boldsymbol{y}) := f(\boldsymbol{x} - \boldsymbol{y}), \quad (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then h is a Borel function, since it is the composition of the Borel function f with the continuous function $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ given by $g(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x} - \boldsymbol{y}$. In turn, the function

$$(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})$$

is Borel measurable. We are now in a position to apply Minkowski's inequality for integrals and Tonelli's theorem to conclude that

$$\begin{aligned}\|f * g\|_{L^p} &= \left\| \int_{\mathbb{R}^N} |f(\cdot - \boldsymbol{y})\, g(\boldsymbol{y})| \; d\boldsymbol{y} \right\|_{L^p} \leq \int_{\mathbb{R}^N} \|f(\cdot - \boldsymbol{y})\, g(\boldsymbol{y})\|_{L^p} \; d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} |g(\boldsymbol{y})| \, \|f(\cdot - \boldsymbol{y})\|_{L^p} \; d\boldsymbol{y} = \|f\|_{L^p} \int_{\mathbb{R}^N} |g(\boldsymbol{y})| \; d\boldsymbol{y},\end{aligned}$$

where in the last equality we have used the fact that the Lebesgue measure is translation invariant. Hence, $f * g$ belongs to $L^p(\mathbb{R}^N)$, and so it is finite $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$. ■

Monday, February 09, 2015

The following is the generalized form of the previous inequality.

Theorem 74 (Young's inequality) *Let $1 \leq p \leq q \leq \infty$ and let $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then $(f * g)(\boldsymbol{x})$ exists for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$ and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{48}$$

Proof. If $p = 1$, then $r = q$ and the result follows from the previous theorem. Thus assume that $p > 1$. Fix $g \in L^q(\mathbb{R}^N)$ and consider the linear operator $T_g(h) := g * h$. By the previous theorem we have that T_g is of strong type $(1, q)$. Moreover, by Hölder's inequality for every $h \in L^{q'}(\mathbb{R}^N)$,

$$\begin{aligned}|T_g(h)(\boldsymbol{x})| &= \left| \int_{\mathbb{R}^N} h(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y}) \; d\boldsymbol{y} \right| \leq \|g\|_{L^q} \|h(\boldsymbol{x} - \cdot)\|_{L^{q'}} \\ &= \|g\|_{L^q} \|h\|_{L^{q'}},\end{aligned}$$

where in the last equality we used the translation invariance of the Lebesgue measure. This shows that T_g is of strong type (q', ∞). Hence, we are in a position to apply the Riesz–Thorin interpolation theorem with $(p_1, q_1) = (1, q)$ and $(p_2, q_2) = (q', \infty)$ and $c_{1,q} = c_{q',\infty} = \|g\|_{L^q}$ to obtain that for every $\theta \in (0,1)$, T_g can be uniquely extended to an operator of strong type (p_3, q_3), where

$$\frac{1}{p_3} = \frac{1-\theta}{1} + \frac{\theta}{q'} = \frac{1-\theta}{1} + \frac{\theta(q-1)}{q} = 1 - \frac{\theta}{q}, \quad \frac{1}{q_3} = \frac{1-\theta}{q} + \frac{\theta}{\infty}.$$

Since $p > 1$, it follows from (48) that $r > q$ and so we can find $0 < \theta < 1$ such that $q = (1-\theta)r$. Hence $q_3 = r$. In turn

$$\frac{1}{p} = 1 - \frac{1}{q} + \frac{1}{r} = 1 - \frac{1}{q} + \frac{1-\theta}{q} = 1 - \frac{\theta}{q},$$

which shows that $p_3 = p$. It follows that

$$\|T_g(f)\|_{L^r} \le \|g\|_{L^q}^{1-\theta} \|g\|_{L^q}^{\theta} \|f\|_{L^p} = \|g\|_{L^q} \|f\|_{L^p}.$$

To conclude the proof it remains to show that the convolution is defined pointwise. This is left as an exercise. ■

In applications it will be important to consider functions of the form $g(\boldsymbol{x}) := |\boldsymbol{x}|^{-\ell}$, $\ell > 0$. As seen in Example 10, these functions are not in any $L^p(\mathbb{R}^N)$ but belong to some weak L^p_w. Hence, we will extend the previous inequality to include these spaces.

Theorem 75 (General Young's inequality) *Let* $1 < p, r, q < \infty$ *and let* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q_w(\mathbb{R}^N)$. *Then* $(f * g)(\boldsymbol{x})$ *exists for* $\mathcal{L}^N$*-a.e.* $\boldsymbol{x} \in \mathbb{R}^N$ *and*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q_w},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{49}$$

Proof. Step 1: Let $1 < \tilde{p} < \infty$, fix $f \in L^{\tilde{p}}(\mathbb{R}^N)$, and consider the linear operator $T_f(h) := f * h$. We want to apply Theorem 25 to T_f. Take $p_1 = q_1 = 1$ and $r_1 = s_1 = \tilde{p}$, then by Young's inequality we have that T_f is of strong type $(1, p)$. Similarly, taking $p_2 = q_2 = \tilde{p}'$ and $r_2 = s_2 = \infty$, again by Young's inequality we have that T_f is of strong type $(\tilde{p}', \infty)$. It follows by Theorem 25 that T_f is bounded from $L^{\tilde{q}}_w(\mathbb{R}^N)$ into $L^{\tilde{r}}_w(\mathbb{R}^N)$ for all $\theta \in (0,1)$, where

$$\frac{1}{\tilde{q}} = \frac{1-\theta}{1} + \frac{\theta}{\tilde{p}'}, \quad \frac{1}{\tilde{r}} = \frac{1-\theta}{\tilde{p}} + \frac{\theta}{\infty}. \tag{50}$$

Note that $1 < \tilde{q} < \tilde{p}'$.

Step 2: Let $1 < \tilde{q} < \infty$, fix $\tilde{g} \in L^{\tilde{q}}_w(\mathbb{R}^N)$ and consider the linear operator $T_{\tilde{g}}(h) := g * h$. Then for every $1 < \tilde{p} < \infty$ such that $1 < \tilde{q} < \tilde{p}'$, by the previous step we have that $T_{\tilde{g}}$ is bounded from $L^{\tilde{p}}(\mathbb{R}^N)$ into $L^{\tilde{r}}_w(\mathbb{R}^N)$ where $\tilde{r}$ is related to $\tilde{p}$ and $\tilde{q}$ by (50).

Step 3: Fix $g \in L_w^q(\mathbb{R}^N)$ and consider the linear operator $T_g(h) := g * h$. Since by hypothesis $1 < p, r, q < \infty$, it follows from (49) that $q < p'$. Let $1 < p_1 < p < p_2 < \infty$ be such that $q < p_2'$. By the previous step T_g is bounded from $L^{p_i}(\mathbb{R}^N)$ into $L_w^{r_i}(\mathbb{R}^N)$, $i = 1, 2$, r_i is related to p_i and q by (50). Take $q_1 = p_1$, $q_2 = p_2$, $s_1 = r_1$ and $s_2 = r_2$. It follows by Theorem 25 with $s = r$ that T_g is bounded from $L^p(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$. It follows that $f * g \in L^r(\mathbb{R}^N)$. ■

Remark 76 *Note that in order for $1 < p, q, r < \infty$ to satisfy (49) it is necessary to have $q < p'$, or equivalently $p < q'$.*

Given $0 < \alpha < N$ and a function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, the *Riesz potential* of f is defined by

$$I_\alpha(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N} \frac{f(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{N-\alpha}}\, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^N.$$

Corollary 77 (Sobolev inequality) *Let $0 < \alpha < N$ and let*

$$1 < p < N/\alpha. \tag{51}$$

Then for all $f \in L^p(\mathbb{R}^N)$,

$$\|I_\alpha(f)\|_{L^r} \leq c(N, \alpha, p)\, \|f\|_{L^p}\,, \tag{52}$$

where

$$r := \frac{Np}{N - \alpha p}.$$

Proof. Let $g(\boldsymbol{x}) := |\boldsymbol{x}|^{-(N-\alpha)}$. By Example 10, we have that $g \in L_w^{N/(N-\alpha)}(\mathbb{R}^N)$. Hence, we are in a position to apply Young's general inequality with $q = N/(N-\alpha)$. Note that the restriction (51) comes from Remark 76. ■

Remark 78 *This gives a different proof of the Sobolev–Gagliardo–Nirenberg embedding theorem for $1 < p < N$, where $\alpha = N - 1$. Indeed, given $f \in W^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, for every $\boldsymbol{x} \in \mathbb{R}^N$ one has (exercise) ,*

$$|f(\boldsymbol{x})| \leq \frac{1}{\beta_N} \int_{\mathbb{R}^N} \frac{|\nabla f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^{N-1}}\, d\boldsymbol{y} = \frac{1}{\beta_N} I_1(|\nabla f|)(\boldsymbol{x}). \tag{53}$$

It follows by Sobolev'sinequality that $I_1(|\nabla f|) \in L^{p^*}(\mathbb{R}^N)$, and hence, so does f.

**********************************This was not done in class, please read it*****************************

Next we define the convolution of a function $f \in \mathcal{S}(\mathbb{R}^N)$ and a functional $T \in \mathcal{S}'(\mathbb{R}^N)$. We begin with the case in which $T = T_g$ for some function $g \in \mathcal{S}(\mathbb{R}^N)$, where we recall that $T_g \in \mathcal{S}'(\mathbb{R}^N)$ is defined by

$$T_g(h) := \int_{\mathbb{R}^N} g(\boldsymbol{x}) h(\boldsymbol{x})\, d\boldsymbol{x}, \quad h \in \mathcal{S}(\mathbb{R}^N).$$

By Fubini's theorem

$$\begin{aligned}\int_{\mathbb{R}^N}(f*g)(\boldsymbol{x})h(\boldsymbol{x})\ d\boldsymbol{x} &= \int_{\mathbb{R}^N} h(\boldsymbol{x})\int_{\mathbb{R}^N} f\left(\boldsymbol{x}-\boldsymbol{y}\right)g\left(\boldsymbol{y}\right)\ d\boldsymbol{y}d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} g\left(\boldsymbol{y}\right)\int_{\mathbb{R}^N} f\left(\boldsymbol{x}-\boldsymbol{y}\right)h(\boldsymbol{x})\,d\boldsymbol{x}d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} g\left(\boldsymbol{y}\right)\int_{\mathbb{R}^N} f\left(-\boldsymbol{\xi}\right)h(\boldsymbol{y}-\boldsymbol{\xi})\,d\boldsymbol{\xi}d\boldsymbol{y} \\ &= \int_{\mathbb{R}^N} g\left(\boldsymbol{y}\right)(\tilde{u}*h)(\boldsymbol{y})d\boldsymbol{y}\end{aligned}$$

where $\boldsymbol{\xi} := \boldsymbol{y}-\boldsymbol{x}$ and $\widetilde{f}(\boldsymbol{x}) := f(-\boldsymbol{x})$. Hence, we have shown that

$$T_{f*g}(h) = T_g(\widetilde{f}*h)$$

for all $h \in \mathcal{S}\left(\mathbb{R}^N\right)$. Motivated by this formula we define:

Definition 79 *If $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ the* convolution *of f and T is the linear functional $f*T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathbb{C}$ defined by*

$$\left(f*T\right)\left(g\right) := T\left(\widetilde{f}*g\right),$$

where

$$\widetilde{f}(\boldsymbol{x}) := f(-\boldsymbol{x}). \tag{54}$$

It actually turns out that $f*T$ is actually a function. Given $\boldsymbol{h} \in \mathbb{R}^N$ and a function $f : \mathbb{R}^N \to \mathbb{C}$ we define the translation operatotor $\tau_{\boldsymbol{h}}$ as

$$\tau_{\boldsymbol{h}}(f)(\boldsymbol{x}) := f(\boldsymbol{x}+\boldsymbol{h}), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

Theorem 80 *If $f \in \mathcal{S}\left(\mathbb{R}^N\right)$ and $T \in \mathcal{S}'\left(\mathbb{R}^N\right)$ then $f*T = T_h$, where h_f is the function given by*

$$h_f(\boldsymbol{x}) := T(\tau_{\boldsymbol{x}}(\widetilde{f})), \quad \boldsymbol{x} \in \mathbb{R}^N.$$

Moreover $h_f \in C^\infty\left(\mathbb{R}^N\right)$ and for every multi-index $\boldsymbol{a}$ there exist $C_{\boldsymbol{a}}$ and $n_{\boldsymbol{a}} \in \mathbb{N}$ such that

$$|\partial^{\boldsymbol{\alpha}} h_f(\boldsymbol{x})| \le C_{\boldsymbol{a}}(1+|\boldsymbol{x}|^2)^{n_{\boldsymbol{a}}}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$.

Proof. Exercise. ■

Example 81 *By Example 50,* $\mathrm{pv}\,\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$. *Hence, given* $f \in \mathcal{S}(\mathbb{R})$ *we can consider the convolution* $f * \mathrm{pv}\,\frac{1}{x}$ *defined by*

$$\begin{aligned}(f * \mathrm{pv}\,\frac{1}{x})(g) &= (\mathrm{pv}\,\frac{1}{x})(\widetilde{f} * g) \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{(\widetilde{f} * g)(t)}{t}\, dt \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{1}{t} \int_{\mathbb{R}} f(x-t) g(x)\, dx dt \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} g(x) \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{1}{t} f(x-t)\, dt dx \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} g(x) \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{y-x}\, dy dx,\end{aligned}$$

where we have made the change of variables $y := x - t$. *In view of the previous theorem we have that*

$$(f * \mathrm{pv}\,\frac{1}{x})(g) = \int_{\mathbb{R}} g(x) h_f(x)\, dx,$$

where

$$h_f(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{f(y)}{y-x}\, dy$$

is a function in $C^\infty(\mathbb{R})$.

Wednesday, February 11, 2015

4 Three Important Singular Integrals

4.1 The Hilbert Transform

Given a function $f : \mathbb{R} \to \mathbb{C}$, the Hilbert transform of f is "defined formally as"

$$H(f)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy.$$

Note that $H(f)$ is the convolution of f with $1/x$. However, we are outside of the range of Sobolev's inequality since in this case $\alpha = N = 1$. Indeed, the integral does not convergence absolutely, even if f is smooth. To see this assume that f is continuous at some $x_0 \in \mathbb{R}$ with $f(x_0) \neq 0$. Then taking $\varepsilon = |f(x_0)|/2$ we can find $\delta > 0$ such that $|f(x) - f(x_0)| \leq |f(x_0)|/2$ for all $x \in \mathbb{R}$ with $|x - x_0| \leq \delta$. It follows that

$$\int_{-\infty}^{+\infty} \left| \frac{f(y)}{x-y} \right| dy \geq \frac{|f(x_0)|}{2} \int_{x_0-\delta}^{x_0+\delta} \frac{1}{|x_0-y|}\, dy = \infty.$$

To remedy this situation assume that $f \in \mathcal{S}(\mathbb{R})$ and define

$$\begin{aligned} H(f)(x) &:= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{f(y)}{x-y}\, dy \\ &= \frac{1}{\pi} \operatorname{pv} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy. \end{aligned}$$

We have seen in Example 50 that the limit exists for every $x \in \mathbb{R}$ and that

$$|H(f)(x)| \le \frac{2}{\pi}(\|f\|_{0,1} + \|f\|_{1,0}).$$

Note that since

Remark 82 *The proof in Example 50 continues to hold if f is Hölder continuous with exponent $0 < \alpha \le 1$ and $\sup_x |xf(x)| < \infty$. In this case $|f(y) - f(x)| \le M|x-y|^\alpha$ and since the function $1/|x|^{1-\alpha}$ is integrable in $(-1,1)$ we can again apply the Lebesgue dominated convergence theorem to pass to the limit.*

Next we show that we cannot expect the Hilbert transform to be of strong type $(1,1)$.

Theorem 83 *Let $f \in \mathcal{S}(\mathbb{R})$. Then there exists*

$$\lim_{|x| \to \infty} xH(f)(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(y)\, dy.$$

In particular, if $\int_{-\infty}^{+\infty} f(y)\, dy \ne 0$, then $H(f) \notin L^1(\mathbb{R})$.

Proof. Exercise. ■

We will show that H is of weak type $(1,1)$.

Next we show that H is of strong type $(2,2)$. The idea consists in consider the family of truncated operators

$$H_\varepsilon(f)(x) := \frac{1}{\pi} \int_{\mathbb{R} \setminus [x-\varepsilon, x+\varepsilon]} \frac{f(y)}{x-y}\, dy,$$

where $\varepsilon > 0$. Note that

$$H_\varepsilon(f)(x) = f * K_\varepsilon,$$

where

$$K_\varepsilon(x) := \frac{1}{\pi} \begin{cases} 1/x & \text{if } \varepsilon < |x|, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 84 *For all $f \in \mathcal{S}(\mathbb{R})$,*

$$\widehat{H(f)}(x) = -\operatorname{sgn} x\, \boldsymbol{i} \widehat{f}(x) \tag{55}$$

for $\mathcal{L}^1$ a.e. x and

$$\|H(f)\|_{L^2} = \|f\|_{L^2}.$$

Proof. Since the function K_ε belongs to $L^p(\mathbb{R})$ for every $1 < p \le \infty$ but not to $L^1(\mathbb{R})$, we further truncate K_ε and consider

$$K_{\varepsilon,R}(x) := \frac{1}{\pi}\begin{cases} 1/x & \text{if } \varepsilon < |x| < R, \\ 0 & \text{otherwise.} \end{cases}$$

Let's estimate $\widehat{K_{\varepsilon,R}}$. For $x \ne 0$, we have

$$\begin{aligned}\widehat{K_{\varepsilon,R}}(x) &= \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i} xy} K_{\varepsilon,R}(y)\, dy = \frac{1}{\pi}\int_{(-R,R)\setminus[-\varepsilon,\varepsilon]} \frac{\cos(2\pi xy)}{y}\, dy \\ &\quad - \frac{\boldsymbol{i}}{\pi}\int_{(-R,R)\setminus[-\varepsilon,\varepsilon]} \frac{\sin(2\pi xy)}{y}\, dy \\ &= -2\boldsymbol{i}\operatorname{sgn} x \int_\varepsilon^R \frac{\sin(2\pi|x|y)}{y}\, dy = -2\boldsymbol{i}\operatorname{sgn} x \int_{2\pi|x|\varepsilon}^{2\pi|x|R} \frac{\sin t}{t}\, dt,\end{aligned}$$

where we have used the fact that cos is even, sin is odd, and the change of variable $t = 2\pi|x|y$. We claim that

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \le 4 \tag{56}$$

for all $0 < a < b$ and that there exists

$$\lim_{a\to 0^+}\lim_{b\to\infty}\int_a^b \frac{\sin t}{t}\, dt = \lim_{a\to 0^+}\int_a^\infty \frac{\sin t}{t}\, dt = \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}. \tag{57}$$

Note that the integral $\int_a^\infty \frac{\sin t}{t}\, dt$ is an improper Riemann integral and not a Lebesgue integral, since it can be shown (exercise) that

$$\int_a^\infty \left|\frac{\sin t}{t}\right|\, dt = \infty.$$

This is why we are working with the kernel $K_{\varepsilon,R}$ instead of K_ε. To prove the claim, write

$$\int_a^b \frac{\sin t}{t}\, dt = \int_a^1 \frac{\sin t}{t}\, dy + \int_1^b \frac{\sin t}{t}\, dt. \tag{58}$$

For $a \le 1$ using the fact that $|\sin t| \le |t|$ for all $t \in \mathbb{R}$, we have that $\left|\frac{\sin t}{t}\right| \le 1$ and so by the Lebesgue dominated convergence theorem,

$$\lim_{a\to 0^+}\int_a^1 \frac{\sin t}{t}\, dy = \int_0^1 \frac{\sin t}{t}\, dt. \tag{59}$$

On the other hand, integrating by parts we get

$$\begin{aligned}\int_1^b \frac{\sin t}{t}\, dt &= \left[-\frac{\cos t}{t}\right]_1^b - \int_1^b \frac{\cos t}{t^2}\, dt \\ &= \left[-\frac{\cos b}{b} + \cos 1\right] - \int_1^b \frac{\cos t}{t^2}\, dt.\end{aligned} \tag{60}$$

Since $\frac{\cos t}{t^2}$ is integrable in $[1, \infty)$, by the Lebesgue dominated convergence theorem (applied to the integral on the right-hand side of (60), not the left-hand side !!!),

$$\lim_{b\to\infty} \int_1^b \frac{\sin t}{t}\, dy = \cos 1 - \int_1^\infty \frac{\cos t}{t^2}\, dt. \tag{61}$$

Hence, by (59) and (61) we have that (57) holds.

It remains to show (56). If $b \le 1$, since $\left|\frac{\sin t}{t}\right| \le 1$, we have

$$\left|\int_a^b \frac{\sin t}{t}\, dy\right| \le b - a \le 1.$$

If $b > 1$ and $a < 1$ by (58) and (60) we

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \le 1 + \left|\frac{\cos b}{b}\right| + \cos 1 + \int_1^\infty \frac{1}{t^2}\, dt \le 4.$$

Finally, if $1 < a < b$, then integrating by parts as in (58) but with a in place of 1 we get

$$\left|\int_a^b \frac{\sin t}{t}\, dt\right| \le \left|\frac{\cos b}{b}\right| + \left|\frac{\cos a}{a}\right| + \int_1^\infty \frac{1}{t^2}\, dt \le 3.$$

Hence, we have shown that

$$|\widehat{K_{\varepsilon,R}}(x)| \le 8 \tag{62}$$

for all $0 < \varepsilon < R$ and all $x \in \mathbb{R}$ and that

$$\lim_{R\to\infty} \widehat{K_{\varepsilon,R}}(x) = -2\boldsymbol{i} \operatorname{sgn} x \int_{2\pi|x|\varepsilon}^\infty \frac{\sin t}{t}\, dt, \tag{63}$$

$$\lim_{\varepsilon\to 0^+} \lim_{R\to\infty} \widehat{K_{\varepsilon,R}}(x) = -\pi \operatorname{sgn} x \boldsymbol{i}. \tag{64}$$

Note that the order in which the limits were taken was not important.

Now, since the function K_ε belongs to $L^2(\mathbb{R})$ and $K_{\varepsilon,R}(x) \to K_\varepsilon(x)$ as $\varepsilon \to 0^+$ for every x, by the Lebesgue dominated convergence theorem, $K_{\varepsilon,R} \to K_\varepsilon$ in $L^2(\mathbb{R})$. In turn, by the Plancherel identity, $\widehat{K_{\varepsilon,R}} \to \widehat{K_\varepsilon}$ in $L^2(\mathbb{R})$. It follows by (63) and (64) that

$$\widehat{K_\varepsilon}(x) = -2\boldsymbol{i} \operatorname{sgn} x \int_{2\pi|x|\varepsilon}^\infty \frac{\sin t}{t}\, dt \to -\pi \operatorname{sgn} x \boldsymbol{i} \tag{65}$$

as $\varepsilon \to 0^+$ for $\mathcal{L}^1$ a.e. x. Moreover, by (62),

$$|\widehat{K_\varepsilon}(x)| \le 8 \tag{66}$$

for $\mathcal{L}^1$ a.e. x and for all $\varepsilon > 0$. Since

$$\widehat{f * K_\varepsilon} = \widehat{f}\widehat{K_\varepsilon}$$

by Theorem 71, by Plancherel identity and (66),

$$\|f * K_\varepsilon\|_{L^2} = \left\|\widehat{f * K_\varepsilon}\right\|_{L^2} = \left\|\widehat{f}\widehat{K_\varepsilon}\right\|_{L^2} \le 8\left\|\widehat{f}\right\|_{L^2} = 8\|f\|_{L^2}$$

for all $\varepsilon > 0$. Moreover, by (65), (66), and the Lebesgue dominated convergence theorem,

$$\left\|\widehat{f * K_\varepsilon} + \pi \operatorname{sgn} x \boldsymbol{i}\widehat{f}\right\|_{L^2} = \left\|\widehat{f}\left(\widehat{K_\varepsilon} + \pi \operatorname{sgn} x \boldsymbol{i}\right)\right\|_{L^2} \to 0$$

as $\varepsilon \to 0^+$, which shows that $\widehat{f * K_\varepsilon} \to -\pi \operatorname{sgn} x \boldsymbol{i}\widehat{f}$ in $L^2(\mathbb{R})$. Hence, taking the inverse Fourier transform, again by Plancherel identity, there exists a function $g_f \in L^2(\mathbb{R})$ such that

$$f * K_\varepsilon \to g_f \quad \text{in } L^2(\mathbb{R})$$

as $\varepsilon \to 0^+$

$$\widehat{g_f}(x) = -\pi \operatorname{sgn} x \boldsymbol{i}\widehat{f}(x).$$

On the other hand, as seen in Example 50,

$$(f * K_\varepsilon)(x) \to H(f)(x)$$

for all x , and so

$$\widehat{H(f)}(x) = -\pi \operatorname{sgn} x \boldsymbol{i}\widehat{f}(x).$$

Since $|\widehat{H(f)}(x)| = |\widehat{f}(x)|$, again by Plancherel identity,

$$\|H(f)\|_{L^2} = \|f\|_{L^2},$$

which completes the proof. ■

Remark 85 *Formula (55) gives an alternative proof of the fact that H is not of $(1,1)$ type. Indeed, since*

Exercise 86 *Hilbert of $\chi_{[a,b]}$.*

4.2 Newton Potential

Given a function $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, $N \ge 3$, consider the function

$$u(\boldsymbol{x}) := c\int_{\mathbb{R}^N} \frac{f(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{N-2}}\,d\boldsymbol{y} = (f * \Gamma)(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^N, \tag{67}$$

where

$$\Gamma(\boldsymbol{x}) := \frac{c}{|\boldsymbol{x}|^{N-2}}, \quad \boldsymbol{x} \in \mathbb{R}^N \setminus \{\boldsymbol{0}\}.$$

For $i, j = 1, \ldots, N$, we have

$$\frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x}) = -c(N-2)\frac{x_i}{|\boldsymbol{x}|^N} \tag{68}$$

and

$$\frac{\partial^2\Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}) = cN(N-2)\frac{x_i x_j}{|\boldsymbol{x}|^{N+2}} \tag{69}$$

for $i \neq j$, and

$$\frac{\partial^2\Gamma}{\partial x_i^2}(\boldsymbol{x}) = c(N-2)\left(\frac{N x_i^2}{|\boldsymbol{x}|^{N+2}} - \frac{1}{|\boldsymbol{x}|^N}\right). \tag{70}$$

Observe that

$$\Delta\Gamma(\boldsymbol{x}) = c(N-2)\sum_{i=1}^{N}\left(\frac{N x_i^2}{|\boldsymbol{x}|^{N+2}} - \frac{1}{|\boldsymbol{x}|^N}\right) = 0 \tag{71}$$

in $\mathbb{R}^N \setminus \{\mathbf{0}\}$.

Using the Lebesgue dominated convergence theorem and the fact that $\frac{x_i}{|\boldsymbol{x}|^N}$ is locally integrable, we can show that for every $i = 1, \ldots, N$,

$$\frac{\partial u}{\partial x_i}(\boldsymbol{x}) = \int_{\mathbb{R}^N} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}.$$

This is left as an exercise.

Friday, February 13, 2015

On the other hand, formally, for $i \neq j$,

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) &= -(N-2)c\int_{\mathbb{R}^N} \frac{\partial}{\partial x_j}\left(\frac{x_i - y_i}{|\boldsymbol{x}-\boldsymbol{y}|^N}\right) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &= c(N-1)(N-2)\int_{\mathbb{R}^N} \frac{(x_i-y_i)(x_j-y_j)}{|\boldsymbol{x}-\boldsymbol{y}|^{N+2}} f(\boldsymbol{y})\, d\boldsymbol{y}.\end{aligned}$$

Note that the kernel

$$K_{i,j}(\boldsymbol{x}) := \frac{\partial^2\Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}) \tag{72}$$

is positively homogeneous of degree $-N$, that is,

$$K_{i,j}(t\boldsymbol{x}) = t^{-N} K_{i,j}(\boldsymbol{x})$$

for all $t > 0$ and $\boldsymbol{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$. Moreover,

$$|K_{i,j}(\boldsymbol{x})| \leq \frac{1}{|\boldsymbol{x}|^N}.$$

However, $K_{i,j}$ is not integrable near the origin. So we cannot use Lebesgue dominated convergence theorem. We will prove the following.

Theorem 87 *Let $f \in \mathcal{S}\left(\mathbb{R}^N\right)$. Then there exists*

$$\begin{aligned}\frac{\partial^2 u}{\partial x_j \partial x_i}(\boldsymbol{x}) &= cf(\boldsymbol{x})(N-2)\int_{\partial B(\mathbf{0},1)} y_j y_i \, d\mathcal{H}^{N-1}(\boldsymbol{y}) \\ &\quad + \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\varepsilon)} \frac{\partial^2\Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}\end{aligned}$$

for every $\boldsymbol{x} \in \mathbb{R}^N$ and for all $i, j = 1, \ldots, N$.

We begin with a preliminary lemma, which is due to Tartar.

Lemma 88 *Let $u \in L^p(\mathbb{R}^N)$, $1 \le p \le \infty$, and let $v \in L^1(\mathbb{R}^N)$ be such that*

$$|v(\boldsymbol{x})| \le g(\boldsymbol{x})$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$, where $g \in L^1(\mathbb{R}^N)$ is a radial function of the form $g(\boldsymbol{x}) = h(|\boldsymbol{x}|)$, with $h : [0,\infty) \to [0,\infty)$ decreasing. Then

$$\left| \int_{\mathbb{R}^N} v(\boldsymbol{x}-\boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y} \right| \le \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(\boldsymbol{x})$$

for $\mathcal{L}^N$-a.e. $\boldsymbol{x} \in \mathbb{R}^N$.

Proof. By the hypotheses on v,

$$\left| \int_{\mathbb{R}^N} v(\boldsymbol{x}-\boldsymbol{y}) u(\boldsymbol{y}) \, d\boldsymbol{y} \right| \le \int_{\mathbb{R}^N} g(\boldsymbol{x}-\boldsymbol{y}) |u(\boldsymbol{y})| \, d\boldsymbol{y}.$$

Step 1: Assume first that $h = \chi_{[0,r)}$, so that $g = \chi_{B(0,r)}$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} g(\boldsymbol{x}-\boldsymbol{y}) |u(\boldsymbol{y})| \, d\boldsymbol{y} &= \int_{B(\boldsymbol{x},r)} |u(\boldsymbol{y})| \, d\boldsymbol{y} \le \mathcal{L}^N(B(\boldsymbol{x},r)) \mathrm{M}(u)(\boldsymbol{x}) \\ &= \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(\boldsymbol{x}).\end{aligned}$$

Step 2: Next, consider the case in which

$$h = \sum_{i=1}^n a_i \chi_{[r_{i-1},r_i)},$$

where $0 =: r_0 < r_1 < \cdots < r_n$ and $a_1 > a_2 > \cdots > a_n$. Set $c_i := a_i - a_{i+1} > 0$, $i = 1, \ldots, n$, where $a_{n+1} := 0$. Then we can write

$$h = \sum_{i=1}^n c_i \chi_{[0,r_i)}$$

and

$$\int_{\mathbb{R}} h(t) \, dt = \sum_{i=1}^n a_i (r_i - r_{i-1}) = \sum_{i=1}^n c_i r_i.$$

In turn,

$$g = \sum_{i=1}^n c_i \chi_{B(0,r_i)}$$

and so

$$\begin{aligned}\int_{\mathbb{R}^N} g(\boldsymbol{x}-\boldsymbol{y}) |u(\boldsymbol{y})| \, d\boldsymbol{y} &\le \sum_{i=1}^n c_i \int_{\mathbb{R}^N} \chi_{B(0,r_i)}(\boldsymbol{x}-\boldsymbol{y}) |u(\boldsymbol{y})| \, d\boldsymbol{y} \\ &\le \sum_{i=1}^n c_i \mathcal{L}^N(B(\boldsymbol{x},r_i)) (\mathrm{M}(u)(\boldsymbol{x})) = \|g\|_{L^1(\mathbb{R}^N)} \mathrm{M}(u)(\boldsymbol{x}),\end{aligned}$$

where in the second inequality we have used Step 1.

Step 3: The general case follows by observing that every increasing function $h : [0, \infty) \to [0, \infty)$ can be approximated from below by an increasing sequence of simple functions of the type given in Step 2. ■

We turn to the proof of Theorem 87.

Proof of Theorem 87. For $t > 0$ and $R > 0$ write

$$\frac{\frac{\partial u}{\partial x_i}(\boldsymbol{x} + t\boldsymbol{e}_j) - \frac{\partial u}{\partial x_i}(\boldsymbol{x})}{t} = \int_{\mathbb{R}^N} \frac{\frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j) - \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})}{t} f(\boldsymbol{y})\, d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, Rt)} \left[\frac{\frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j) - \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})}{t} - \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y}) \right] f(\boldsymbol{y})\, d\boldsymbol{y}$$
$$+ \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, Rt)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} + \frac{1}{t} \int_{B(\boldsymbol{x}, Rt)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j) f(\boldsymbol{y})\, d\boldsymbol{y}$$
$$+ \frac{1}{t} \int_{B(\boldsymbol{x}, Rt)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}$$
$$=: I_{t,R}(f)(\boldsymbol{x}) + II_{t,R}(f)(\boldsymbol{x}) + III_{t,R}(f)(\boldsymbol{x}) + IV_{t,R}(f)(\boldsymbol{x}).$$

We now estimate each of the term on the right-hand side of the previous identity.

Step 1: We claim that

$$\lim_{R \to \infty} \lim_{t \to 0^+} I_{t,R}(f)(\boldsymbol{x}) = 0$$

By the mean value theorem, applied twice

$$\frac{\frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j) - \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y})}{t} - \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y})$$
$$= \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y} + \theta_1 t\boldsymbol{e}_j) - \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y})$$
$$= \frac{\partial^3 \Gamma}{\partial x_j^2 \partial x_i}(\boldsymbol{x} - \boldsymbol{y} + \theta_2 \theta_1 t\boldsymbol{e}_j)\theta t,$$

where $|\theta_i| \le 1$, $i = 1, 2$. Note that $|\theta_2 \theta_1 t\boldsymbol{e}_j| \le t \le \frac{Rt}{2}$ for $R \ge 2$, and so for $\boldsymbol{y} \in \mathbb{R}^N \setminus B(\boldsymbol{x}, Rt)$,

$$|\boldsymbol{x} - \boldsymbol{y} + \theta_2 \theta_1 t\boldsymbol{e}_j| \ge |\boldsymbol{x} - \boldsymbol{y}| - t \ge |\boldsymbol{x} - \boldsymbol{y}| - \frac{Rt}{2} \ge \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|. \tag{73}$$

Using the fact that $\left|\frac{\partial^3 \Gamma}{\partial x_j^2 \partial x_i}(\boldsymbol{\xi})\right| \le C/|\boldsymbol{\xi}|^{N+1}$, it follows that

$$|I_t(f)(\boldsymbol{x})| \le Ct \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, Rt)} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^{N+1}}\, d\boldsymbol{y}. \tag{74}$$

■

Monday, February 16, 2015

Proof. Consider the function

$$g(\boldsymbol{x}) = h(|\boldsymbol{x}|), \quad h(r) := \begin{cases} 1/(Rt)^{N+1} & \text{if } 0 \le r \le Rt, \\ 1/r^{N+1} & \text{if } r > Rt. \end{cases}$$

Then, using spherical coordinates,

$$\begin{aligned} \|g\|_{L^1(\mathbb{R}^N)} &= \beta_N \int_0^{Rt} \frac{s^{N-1}}{(Rt)^{N+1}}\, ds + \beta_N \int_{Rt}^{\infty} \frac{s^{N-1}}{s^{N+1}}\, ds \\ &= \beta_N \frac{(Rt)^N}{N(Rt)^{N+1}} + \beta_N \frac{2}{Rt} = \frac{\beta_N}{Rt}(N^{-1}+2). \end{aligned}$$

Hence, by Lemma 88,

$$|I_{t,R}(f)(\boldsymbol{x})| \le Ct\, \|g\|_{L^1(\mathbb{R}^N)} \operatorname{M}(f)(\boldsymbol{x}) = C\frac{\beta_N}{R}(N^{-1}+2) \operatorname{M}(f)(\boldsymbol{x}), \tag{75}$$

and it now suffices to let $t \to 0^+$ and then $R \to \infty$.
Step 2: We claim that there exists

$$\begin{aligned} \lim_{t\to 0^+} II_{t,R}(f)(\boldsymbol{x}) = \lim_{\varepsilon \to 0^+} &\int_{B(\boldsymbol{x},1)\setminus B(\boldsymbol{x},\varepsilon)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y})(f(\boldsymbol{y})-f(\boldsymbol{x}))\, d\boldsymbol{y} \\ &+ \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},1)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}. \end{aligned}$$

Here we use a cancellation property of the kernels, namely, the fact that

$$0 = \int_{B(\boldsymbol{x},1)\setminus B(\boldsymbol{x},Rt)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = \int_{B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},Rt)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

For $i \ne j$, this follows from the fact that $\frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{\xi})$ is odd in the variable ξ_i and the annulus is symmetric with respect to the hyperplane $\xi_i = 0$ (see (69)). On the other hand, for $i = j$, since

$$N \int_{B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},Rt)} \frac{\xi_i^2}{|\boldsymbol{\xi}|^{N+2}}\, d\boldsymbol{\xi} = \sum_{k=1}^{N} \int_{B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},Rt)} \frac{\xi_k^2}{|\boldsymbol{\xi}|^{N+2}}\, d\boldsymbol{\xi} = \int_{B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},Rt)} \frac{1}{|\boldsymbol{\xi}|^N}\, d\boldsymbol{\xi},$$

by (70) we have that

$$\int_{B(\boldsymbol{0},1)\setminus B(\boldsymbol{0},Rt)} \frac{\partial^2 \Gamma}{\partial x_i^2}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0.$$

Hence, for all $i, j = 1, \ldots, N$ we can write

$$\begin{aligned} II_{t,R}(f)(\boldsymbol{x}) &= \int_{B(\boldsymbol{x},1)\setminus B(\boldsymbol{x},Rt)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y})(f(\boldsymbol{y})-f(\boldsymbol{x}))\, d\boldsymbol{y} \\ &\quad + \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},1)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x}-\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y} \\ &=: II_1 + II_2. \end{aligned}$$

Since $f \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$|f(\boldsymbol{y}) - f(\boldsymbol{x})| \leq \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} |\boldsymbol{y} - \boldsymbol{x}|. \tag{76}$$

In turn,

$$\left|\frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y})(f(\boldsymbol{y}) - f(\boldsymbol{x}))\right| \leq \frac{C \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))}}{|\boldsymbol{y} - \boldsymbol{x}|^{N-1}},$$

which is integrable in $B(\boldsymbol{x}, 1)$. Hence, by the Lebesgue dominated convergence theorem, there exists

$$\lim_{t \to 0^+} II_1 = \lim_{\varepsilon \to 0^+} \int_{B(\boldsymbol{x},1) \setminus B(\boldsymbol{x},\varepsilon)} \frac{\partial^2 \Gamma}{\partial x_j \partial x_i}(\boldsymbol{x} - \boldsymbol{y})(f(\boldsymbol{y}) - f(\boldsymbol{x}))\, d\boldsymbol{y}.$$

On the other, hand since f decreases faster than any polynomial, it follows that II_2 is well-defined and finite.

Step 3: We claim that there exists

$$\lim_{R \to \infty} \lim_{t \to 0^+} III_{t,R}(f)(\boldsymbol{x}) = c f(\boldsymbol{x})(2 - N) \int_{\partial B(\boldsymbol{0},1)} y_j y_i \, d\mathcal{H}^{N-1}(\boldsymbol{y}).$$

Write

$$\begin{aligned} III_{t,R}(f)(\boldsymbol{x}) &= \frac{1}{t} \int_{B(\boldsymbol{x},Rt)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j)(f(\boldsymbol{y}) - f(\boldsymbol{x} + t\boldsymbol{e}_j))\, d\boldsymbol{y} \\ &\quad + \frac{f(\boldsymbol{x} + t\boldsymbol{e}_j)}{t} \int_{B(\boldsymbol{x},Rt)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j)\, d\boldsymbol{y} \\ &=: III_1 + III_2. \end{aligned}$$

By (68) and (76),

$$\begin{aligned} |III_1| &\leq \frac{1}{t} \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} c(N-2) \int_{B(\boldsymbol{x},Rt)} \frac{|\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j|}{|\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j|^{N-1}}\, d\boldsymbol{y} \\ &\leq \frac{2^{N-1}}{t} \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} c(N-2) \int_{B(\boldsymbol{x}+t\boldsymbol{e}_j,2Rt)} \frac{1}{|\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j|^{N-2}}\, d\boldsymbol{y} \\ &= \frac{2^{N-1}}{t} \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} c(N-2)\alpha_N \int_0^{Rt} \frac{r^{N-1}}{r^{N-2}}\, dr \\ &= 2^{N-2} \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} c(N-2)\alpha_N R^2 t, \end{aligned}$$

where we used the fact that $B(\boldsymbol{x}, Rt) \subseteq B(\boldsymbol{x} + t\boldsymbol{e}_j, 2Rt)$. Hence,

$$\lim_{t \to 0^+} III_1 = 0.$$

On the other hand, by the fact that $\frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j) = -\frac{\partial \Gamma}{\partial y_i}(\boldsymbol{x} - \boldsymbol{y} + t\boldsymbol{e}_j)$ and integration by parts

$$\begin{aligned}
&\int_{B(\boldsymbol{x},Rt)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{x}-\boldsymbol{y}+t\boldsymbol{e}_j)\, d\boldsymbol{y} = -\int_{\partial B(\boldsymbol{x},Rt)} \Gamma(\boldsymbol{x}-\boldsymbol{y}+t\boldsymbol{e}_j)\nu_i \, d\mathcal{H}^{N-1}(\boldsymbol{y}) \\
&= -(Rt)^{N-1}\int_{\partial B(\mathbf{0},1)} \Gamma(Rt\boldsymbol{\xi}+t\boldsymbol{e}_j)\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\
&= -\frac{(Rt)^{N-1}}{(Rt)^{N-2}}\int_{\partial B(\mathbf{0},1)} \Gamma(\boldsymbol{\xi}+R^{-1}\boldsymbol{e}_j)\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\
&= -Rt\int_{\partial B(\mathbf{0},1)} (\Gamma(\boldsymbol{\xi}+R^{-1}\boldsymbol{e}_j)-\Gamma(\boldsymbol{\xi}))\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}),
\end{aligned}$$

where the last identity follows from the fact that $\int_{\partial B(\mathbf{0},1)} \Gamma(\boldsymbol{\xi}))\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) = 0$. Hence, by the Lebesgue dominated convergence theorem

$$\begin{aligned}
III_2 &= -f(\boldsymbol{x}+t\boldsymbol{e}_j)\int_{\partial B(\mathbf{0},1)} \frac{\Gamma(\boldsymbol{\xi}+R^{-1}\boldsymbol{e}_j)-\Gamma(\boldsymbol{\xi})}{R^{-1}}\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\
&\to -f(\boldsymbol{x})\int_{\partial B(\mathbf{0},1)} \frac{\partial \Gamma}{\partial x_i}(\boldsymbol{\xi})\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\
&= cf(\boldsymbol{x})(N-2)\int_{\partial B(\mathbf{0},1)} \xi_j\xi_i \, d\mathcal{H}^{N-1}(\boldsymbol{\xi})
\end{aligned}$$

as $t \to 0$ followed by $R \to \infty$.
Step 4: We claim that there exists

$$\lim_{R\to\infty}\lim_{t\to 0^+} IV_{t,R}(f)(\boldsymbol{x}) = cf(\boldsymbol{x})(2-N)\int_{\partial B(\mathbf{0},1)} y_j y_i \, d\mathcal{H}^{N-1}(\boldsymbol{y}).$$

The proof is similar to the one of Step 3 and we omit it.

The conclusion follows by combining Steps 1–3. ■

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In view of (71) it follows that u solves the Poisson equation

$$\Delta u = f \quad \text{in } \mathbb{R}^N. \tag{77}$$

To be precise, we have the following result.

Theorem 89 *Let $f \in \mathcal{S}\left(\mathbb{R}^N\right)$, $N \geq 3$, and let u be the function in* (67)*, where*

$$c := \frac{1}{2(N-2)N\beta_N}. \tag{78}$$

Then u solves (77).

Proof. By (71) and Theorem 87 it follows that

$$\begin{aligned}\Delta u(\boldsymbol{x}) &= cf(\boldsymbol{x})(N-2)\sum_{i=1}^{N}\int_{\partial B(\boldsymbol{0},1)} y_i^2 \, d\mathcal{H}^{N-1}(\boldsymbol{y}) \\ &\quad + \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\varepsilon)} \Delta\Gamma(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y}) \, d\boldsymbol{y} \\ &= cf(\boldsymbol{x})(N-2)N\beta_N + 0 = f(\boldsymbol{x}),\end{aligned}$$

where $\beta_N := \mathcal{H}^{N-1}(\partial B(\boldsymbol{0},1))$. ■

Using the theory of singular integrals we are going to show that the operator

$$T^{\#}(f)(\boldsymbol{x}) := \sup_{\varepsilon>0}\left|\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\varepsilon)} \frac{\partial^2\Gamma}{\partial x_j\partial x_i}(\boldsymbol{x}-\boldsymbol{y})f(\boldsymbol{y}) \, d\boldsymbol{y}\right|$$

is of strong type (p,p) for all $1 < p < \infty$. As a consequence, we will get (interior) regularity for the Poisson equation.

Note that up to a multiplicative constant u is nothing else than the Riesz potential $I_2(f)$, and thus by Corollary 77, for all $1 < p < N/2$, and all $f \in L^p(\mathbb{R}^N)$, we have that

$$\|u\|_{L^r} \le c(N,\alpha,p)\,\|f\|_{L^p},$$

where

$$r := \frac{Np}{N-2p}.$$

Theorem 90 (L^p regularity) *Let $f \in L^p(\mathbb{R}^N)$, $1 < p < N/2$, $N \ge 3$. Then there exists a unique weak solution u of* (77)*. Moreover u belongs to $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ with*

$$\left\|\frac{\partial^2 u}{\partial x_j\partial x_i}\right\|_{L^p} \le C\,\|f\|_{L^p}$$

for all $i,j = 1,\dots,N$ and for some constant $C = C(N,p) > 0$.

4.3 The Cauchy Integral

Let $\Omega \subseteq \mathbb{C}$ be a simply connected open set and let $f : \Omega \to \mathbb{C}$ be an holomorphic function. *Cauchy's integral formula* states that for every $\boldsymbol{z} \in \Omega$ and for every simple closed oriented rectifiable curve γ in Ω,

$$f(\boldsymbol{z}) = \frac{1}{2\pi\boldsymbol{i}}\int_\gamma \frac{f(\boldsymbol{\zeta})}{\boldsymbol{\zeta}-\boldsymbol{z}} \, d\boldsymbol{\zeta},$$

where the orientation is in the counterclockwise sense.

Assume now that γ is a simple closed oriented rectifiable curve γ in $\mathbb{C}$ or a simple oriented locally rectifiable curve through infinity (for example, the unit circle and the real axis). In what follows we identify the curve γ with its support. Given a function $f \in L^1(\gamma)$, the *Cauchy integral of* f is the operator defined

$$\mathcal{C}(f)(\boldsymbol{z}) := \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta})}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta} \tag{79}$$

for $\boldsymbol{z} \in \mathbb{C} \setminus \gamma$. Given a point $\boldsymbol{z}_0 \in \gamma$, we are interested in studying the behavior of $\mathcal{C}(f)(\boldsymbol{z})$ as $\boldsymbol{z} \in \mathbb{C} \setminus \gamma$ approaches $\boldsymbol{z}_0$. The integral is not absolutely convergent, so as usual we consider the truncated integrals

$$\mathcal{C}_\varepsilon(f)(\boldsymbol{z}) := \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma \setminus B(\boldsymbol{z},\varepsilon)} \frac{f(\boldsymbol{\zeta})}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta}, \quad \boldsymbol{z} \in \gamma, \tag{80}$$

which are absolutely convergent. We set

$$\mathcal{C}(f)(\boldsymbol{z}) := \lim_{\varepsilon \to 0^+} \mathcal{C}_\varepsilon(f)(\boldsymbol{z}), \quad \boldsymbol{z} \in \gamma, \tag{81}$$

whenever the limit exists. Hence, $\mathcal{C}(f)(\boldsymbol{z})$ is the principal value of the Cauchy integral.

Assume that f is the restriction to γ of a C^1 function defined on the entire $\mathbb{C}$. Consider the case in which γ is a simple closed oriented rectifiable curve γ in $\mathbb{C}$. It can be shown that γ sudivide $\mathbb{C} \setminus \gamma$ into two open connected sets Ω^+ and Ω^-, one bounded and one unbounded. Assume that Ω^+ is the bounded open set.

Remark 91 *In what follows we will use the fact that*

$$\frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta} = \begin{cases} 1 & \text{if } \boldsymbol{z} \in \Omega^+, \\ 0 & \text{if } \boldsymbol{z} \in \Omega^-. \end{cases}$$

To see this, let γ be parametrized by a Lipschitz function $t \in [a,b] \mapsto \boldsymbol{\zeta}(t)$, (since γ can be parametrized , this is always possible). Then

$$\int_\gamma \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta} = \int_a^b \frac{\boldsymbol{\zeta}'(t)}{\boldsymbol{\zeta}(t) - \boldsymbol{z}} \, dt.$$

Consider the function

$$g_{\boldsymbol{z}}(s) := \int_a^s \frac{\boldsymbol{\zeta}'(t)}{\boldsymbol{\zeta}(t) - \boldsymbol{z}} \, dt. \tag{82}$$

The function $g_{\boldsymbol{z}}$ is absolutely continuous as a function of s, and so for $\mathcal{L}^1$ a.e. $s \in [a,b]$,

$$g'_{\boldsymbol{z}}(s) = \frac{\boldsymbol{\zeta}'(s)}{\boldsymbol{\zeta}(s) - \boldsymbol{z}}.$$

Hence, the

$$(e^{-g_{\boldsymbol{z}}(s)}(\boldsymbol{\zeta}(s) - \boldsymbol{z}))' = -g'_{\boldsymbol{z}}(s) e^{-g_{\boldsymbol{z}}(s)}(\boldsymbol{\zeta}(s) - \boldsymbol{z}) + \boldsymbol{\zeta}'(s) e^{-g_{\boldsymbol{z}}(s)} = 0$$

$\mathcal{L}^1$ a.e. $s \in [a, b]$. Since $g_{\boldsymbol{z}}$ and $\boldsymbol{\zeta}$ are absolutely continuous, it follows that the function must be constant, $e^{-g_{\boldsymbol{z}}(s)}(\boldsymbol{\zeta}(s) - \boldsymbol{z}) = e^0(\boldsymbol{\zeta}(a) - \boldsymbol{z})$, that is,

$$e^{g_{\boldsymbol{z}}(s)} = \frac{\boldsymbol{\zeta}(s) - \boldsymbol{z}}{\boldsymbol{\zeta}(a) - \boldsymbol{z}}.$$

Since $g_{\boldsymbol{z}}(a) = g_{\boldsymbol{z}}(b)$, we get $e^{g_{\boldsymbol{z}}(b)} = 1$, which implies that $g_{\boldsymbol{z}}(b)$ is a multiple of $2\pi \boldsymbol{i}$. On the other hand, since $g_{\boldsymbol{z}}$ is continuous as a function of $\boldsymbol{z}$ in $\mathbb{C} \setminus \gamma$, we must have that $g_{\boldsymbol{z}}(b)$ must be constant in Ω^+ and in Ω^-. Letting $|\boldsymbol{z}| \to \infty$ in (82) shows that $g_{\boldsymbol{z}}(b) = 0$ for all $\boldsymbol{z} \in \Omega^-$. The fact that $g_{\boldsymbol{z}}(b) = 1$ for all $\boldsymbol{z} \in \Omega^+$ is left as an exercise.

Fix $\boldsymbol{z}_0 \in \gamma$. Let $\gamma_\varepsilon := \gamma \setminus B(\boldsymbol{z}_0, \varepsilon)$ and $\Gamma_\varepsilon := \Omega^+ \cap \partial B(\boldsymbol{z}_0, \varepsilon)$ oriented counterclockwise. Since $\boldsymbol{z}_0$ belongs to the unbounded region outside the closed curve $\gamma_\varepsilon \cup \Gamma_\varepsilon^-$, by Remark 91 we have that

$$0 = \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma_\varepsilon} \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta} - \frac{1}{2\pi \boldsymbol{i}} \int_{\Gamma_\varepsilon} \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta},$$

and so

$$\begin{aligned} \mathcal{C}_\varepsilon(f)(\boldsymbol{z}_0) &= \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma_\varepsilon} \frac{f(\boldsymbol{\zeta})}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta} \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_{\gamma_\varepsilon} \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z}_0)}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta} + f(\boldsymbol{z}_0) \frac{1}{2\pi \boldsymbol{i}} \int_{\Gamma_\varepsilon} \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta}. \end{aligned}$$

By the regularity of f, we can find a constant $L > 0$ such that $|f(\boldsymbol{\zeta}) - f(\boldsymbol{z})| \le L|\boldsymbol{\zeta} - \boldsymbol{z}|$ for all $\boldsymbol{\zeta} \in \gamma$ and all $\boldsymbol{z}$ close to $\boldsymbol{z}_0$. Hence, by the Lebesgue dominated convergence theorem the first term converges to $\frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z}_0)}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta}$. On the other hand, if γ is differentiable at $\boldsymbol{z}$ then $\frac{1}{2\pi \boldsymbol{i}} \int_{\Gamma_\varepsilon} \frac{1}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta} \to \frac{1}{2}$. Hence,

$$\begin{aligned} \mathcal{C}(f)(\boldsymbol{z}_0) &= \lim_{\varepsilon \to 0^+} \mathcal{C}_\varepsilon(f)(\boldsymbol{z}_0) \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z}_0)}{\boldsymbol{\zeta} - \boldsymbol{z}_0} \, d\boldsymbol{\zeta} + \frac{f(\boldsymbol{z}_0)}{2}. \end{aligned} \tag{83}$$

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On the other hand, again by Remark 91, for $\boldsymbol{z} \in \Omega^+$ we can write

$$\mathcal{C}(f)(\boldsymbol{z}) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z})}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta} + f(\boldsymbol{z}),$$

while for for $\boldsymbol{z} \in \Omega^-$,

$$\mathcal{C}(f)(\boldsymbol{z}) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z})}{\boldsymbol{\zeta} - \boldsymbol{z}} \, d\boldsymbol{\zeta}.$$

Hence, using Lebesgue dominated convergence theorem,

$$\mathcal{C}^+(f)(\boldsymbol{z}_0) := \lim_{\boldsymbol{z}\to\boldsymbol{z}_0,\, \boldsymbol{z}\in\Omega^+} \mathcal{C}(f)(\boldsymbol{z}) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z}_0)}{\boldsymbol{\zeta} - \boldsymbol{z}_0}\, d\boldsymbol{\zeta} + f(\boldsymbol{z}_0),$$
$$\mathcal{C}^-(f)(\boldsymbol{z}_0) := \lim_{\boldsymbol{z}\to\boldsymbol{z}_0,\, \boldsymbol{z}\in\Omega^-} \mathcal{C}(f)(\boldsymbol{z}) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\boldsymbol{\zeta}) - f(\boldsymbol{z}_0)}{\boldsymbol{\zeta} - \boldsymbol{z}_0}\, d\boldsymbol{\zeta}.$$

Comparing these two expressions with (83) we obtain the Sokhotskyi–Plemelj formula

$$\begin{cases} \mathcal{C}^+(f)(\boldsymbol{z}_0) = \mathcal{C}(f)(\boldsymbol{z}_0) + \frac{f(\boldsymbol{z}_0)}{2}, \\ \mathcal{C}^-(f)(\boldsymbol{z}_0) = \mathcal{C}(f)(\boldsymbol{z}_0) - \frac{f(\boldsymbol{z}_0)}{2}, \end{cases} \tag{84}$$

which hold at all $\boldsymbol{z}_0 \in \gamma$ for which γ has a tangent at $\boldsymbol{z}_0$. Since γ is rectifiable, this happens for $\mathcal{H}^1$ a.e. $\boldsymbol{z}_0 \in \gamma$.

If we now drop the assumption that f is smooth and only require $f \in L^1(\gamma)$, then testing with some simple cases (exercise) shows that the way $\boldsymbol{z} \in \mathbb{C} \setminus \gamma$ approaches $\boldsymbol{z}_0 \in \gamma$ is important. If one wants limits to exist, only non-tangential limits are allowed. Let $\boldsymbol{z}_0 \in \gamma$ be such that γ has a tangent at $\boldsymbol{z}_0$. Then we can find two cones $C^-(\boldsymbol{z}_0)$ and $C^+(\boldsymbol{z}_0)$ with vertex at $\boldsymbol{z}_0$ and contained (except for the vertex) in Ω^- and in Ω^+, respectively. Then we define the non-tangential limits

$$\mathcal{C}^+(f)(\boldsymbol{z}_0) := \lim_{\boldsymbol{z}\to\boldsymbol{z}_0,\, \boldsymbol{z}\in C^+(\boldsymbol{z}_0)} \mathcal{C}(f)(\boldsymbol{z}),$$
$$\mathcal{C}^-(f)(\boldsymbol{z}_0) := \lim_{\boldsymbol{z}\to\boldsymbol{z}_0,\, \boldsymbol{z}\in C^-(\boldsymbol{z}_0)} \mathcal{C}(f)(\boldsymbol{z}),$$

whenever they exist. In 1950 Privalov proved the following fundamental result:

Theorem 92 (Privalov) *Let γ be a simple closed oriented rectifiable curve γ in $\mathbb{C}$ and let $f \in L^1(\gamma)$. Then for $\mathcal{H}^1$ a.e. $\boldsymbol{z} \in \gamma$, the non-tangential limits $\mathcal{C}^+(f)(\boldsymbol{z})$ and $\mathcal{C}^-(f)(\boldsymbol{z})$ exist if and only if the principal value of the Cauchy integral $\mathcal{C}(f)(\boldsymbol{z})$ exists. Moreover, in this case, the Sokhotskyi–Plemelj formula hold for $\mathcal{H}^1$ a.e. $\boldsymbol{z} \in \gamma$.*

Thus, once again we are interested in the existence of $\mathcal{C}(f)(\boldsymbol{z})$ for $\mathcal{H}^1$ a.e. $\boldsymbol{z} \in \gamma$.

Note that if γ is a simple closed oriented rectifiable curve γ in $\mathbb{C}$ or a simple oriented locally rectifiable curve through infinity, then we can parametrize γ with a Lipschitz function $t \in I \mapsto \boldsymbol{\zeta}(t)$, where I is an interval. Hence, we can rewrite (79) as

$$\begin{aligned} \mathcal{C}(f)(\boldsymbol{z}) &= \frac{1}{2\pi \boldsymbol{i}} \int_I \frac{f(\boldsymbol{\zeta}(t))\boldsymbol{\zeta}'(t)}{\boldsymbol{\zeta}(t) - \boldsymbol{z}}\, dt = \frac{1}{2\pi \boldsymbol{i}} \int_I \frac{1}{\boldsymbol{\zeta}(t) - \boldsymbol{z}} g(t)\, dt \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_I \frac{\overline{\boldsymbol{\zeta}(t) - \boldsymbol{z}}}{|\boldsymbol{\zeta}(t) - \boldsymbol{z}|^2} g(t)\, dt \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_I \frac{(x(t) - x) - \boldsymbol{i}(y(t) - y)}{(x(t) - x)^2 + (y(t) - y)^2} g(t)\, dt \end{aligned}$$

for $\boldsymbol{z} = x + \boldsymbol{i}y \in \mathbb{C} \setminus \gamma$ and $\boldsymbol{\zeta}(t) = x(t) + \boldsymbol{i}y(t)$, $t \in I$, and where $g(t) := f(\boldsymbol{\zeta}(t))\boldsymbol{\zeta}'(t)$. Hence, this important singular operator is not of convolution type.

5 Singular Integrals

Singular integrals operators are defined, at least formally, as integrals

$$T(f)(\boldsymbol{x}) = \int K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y},$$

where K is the kernel which is singular in some way. We are interested in finding hypotheses on the kernel K that will guarantee that T is of strong type (p, p) for all $1 < p < \infty$.

Let

$$D := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2N} : \ \boldsymbol{x} = \boldsymbol{y}\}.$$

Definition 93 *We will say that $K : \mathbb{R}^{2N} \setminus D \to \mathbb{C}$ is a* standard kernel *if $K \in C^1(\mathbb{R}^{2N} \setminus D)$ and*

$$|K(\boldsymbol{x}, \boldsymbol{y})| \le \frac{C_0}{|\boldsymbol{x} - \boldsymbol{y}|^N}, \tag{85}$$

$$|K(\boldsymbol{x}, \boldsymbol{y}_1) - K(\boldsymbol{x}, \boldsymbol{y})| \le \frac{C_0 |\boldsymbol{y}_1 - \boldsymbol{y}|^\alpha}{|\boldsymbol{x} - \boldsymbol{y}|^{N+\alpha}} \quad \text{if } |\boldsymbol{y}_1 - \boldsymbol{y}| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|, \tag{86}$$

$$|K(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}_1, \boldsymbol{y})| \le \frac{C_0 |\boldsymbol{x}_1 - \boldsymbol{x}|^\alpha}{|\boldsymbol{x} - \boldsymbol{y}|^{N+\alpha}} \quad \text{if } |\boldsymbol{x}_1 - \boldsymbol{x}| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| \tag{87}$$

for all $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{y}_1), (\boldsymbol{x}_1, \boldsymbol{y}) \in \mathbb{R}^{2N} \setminus D$ and for some constant $C_0 > 0$ and some $\alpha \in (0, 1]$.

Example 94 *Assume that $K \in C^1(\mathbb{R}^{2N} \setminus D)$ satisfies (85) and*

$$|\nabla_{\boldsymbol{x}} K(\boldsymbol{x}, \boldsymbol{y})| + |\nabla_{\boldsymbol{y}} K(\boldsymbol{x}, \boldsymbol{y})| \le \frac{C}{|\boldsymbol{x} - \boldsymbol{y}|^{N+1}} \tag{88}$$

for all $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2N} \setminus D$ and for some constant $C > 0$. Then K is a standard kernel. To see this, let $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{y}_1) \in \mathbb{R}^{2N} \setminus D$ be such that $|\boldsymbol{y}_1 - \boldsymbol{y}| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|$. By the mean value theorem

$$K(\boldsymbol{x}, \boldsymbol{y}_1) - K(\boldsymbol{x}, \boldsymbol{y}) = \nabla_{\boldsymbol{y}} K(\boldsymbol{x}, \boldsymbol{\theta}) \cdot (\boldsymbol{y}_1 - \boldsymbol{y})$$

for some $\boldsymbol{\theta}$ in the segment of endpoints $\boldsymbol{y}_1$ and $\boldsymbol{y}$. Hence, by (88),

$$|K(\boldsymbol{x}, \boldsymbol{y}_1) - K(\boldsymbol{x}, \boldsymbol{y})| \le \frac{C}{|\boldsymbol{x} - \boldsymbol{\theta}|^{N+1}} |\boldsymbol{y} - \boldsymbol{y}_1|.$$

Since $|\boldsymbol{y}_1 - \boldsymbol{y}| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|$, then

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{\theta}| &\ge |\boldsymbol{x} - \boldsymbol{y}| - |\boldsymbol{\theta} - \boldsymbol{y}| \ge \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| + \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| - |\boldsymbol{y}_1 - \boldsymbol{y}| \\ &\ge \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|. \end{aligned} \tag{89}$$

It follows that

$$|K(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y}_1)| \le \frac{2^{N+1} C |\boldsymbol{y} - \boldsymbol{y}_1|}{|\boldsymbol{x} - \boldsymbol{y}|^{N+1}}$$

and so (86) holds. Similarly, we can show that (87) is satisfied.

Monday, February 23, 2015

The growth condition (85) is what makes the operator T singular. The hypotheses (85), (86), and (87) are not enough. In all the examples we have seen there was always some cancellation hypothesis.

Indeed, the kernel $K(\boldsymbol{x}, \boldsymbol{y}) = 1/|\boldsymbol{x} - \boldsymbol{y}|^N$ satisfies both hypotheses, but the corresponding operator T cannot map L^2 into L^2, even if we consider some kind of principal value. We will come back to the cancellation hypothesis later on.

5.1 Truncated Singular Operators

In the three examples of singular integrals in Section 4 we have seen that the *truncated operator*

$$T_\varepsilon(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} \tag{90}$$

plays an important role. Note that $T_\varepsilon(f)(\boldsymbol{x})$ is well-defined for a function $f \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$. Indeed, if $1 < p < \infty$, by Hölder's inequality and (85),

$$\begin{aligned}
|T_\varepsilon(f)(\boldsymbol{x})| &\le \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} |K(\boldsymbol{x}, \boldsymbol{y})| |f(\boldsymbol{y})| \, d\boldsymbol{y} \\
&\le C_0 \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^N} \, d\boldsymbol{y} \le C_0 \|f\|_{L^p} \left(\int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{Np'}} \, d\boldsymbol{y} \right)^{1/p'} \\
&= C_0 \|f\|_{L^p} \left(\alpha_N \int_\varepsilon^\infty \frac{r^{N-1}}{r^{Np'}} \, dr \right)^{1/p'} = C_0 \|f\|_{L^p} \alpha_N^{1/p'} \frac{1}{N - Np'} \left[r^{N - Np'} \right]_\varepsilon^\infty \\
&= \|f\|_{L^p} \frac{C_0 \alpha_N^{1/p'}}{Np' - N} \varepsilon^{N - Np'},
\end{aligned}$$

where we have used the fact that $p' > 1$. On the other hand, if $f \in L^1(\mathbb{R}^N)$,

$$\begin{aligned}
|T_\varepsilon(f)(\boldsymbol{x})| &\le \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} |K(\boldsymbol{x}, \boldsymbol{y})| |f(\boldsymbol{y})| \, d\boldsymbol{y} \\
&\le C_0 \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^N} \, d\boldsymbol{y} \\
&\le \frac{C}{\varepsilon^N} \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)} |f(\boldsymbol{y})| \, d\boldsymbol{y} \le \frac{C}{\varepsilon^N} \|f\|_{L^1}.
\end{aligned}$$

The truncated kernel

$$K_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) := K(\boldsymbol{x}, \boldsymbol{y}) \chi_{\mathbb{R}^N \setminus B(\boldsymbol{x}, \varepsilon)}(\boldsymbol{y}) \tag{91}$$

continues to satisfy hypothesis (85) but it is no longer smooth, so it cannot satisfy (86) and (87). To circumvent this problem we will introduce a smooth truncation.

Construct a function φ such that

$$\begin{gathered}\varphi \in C_c^\infty(\mathbb{R}^N), \quad 0 \le \varphi \le 1, \quad \varphi \text{ even}, \\ \varphi(\boldsymbol{x}) = 1 \text{ if } |\boldsymbol{x}| \ge 2, \quad \varphi(\boldsymbol{x}) = 0 \text{ if } |\boldsymbol{x}| \le 1.\end{gathered} \tag{92}$$

Consider the smooth kernel

$$K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) := K(\boldsymbol{x}, \boldsymbol{y})\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) \tag{93}$$

Proposition 95 *Let K be a standard kernel and let φ be as in (92). Then K_{φ_ε} satisfies (85), (86), and (87) with C_0 replaced by*

$$C_\varphi := C_0(1 + 2^{2-\alpha} \max_{1 \le |\boldsymbol{x}| \le 2} |\nabla\varphi(\boldsymbol{x})|).$$

Proof. Since $0 \le \varphi \le 1$, by (85),

$$|K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y})| \le |K(\boldsymbol{x}, \boldsymbol{y})| \le \frac{C_0}{|\boldsymbol{x}-\boldsymbol{y}|^N}.$$

Let $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{y}_1) \in \mathbb{R}^{2N} \setminus D$ be such that $|\boldsymbol{y}_1 - \boldsymbol{y}| \le \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|$. Then

$$\begin{aligned}K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}_1) = K(\boldsymbol{x}, \boldsymbol{y}) &\left[\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) - \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}_1}{\varepsilon}\right)\right] \\ &+ \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}_1}{\varepsilon}\right)(K(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{y}_1)).\end{aligned}$$

By the mean value theorem

$$\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) - \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}_1}{\varepsilon}\right) = \frac{1}{\varepsilon}\nabla_{\boldsymbol{x}}\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{\theta}}{\varepsilon}\right) \cdot (\boldsymbol{y}_1 - \boldsymbol{y})$$

for some $\boldsymbol{\theta}$ in the segment of endpoints $\boldsymbol{y}_1$ and $\boldsymbol{y}$. Note that by (92), $\nabla_{\boldsymbol{x}}\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{\theta}}{\varepsilon}\right) = 0$ for $|\boldsymbol{x}-\boldsymbol{\theta}|/\varepsilon \le 1$ and $|\boldsymbol{x}-\boldsymbol{\theta}|/\varepsilon \ge 2$, and so

$$\begin{aligned}\left|\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) - \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}_1}{\varepsilon}\right)\right| &\le \frac{1}{\varepsilon}\left|\nabla_{\boldsymbol{x}}\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\right||\boldsymbol{y}_1 - \boldsymbol{y}| \\ &\le \frac{|\boldsymbol{y}_1 - \boldsymbol{y}|}{|\boldsymbol{x}-\boldsymbol{\theta}|}\max_{1 \le |\boldsymbol{\xi}| \le 2}|\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{\xi})| \\ &\le \frac{|\boldsymbol{y}_1 - \boldsymbol{y}|^\alpha|\boldsymbol{x}-\boldsymbol{y}|^{1-\alpha}}{|\boldsymbol{x}-\boldsymbol{y}|}2^{2-\alpha}\max_{1 \le |\boldsymbol{\xi}| \le 2}|\nabla_{\boldsymbol{x}}\varphi(\boldsymbol{\xi})|.\end{aligned}$$

where we have used (89) and the fact that $|\boldsymbol{y}_1 - \boldsymbol{y}| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|$. Hence, by (86), and the fact that $0 \le \varphi \le 1$,

$$\begin{aligned}|K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}_1)| \le{}& \frac{C_0}{|\boldsymbol{x} - \boldsymbol{y}|^N} \frac{|\boldsymbol{y}_1 - \boldsymbol{y}|^\alpha}{|\boldsymbol{x} - \boldsymbol{y}|^\alpha} 2^{2-\alpha} \max_{1 \le |\boldsymbol{\xi}| \le 2} |\nabla_{\boldsymbol{x}} \varphi(\boldsymbol{\xi})| . \\ &+ \frac{C_0 |\boldsymbol{y}_1 - \boldsymbol{y}|^\alpha}{|\boldsymbol{x} - \boldsymbol{y}|^{N+\alpha}}.\end{aligned}$$

Similarly, we can show that (87) is satisfied. ■

Consider the *smoothly truncated operator*

$$T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N} K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y}. \tag{94}$$

Note that $T_{\varphi_\varepsilon}(f)(\boldsymbol{x})$ is well-defined for all $f \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$.

Theorem 96 *Let K be a standard kernel and let φ be as in (92). Then for all $f \in L^p(\mathbb{R}^N)$, $1 \le p < \infty$,*

$$|T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) - T_\varepsilon(f)(\boldsymbol{x})| \le C(N) \, \mathrm{M}(f)(\boldsymbol{x}) .$$

In particular, if T_{φ_ε} (or T_ε) is of weak $(1,1)$ type and/or of strong (p,p) type, $1 < p < \infty$, then so is T_ε (or T_{φ_ε}).

Proof. We have

$$T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) - T_\varepsilon(f)(\boldsymbol{x}) = \int_{\mathbb{R}^N} K(\boldsymbol{x}, \boldsymbol{y}) \left(\chi_{\mathbb{R}^N \setminus B(\boldsymbol{x},\varepsilon)}(\boldsymbol{y}) - \varphi\left(\frac{\boldsymbol{x} - \boldsymbol{y}}{\varepsilon}\right)\right) f(\boldsymbol{y}) \, d\boldsymbol{y}.$$

By (92),

$$\chi_{\mathbb{R}^N \setminus B(\boldsymbol{x},\varepsilon)}(\boldsymbol{y}) - \varphi\left(\frac{\boldsymbol{x} - \boldsymbol{y}}{\varepsilon}\right) = 0$$

if $|\boldsymbol{x} - \boldsymbol{y}| \le \varepsilon$ and $|\boldsymbol{x} - \boldsymbol{y}| \ge 2\varepsilon$, and so by (85),

$$\begin{aligned}|T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) - T_\varepsilon(f)(\boldsymbol{x})| &\le \int_{B(\boldsymbol{x},2\varepsilon) \setminus B(\boldsymbol{x},\varepsilon)} \frac{C_0}{|\boldsymbol{x} - \boldsymbol{y}|^N} |f(\boldsymbol{y})| \, d\boldsymbol{y} \\ &\le \frac{C_0}{\varepsilon^N} \int_{B(\boldsymbol{x},2\varepsilon)} |f(\boldsymbol{y})| \, d\boldsymbol{y} \le \alpha_N 2^N C_0 \, \mathrm{M}(f)(\boldsymbol{x}) .\end{aligned}$$

The second statement follows from Theorem 33. ■

Theorem 97 (Calderón–Zygmund) *Let $f \in L^1(\mathbb{R}^N)$ be a nonnegative function and let $t > 0$. Then there exists a countable family $\{Q_n\}$ of open mutually disjoint cubes such that*

$$f(\boldsymbol{x}) \le t \quad \textit{for } \mathcal{L}^N \textit{ a.e. } \boldsymbol{x} \in \mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}, \tag{95}$$

and for every $n \in \mathbb{N}$,

$$t < f_{Q_n} \le 2^N t. \tag{96}$$

Proof. Fix $t > 0$ and choose $L > 0$ large enough that

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\, d\boldsymbol{x} \leq tL^N.$$

Decompose $\mathbb{R}^N$ into a rectangular grid such that each cube Q of the partition has side length L, and thus

$$f_Q \leq t. \tag{97}$$

Fix one such cube Q and subdivide it into 2^N congruent subcubes. Let Q' be one of these subcubes. If

$$f_{Q'} > t,$$

and in view of the fact that

$$f_{Q'} = \frac{1}{\mathcal{L}^N(Q')} \int_{Q'} f(\boldsymbol{x})\, d\boldsymbol{x} \leq \frac{2^N}{\mathcal{L}^N(Q)} \int_Q f(\boldsymbol{x})\, d\boldsymbol{x} = 2^N f_Q \leq 2^N t,$$

then (96) is satisfied and therefore Q' will be selected as one of the Q_n. On the other hand, if

$$f_{Q'} \leq t$$

(note that by (97) there is at least one), then we subdivide Q' into 2^N congruent subcubes and we repeat the process.

In this way we construct a family of cubes $\{Q_n\}$ for which (96) is satisfied, and it remains to prove (95). It can be seen from the construction that the cubes that were not selected to belong to the family $\{Q_n\}$ form a fine covering $\mathcal{F}$ of $\mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \overline{Q_n}$. Therefore if $x \in \mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \overline{Q_n}$ is a Lebesgue point for f, then by properties of Lebesgue points, we have

$$f(\boldsymbol{x}) = \limsup_{\operatorname{diam} F \to 0,\ F \in \mathcal{F},\ \boldsymbol{x} \in F} \frac{1}{\mathcal{L}^N(F)} \int_F f(\boldsymbol{y})\, d\boldsymbol{y} \leq t$$

and the proof is completed. ■

Remark 98 *A local version of the previous theorem holds. Precisely, if Q is a cube, $f \in L^1(Q)$ is nonnegative, and if*

$$t \geq f_Q,$$

then it follows from the above proof that there exists a countable family $\{Q_n\} \subset Q$ of open mutually disjoint cubes such that

$$f(\boldsymbol{x}) \leq t \quad \text{for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x} \in Q \setminus \bigcup_{n=1}^{\infty} \overline{Q_n},$$

and for every $n \in \mathbb{N}$,

$$t < f_{Q_n} \leq 2^N t.$$

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Theorem 99 (Calderón–Zygmund decomposition) *Let $f \in L^1\left(\mathbb{R}^N\right)$ be a nonnegative function and let $t > 0$. Then we can decompose*

$$f = g + h,$$

where

$$g(\boldsymbol{x}) := \begin{cases} f_{Q_n} & \text{if } \boldsymbol{x} \in \overline{Q_n} \text{ for some } n, \\ f(\boldsymbol{x}) & \text{otherwise.} \end{cases} \qquad h(\boldsymbol{x}) := f(\boldsymbol{x}) - g(\boldsymbol{x}).$$

satisfy

$$\int_{\mathbb{R}^N} |g| \, d\boldsymbol{x} = \int_{\mathbb{R}^N} f \, d\boldsymbol{x}, \quad \int_{\mathbb{R}^N} g^2 \, d\boldsymbol{x} \le 2^N t \int_{\mathbb{R}^N} f \, d\boldsymbol{x}, \tag{98}$$

and

$$h = 0 \text{ on } \mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \overline{Q_n}, \quad \int_{Q_n} h \, d\boldsymbol{x} = 0, \quad \int_{\mathbb{R}^N} |h| \, d\boldsymbol{x} \le 2 \int_{\mathbb{R}^N} f \, d\boldsymbol{x}, \tag{99}$$

where $\{Q_n\}$ is the family of open mutually disjoint cubes given in Theorem 97.

Proof. By the definition of g,

$$\int_{\mathbb{R}^N} g^2 \, d\boldsymbol{x} = \int_{\bigcup_{n=1}^\infty \overline{Q_n}} g^2 \, d\boldsymbol{x} + \int_{\mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}} f^2 \, d\boldsymbol{x}.$$

By (95), $0 \le f \le t$ on $\bigcup_{n=1}^\infty \overline{Q_n}$ and so

$$\int_{\mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}} f^2 \, d\boldsymbol{x} \le t \int_{\mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}} f \, d\boldsymbol{x}.$$

On the other hand, since the interiors of the cubes are pairwise disjoint and $f_{Q_n} \le 2^N t$ by (96)

$$\begin{aligned} \int_{\bigcup_{n=1}^\infty \overline{Q_n}} g^2 \, d\boldsymbol{x} &= \sum_{n=1}^{\infty} \int_{\overline{Q_n}} f_{Q_n}^2 \, d\boldsymbol{x} \le 2^N t \sum_{n=1}^{\infty} \int_{\overline{Q_n}} f_{Q_n} \, d\boldsymbol{x} \\ &= 2^N t \sum_{n=1}^{\infty} \int_{Q_n} f \, d\boldsymbol{x} = 2^N t \int_{\bigcup_{n=1}^\infty \overline{Q_n}} f \, d\boldsymbol{x}. \end{aligned}$$

Hence (98) holds.

To prove (99) observe that $\mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}$ we have that $h = f - g = f - f = 0$, while

$$\int_{Q_n} h \, d\boldsymbol{x} = \int_{Q_n} f \, d\boldsymbol{x} - \int_{Q_n} g \, d\boldsymbol{x} = \int_{Q_n} f \, d\boldsymbol{x} - \int_{Q_n} f_{Q_n} \, d\boldsymbol{x} = 0.$$

Finally, since

$$\begin{aligned}\int_{\mathbb{R}^N} g\ d\boldsymbol{x} &= \int_{\bigcup_{n=1}^\infty \overline{Q_n}} g\ d\boldsymbol{x} + \int_{\mathbb{R}^N\setminus\bigcup_{n=1}^\infty \overline{Q_n}} f\ d\boldsymbol{x} \\ &= \sum_{n=1}^\infty \int_{\overline{Q_n}} f_{Q_n}\ d\boldsymbol{x} + \int_{\mathbb{R}^N\setminus\bigcup_{n=1}^\infty \overline{Q_n}} f\ d\boldsymbol{x} \\ &= \int_{\bigcup_{n=1}^\infty \overline{Q_n}} f\ d\boldsymbol{x} + \int_{\mathbb{R}^N\setminus\bigcup_{n=1}^\infty \overline{Q_n}} f\ d\boldsymbol{x},\end{aligned}$$

we have

$$\int_{\mathbb{R}^N} |h|\ d\boldsymbol{x} \le \int_{\mathbb{R}^N} f\ d\boldsymbol{x} + \int_{\mathbb{R}^N} g\ d\boldsymbol{x} = 2\int_{\mathbb{R}^N} f\ d\boldsymbol{x}.$$

This concludes the proof. ■

The function g is called the *good function*, while the function h is called the *bad function*.

In all the theorems in this section we will assume that T_{φ_ε} satisfies a uniform $(2,2)$ estimate, that is, that there exists a constant $C_1 > 0$ such that

$$\|T_{\varphi_\varepsilon}(f)\|_{L^2} \le C_1 \|f\|_{L^2} \tag{100}$$

for all $f \in L^2(\mathbb{R}^N)$ and all $\varepsilon > 0$. We will see that this hypothesis is enough to guarantee that T_{φ_ε} is of uniform type (p,p) for all $1 < p < \infty$ and of uniform weak type $(1,1)$.

Using the Calderón–Zygmund decomposition we can prove that the operator T_{φ_ε} is of weak type $(1,1)$.

Theorem 100 (Weak (1,1) estimate) *Let K be a standard kernel and let φ be as in (92). If T_{φ_ε} satisfies (100), then there exists a constant $C = C(N) > 0$ such that*

$$t\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N :\ |T_{\varphi_\varepsilon}(f)(\boldsymbol{x})| > t\}) \le C \max\{C_\varphi, C_1\} \|f\|_{L^1}$$

for all $\varepsilon, t > 0$ and for all $f \in L^1(\mathbb{R}^N)$.

Proof. Given $f \in L^1(\mathbb{R}^N)$, we can write

$$\begin{aligned} f &= (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + \boldsymbol{i}(\operatorname{Im} f)^+ - \boldsymbol{i}(\operatorname{Im} f)^- \\ &=: f_1 - f_2 + \boldsymbol{i}(f_3 - f_4).\end{aligned}$$

Since T_{φ_ε} is linear, we have

$$|T_{\varphi_\varepsilon}(f)(\boldsymbol{x})| \le \sum_{n=1}^4 |T_{\varphi_\varepsilon}(f_n)(\boldsymbol{x})|,$$

and so for every $t > 0$,

$$\{\boldsymbol{x} \in \mathbb{R}^N :\ |T_{\varphi_\varepsilon}(f)(\boldsymbol{x})| > t\} \subset \bigcup_{n=1}^4 \{\boldsymbol{x} \in \mathbb{R}^N :\ |T_{\varphi_\varepsilon}(f_n)(\boldsymbol{x})| > t/4\}.$$

Hence, it is enough to prove a weak $(1,1)$ estimate for a nonnegative function. In what follows we will assume that $f \geq 0$.

We apply Theorem 97 to f to obtain a family $\{Q_n\}$ of open mutually disjoint cubes for which (95) and (96) hold. Let $\boldsymbol{x}_n$ and r_n be the center and side-length of Q_n, so that $Q_n = Q(\boldsymbol{x}_n, r_n)$, and define $mQ_n := Q(\boldsymbol{x}_n, mr_n)$, where $m \geq 3$ is to be determined. Let

$$D := \bigcup_{n=1}^{\infty} \overline{Q_n}, \quad D^* := \bigcup_{n=1}^{\infty} \overline{mQ_n}. \tag{101}$$

Note that

$$\mathcal{L}^N(D^*) \leq \sum_{n=1}^{\infty} \mathcal{L}^N(\overline{mQ_n}) = c(N)^N \sum_{n=1}^{\infty} \mathcal{L}^N(Q_n). \tag{102}$$

Hence,

$$\begin{aligned} &\mathcal{L}^N(\{x \in \mathbb{R}^N : \ |T_{\varphi_\varepsilon}(f)(x)| > t\} \cap D^*) \leq \mathcal{L}^N(D^*) \\ &\leq c(N)^N \sum_{n=1}^{\infty} \frac{t}{t} \mathcal{L}^N(Q_n) \leq \frac{c(N)^N}{t} \sum_{n=1}^{\infty} \int_{Q_n} f \, d\boldsymbol{x} \\ &= \frac{c(N)^N}{t} \int_D f \, d\boldsymbol{x}, \end{aligned} \tag{103}$$

where we used the fact that $t\mathcal{L}^N(Q_n) < \int_{Q_n} f \, d\boldsymbol{x}$ by (96).

Thus, it remains to estimate $\mathbb{R}^N \setminus D^*$. Apply Theorem 99 to write $f = g + h$, where g and h satisfy (98) and (99) hold. Since T_{φ_ε} is linear, we have

$$|T_{\varphi_\varepsilon}(f)(\boldsymbol{x})| \leq |T_{\varphi_\varepsilon}(g)(\boldsymbol{x})| + |T_{\varphi_\varepsilon}(g)(\boldsymbol{x})|, \tag{104}$$

and so for every $t > 0$,

$$\begin{aligned} &\{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ |T_{\varphi_\varepsilon}(f)(\boldsymbol{x})| > t\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^N : \ |T_{\varphi_\varepsilon}(g)(\boldsymbol{x})| > t/2\} \\ &\cup \{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ |T_{\varphi_\varepsilon}(h)(\boldsymbol{x})| > t/2\}. \end{aligned}$$

In view of and (98) and (100),

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ |T_{\varphi_\varepsilon}(g)(\boldsymbol{x})| > t/2\}) &\leq \frac{4}{t^2} \int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(g)|^2 \, d\boldsymbol{x} \\ &\leq \frac{4C_1}{t^2} \int_{\mathbb{R}^N} g^2 \, d\boldsymbol{x} \leq \frac{2^{N+2} C_1 t}{t^2} \int_{\mathbb{R}^N} f \, d\boldsymbol{x}. \end{aligned} \tag{105}$$

On the other hand, since $h = \sum_{n=1}^{\infty} h\chi_{Q_n}$ by (98), using the linearity of T_{φ_ε} we have

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ |T_{\varphi_\varepsilon}(h)(\boldsymbol{x})| > t/2\}) &\leq \frac{2}{t} \int_{\mathbb{R}^N \setminus D^*} |T_{\varphi_\varepsilon}(h)| \, d\boldsymbol{x} \\ &\leq \sum_{n=1}^{\infty} \frac{2}{t} \int_{\mathbb{R}^N \setminus mQ_n} |T_{\varphi_\varepsilon}(h\chi_{Q_n})| \, d\boldsymbol{x}. \end{aligned} \tag{106}$$

Since $\int_{Q_n} h\, d\boldsymbol{x} = 0$ by (98), for $\boldsymbol{x} \in \mathbb{R}^N \setminus mQ_n$ we can write

$$\begin{aligned} T_{\varphi_\varepsilon}(h\chi_{Q_n})(\boldsymbol{x}) &= \int_{Q_n} K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) h(\boldsymbol{y})\, d\boldsymbol{y} \\ &= \int_{Q_n} (K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{x}_n)) h(\boldsymbol{y})\, d\boldsymbol{y}, \end{aligned}$$

where we recall that $Q_n = Q(\boldsymbol{x}_n, r_n)$ and we have used (94). For $\boldsymbol{x} \in \mathbb{R}^N \setminus mQ_n$ and $\boldsymbol{y} \in Q_n$,

$$|\boldsymbol{y} - \boldsymbol{x}_n| \le \frac{r_n}{2}\sqrt{N} \le (m-1)\frac{r_n}{2} \le \frac{1}{2}|\boldsymbol{y} - \boldsymbol{x}| \tag{107}$$

provided $m := \lceil \sqrt{N} + 1 \rceil$. Hence, by Proposition 95,

$$|K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{x}_n)| \le \frac{C_0 |\boldsymbol{y} - \boldsymbol{x}_n|^\alpha}{|\boldsymbol{x} - \boldsymbol{y}|^{N+\alpha}} \le \frac{C_\varphi r_n^\alpha N^{\alpha/2} 2^{N+\alpha}}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \tag{108}$$

and so, since $B(\boldsymbol{x}_n, mr_n/2) \subset mQ_n = Q(\boldsymbol{x}_n, mr_n)$,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus Q_n^*} |T_{\varphi_\varepsilon}(h\chi_{Q_n})|\, d\boldsymbol{x} &\le cC_\varphi r_n^\alpha \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}_n, mr_n/2)} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}}\, d\boldsymbol{x} \int_{Q_n} |h|\, d\boldsymbol{y} \\ &\le cC_\varphi r_n^\alpha \int_{mr_n/2}^\infty \frac{1}{r^{1+\alpha}} dr \int_{Q_n} |h|\, d\boldsymbol{y} \\ &= cC_\varphi \int_{Q_n} |h|\, d\boldsymbol{y}. \end{aligned}$$

In turn,

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* :\ |T_{\varphi_\varepsilon}(h)(\boldsymbol{x})| > t/2\}) &\le \frac{2}{t} \int_{\mathbb{R}^N \setminus D^*} |T_{\varphi_\varepsilon}(h)|\, d\boldsymbol{x} \\ \le \sum_{n=1}^\infty \frac{cC_\varphi}{t} \int_{Q_n} |h|\, d\boldsymbol{y} &\le \frac{cC_\varphi}{t} \int_{\mathbb{R}^N} |f|\, d\boldsymbol{y}, \end{aligned}$$

where in the last inequality we have used (99). ■

Theorem 101 (Strong (p,p) estimate) *Let K be a standard kernel and let φ be as in (92). If T_{φ_ε} satisfies (100), then for every $1 < p < \infty$ there exists a constant $c(N,p) > 0$ such that*

$$\|T_{\varphi_\varepsilon}(f)\|_{L^p} \le c(\max\{C_\varphi, C_1\})^{2/p} \|f\|_{L^p} \tag{109}$$

for all $\varepsilon > 0$ and for all $f \in L^p(\mathbb{R}^N)$.

Proof. By Theorem 100 the linear operator T_{φ_ε} is of weak type $(1,1)$ and by hypothesis (100) is also of strong type $(2,2)$. Hence, by Theorem 22, T_{φ_ε} is of strong type (p,p) with constant

$$C_p \le C(N,p) \left[\max\{C_\varphi, C_1\}\right]^{2/p}$$

for every $1 < p < 2$.

To prove that T_{φ_ε} is of strong type (p,p) for $2 < p < \infty$, we use a duality argument. By (1) for $f \in L^p(\mathbb{R}^N)$,

$$\|T_{\varphi_\varepsilon}(f)\|_{L^p} = \sup\left\{\left|\int_{\mathbb{R}^N} T_{\varphi_\varepsilon}(f) g \, d\boldsymbol{x}\right| : \ g \in L^{p'}(\mathbb{R}^N), \ \ \|g\|_{L^{p'}} \le 1\right\}.$$

Given $g \in L^{p'}(\mathbb{R}^N)$, by Fubini's theorem and (94) we have

$$\begin{aligned}\int_{\mathbb{R}^N} T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) g(\boldsymbol{x}) \, d\boldsymbol{x} &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} K_{\varphi_\varepsilon}(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y}\right) g(\boldsymbol{x}) \, d\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} f(\boldsymbol{y}) \left(\int_{\mathbb{R}^N} K_{\varphi_\varepsilon}(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{x}) \, d\boldsymbol{x}\right) \, d\boldsymbol{y} \qquad (110) \\ &= \int_{\mathbb{R}^N} f(\boldsymbol{y}) T^*_{\varphi_\varepsilon}(g)(\boldsymbol{y}) \, d\boldsymbol{y},\end{aligned}$$

where

$$K^*(\boldsymbol{x},\boldsymbol{y}) := K(\boldsymbol{y},\boldsymbol{x}).$$

Note that K^* is still a standard kernel. Moreover, since φ is even

$$K^*_{\varphi_\varepsilon}(\boldsymbol{x},\boldsymbol{y}) = K^*(\boldsymbol{x},\boldsymbol{y}) \varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right) = K(\boldsymbol{y},\boldsymbol{x}) \varphi\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\varepsilon}\right) = K_{\varphi_\varepsilon}(\boldsymbol{y},\boldsymbol{x}).$$

In turn the operator

$$T^*_{\varphi_\varepsilon}(g)(\boldsymbol{y}) := \int_{\mathbb{R}^N} K_{\varphi_\varepsilon}(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{x}) \, d\boldsymbol{x}$$

is a smoothly truncated operator. Since

$$\left\|T^*_{\varphi_\varepsilon}(g)\right\|_{L^2} = \|T_{\varphi_\varepsilon}(g)\|_{L^2} \le C_1 \|g\|_{L^2}$$

for $g \in L^2$ by (110), we can apply the first part of the theorem to conclude that $T^*_{\varphi_\varepsilon}$ is of strong type (q,q) for all $1 < q < 2$. In particular, if $2 < p < \infty$, then $1 < p' < 2$. It follows from (110) and Hölder's inequality that for $f \in L^p(\mathbb{R}^N)$ and $g \in L^{p'}(\mathbb{R}^N)$, with $\|g\|_{L^{p'}} \le 1$,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) g(\boldsymbol{x}) \, d\boldsymbol{x}\right| &= \left|\int_{\mathbb{R}^N} f(\boldsymbol{y}) T^*_{\varphi_\varepsilon}(g)(\boldsymbol{y}) \, d\boldsymbol{y}\right| \le \|f\|_{L^p} \left\|T^*_{\varphi_\varepsilon}(g)\right\|_{L^{p'}} \\ &\le C'_{p'} \|g\|_{L^{p'}} \|f\|_{L^p} \le C'_{p'} \|f\|_{L^p}.\end{aligned}$$

Taking the supremum over all g gives

$$\|T_{\varphi_\varepsilon}(f)\|_{L^p} \le C'_{p'} \|f\|_{L^p}$$

for all $L^p(\mathbb{R}^N)$, which completes the proof. ∎

Friday, February 27, 2015

5.2 Maximal Singular Operators

In the previous subsection we have proved some L^p estimates for the truncated operators T_ε and T_{φ_ε}. The goal of this subsection is to show that we can pass to the limit as $\varepsilon \to 0^+$.

Define the maximal singular operator

$$T^{\#}(f)(\boldsymbol{x}) := \sup_{\varepsilon>0} |T_\varepsilon(f)(\boldsymbol{x})|. \tag{111}$$

Theorem 102 (Strong (p,p) estimate) *Let K be a standard kernel and let φ be as in (92). If T_{φ_ε} satisfies (100), then $T^{\#}$ is of strong type (p,p) for all $1<p<\infty$.*

Proof. Step 1: For $\varepsilon > 0$ define

$$T_\varepsilon^{\#}(f)(\boldsymbol{x}) := \sup_{\delta>\varepsilon} |T_\delta(f)(\boldsymbol{x})|.$$

We claim that for $\varepsilon > 0$ and $1 < q < p < \infty$,

$$T_\varepsilon^{\#}(f)(\boldsymbol{x}) \le c(N,q)[\mathrm{M}(f)(\boldsymbol{x}) + \mathrm{M}(T_\varepsilon(f))(\boldsymbol{x}) + (\mathrm{M}(|f|^q)(\boldsymbol{x}))^{1/q}] \tag{112}$$

for every $f \in L^p(\mathbb{R}^N)$ and for all $\boldsymbol{x} \in \mathbb{R}^N$.

Let $\delta > \varepsilon$. Fix $\boldsymbol{x} \in \mathbb{R}^N$ and let $\boldsymbol{x}_1 \in B(\boldsymbol{x}, \delta/2)$. By (90) we can write

$$\begin{aligned} T_\delta(f)(\boldsymbol{x}) = &\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)} (K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x}_1,\boldsymbol{y}))f(\boldsymbol{y})\, d\boldsymbol{y} \\ &+ \int_{\mathbb{R}^N} K(\boldsymbol{x}_1,\boldsymbol{y})f(\boldsymbol{y})\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}(\boldsymbol{y})\, d\boldsymbol{y}. \end{aligned}$$

Since $\delta > \varepsilon$, $\mathbb{R}^N \setminus B(\boldsymbol{x},\varepsilon) \supset \mathbb{R}^N \setminus B(\boldsymbol{x},\delta)$, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} K(\boldsymbol{x}_1,\boldsymbol{y})f(\boldsymbol{y})\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}(\boldsymbol{y})\, d\boldsymbol{y} = \int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x}_1,\boldsymbol{y})f(\boldsymbol{y})\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}(\boldsymbol{y})\, d\boldsymbol{y} \\ &= T_\varepsilon(f\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)})(\boldsymbol{x}_1) = T_\varepsilon(f)(\boldsymbol{x}_1) - T_\varepsilon(f\chi_{B(\boldsymbol{x},\delta)})(\boldsymbol{x}_1). \end{aligned}$$

Hence,

$$\begin{aligned} |T_\delta(f)(\boldsymbol{x})| \le &\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)} |K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x}_1,\boldsymbol{y})||f(\boldsymbol{y})|\, d\boldsymbol{y} \\ &+ |T_\varepsilon(f)(\boldsymbol{x}_1)| + |T_\varepsilon(f\chi_{B(\boldsymbol{x},\delta)})(\boldsymbol{x}_1)|. \end{aligned}$$

We now average both sides in $\boldsymbol{x}_1$ over the ball $B(\boldsymbol{x},\delta/2)$ to get

$$\begin{aligned} |T_\delta(f)(\boldsymbol{x})| \le &\frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))} \int_{B(\boldsymbol{x},\delta/2)} \int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)} |K(\boldsymbol{x},\boldsymbol{y}) - K(\boldsymbol{x}_1,\boldsymbol{y})||f(\boldsymbol{y})|\, d\boldsymbol{y} d\boldsymbol{x}_1 \\ &+ \frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))} \int_{B(\boldsymbol{x},\delta/2)} |T_\varepsilon(f)(\boldsymbol{x}_1)|\, d\boldsymbol{x}_1 \\ &+ \frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))} \int_{B(\boldsymbol{x},\delta/2)} |T_\varepsilon(f\chi_{B(\boldsymbol{x},\delta)})(\boldsymbol{x}_1)|\, d\boldsymbol{x}_1 =: I_1 + I_2 + I_3. \end{aligned}$$

By Fubini's theorem,

$$I_1 = \frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\int_{B(\boldsymbol{x},\delta/2)} |K(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x}_1,\boldsymbol{y})||f(\boldsymbol{y})|\; d\boldsymbol{x}_1 d\boldsymbol{y}.$$

For every $\boldsymbol{x}_1 \in B(\boldsymbol{x},\delta/2)$ and $\boldsymbol{y}\in\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)$, we have

$$|\boldsymbol{x}_1-\boldsymbol{x}| \le \frac{\delta}{2} \le \frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|,$$

and so by (87),

$$|K(\boldsymbol{x},\boldsymbol{y})-K(\boldsymbol{x}_1,\boldsymbol{y})| \le \frac{C_0|\boldsymbol{x}_1-\boldsymbol{x}|^\alpha}{|\boldsymbol{x}-\boldsymbol{y}|^{N+\alpha}} \le \frac{C_0(\delta/2)^\alpha}{|\boldsymbol{x}-\boldsymbol{y}|^N\delta^\alpha}.$$

In turn,

$$I_1 \le 2^{-\alpha}C_0\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^N}\; d\boldsymbol{y} \le 2^{-\alpha}C_0\,\mathrm{M}\,(f)\,(\boldsymbol{x})\,,$$

where the last inequality follows as in the proof of (75) starting from (74) with $R=1$.

On the other hand,

$$I_2 \le \mathrm{M}\,(T_\varepsilon(f))\,(\boldsymbol{x})$$

by the definition of maximal function (see (**??**)). Finally, by Hölder's inequality and (109),

$$\begin{aligned}I_3 &\le \frac{\left(\mathcal{L}^N(B(\boldsymbol{x},\delta/2))\right)^{1/q'}}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\left(\int_{B(\boldsymbol{x},\delta/2)}|T_\varepsilon(f\chi_{B(\boldsymbol{x},\delta)})(\boldsymbol{x}_1)|^q\; d\boldsymbol{x}_1\right)^{1/q}\\ &\le \frac{c(N,q)(\max\{C_\varphi,C_1\})^{2/q}}{\left(\mathcal{L}^N(B(\boldsymbol{x},\delta/2))\right)^{1-1/q'}}\left(\int_{B(\boldsymbol{x},\delta)}|f(\boldsymbol{x}_1)|^q\; d\boldsymbol{x}_1\right)^{1/q}\\ &\le c(N,q)(\max\{C_\varphi,C_1\})^{2/q}2^{N/q}(\mathrm{M}\,(|f|^q)\,(\boldsymbol{x}))^{1/q}.\end{aligned}$$

This completes the proof of (112). ■

Monday, March 01, 2015

Proof. Step 2: We show that $T^\#$ is of strong type (p,p) for every $1<p<\infty$. By (112),

$$\left\|T_\varepsilon^\#(f)\right\|_{L^p} \le c[\|\mathrm{M}\,(f)\|_{L^p} + \|\mathrm{M}\,(T_\varepsilon(f))\|_{L^p} + \left\|(\mathrm{M}\,(|f|^q))^{1/q}\right\|_{L^p}].$$

By Theorem 33 and (109) the first two terms are bounded by $c\,\|f\|_{L^p}$. On the other hand, by Theorem 33 with exponent $p/q>1$,

$$\begin{aligned}\left\|(\mathrm{M}\,(|f|^q)^{1/q}\right\|_{L^p} &= \left(\int_{\mathbb{R}^N}(\mathrm{M}\,(|f|^q))^{p/q}\; d\boldsymbol{x}\right)^{1/p}\\ &\le c\left(\int_{\mathbb{R}^N}(|f|^q)^{p/q}\; d\boldsymbol{x}\right)^{1/p} = c\,\|f\|_{L^p}\,.\end{aligned}$$

Hence,

$$\left\| T_{\varepsilon}^{\#}(f) \right\|_{L^p} \le c \left\| f \right\|_{L^p}.$$

Since $\varepsilon \mapsto T_{\varepsilon}^{\#}(f)$ is increasing, by letting $\varepsilon \to 0^+$ it follows from Fatou's lemma that

$$\left\| T^{\#}(f) \right\|_{L^p} \le c \left\| f \right\|_{L^p}.$$

This concludes the proof. ∎

Remark 103 *The inequality (112) is known as* Cotlar's inequality*.*

Theorem 104 (Weak (1,1) estimate) *Let K be a standard kernel and let φ be as in (92). If T_{φ_ε} satisfies (100), then $T^{\#}$ is of weak type* $(1,1)$*.*

Proof. The proof is similar to the one of Theorem 100. We only indicate the main changes. Given $f \in L^1(\mathbb{R}^N)$ as before we can assume, without loss of generality, that $f \ge 0$. We define Q_n, Q_n^*, D, D^* as before and write $f = g + h$. Since $T^{\#}$ is sublinear, (104) continues to hold for $T^{\#}$. Also $(1,1)$ weak estimate (105) for g follows using Theorem 102.

It remains to study $T^{\#}(h)$. Since $h = \sum_{n=1}^{\infty} h\chi_{Q_n}$ and T_ε is sublinear, we have that

$$|T^{\#}(h)(\boldsymbol{x})| \le \sup_{\varepsilon>0} |T_\varepsilon(h)(\boldsymbol{x})| \le \sup_{\varepsilon>0} \sum_{n=1}^{\infty} |T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})|. \tag{113}$$

Fix $\boldsymbol{x} \in \mathbb{R}^N \setminus D^*$ and $\varepsilon > 0$. There are three possibilities.

Case 1: For all $\boldsymbol{y} \in Q_n$ we have that $|\boldsymbol{x} - \boldsymbol{y}| > \varepsilon$. In this case, by 90,

$$\begin{aligned} T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x}) &= \int_{Q_n} K_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) h(\boldsymbol{y}) \, d\, \boldsymbol{y} = \int_{Q_n} K(\boldsymbol{x}, \boldsymbol{y}) h(\boldsymbol{y}) \, d\boldsymbol{y} \\ &= \int_{Q_n} (K(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{x}_n)) h(\boldsymbol{y}) \, d\boldsymbol{y}, \end{aligned}$$

where $\boldsymbol{x}_n$ is the center of Q_n and we have used the fact that $\int_{Q_n} h(\boldsymbol{y}) \, d\boldsymbol{y} = 0$.

We can now continue as in the proof of (108) to conclude that

$$|K(\boldsymbol{x}, \boldsymbol{y}) - K(\boldsymbol{x}, \boldsymbol{x}_n)| \le \frac{cC_0 r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}}$$

so that

$$|T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| \le \frac{cC_0 r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \int_{Q_n} |h(\boldsymbol{y})| \, d\boldsymbol{y}.$$

We let I_1 be the collection of all n such that Q_n is as in Case 1.

Case 2: There exist $\boldsymbol{y}_1, \boldsymbol{y}_2 \in Q_n = Q(\boldsymbol{x}_n, r_n)$ such that $|\boldsymbol{x} - \boldsymbol{y}_1| > \varepsilon$ and $|\boldsymbol{x} - \boldsymbol{y}_2| \le \varepsilon$. In this case, we claim that

$$\frac{\varepsilon}{2} \le |\boldsymbol{x} - \boldsymbol{y}| \le 2\varepsilon \tag{114}$$

for all $\boldsymbol{y} \in Q_n$. Indeed, recalling that $mQ_n = Q(\boldsymbol{x}_n, mr_n)$, where $m := \lceil \sqrt{N} + 1 \rceil$, and that $\boldsymbol{x} \in \mathbb{R}^N \setminus mQ_n$, for every $\boldsymbol{y} \in Q_n$, we have

$$\begin{aligned}|\boldsymbol{x} - \boldsymbol{y}| &\geq |\boldsymbol{x} - \boldsymbol{y}_1| - |\boldsymbol{y} - \boldsymbol{y}_1| \geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}_1| + \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}_1| - \frac{r_n}{2}\sqrt{N} \\ &\geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}_1| + \frac{1}{2}(m-1)r_n - \frac{r_n}{2}\sqrt{N} \\ &\geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}_1| \geq \frac{\varepsilon}{2},\end{aligned}$$

while

$$\begin{aligned}\varepsilon &\geq |\boldsymbol{x} - \boldsymbol{y}_2| \geq |\boldsymbol{x} - \boldsymbol{y}| - |\boldsymbol{y} - \boldsymbol{y}_2| \geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| + \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| - \frac{r_n}{2}\sqrt{N} \\ &\geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}| + \frac{1}{2}(m-1)r_n - \frac{r_n}{2}\sqrt{N} \geq \frac{1}{2}|\boldsymbol{x} - \boldsymbol{y}|.\end{aligned}$$

This proves the claim. In turn, by (85), 90,

$$|T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| \leq C_0 \int_{Q_n} \frac{|h(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^N}\, d\boldsymbol{y} \leq \frac{2^N C_0}{\varepsilon^N} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y}.$$

We let I_2 be the collection of all n such that Q_n is as in Case 2.
Case 3: For all $\boldsymbol{y} \in Q_n$ we have that $|\boldsymbol{x} - \boldsymbol{y}| \leq \varepsilon$. In this case $K_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in Q_n$ and so $T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x}) = 0$.

Combining the three cases, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} |T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| &= \sum_{n\in I_1} |T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| + \sum_{n\in I_2} |T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| \\ &\leq \sum_{n\in I_1} \frac{cC_0 r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y} + \sum_{n\in I_2} \frac{2^N C_0}{\varepsilon^N} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y}.\end{aligned}$$

Since the cubes Q_n are disjoint, we have that

$$\begin{aligned}\sum_{n\in I_2} \frac{2^N C_0}{\varepsilon^N} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y} &= \frac{2^N C_0}{\varepsilon^N} \int_{\bigcup_{n\in I_2} Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y} \\ &\leq \frac{2^N C_0}{\varepsilon^N} \int_{B(\boldsymbol{x}, 2\varepsilon)} |h(\boldsymbol{y})|\, d\boldsymbol{y} \leq 4^N \alpha_N C_0 \,\mathrm{M}\,(h)\,(\boldsymbol{x}).\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} |T_\varepsilon(h\chi_{Q_n})(\boldsymbol{x})| \leq \sum_{n=1}^{\infty} \frac{cC_0 r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y} + 4^N \alpha_N C_0 \,\mathrm{M}\,(h)\,(\boldsymbol{x}).$$

It follows from (113),

$$\begin{aligned}|T^{\#}(h)(\boldsymbol{x})| &\leq \sum_{n=1}^{\infty} \frac{cC_0 r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \int_{Q_n} |h(\boldsymbol{y})|\, d\boldsymbol{y} + 4^N \alpha_N C_0 \,\mathrm{M}\,(h)\,(\boldsymbol{x}) \\ &=: H(\boldsymbol{x}) + 4^N \alpha_N C_0 \,\mathrm{M}\,(h)\,(\boldsymbol{x}).\end{aligned}$$

As usual for every $t > 0$, we have that

$$\{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ |T^\#(h)(\boldsymbol{x})| > t\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^N : \ 4^N \alpha_N C_0 \operatorname{M}(h)(\boldsymbol{x}) > t/2\}$$
$$\cup \{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ H(\boldsymbol{x}) > t/2\}.$$

By Theorem 33,

$$\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ 4^N \alpha_N C_0 \operatorname{M}(h)(\boldsymbol{x}) > t/2\}) \leq \frac{c}{t} \int_{\mathbb{R}^N} |h| \, d\boldsymbol{x}$$
$$\leq \frac{c}{t} \int_{\mathbb{R}^N} |f| \, d\boldsymbol{x},$$

where in the last inequality we have used (99). On the other hand,

$$\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N \setminus D^* : \ H(\boldsymbol{x}) > t/2\}) \leq \frac{2}{t} \int_{\mathbb{R}^N \setminus D^*} H \, d\boldsymbol{x}$$
$$\leq \frac{cC_0}{t} \sum_{n=1}^{\infty} \int_{Q_n} |h(\boldsymbol{y})| \, d\boldsymbol{y} \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}_n, mr_n/2)} \frac{r_n^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_n|^{N+\alpha}} \, d\boldsymbol{x}$$
$$= \frac{cC_0}{t} \sum_{n=1}^{\infty} \int_{Q_n} |h(\boldsymbol{y})| \, d\boldsymbol{y} r_n^\alpha \int_{mr_n/2}^{\infty} \frac{1}{r^{1+\alpha}} dr \leq \frac{cC_0}{t} \int_{\mathbb{R}^N} |h| \, d\boldsymbol{x}$$
$$\leq \frac{cC_0}{t} \int_{\mathbb{R}^N} |f| \, d\boldsymbol{x}.$$

This completes the proof. ■

Wednesday, March 03, 2015

Theorem 105 *Let K be a standard kernel and let φ be as in (92). Assume also that there exists*

$$\lim_{\varepsilon \to 0^+} \int_{B(\boldsymbol{x},1) \setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} \tag{115}$$

for every $\boldsymbol{x} \in \mathbb{R}^N$. If T_{φ_ε} satisfies (100), then for all $1 \leq p < \infty$ and all $f \in L^p(\mathbb{R}^N)$ there exists

$$\lim_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) =: T(f)(\boldsymbol{x}) \tag{116}$$

for $\mathcal{L}^N$ a.e $\boldsymbol{x} \in \mathbb{R}^N$ and the operator T is of weak type $(1,1)$ and of strong type (p,p) for every $1 < p < \infty$.

Proof. Step 1: We first prove that the limit exists for smooth functions. Let $f \in C_c^1(\mathbb{R}^N)$ or $\mathcal{S}(\mathbb{R}^N)$. Then we can write

$$T_\varepsilon(f)(\boldsymbol{x}) = \int_{B(\boldsymbol{x},1) \setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x}, \boldsymbol{y})(f(\boldsymbol{y}) - f(\boldsymbol{x})) \, d\boldsymbol{y}$$
$$+ f(\boldsymbol{x}) \int_{B(\boldsymbol{x},1) \setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} + \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},1)} K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y}$$
$$=: I_1 + I_2 + I_3.$$

Since f is smooth $\mathcal{S}\left(\mathbb{R}^N\right)$,

$$|f(\boldsymbol{y}) - f(\boldsymbol{x})| \leq \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))} |\boldsymbol{y} - \boldsymbol{x}|.$$

In turn, by (85),

$$|K(\boldsymbol{x}, \boldsymbol{y})(f(\boldsymbol{y}) - f(\boldsymbol{x}))| \leq \frac{C_0 \|\nabla f\|_{L^\infty(B(\boldsymbol{x},1))}}{|\boldsymbol{y} - \boldsymbol{x}|^{N-1}},$$

which is integrable in $B(\boldsymbol{x}, 1)$. Hence, by the Lebesgue dominated convergence theorem, there exists $\lim_{\varepsilon \to 0^+} I_1$.

For the term I_2 we use the hypothesis (115). Finally, by (85),

$$\left| \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},1)} K(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} \right| \leq C_0 \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},1)} \frac{|f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^N} \, d\boldsymbol{y}.$$

Since $f \in \mathcal{S}(\mathbb{R}^N)$, it follows that I_3 is well-defined and finite.
Step 2: Let now $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^N)$. By decomposing f and K in their real and imaginary parts and using the linearity of T_ε, (in f and K), without loss of generality, we can assume that both f and K are real-valued.

By Theorem 102 for $1 < p < \infty$ and Theorem 104 for $p = 1$, we have that the function $T^\#(f)$ is finite for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$. In turn, for any such $\boldsymbol{x}$,

$$\left| \liminf_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) \right| \leq T^\#(f)(\boldsymbol{x}) < \infty, \tag{117}$$
$$\left| \limsup_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) \right| \leq T^\#(f)(\boldsymbol{x}) < \infty,$$

and so the operator

$$\begin{aligned} L(f)(\boldsymbol{x}) := & \limsup_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) \\ & - \limsup_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) \end{aligned}$$

is well-defined (and in turn finite) for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$. Moreover, for every $t > 0$,

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : L(f)(\boldsymbol{x}) > t\}) &\leq \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : T^\#(f)(\boldsymbol{x}) > t/2\}) \\ &\leq \frac{c}{t^p} \int_{\mathbb{R}^N} |f|^p \, d\boldsymbol{y}. \end{aligned}$$

If we now replace in the previous inequality f with $f - g$, where $g \in C_c^1\left(\mathbb{R}^N\right)$, and observe that $L(f)(\boldsymbol{x}) = L(f - g)(\boldsymbol{x})$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$ by Step 1, by the previous inequality we have that

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : L(f)(\boldsymbol{x}) > 2t\}) &\leq \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : T^\#(f - g)(\boldsymbol{x}) > 2t\}) \\ &\leq \frac{c}{t^p} \int_{\mathbb{R}^N} |f - g|^p \, d\boldsymbol{y}. \end{aligned}$$

Since $C_c^1(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, the right-hand side of the previous inequality can be made arbitrarily small, and so

$$\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : L(f)(\boldsymbol{x}) > 2t\}) = 0$$

for all $t > 0$. Hence $L(f)(\boldsymbol{x}) = 0$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$, and in turn, by (117) we have that there exists

$$\lim_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) =: T(f)(\boldsymbol{x}) \in \mathbb{R} \tag{118}$$

for all $x \in \mathbb{R}^N \setminus E$, where E is a set of Lebesgue measure zero.

Using the fact that $|T(f)(\boldsymbol{x})| \leq T^{\#}(f)(\boldsymbol{x})$, it follows from Theorems 102 and 104 that T is of weak type $(1,1)$ and of strong type (p,p) for every $1 < p < \infty$. ■

The operator T is called the *Calderon-Zygmund operator* associated to the kernel K.

5.3 The Crux of the Matter, Part I

In all the theorems in the last two subsectiones we assumed that T_{φ_ε} satisfies (100). In the next theorem we show that (100) holds for homogeneous kernels of convolution type.

Theorem 106 *Let $K \in C^1(\mathbb{R}^N \setminus \{\mathbf{0}\})$ be such that*

$$K(\lambda \boldsymbol{x}) = \lambda^{-N} K(\boldsymbol{x}) \tag{119}$$

for all $\lambda > 0$ and $\boldsymbol{x} \in \mathbb{R}^N \setminus \{\mathbf{0}\}$ and

$$\int_{\partial B} K(\boldsymbol{x})\ d\mathcal{H}^{N-1}(\boldsymbol{x}) = 0. \tag{120}$$

For $0 < \varepsilon < R$ consider the operator

$$T_{\varepsilon,R}(f)(\boldsymbol{x}) := \int_{B(\boldsymbol{x},R) \setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\ d\boldsymbol{y}.$$

Then there exists a constant $C > 0$ such that

$$\|T_{\varepsilon,R}(f)\|_{L^2} \leq C \|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R}^N)$ and all $\varepsilon, R > 0$.

Proof. Let $K_{\varepsilon,R} := K\chi_{B(\mathbf{0},R) \setminus B(\mathbf{0},\varepsilon)}$. Let's compute the Fourier transform of $K_{\varepsilon,R}$. Using spherical coordinates if $\boldsymbol{x} = \mathbf{0}$ by (119) and (120) we have

$$\begin{aligned}
\widehat{K_{\varepsilon,R}}(\boldsymbol{x}) &= \int_\varepsilon^R r^{N-1} \int_{\partial B(\mathbf{0},1)} K(r\boldsymbol{\xi})\ d\mathcal{H}^{N-1}(\boldsymbol{\xi}) dr \\
&= \int_\varepsilon^R \frac{r^{N-1}}{r^N} \int_{\partial B(\mathbf{0},1)} K(\boldsymbol{\xi})\ d\mathcal{H}^{N-1}(\boldsymbol{\xi}) dr = 0
\end{aligned}$$

while if $\boldsymbol{x} \neq \mathbf{0}$ by (120) and Fubini's theorem we have

$$\begin{aligned}\widehat{K_{\varepsilon,R}}(\boldsymbol{x}) &= \int_{\varepsilon}^{R} r^{N-1} \int_{\partial B(\mathbf{0},1)} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot r\boldsymbol{\xi}} K(r\boldsymbol{\xi})\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) dr \\ &= \int_{\partial B(\mathbf{0},1)} K(\boldsymbol{\xi}) \int_{\varepsilon}^{R} \frac{r^{N-1}}{r^N} e^{-2\pi \boldsymbol{i}\boldsymbol{x}\cdot r\boldsymbol{\xi}}\, dr d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\ &= \int_{\partial B(\mathbf{0},1)} K(\boldsymbol{\xi}) \int_{\varepsilon|\boldsymbol{x}|}^{R|\boldsymbol{x}|} \frac{1}{s} e^{-2\pi \boldsymbol{i} s(\boldsymbol{\xi}\cdot\boldsymbol{x})/|\boldsymbol{x}|}\, ds d\mathcal{H}^{N-1}(\boldsymbol{\xi}),\end{aligned}$$

where we have made the change of variables $s = r|\boldsymbol{x}|$.

We now distinguish a few cases. If $R|\boldsymbol{x}| \leq 1$ let $g(s) := e^{-2\pi \boldsymbol{i} s(\boldsymbol{\xi}\cdot\boldsymbol{x})/|\boldsymbol{x}|}$. By the mean value theorem,

$$\frac{g(s) - g(0)}{s} = g'(\theta) = -2\pi \boldsymbol{i}(\boldsymbol{\xi}\cdot\boldsymbol{x}) e^{-2\pi \boldsymbol{i}\theta(\boldsymbol{\xi}\cdot\boldsymbol{x})/|\boldsymbol{x}|}/|\boldsymbol{x}|$$

and so

$$\left|\frac{g(s) - g(0)}{s}\right| \leq 2\pi.$$

In turn, by (120),

$$\begin{aligned}|\widehat{K_{\varepsilon,R}}(\boldsymbol{x})| &= \left|\int_{\partial B(\mathbf{0},1)} K(\boldsymbol{\xi}) \int_{\varepsilon|\boldsymbol{x}|}^{R|\boldsymbol{x}|} \frac{g(s) - g(0)}{s}\, ds d\mathcal{H}^{N-1}(\boldsymbol{\xi})\right| \\ &\leq 2\pi R|\boldsymbol{x}| \int_{\partial B(\mathbf{0},1)} |K(\boldsymbol{\xi})|\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\ &\leq 2\pi \int_{\partial B(\mathbf{0},1)} |K(\boldsymbol{\xi})|\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}).\end{aligned}$$

If $\varepsilon|\boldsymbol{x}| < 1 < R|\boldsymbol{x}|$, then

$$\int_{\varepsilon|\boldsymbol{x}|}^{R|\boldsymbol{x}|} = \int_{\varepsilon|\boldsymbol{x}|}^{1} + \int_{1}^{R|\boldsymbol{x}|}.$$

The first integral on the right-hand side can be treated as in the previous case. We will estimate the second later.

Finally if $1 \leq \varepsilon|\boldsymbol{x}| \leq R|\boldsymbol{x}|$, then

$$\int_{\varepsilon|\boldsymbol{x}|}^{R|\boldsymbol{x}|} = -\int_{1}^{\varepsilon|\boldsymbol{x}|} + \int_{1}^{R|\boldsymbol{x}|}.$$

Thus, it remains to estimate the integral

$$\int_1^\ell \frac{1}{s} e^{-2\pi \boldsymbol{i} s\theta}\, ds = \int_1^\ell \frac{1}{s} \cos(2\pi s\theta)\, ds - \boldsymbol{i} \int_1^\ell \frac{1}{s} \sin(2\pi s\theta)\, ds$$

for $\ell > 1$ and where $\theta \in [-1, 1]$. We have already seen the second integral is bounded by 3. To estimate the first, when $\theta \neq 0$, we make the change of variables $s = 2\pi|\theta|\tau$. Then

$$\int_1^\ell \frac{1}{s} \cos(2\pi s\theta)\, ds = \int_{2\pi|\theta|}^{2\pi\ell|\theta|} \frac{1}{\tau} \cos\tau\, d\tau.$$

We claim (exercise) that for all $0 < a < b$,

$$\left| \int_a^b \frac{1}{\tau} \cos\tau\, d\tau \right| \leq c(1 + |\log a|).$$

Using this estimate we get

$$\begin{aligned} |\widehat{K_{\varepsilon,R}}(\boldsymbol{x})| &\leq c \, \|K\|_{L^\infty(\partial B(\mathbf{0},1))} \int_{\partial B(\mathbf{0},1)} (1 + |\log(2\pi|\boldsymbol{\xi} \cdot \boldsymbol{x}|/|\boldsymbol{x}|)|)\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \\ &= c \, \|K\|_{L^\infty(\partial B(\mathbf{0},1))} \int_{\partial B(\mathbf{0},1)} (1 + |\log(2\pi|\boldsymbol{\xi} \cdot \boldsymbol{e}_N|)|)\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}), \end{aligned}$$

where in the last equality we used the fact that the integral on the sphere is invariant by rotation. We leave as an exercise to check that

$$\int_{\partial B(\mathbf{0},1)} (1 + |\log(2\pi|\boldsymbol{\xi} \cdot \boldsymbol{e}_N|)|)\, d\mathcal{H}^{N-1}(\boldsymbol{\xi}) \leq c(N).$$

In conclusion we have shown that in all cases

$$|\widehat{K_{\varepsilon,R}}(\boldsymbol{x})| \leq c \, \|K\|_{L^\infty(\partial B(\mathbf{0},1))} .$$

Since for $f \in L^2(\mathbb{R}^N)$,

$$\widehat{f * K_{\varepsilon,R}} = \widehat{f}\widehat{K_{\varepsilon,R}}$$

by Theorem 71, by Plancherel identity and the previous inequality we have

$$\begin{aligned} \|T_{\varepsilon,R}(f)\|_{L^2} &= \|f * K_{\varepsilon,R}\|_{L^2} = \left\|\widehat{f * K_{\varepsilon,R}}\right\|_{L^2} \\ &= \left\|\widehat{f}\widehat{K_{\varepsilon,R}}\right\|_{L^2} \leq c \left\|\widehat{f}\right\|_{L^2} = c\, \|f\|_{L^2} \end{aligned}$$

for all $\varepsilon > 0$ and $R > 0$. ■

Corollary 107 *Let $K \in C^1(\mathbb{R}^N \setminus \{\mathbf{0}\})$ satisfy (119) and (120) and let*

$$T_\varepsilon(f)(\boldsymbol{x}) := \int_{\mathbb{R}^N \setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}.$$

Then for all $1 \leq p < \infty$ and all $f \in L^p\left(\mathbb{R}^N\right)$ there exists

$$\lim_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) =: T(f)(\boldsymbol{x}) \tag{121}$$

for $\mathcal{L}^N$ a.e $\boldsymbol{x} \in \mathbb{R}^N$ and the operator T is of weak type $(1,1)$ and of strong type (p,p) for every $1 < p < \infty$.

Proof. By Fatou's lemma and the previous theorem,

$$\|T_\varepsilon(f)\|_{L^2} \le \liminf_{R\to\infty} \|T_{\varepsilon,R}(f)\|_{L^2} \le c\,\|f\|_{L^2}\,.$$

Moreover, using spherical coordinates, (119), and (120) we have

$$\begin{aligned}\int_{B(\boldsymbol{x},1)\setminus B(\boldsymbol{x},\varepsilon)} K(\boldsymbol{x},\boldsymbol{y})\ d\boldsymbol{y} &= \int_\varepsilon^1 r^{N-1}\int_{\partial B(\mathbf{0},1)} K(r\boldsymbol{\xi})\ d\mathcal{H}^{N-1}(\boldsymbol{\xi})dr \\ &= \int_\varepsilon^1 \frac{r^{N-1}}{r^N}\int_{\partial B(\mathbf{0},1)} K(\boldsymbol{\xi})\ d\mathcal{H}^{N-1}(\boldsymbol{\xi})dr = 0\end{aligned}$$

and so (115) holds. Thus we are in a position to apply Theorem 105 are satisfied. ■

Corollary 108 *Theorem 90 holds.*

Proof. The kernel $K_{i,j}$ be given as in (72) satisfies all the hypotheses of Theorem 106. Hence, we can apply the previous corollary. ■

The hypotheses on the kernel in Theorem 106 are very restrictive.

Monday, March 16, 2015

6 BMO Spaces

6.1 BMO Spaces

Definition 109 *We say that a function* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *has* bounded mean oscillation*, and we write* $f \in \mathrm{BMO}(\mathbb{R}^N)$*, if*

$$|f|_{\mathrm{BMO}} := \sup \frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x}) - f_Q|\ d\boldsymbol{x} < \infty, \tag{122}$$

where the supremum is taken over all cubes Q*.*

Here we are using the notation

$$f_E := \frac{1}{\mathcal{L}^N(E)}\int_E f(\boldsymbol{x})\ d\boldsymbol{x}. \tag{123}$$

Remark 110 *The quantity* $\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x}) - f_Q|\ d\boldsymbol{x}$ *is called the* mean oscillation *of* f *over* Q*. It measures the average distance of* f *from* f_Q *over the same cube.*

Remark 111 *The space* $\mathrm{BMO}(\mathbb{R}^N)$ *is a vector space and* $|\cdot|_{\mathrm{BMO}}$ *is a seminorm. Since* $|f|_{\mathrm{BMO}} = 0$ *if and only if* $f \equiv$*const,* $|\cdot|_{\mathrm{BMO}}$ *is a norm in the quotient space*

$$\mathrm{BMO}(\mathbb{R}^N)/\mathbb{R}.$$

Remark 112 *In the sequel we will use the fact that* $f \in L^1_{\text{loc}}\left(\mathbb{R}^N\right)$ *belongs to* $\text{BMO}\left(\mathbb{R}^N\right)$ *if and only if for every cube* Q *there exists* $c_{Q,f} \in \mathbb{C}$ *such that*

$$\sup \frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - c_{Q,f}| \, d\boldsymbol{x} < \infty.$$

To see this assume that the latter holds. Then

$$f(\boldsymbol{x}) - f_Q = f(\boldsymbol{x}) - c_{Q,f} + \frac{1}{\mathcal{L}^N(Q)} \int_Q (f(\boldsymbol{y}) - c_{Q,f}) \, d\boldsymbol{y}$$

and so

$$|f(\boldsymbol{x}) - f_Q| \leq |f(\boldsymbol{x}) - c_{Q,f}| + \frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{y}) - c_{Q,f}| \, d\boldsymbol{y}.$$

By averaging over Q *in* $\boldsymbol{x}$ *we get*

$$\frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - f_Q| \, d\boldsymbol{x} \leq 2 \frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - c_{Q,f}| \, d\boldsymbol{x}. \tag{124}$$

Note that if we define

$$|f|^*_{\text{BMO}} := \sup \frac{1}{\mathcal{L}^N(Q)} \inf_{c_Q \in \mathbb{C}} \int_Q |f(\boldsymbol{x}) - c_Q| \, d\boldsymbol{x},$$

then by (124),

$$\frac{1}{2} |f|_{\text{BMO}} \leq |f|^*_{\text{BMO}} \leq |f|_{\text{BMO}}.$$

Remark 113 $|f|^*_{\text{BMO}}$ *is a seminorm. Indeed, if* $f, g \in \text{BMO}\left(\mathbb{R}^N\right)$, *then given a cube* Q *and* $d_Q, e_Q \in \mathbb{C}$, *we have*

$$\begin{aligned} \inf_{c_Q \in \mathbb{C}} \int_Q |f + g - c_Q| \, d\boldsymbol{x} &\leq \int_Q |f + g - d_Q - e_Q| \, d\boldsymbol{x} \\ &\leq \int_Q |f - d_Q| \, d\boldsymbol{x} + \int_Q |g - e_Q| \, d\boldsymbol{x} \end{aligned}$$

and taking the infimum over all d_Q *and* e_Q *gives*

$$\begin{aligned} \frac{1}{\mathcal{L}^N(Q)} \inf_{c_Q \in \mathbb{C}} \int_Q |f + g - c_Q| \, d\boldsymbol{x} &\leq \frac{1}{\mathcal{L}^N(Q)} \inf_{d_Q \in \mathbb{C}} \int_Q |f - d_Q| \, d\boldsymbol{x} + \frac{1}{\mathcal{L}^N(Q)} \inf_{e_Q \in \mathbb{C}} \int_Q |g - e_Q| \, d\boldsymbol{x} \\ &\leq |f|^*_{\text{BMO}} + |g|^*_{\text{BMO}}. \end{aligned}$$

By taking the supremum over all cubes we obtain

$$|f + g|^*_{\text{BMO}} \leq |f|^*_{\text{BMO}} + |g|^*_{\text{BMO}}.$$

On the other hand, given $t \in \mathbb{R}$ *and* $d_Q \in \mathbb{C}$, *we have*

$$\inf_{c_Q \in \mathbb{C}} \int_Q |tf - c_Q| \, d\boldsymbol{x} \leq \int_Q |tf - td_Q| \, d\boldsymbol{x} = |t| \int_Q |f - d_Q| \, d\boldsymbol{x}$$

and taking the infimum over all d_Q and e_Q gives

$$\frac{1}{\mathcal{L}^N(Q)}\inf_{c_Q\in\mathbb{C}}\int_Q |tf-c_Q|\,d\boldsymbol{x} \le |t|\frac{1}{\mathcal{L}^N(Q)}\inf_{d_Q\in\mathbb{C}}\int_Q |f-d_Q|\,d\boldsymbol{x} \\ \le |t|\,|f|^*_{\mathrm{BMO}}\,.$$

Hence $|tf|^*_{\mathrm{BMO}} \le |t|\,|f|^*_{\mathrm{BMO}}$*. Similarly, given* $t\in\mathbb{R}$*,* $t\neq 0$*, and* $d_Q\in\mathbb{C}$*, we have*

$$\inf_{c_Q\in\mathbb{C}}\int_Q |f-c_Q|\,d\boldsymbol{x} \le \int_Q \left|f-\frac{1}{t}d_Q\right|d\boldsymbol{x} = \frac{1}{|t|}\int_Q |tf-d_Q|\,d\boldsymbol{x}$$

and taking the infimum over all d_Q and e_Q gives

$$\frac{1}{\mathcal{L}^N(Q)}\inf_{c_Q\in\mathbb{C}}\int_Q |f-c_Q|\,d\boldsymbol{x} \le \frac{1}{|t|}\frac{1}{\mathcal{L}^N(Q)}\inf_{d_Q\in\mathbb{C}}\int_Q |tf-d_Q|\,d\boldsymbol{x} \\ \le \frac{1}{|t|}\,|tf|^*_{\mathrm{BMO}}\,.$$

This shows that $|f|^*_{\mathrm{BMO}} \le \frac{1}{|t|}|tf|^*_{\mathrm{BMO}}$ *and so* $|tf|^*_{\mathrm{BMO}} = |t|\,|f|^*_{\mathrm{BMO}}$*.*

Using the previous property we can show that $L^\infty(\mathbb{R}^N) \subsetneqq \mathrm{BMO}(\mathbb{R}^N)$.

Theorem 114 *The space $L^\infty(\mathbb{R}^N)$ is strictly contained in* $\mathrm{BMO}(\mathbb{R}^N)$*.*

Proof. Given $f\in L^\infty(\mathbb{R}^N)$ take $c_{Q,f}:=0$. Then

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x})-0|\,d\boldsymbol{x} \le \|f\|_{L^\infty}\,,$$

which, by Remark 112, implies that $f\in\mathrm{BMO}(\mathbb{R}^N)$.

Next we show that the unbounded function $f(\boldsymbol{x}):=\log|\boldsymbol{x}|$ belongs to $\mathrm{BMO}(\mathbb{R}^N)$. Let $Q=Q(\boldsymbol{x}_0,r)$ for some $\boldsymbol{x}_0$ and $r>0$. If $Q\cap B(\boldsymbol{0},2r)=\emptyset$, take $c_{Q,f}:=\log|\boldsymbol{x}_0|$. By the mean value theorem for $\boldsymbol{x}\in Q$,

$$\log|\boldsymbol{x}|-c_{Q,f} = \log|\boldsymbol{x}|-\log|\boldsymbol{x}_0| = \frac{\boldsymbol{y}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)}{|\boldsymbol{y}|^2}$$

for some $\boldsymbol{y}$ between $\boldsymbol{x}$ and $\boldsymbol{x}_0$, and so

$$|\log|\boldsymbol{x}|-c_{Q,f}| \le \frac{|\boldsymbol{x}-\boldsymbol{x}_0|}{|\boldsymbol{y}|} \le \frac{r\sqrt{N}}{2r} = \frac{\sqrt{N}}{2}.$$

It follows that

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x})-c_{Q,f}|\,d\boldsymbol{x} \le \frac{\sqrt{N}}{2}.$$

On the other hand, if $Q \cap B(\mathbf{0}, 2r) \neq \emptyset$, then $Q \subset B(\mathbf{0}, 2r + \sqrt{N}r)$. In this case take $c_{Q,f} := \log r$. Then

$$\begin{aligned}
\frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - c_{Q,f}| \, d\boldsymbol{x} &= \frac{1}{r^N} \int_Q |\log|\boldsymbol{x}| - \log r| \, d\boldsymbol{x} \\
&\leq \frac{1}{r^N} \int_{B(\mathbf{0}, 2r+\sqrt{N}r)} |\log|\boldsymbol{x}|/r| \, d\boldsymbol{x} = \frac{\alpha_N}{r^N} \int_0^{2r+\sqrt{N}r} s^{N-1} |\log s/r| \, ds \\
&= \alpha_N \int_0^{2+\sqrt{N}} t^{N-1} |\log t| \, dt < \infty,
\end{aligned}$$

where $rt = s$. Hence, f belongs to $\mathrm{BMO}(\mathbb{R}^N)$. ■

Exercise 115 *Prove that for $N = 1$, the function*

$$g(x) = \begin{cases} \log x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

does not belong to $\mathrm{BMO}(\mathbb{R})$. *So multiplying a* BMO *function by a characteristic function does not preserve* BMO.

Exercise 116 *Prove that* $W^{1,N}(\mathbb{R}^N) \subset \mathrm{BMO}(\mathbb{R}^N)$.

Theorem 117 *Lef $f, g \in \mathrm{BMO}(\mathbb{R}^N)$. Then*

(i) the function $h := |f|$ belongs to $\mathrm{BMO}(\mathbb{R}^N)$ *and*

$$|h|_{\mathrm{BMO}} \leq 2|f|_{\mathrm{BMO}}, \quad |h|_{\mathrm{BMO}}^* \leq |f|_{\mathrm{BMO}}^*$$

(ii) the functions $\min\{f, g\}$ *and* $\max\{f, g\}$ *belong to* $\mathrm{BMO}(\mathbb{R}^N)$ *with*

$$\begin{aligned}
|\min\{f,g\}|_{\mathrm{BMO}} &\leq \frac{3}{2}|f|_{\mathrm{BMO}} + \frac{3}{2}|g|_{\mathrm{BMO}}, \quad |\max\{f,g\}|_{\mathrm{BMO}} \leq \frac{3}{2}|f|_{\mathrm{BMO}} + \frac{3}{2}|g|_{\mathrm{BMO}}, \\
|\min\{f,g\}|_{\mathrm{BMO}}^* &\leq |f|_{\mathrm{BMO}}^* + |g|_{\mathrm{BMO}}^*, \quad |\max\{f,g\}|_{\mathrm{BMO}}^* \leq |f|_{\mathrm{BMO}}^* + |g|_{\mathrm{BMO}}^*.
\end{aligned}$$

(iii) for every $t > 0$ the truncated function

$$f_t(\boldsymbol{x}) := \begin{cases} t & \text{if } f(\boldsymbol{x}) \geq t, \\ f(\boldsymbol{x}) & \text{if } |f(\boldsymbol{x})| \leq t, \\ -t & \text{if } f(\boldsymbol{x}) \leq -t, \end{cases}$$

belongs to $\mathrm{BMO}(\mathbb{R}^N)$ *and*

$$|f_t|_{\mathrm{BMO}} \leq \frac{9}{4}|f|_{\mathrm{BMO}}, \quad |f_t|_{\mathrm{BMO}}^* \leq |f|_{\mathrm{BMO}}^*.$$

Proof. (i) Take $c_{Q,h} := |c_{Q,f}|$. Then

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |h(\boldsymbol{x}) - c_{Q,h}|\, d\boldsymbol{x} = \frac{1}{\mathcal{L}^N(Q)}\int_Q ||f(\boldsymbol{x})| - |c_{Q,f}||\, d\boldsymbol{x}$$
$$\leq \frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x}) - c_{Q,f}|\, d\boldsymbol{x}.$$

(ii) Write

$$\min\{f,g\} = \frac{f+g-|f-g|}{2}, \quad \max\{f,g\} = \frac{f+g+|f-g|}{2}.$$

Since $|\cdot|_{\mathrm{BMO}}$ is a seminorm and by part (i),

$$|\min\{f,g\}|^*_{\mathrm{BMO}} = \left|\frac{f+g-|f-g|}{2}\right|_{\mathrm{BMO}} \leq \frac{1}{2}|f+g|^*_{\mathrm{BMO}} + \frac{1}{2}|-|f-g||^*_{\mathrm{BMO}}$$
$$\leq \frac{1}{2}|f+g|^*_{\mathrm{BMO}} + \frac{1}{2}|f-g|^*_{\mathrm{BMO}} \leq |f|^*_{\mathrm{BMO}} + |g|^*_{\mathrm{BMO}}.$$

(iii) Write $f_t = \min\{t, \max\{f,-t\}\}$. Then by part (ii),

$$|f_t|^*_{\mathrm{BMO}} \leq |\max\{f,-t\}|^*_{\mathrm{BMO}} + |t|^*_{\mathrm{BMO}} = |\max\{f,-t\}|^*_{\mathrm{BMO}} + 0$$
$$\leq |f|^*_{\mathrm{BMO}} + |-t|^*_{\mathrm{BMO}} = |f|^*_{\mathrm{BMO}} + 0,$$

where we used the fact that the seminorm of constant functions is zero. ■

Theorem 118 *The space* $\mathrm{BMO}(\mathbb{R}^N)/\mathbb{R}$ *is a Banach space.*

Proof. Exercise. ■

Theorem 119 (John–Nirenberg) *Let* $f \in \mathrm{BMO}(\mathbb{R}^N)$*. Then there exist two constant* $c_1, c_2 > 0$ *depending only on* N *such that for every* $t > 0$ *and every cube* Q*,*

$$\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t\}) \leq c_1 \mathcal{L}^N(Q)\, e^{-c_2 t/|f|_{\mathrm{BMO}}}. \tag{125}$$

Proof. If $|f|_{\mathrm{BMO}} = 0$, then f is a constant and so there is nothing to prove. Thus assume that $|f|_{\mathrm{BMO}} \neq 0$.
Step 1: Assume that $|f|_{\mathrm{BMO}} = 1$. Fix a cube Q_0. For $t > 0$ let $g(t)$ denote the least number such that

$$\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t\}) \leq g(t)\mathcal{L}^N(Q) \tag{126}$$

for all cubes $Q \subseteq Q_0$. Note that $g(t) \leq 1$ and g is decreasing.

Consider a cube $Q \subseteq Q_0$. By (122),

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x}) - f_Q|\, d\boldsymbol{x} \leq |f|_{\mathrm{BMO}} = 1 < 2^N. \tag{127}$$

Hence, we are in a position to apply Remark 98 to $|f - f_Q|$ to find a countable family $\{Q_n\} \subset Q$ of open mutually disjoint cubes such that

$$|f(\boldsymbol{x}) - f_Q| \le 2^N \quad \text{for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x} \in Q \setminus \bigcup_{n=1}^{\infty} \overline{Q_n}, \tag{128}$$

and for every $n \in \mathbb{N}$,

$$2^N < \frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} |f - f_Q| \, d\boldsymbol{x} \le 4^N. \tag{129}$$

By (129),

$$\begin{aligned} \sum_{n=1}^{\infty} \mathcal{L}^N(Q_n) &\le \frac{1}{2^N} \sum_{n=1}^{\infty} \int_{Q_n} |f - f_Q| \, d\boldsymbol{x} = \frac{1}{2^N} \int_{\bigcup_{n=1}^{\infty} Q_n} |f - f_Q| \, d\boldsymbol{x} \\ &\le \frac{1}{2^N} \int_Q |f - f_Q| \, d\boldsymbol{x} \le \frac{1}{2^N} \mathcal{L}^N(Q). \end{aligned} \tag{130}$$

Moreover, by (129),

$$|f_{Q_n} - f_Q| = \left| \frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} (f - f_Q) \, d\boldsymbol{x} \right| \le \frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} |f - f_Q| \, d\boldsymbol{x} \le 4^N. \tag{131}$$

Hence, if $t > 4^N$ and if $\boldsymbol{x} \in Q_n$ is such that $|f(\boldsymbol{x}) - f_Q| > t$, then

$$|f(\boldsymbol{x}) - f_{Q_n}| \ge |f(\boldsymbol{x}) - f_Q| - |f_{Q_n} - f_Q| > t - 4^N.$$

It follows from (128) and (130) that for $t > 4^N$,

$$\begin{aligned} &\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t\}) \\ &\quad = \mathcal{L}^N\left(\left\{\boldsymbol{x} \in \bigcup_{n=1}^{\infty} Q_n : |f(\boldsymbol{x}) - f_Q| > t\right\}\right) \\ &\quad \le \sum_{n=1}^{\infty} \mathcal{L}^N(\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x}) - f_Q| > t\}) \\ &\quad \le \sum_{n=1}^{\infty} \mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x}) - f_{Q_n}| > t - 4^N\right\}\right) \\ &\quad = \sum_{n=1}^{\infty} \frac{\mathcal{L}^N(Q_n)}{\mathcal{L}^N(Q_n)} \mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x}) - f_{Q_n}| > t - 4^N\right\}\right) \\ &\quad \le g(t - 4^N) \sum_{n=1}^{\infty} \mathcal{L}^N(Q_n) \le \frac{1}{2^N} g(t - 4^N) \mathcal{L}^N(Q). \end{aligned}$$

This shows that

$$g(t) \le \frac{1}{2^N} g(t - 4^N) \tag{132}$$

for all $t > 4^N$. ■

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Proof. Now for any $t > 4^N$, let n be the integer part of $(t-1)4^{-N}$. Then $s := 1 + n4^N \le t$ and so, since g is decreasing, by iterating (132) n times,

$$g(t) \le g(s) = g(1 + n4^N) \le \frac{1}{2^{nN}},$$

where we used the fact that $g \le 1$. Since $n - 1 \le (t-1)4^{-N} - 1 < n$, we have

$$\frac{1}{2^{nN}} \le \frac{1}{2^{(t-1)N4^{-N}-N}} = 2^{N(4^{-N}+1)} e^{\log 2^{-tN4^{-N}}} = 2^{N(4^{-N}+1)} e^{-tN4^{-N}\log 2}.$$

Set $c_1 := 2^{N(4^{-N}+1)}$ and $c_2 := N4^{-N}\log 2$. Then we have shown that for all $t > 4^N$,

$$g(t) \le c_1 e^{-c_2 t}.$$

On the other hand, if $0 \le t \le 4^N$, then

$$g(t) \le 1 \le c_1 e^{-c_2 t},$$

since $e^{c_2 t} \le e^{N\log 2} = 2^N$. Thus, by (126),

$$\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t\}) \le c_1 e^{-c_2 t}\mathcal{L}^N(Q)$$

for all $t > 0$.

Step 2: Given a function $f \in \mathrm{BMO}(\mathbb{R}^N)$ with f not constant, we apply Step 1 to $f/|f|_{\mathrm{BMO}}$ to obtain

$$\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t|f|_{\mathrm{BMO}}\}) \le c_1 e^{-c_2 t}\mathcal{L}^N(Q).$$

Taking $s := t|f|_{\mathrm{BMO}}$ gives

$$\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > s\}) \le c_1 e^{-c_2 s/|f|_{\mathrm{BMO}}}\mathcal{L}^N(Q).$$

Since $t > 0$ is arbitrary, the previous inequality holds for all $s > 0$. ■

Exercise 120 *Mistake in file John-Nirenberg-mistake.pdf*

Corollary 121 *Let $f \in \mathrm{BMO}(\mathbb{R}^N)$. Then for every $1 \le p < \infty$ the function f belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ and there exists a constant $C_p > 0$ such that for every cube Q,*

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x}) - f_Q|^p \, d\boldsymbol{x} \le C_p |f|^p_{\mathrm{BMO}}.$$

Proof. By Theorem 4, and Theorem 119,

$$\begin{aligned}\int_Q |f(\boldsymbol{x}) - f_Q|^p \, d\boldsymbol{x} &= p\int_0^\infty t^{p-1}\mathcal{L}^N(\{\boldsymbol{x} \in Q : |f(\boldsymbol{x}) - f_Q| > t\}) \, dt \\ &\le pc_1\mathcal{L}^N(Q)\int_0^\infty t^{p-1}e^{-c_2 t/|f|_{\mathrm{BMO}}} \, dt \\ &= pc_1\mathcal{L}^N(Q)|f|^p_{\mathrm{BMO}}\int_0^\infty s^{p-1}e^{-c_2 s} \, ds,\end{aligned}$$

where we have made the change of variables $|f|_{\mathrm{BMO}}\, s = t$. ■

Remark 122 *It follows by the previous corollary and Hölder's inequality that*

$$\frac{1}{C_p^{1/p}} \sup_Q \left(\frac{1}{\mathcal{L}^N(Q)} \int_Q |f - f_Q|^p \, d\boldsymbol{x} \right)^{1/p} \le |f|_{\mathrm{BMO}} \le \sup_Q \left(\frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - f_Q|^p \, d\boldsymbol{x} \right)^{1/p},$$

so $\sup_Q \left(\frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - f_Q|^p \, d\boldsymbol{x} \right)^{1/p}$ *is an equivalent seminorm in* BMO *for all* $1 \le p < \infty$.

6.2 Interpolation, $p = \infty$

We now use the John–Nirenberg's theorem to prove an interpolation result between $L^p(\mathbb{R}^N)$ and $\mathrm{BMO}(\mathbb{R}^N)$.

Theorem 123 *Let* $1 \le p < \infty$ *and let* $f \in L^p(\mathbb{R}^N) \cap \mathrm{BMO}(\mathbb{R}^N)$. *Then* f *belongs to* $L^q(\mathbb{R}^N)$ *for all* $p \le q < \infty$, *with*

$$\|f\|_{L^q} \le c \, \|f\|_{L^p}^{p/q} \, |f|_{\mathrm{BMO}}^{1-p/q}$$

for some constant $c = c(p, q, N) > 0$.

Proof. If $|f|_{\mathrm{BMO}} = 0$, then f must be zero and so there is nothing to prove. Thus assume that $|f|_{\mathrm{BMO}} \ne 0$. We apply Theorem 97 to the function $|f|^p \in L^1(\mathbb{R}^N)$ and with $t := |f|_{\mathrm{BMO}}^p$ to find a countable family $\{Q_n\}$ of open mutually disjoint cubes such that

$$|f(\boldsymbol{x})|^p \le |f|_{\mathrm{BMO}}^p \quad \text{for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x} \in \mathbb{R}^N \setminus \bigcup_{n=1}^{\infty} \overline{Q_n}, \tag{133}$$

and for every $n \in \mathbb{N}$,

$$|f|_{\mathrm{BMO}}^p < \frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} |f|^p \, d\boldsymbol{x} \le 2^N \, |f|_{\mathrm{BMO}}^p \, . \tag{134}$$

By (134),

$$\sum_{n=1}^{\infty} \mathcal{L}^N(Q_n) \le \frac{1}{|f|_{\mathrm{BMO}}^p} \sum_{n=1}^{\infty} \int_{Q_n} |f|^p \, d\boldsymbol{x} = \frac{1}{|f|_{\mathrm{BMO}}^p} \int_{\bigcup_{n=1}^{\infty} Q_n} |f|^p \, d\boldsymbol{x} \le \frac{\|f\|_{L^p}^p}{|f|_{\mathrm{BMO}}^p}, \tag{135}$$

while by Hölder's inequality

$$|f|_{Q_n} = \frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} |f| \, d\boldsymbol{x} \le \left(\frac{1}{\mathcal{L}^N(Q_n)} \int_{Q_n} |f|^p \, d\boldsymbol{x} \right)^{1/p} \le 2^{N/p} |f|_{\mathrm{BMO}} . \tag{136}$$

By Theorem 119 and (133), for all $t > 2^{N/p} |f|_{\mathrm{BMO}}$,

$$\begin{aligned}
&\mathcal{L}^N\left(\left\{\boldsymbol{x} \in \mathbb{R}^N : |f(\boldsymbol{x})| > t\right\}\right) \\
&\quad = \mathcal{L}^N(\{\boldsymbol{x} \in \textstyle\bigcup_n Q_n : |f(\boldsymbol{x})| > t\}) \\
&\quad \le \sum_{n=1}^{\infty} \mathcal{L}^N(\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x})| > t\}) \\
&\quad \le \sum_{n=1}^{\infty} \mathcal{L}^N(\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x}) - f_{Q_n}| > t - |f|_{Q_n}\}) \\
&\quad \le \sum_{n=1}^{\infty} \frac{\mathcal{L}^N(Q_n)}{\mathcal{L}^N(Q_n)} \mathcal{L}^N\left(\left\{\boldsymbol{x} \in Q_n : |f(\boldsymbol{x}) - f_{Q_n}| > t - 2^{N/p} |f|_{\mathrm{BMO}}\right\}\right) \\
&\quad \le c_1 e^{[-c_2(t - 2^{N/p}|f|_{\mathrm{BMO}})/|f|_{\mathrm{BMO}}]} \sum_{n=1}^{\infty} \mathcal{L}^N(Q_n) \\
&\quad \le c_1 e^{[-c_2(t - 2^{N/p}|f|_{\mathrm{BMO}})/|f|_{\mathrm{BMO}}]} \frac{\|f\|_{L^p}^p}{|f|_{\mathrm{BMO}}^p},
\end{aligned}$$

where in the last inequality we used (135). By Theorem 4, and Theorem 119,

$$\begin{aligned}
\int_{\mathbb{R}^N} |f|^q \, d\boldsymbol{x} &= q \int_0^{\infty} t^{q-1} \mathcal{L}^N(\{|f| > t\}) \, dt \\
&= q \int_0^{2^{N/p}|f|_{\mathrm{BMO}}} t^{q-1} \mathcal{L}^N(\{|f| > t\}) \, dt + q \int_{2^{N/p}|f|_{\mathrm{BMO}}}^{\infty} t^{q-1} \mathcal{L}^N(\{|f| > t\}) \, dt \\
&\le q \int_0^{2^{N/p}|f|_{\mathrm{BMO}}} t^{q-p-1} \|f\|_{L^p}^p \, dt + q c_1 \frac{\|f\|_{L^p}^p}{|f|_{\mathrm{BMO}}^p} \int_{2^{N/p}|f|_{\mathrm{BMO}}}^{\infty} t^{q-1} e^{[-c_2(t - 2^{N/p}|f|_{\mathrm{BMO}})/|f|_{\mathrm{BMO}}]} \, dt \\
&= \frac{q 2^{N(q-p)/p}}{q-p} \|f\|_{L^p}^p |f|_{\mathrm{BMO}}^{q-p} + q c_1 \frac{\|f\|_{L^p}^p}{|f|_{\mathrm{BMO}}^p} |f|_{\mathrm{BMO}}^q \int_0^{\infty} (s + 2^{N/p})^{q-1} e^{-c_2 s} \, ds,
\end{aligned}$$

where we have made the change of variables $s = (t - 2^{N/p} |f|_{\mathrm{BMO}}) / |f|_{\mathrm{BMO}}$, or, equivalently $(s + 2^{N/p}) |f|_{\mathrm{BMO}} = t$. This completes the proof. ■

Friday, March 20, 2015

In what follows we concentrate on the Lebesgue measure. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ we define the *sharp maximal function* of f,

$$\mathrm{M}^{\#}(f)(\boldsymbol{x}) := \sup_{Q:\ \boldsymbol{x} \in Q} \frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{y}) - f_Q| \, d\boldsymbol{y}, \tag{137}$$

Remark 124 *Note that* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *belongs to* $\mathrm{BMO}(\mathbb{R}^N)$ *if and only if* $\mathrm{M}^{\#}(f) \in L^{\infty}(\mathbb{R}^N)$.

Remark 125 *Since*

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{y}) - f_Q|\, d\boldsymbol{y} \le \frac{2}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{y})|\, d\boldsymbol{y} \le 2\,\mathrm{N}^{nc}(f)(\boldsymbol{x}),$$

where $\mathrm{N}^{nc}(f)$ *is the uncentered maximal function for cubes, we have that*

$$\mathrm{M}^{\#}(f)(\boldsymbol{x}) \le 2\,\mathrm{N}^{nc}(f)(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^N$*.*

We also define the *dyadic maximal function*

$$\mathrm{M}^d(f)(\boldsymbol{x}) := \sup_{Q \text{ dyadic: } \boldsymbol{x}\in Q} \frac{1}{\mathcal{L}^N(Q)}\int_Q |f(\boldsymbol{x})|\, d\boldsymbol{x},$$

where a dyadic cube Q has the form $Q = 2^k(\boldsymbol{j} + [0,1)^N)$, where $k \in \mathbb{Z}$ and $\boldsymbol{j} \in \mathbb{Z}^N$.

Exercise 126 *Prove that* M^d *is of weak type* $(1,1)$ *and strong type* (p,p) *for all* $1 < p \le \infty$*.*

Theorem 127 (Fefferman–Stein) *If* $1 < p \le q < \infty$ *and* $f \in L^p(\mathbb{R}^N)$*, then*

$$\left\|\mathrm{M}^d(f)\right\|_{L^q} \le C(N,q)\left\|\mathrm{M}^{\#}(f)\right\|_{L^q}.$$

Proof. Step 1: Let $f \in L^p(\mathbb{R}^N)$ and let $t, \delta > 0$. Assume that $f \ge 0$. We claim that

$$\begin{aligned}&\mathcal{L}^N(\{\boldsymbol{x}\in\mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > 2t,\ \mathrm{M}^{\#}(f)(\boldsymbol{x}) \le \delta t\})\\ &\quad\le 2^N\delta\mathcal{L}^N(\{\boldsymbol{x}\in\mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t\}).\end{aligned}$$

Let $E_t := \left\{\boldsymbol{x}\in\mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t\right\}$. If $\boldsymbol{x} \in E_t$, then there exists a dyadic cube Q such that $\boldsymbol{x} \in Q$ and $f_Q > t$. In turn, $\mathrm{M}^d(f)(\boldsymbol{y}) > t$ for all $\boldsymbol{y} \in Q$, and so $Q \subseteq E_t$. Let Q be the largest such dyadic cube containing $\boldsymbol{x}$. This shows that E_t can be written as a countable disjoint union of maximal dyadic cubes.

Fix one of these cubes $Q = Q(\boldsymbol{x}_0, r)$. It is enough to show that

$$\mathcal{L}^N(\{\boldsymbol{x}\in Q : \ \mathrm{M}^d(f)(\boldsymbol{x}) > 2t,\ \mathrm{M}^{\#}(f)(\boldsymbol{x}) \le \delta t\}) \le \delta(2r)^N.$$

If the right-hand side of the previous inequality is zero, then there is nothing to prove. Thus, assume that there exists $\boldsymbol{x}_1 \in Q$ such that $\mathrm{M}^d(f)(\boldsymbol{x}_1) > 2t$ and $\mathrm{M}^{\#}(f)(\boldsymbol{x}_1) \le \delta t$. Define $2Q := Q(\boldsymbol{x}_0, 2r)$. By the maximality of Q and the definition of $\mathrm{M}^d(f)$, we have that $f_{2Q} \le t$. If $\boldsymbol{x} \in Q$ and $\mathrm{M}^d(f)(\boldsymbol{x}) > 2t$, then $\mathrm{M}^d(f\chi_Q)(\boldsymbol{x}) > 2t$. Hence,

$$\mathrm{M}^d((f - f_{2Q})\chi_Q)(\boldsymbol{x}) \ge \mathrm{M}^d(f\chi_Q)(\boldsymbol{x}) - f_{2Q} > 2t - t = t,$$

where we used the fact the sublinearity of M^d and the fact that $\mathrm{M}^d(c) = c$. This shows that if $\mathrm{M}^d(f)(\boldsymbol{x}) > 2t$, then $\mathrm{M}^d((f - f_{2Q})\chi_Q)(\boldsymbol{x}) > t$. By the weak L^1 inequality for M^d,

$$\begin{aligned}
&\mathcal{L}^N(\{\boldsymbol{x} \in Q : \ \mathrm{M}^d(f)(\boldsymbol{x}) > 2t, \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) \leq \delta t\}) \\
&\leq \mathcal{L}^N(\{\boldsymbol{x} \in Q : \ \mathrm{M}^d((f - f_{2Q})\chi_Q)(\boldsymbol{x}) > t\}) \\
&\leq \frac{1}{t}\int_Q |f - f_{2Q}| \ d\boldsymbol{x} = \frac{(2r)^N}{t}\frac{1}{(2r)^N}\int_{2Q} |f - f_{2Q}| \ d\boldsymbol{x} \\
&\leq \frac{(2r)^N}{t}\mathrm{M}^{\#}(f)(\boldsymbol{x}_1) \leq (2r)^N \delta.
\end{aligned}$$

Hence, the claim is proved.
Step 2: Let $f \in L^p(\mathbb{R}^N)$. Then $\mathrm{M}^d(f) \in L^p(\mathbb{R}^N)$ and by Theorem 4

$$\left\|\mathrm{M}^d(f)\right\|^p_{L^p(\mathbb{R}^N)} = p\int_0^\infty t^{p-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t\}) \ dt < \infty.$$

In particular, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
I_n &:= q\int_0^n t^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t)\} \ dt \\
&\leq qn^{q-p}\int_0^n t^{p-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t\}) \ dt < \infty.
\end{aligned}$$

On the other hand, by changing variables and Step 1,

$$\begin{aligned}
I_n &= 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > 2t\}) \ dt \\
&\leq 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > 2t, \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) \leq \delta t\}) \ dt \\
&\quad + 2^q q\int_0^{n/2} t^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) > \delta t\}) \ dt \\
&\leq \delta 2^{q+N} q\int_0^{n/2} t^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^d(f)(\boldsymbol{x}) > t\}) \ dt \\
&\quad + \frac{2^q}{\delta^q} q\int_0^{\delta n/2} s^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) > s\}) \ ds \\
&= \delta 2^{q+N} I_n + \frac{2^q}{\delta^q} q\int_0^{\delta n/2} s^{q-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) > s\}) \ ds.
\end{aligned}$$

Let δ be such that $\delta 2^{q+N} = \frac{1}{2}$. Since $I_n < \infty$, it follows that

$$\begin{aligned}\frac{1}{2} I_n &\leq \frac{2^q}{\delta^q} q \int_0^{\delta n/2} s^{q-1} \mathcal{L}^N \left(\left\{ \boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) > s \right\} \right) \, ds \\ &\leq \frac{2^q}{\delta^q} q \int_0^{\infty} s^{q-1} \mathcal{L}^N \left(\left\{ \boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{M}^{\#}(f)(\boldsymbol{x}) > s \right\} \right) \, ds \\ &= \frac{2^q}{\delta^q} \left\| \mathrm{M}^{\#}(f) \right\|_{L^q(\mathbb{R}^N)}^q .\end{aligned}$$

Letting $n \to \infty$, we get the desired result. ■

Exercise 128 *Prove that the previous theorem continues to hold if we assume that* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *is such that* $\inf\{1, \mathrm{M}^d(f)\} \in L^q(\mathbb{R}^N)$.

Monday, March 23, 2015

Theorem 129 (Stampacchia Interpolation Theorem) *Let* $1 < p < \infty$, *and let* $T : V \to L^0(\mathbb{R}^N)$ *be a linear operator such that* T *is of strong type* (p, p) *and*

$$|T(f)|_{\mathrm{BMO}} \leq c \|f\|_{L^\infty}$$

for all $f \in V \cap L^\infty(\mathbb{R}^N)$, *where* V *is a subspace of* $L^\infty(\mathbb{R}^N) + L^p(\mathbb{R}^N)$ *closed by truncation. Then* T *is of strong type* (q, q) *for all* $p \leq q < \infty$.

Proof. Define $T_1 := \mathrm{M}^{\#} \circ T$. Then T_1 is sublinear. Indeed, for $f_1,\ f_2 \in V$ we have

$$(T(f_1 + f_2))_Q = (T(f_1) + T(f_2))_Q = (T(f_1))_Q + (T(f_2))_Q$$

and

$$\begin{aligned}&\int_Q \left| T(f_1 + f_2)(\boldsymbol{x}) - (T(f_1 + f_2))_Q \right| d\boldsymbol{x} \\ &= \int_Q \left| T(f_1)(\boldsymbol{x}) + T(f_2)(\boldsymbol{x}) - (T(f_1))_Q - (T(f_2))_Q \right| d\boldsymbol{x} \\ &\leq \int_Q \left| T(f_1)(\boldsymbol{x}) - (T(f_1))_Q \right| d\boldsymbol{x} + \int_Q \left| T(f_2)(\boldsymbol{x}) - (T(f_2))_Q \right| d\boldsymbol{x},\end{aligned}$$

which shows that

$$\left(\mathrm{M}^{\#} \circ T \right)(f_1 + f_2)(\boldsymbol{x}) \leq \left(\mathrm{M}^{\#} \circ T \right)(f_1)(\boldsymbol{x}) + \left(\mathrm{M}^{\#} \circ T \right)(f_2)(\boldsymbol{x}),$$

while by the linearity of T and of integration $|T(\lambda f)(\boldsymbol{x})| = |\lambda| \, |T(f)(\boldsymbol{x})|$.

Moreover, for $f \in L^\infty(\mathbb{R}^N) \cap V$,

$$\sup_{\boldsymbol{x} \in \mathbb{R}^N} \left| \left(\mathrm{M}^{\#} \circ T \right)(f)(\boldsymbol{x}) \right| = |T(f)|_{\mathrm{BMO}} \leq C \|f\|_{L^\infty},$$

which shows that $\mathrm{M}^{\#} \circ T : L^{\infty}\left(\mathbb{R}^N\right) \cap V \to L^{\infty}\left(\mathbb{R}^N\right)$ is bounded. On the other hand, by Hölder's inequality,

$$\int_Q \left|T(f)(\boldsymbol{x}) - (T(f))_Q\right| \, d\boldsymbol{x} \leq 2 \int_Q |T(f)(\boldsymbol{x})| \, d\boldsymbol{x},$$

and so

$$\begin{aligned}\left|\left(\mathrm{M}^{\#} \circ T\right)(f)(\boldsymbol{x})\right| &\leq 2 \sup_{Q:\ \boldsymbol{x}\in Q} \frac{1}{\mathcal{L}^N(Q)} \int_Q |T(f)(\boldsymbol{x})| \, d\boldsymbol{x} \\ &= 2\left(\left(\mathrm{N}^{nc} \circ T\right)(f)\right)(\boldsymbol{x})\end{aligned}$$

where N^{nc} is the uncentered maximal operator for cubes (see (35)). Since $\|\mathrm{N}^{nc}(g)\|_{L^p(\mathbb{R}^N)} \leq C(p,n)\|g\|_{L^p(\mathbb{R}^N)}$ for $p > 1$, we get

$$\begin{aligned}\left\|\left(\mathrm{M}^{\#} \circ T\right)(f)\right\|_{L^p(\mathbb{R}^N)} &\leq 2\left\|\left(\mathrm{N}^{nc} \circ T\right)(f)\right\|_{L^p(\mathbb{R}^N)} \\ &\leq C(p,n)\|Tf\|_{L^p(\mathbb{R}^N)} \\ &\leq C(p,n)\|f\|_{L^p(\mathbb{R}^N)}.\end{aligned}$$

Hence, we are in a position to apply Marcinkiewicz Interpolation Theorem to conclude that $\mathrm{M}^{\#} \circ T$ is bounded in $L^q\left(\mathbb{R}^N\right)$ for all $p < q < \infty$.

Let $f \in L^q\left(\mathbb{R}^N\right) \cap V$ with compact support. Then $f \in L^p\left(\mathbb{R}^N\right)$, and so $T(f) \in L^p\left(\mathbb{R}^N\right)$. By Theorem 127, and the fact that $|v| \leq \mathrm{M}^d(v)\ \mathcal{L}^N$ a.e. in $\mathbb{R}^N$,

$$\begin{aligned}\|T(f)\|_{L^q(\mathbb{R}^N)} &\leq \left\|\mathrm{M}^d \circ T(f)\right\|_{L^q(\mathbb{R}^N)} \leq C(N,q)\left\|\mathrm{M}^{\#} \circ T(f)\right\|_{L^q(\mathbb{R}^N)} \\ &\leq C(N,q)\|f\|_{L^q(\mathbb{R}^N)}.\end{aligned}$$

This concludes the proof. ■

NOTE: $p = 1$ need weak type $(1,1)$ for T_1.

Remark 130 *Note that for all $f \in V \cap L^p\left(\mathbb{R}^N\right) \cap L^{\infty}\left(\mathbb{R}^N\right)$ we have that $T(f) \in L^p\left(\mathbb{R}^N\right) \cap \mathrm{BMO}\left(\mathbb{R}^N\right)$, and so by Theorem 123,*

$$\begin{aligned}\|T(f)\|_{L^q} &\leq c\|T(f)\|_{L^p}^{p/q}|T(f)|_{\mathrm{BMO}}^{1-p/q} \\ &\leq c\|f\|_{L^p}^{p/q}\|f\|_{L^{\infty}}^{1-p/q}.\end{aligned}$$

Next we study singular integrals for functions in $L^{\infty}(\mathbb{R}^N)$. We begin by observe that the truncated operators T_{φ_ε} and T_ε are not well-defined for functions f in $L^{\infty}(\mathbb{R}^N)$, since the truncated kernels are not in $L^1(\mathbb{R}^N)$. Thus we need to further truncate the kernel.

Let φ be as in (92). For $0 < \varepsilon < R$ consider the smooth kernel

$$K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y}) := K(\boldsymbol{x},\boldsymbol{y})\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\varepsilon}\right)\left(1-\varphi\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{R}\right)\right). \tag{138}$$

Note that $K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})=0$ is either $|\boldsymbol{x}-\boldsymbol{y}|\leq\varepsilon$ or $|\boldsymbol{x}-\boldsymbol{y}|\geq 2R$. Reasoning as in Proposition 95 we can show that $K_{\varphi_{\varepsilon,R}}$ satisfies the inequalities (85)–(87).

Consider the *smoothly truncated operator*

$$T_{\varphi_{\varepsilon,R}}(f)(\boldsymbol{x}):=\int_{\mathbb{R}^N}K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}. \tag{139}$$

It is well-defined for every function $f\in L^p(\mathbb{R}^N)$, $1\leq p\leq\infty$. We will assume that $T_{\varphi_{\varepsilon,R}}$ satisfies a uniform $(2,2)$ estimate, that is, that there exists a constant $C_2>0$ such that

$$\left\|T_{\varphi_{\varepsilon,R}}(f)\right\|_{L^2}\leq C_2\left\|f\right\|_{L^2} \tag{140}$$

for all $f\in L^2(\mathbb{R}^N)$ and all $0<\varepsilon<R$.

Theorem 131 *Let K be a standard kernel and let φ be as in (92). If $T_{\varphi_{\varepsilon,R}}$ satisfies (140), then there exists a constant $c(N)>0$ such that*

$$\left|T_{\varphi_{\varepsilon,R}}(f)\right|_{\mathrm{BMO}}\leq c\max\{C_\varphi,C_2\}\left\|f\right\|_{L^\infty} \tag{141}$$

for all $0<\varepsilon<R$ and for all $f\in L^\infty(\mathbb{R}^N)$. The same estimate holds for the operators T_ε, $T^{\#}$, and T given in (90), (111), and (121).

Proof. Fix a cube $Q=Q(\boldsymbol{x}_0,r)$ and let $mQ:=Q(\boldsymbol{x}_0,mr)$, where $m:=\lceil\sqrt{N}+1\rceil$. For $\boldsymbol{x}\in Q$ write

$$\begin{aligned}T_{\varphi_{\varepsilon,R}}(f)(\boldsymbol{x})=&\int_{mQ}K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}\\&+\int_{\mathbb{R}^N\setminus mQ}(K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})-K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y}))f(\boldsymbol{y})\,d\boldsymbol{y}\\&+\int_{\mathbb{R}^N\setminus mQ}K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}.\end{aligned}$$

Take

$$c_Q:=\int_{\mathbb{R}^N\setminus mQ}K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}.$$

Then, by Fubini's theorem

$$\begin{aligned}\int_Q\left|T_{\varphi_{\varepsilon,R}}(f)(\boldsymbol{x})-c_Q\right|\,d\boldsymbol{x}\leq&\int_Q\left|T_{\varphi_{\varepsilon,R}}(\chi_{mQ}f)(\boldsymbol{x})\right|\,d\boldsymbol{x}\\&+\left\|f\right\|_{L^\infty}\int_{\mathbb{R}^N\setminus mQ}\int_Q|K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})-K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y})|\,d\boldsymbol{x}d\boldsymbol{y}.\end{aligned}$$

By Hölder's inequality and (140),

$$\begin{aligned}\int_Q \left|T_{\varphi_{\varepsilon,R}}(\chi_{mQ}f)\right| \, d\boldsymbol{x} &\leq (\mathcal{L}^N(Q))^{1/2}\left(\int_{\mathbb{R}^N}\left|T_{\varphi_{\varepsilon,R}}(\chi_{mQ}f)\right|^2 \, d\boldsymbol{x}\right)^{1/2} \\ &\leq C_2(\mathcal{L}^N(Q))^{1/2}\left(\int_{\mathbb{R}^N}|\chi_{mQ}f|^2 \, d\boldsymbol{x}\right)^{1/2} \\ &\leq C_2\,\|f\|_{L^\infty}(\mathcal{L}^N(Q))^{1/2}(\mathcal{L}^N(mQ))^{1/2} \\ &= m^{N/2}C_2\,\|f\|_{L^\infty}\mathcal{L}^N(Q).\end{aligned}$$

As in (107), for $\boldsymbol{x}\in Q$ and $\boldsymbol{y}\in\mathbb{R}^N\setminus mQ$, we have $|\boldsymbol{x}-\boldsymbol{x}_0|\leq\frac{1}{2}|\boldsymbol{y}-\boldsymbol{x}_0|$ provided $m\geq\sqrt{N}+1$. Hence, by (87),

$$|K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})-K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y})|\leq\frac{C_\varphi|\boldsymbol{x}-\boldsymbol{x}_0|^\alpha}{|\boldsymbol{y}-\boldsymbol{x}_0|^{N+\alpha}}\leq\frac{cC_\varphi r^\alpha}{|\boldsymbol{y}-\boldsymbol{x}_0|^{N+\alpha}}.$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus mQ}\int_Q|K_{\varphi_{\varepsilon,R}}(\boldsymbol{x},\boldsymbol{y})-K_{\varphi_{\varepsilon,R}}(\boldsymbol{x}_0,\boldsymbol{y})|\,d\boldsymbol{x}d\boldsymbol{y} \\ &\quad\leq cC_\varphi r^\alpha\int_{\mathbb{R}^N\setminus mQ}\int_Q\frac{1}{|\boldsymbol{y}-\boldsymbol{x}_0|^{N+\alpha}}\,d\boldsymbol{x}d\boldsymbol{y} \\ &\quad\leq cC_\varphi r^\alpha\mathcal{L}^N(Q)\int_{\mathbb{R}^N\setminus B(\boldsymbol{x}_0,r)}\frac{1}{|\boldsymbol{y}-\boldsymbol{x}_0|^{N+\alpha}}\,d\boldsymbol{y} \\ &\quad= cC_\varphi r^\alpha\mathcal{L}^N(Q)\int_r^\infty\frac{s^{N-1}}{s^{N+\alpha}}\,ds=cC_\varphi\mathcal{L}^N(Q).\end{aligned}$$

Hence,

$$\frac{1}{\mathcal{L}^N(Q)}\int_Q\left|T_{\varphi_{\varepsilon,R}}(f)(\boldsymbol{x})-c_Q\right|\,d\boldsymbol{x}\leq(m^{N/2}C_2+C_\varphi 2^{N+1}\sqrt{N}\alpha_N)\,\|f\|_{L^\infty},$$

which concludes the proof. ■

Wednesday, March 25, 2015

7 Hardy Spaces

7.1 Hardy Spaces

There are several equivalent definitions for Hardy spaces. Here we give the one which will be most useful to study singular integrals.

Definition 132 *Given $1<p\leq\infty$, a measurable function $f:\mathbb{R}^N\to\mathbb{C}$ is called a p-*atom *if there exists a cube $Q\subset\mathbb{R}^N$ such that*

$$\operatorname{supp} f\subseteq Q,\quad\|f\|_{L^p(\mathbb{R}^N)}\leq(\mathcal{L}^N(Q))^{1/p-1} \tag{142}$$

and

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\, d\boldsymbol{x} = 0. \tag{143}$$

Definition 133 *Given $1 < p \le \infty$, a measurable function $f : \mathbb{R}^N \to \mathbb{C}$ is said to belong to the* Hardy space $\mathbb{H}^{1,p}(\mathbb{R}^N)$ *if there exist a sequence $\{f_n\}$ of p-atoms and $\{c_n\} \subset \mathbb{C}$ such that*

$$f(\boldsymbol{x}) = \sum_{n=1}^{\infty} c_n f_n(\boldsymbol{x}) \tag{144}$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$ and

$$\sum_{n=1}^{\infty} |c_n| < \infty.$$

We define

$$\|f\|_{\mathbb{H}^{1,p}} := \inf \sum_{n=1}^{\infty} |c_n|,$$

where the infimum is taken over all decompositions (144) of f.

Remark 134 *If $1 < p < q \le \infty$, then a q-atom f is also a p-atom. Indeed, let f satisfy (142) and (143) (with q in place of p). Then by Hölder's inequality with exponent q/p,*

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^N)} &= \|f\|_{L^p(Q)} \le \|f\|_{L^q(Q)} (\mathcal{L}^N(Q))^{1/p-1/q} \\ &\le (\mathcal{L}^N(Q))^{1/q-1} (\mathcal{L}^N(Q))^{1/p-1/q} = (\mathcal{L}^N(Q))^{1/p-1},\end{aligned}$$

where we used the fact that $(\frac{q}{p})' = \frac{q}{q-p}$. Hence,

$$\mathbb{H}^{1,\infty}(\mathbb{R}^N) \subseteq \mathbb{H}^{1,q}(\mathbb{R}^N) \subseteq \mathbb{H}^{1,p}(\mathbb{R}^N).$$

We will prove that all these spaces coincide and define the Hardy space $\mathbb{H}^1(\mathbb{R}^N)$ *as*

$$\mathbb{H}^1(\mathbb{R}^N) := \mathbb{H}^{1,\infty}(\mathbb{R}^N).$$

Theorem 135 *Let $1 < p \le \infty$. Then*

(i) $\mathbb{H}^{1,p}(\mathbb{R}^N) \subseteq \left\{ f \in L^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} f(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\}$*, with*

$$\|f\|_{L^1} \le \|f\|_{\mathbb{H}^{1,p}}$$

for all $f \in \mathbb{H}^{1,p}(\mathbb{R}^N)$;

(ii) $\mathbb{H}^{1,p}(\mathbb{R}^N)$ *is a Banach space;*

(iii) the set $\left\{ f \in C_c^{\infty}(\mathbb{R}^N) : \int_{\mathbb{R}^N} f(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\}$ *is dense in* $\mathbb{H}^{1,p}(\mathbb{R}^N)$.

Proof. (i) Note that if f is an atom, then by Hölder's inequality,

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|\,d\boldsymbol{x} = \int_Q |f(\boldsymbol{x})|\,d\boldsymbol{x} \le \|f\|_{L^p(\mathbb{R}^N)} (\mathcal{L}^N(Q))^{p/p'} \le 1.$$

Now let $f \in \mathbb{H}^{1,p}(\mathbb{R}^N)$ and write $f = \sum_{n=1}^{\infty} c_n f_n$, where $\{f_n\}$ is a sequence of p-atoms and $\sum_{n=1}^{\infty} |c_n| < \infty$. Then

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|\,d\boldsymbol{x} \le \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} |c_n f_n(\boldsymbol{x})|\,d\boldsymbol{x} \le \sum_{n=1}^{\infty} |c_n|\,1$$

and so taking the infimum over all possible decompositions, we get $\|f\|_{L^1} \le \|f\|_{\mathbb{H}^{1,p}}$. Moreover,

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\; d\boldsymbol{x} = \int_{\mathbb{R}^N} \sum_{n=1}^{\infty} c_n f_n(\boldsymbol{x})\; d\boldsymbol{x} = \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^N} f_n(\boldsymbol{x})\; d\boldsymbol{x} = 0.$$

(ii) Given a normed space X to prove that it is complete it is enough to show that for every sequence $\{x_n\} \subset X$ such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, we have that the series $\sum_{n=1}^{\infty} x_n$ converges in X. Let $\{f_n\} \subset \mathbb{H}^{1,p}(\mathbb{R}^N)$ be such that $\sum_{n=1}^{\infty} \|f_n\|_{\mathbb{H}^{1,p}} < \infty$. Then by part (a), $\sum_{n=1}^{\infty} \|f_n\|_{L^1} < \infty$, and so $f := \sum_{n=1}^{\infty} f_n$ belongs to $L^1(\mathbb{R}^N)$. Moreover, since

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|\,d\boldsymbol{x} \le \int_{\mathbb{R}^N} \sum_{n=1}^{\infty} |f_n(\boldsymbol{x})|\,d\boldsymbol{x} < \infty,$$

we have that $\sum_{n=1}^{\infty} |f_n(\boldsymbol{x})|\,d\boldsymbol{x} < \infty$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$.

Since $f_n \in \mathbb{H}^{1,p}(\mathbb{R}^N)$, given $\varepsilon > 0$ we can find a sequence $\left\{g_k^{(n)}\right\}$ is of p-atoms and $\sum_{k=1}^{\infty} \left|c_k^{(n)}\right| < \infty$ such that $f_n = \sum_{k=1}^{\infty} c_k^{(n)} f_k^{(n)}$ and

$$\sum_{k=1}^{\infty} \left|c_k^{(n)}\right| \le \|f_n\|_{\mathbb{H}^{1,p}} + \frac{\varepsilon}{2^n}.$$

Then

$$f = \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} c_k^{(n)} f_k^{(n)}$$

and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |c_k^{(n)}| \le \sum_{n=1}^{\infty} \left(\|f_n\|_{\mathbb{H}^{1,p}} + \frac{\varepsilon}{2^n} \right) < \infty.$$

This shows that f belongs to $\mathbb{H}^{1,p}(\mathbb{R}^N)$. Moreover,

$$f - \sum_{n=1}^{m} f_n = \sum_{n=m+1}^{\infty} f_n = \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} c_k^{(n)} f_k^{(n)}$$

and so

$$\left\| f - \sum_{n=1}^{m} f_n \right\|_{\mathbb{H}^{1,p}} \leq \sum_{n=m+1}^{\infty} \sum_{k=1}^{\infty} |c_k^{(n)}| \leq \sum_{n=m+1}^{\infty} \left(\|f_n\|_{\mathbb{H}^{1,p}} + \frac{\varepsilon}{2^n} \right) \to 0$$

as $m \to \infty$. This shows that $f = \sum_{n=1}^{\infty} f_n$ in $\mathbb{H}^{1,p}(\mathbb{R}^N)$.

(iii) Given $f \in \mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ and $\varepsilon > 0$ we can find a sequence $\{f_n\}$ is of p-atoms and $\sum_{n=1}^{\infty} |c_n| < \infty$ such that $f = \sum_{n=1}^{\infty} c_n f_n$ and

$$\sum_{n=1}^{\infty} |c_n| \leq \|f\|_{\mathbb{H}^{1,p}} + \varepsilon.$$

■

Remark 136 *Note that in view of the previous theorem, in the definition of* $\mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ *we could have asked for the series in (144) to converge to* f *in* $L^1\left(\mathbb{R}^N\right)$ *and not just pointwise almost everywhere.*

Theorem 137 *Let* $1 < q \leq \infty$ *and let* $f \in L^q\left(\mathbb{R}^N\right)$ *be such that the support of* f *is contained in some cube* Q *and* $\int_{\mathbb{R}^N} f(\boldsymbol{x})\ d\boldsymbol{x} = 0$. *Then* $f \in \mathbb{H}^{1,q}\left(\mathbb{R}^N\right)$ *with*

$$\|f\|_{\mathbb{H}^{1,q}(\mathbb{R}^N)} \leq \|f\|_{L^q(\mathbb{R}^N)} \left(\mathcal{L}^N(Q)\right)^{1/q'}.$$

Moreover, $f \in \mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ *for all* $1 < p \leq q$.

Proof. Assume $f \neq 0$. Then

$$\|tf\|_{L^q(\mathbb{R}^N)} = t\,\|f\|_{L^q(Q)} \leq \left(\mathcal{L}^N(Q)\right)^{1/q-1},$$

provided we take

$$t := \left(\mathcal{L}^N(Q)\right)^{1/q-1} / \|f\|_{L^q(Q)}.$$

It follows that the function tf is a q-atom. In turn, $f \in \mathbb{H}^{1,q}\left(\mathbb{R}^N\right)$ with

$$\|f\|_{\mathbb{H}^{1,q}(\mathbb{R}^N)} = \|(tf)1/t\|_{\mathbb{H}^{1,q}(\mathbb{R}^N)} \leq 1/t = \|f\|_{L^q(\mathbb{R}^N)} \left(\mathcal{L}^N(Q)\right)^{1/q'}.$$

The last part of the result follows from by Remark 134. ■

Theorem 138 *If* $f \in \mathbb{H}^1\left(\mathbb{R}^N\right)$ *is nonnegative in an open set* Ω *then* $f\ \log(2+f) \in L^1_{\mathrm{loc}}(\Omega)$.

Example 139 *Property (v) in Theorem 135 is sharp. Indeed the function*

$$f(x) := \begin{cases} \frac{1}{|x-1|\log^2|x-1|} & 0 < x < 2,\ x \neq 1, \\ \frac{-11}{|x-1|\log^2|x-1|} & -2 < x < 0,\ x \neq -1, \\ 0 & \textit{elsewhere} \end{cases}$$

is in $L^1(\mathbb{R})$, *the support of* f *is compact and* $\int_{\mathbb{R}} f(x)\ dx = 0$, *but* $f \notin \mathbb{H}^1\left(\mathbb{R}^N\right)$ *by Theorem 138.*

Friday, March 27, 2015

7.2 Duals

Let $1 < p \leq \infty$ and let $\mathbb{H}_a^{1,p}\left(\mathbb{R}^N\right)$ be the subspace of $\mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ given by all finite linear combinations of atoms. Note that $\mathbb{H}_a^{1,p}\left(\mathbb{R}^N\right)$ is given by all functions g in $L^p\left(\mathbb{R}^N\right)$ with compact support and $\int_{\mathbb{R}^N} g(\boldsymbol{x})\, d\boldsymbol{x} = 0$. Hence, $\mathbb{H}_a^{1,p}\left(\mathbb{R}^N\right)$ is dense in $\mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ by Theorem 135(iii).

Given $g \in \mathrm{BMO}\left(\mathbb{R}^N\right)$ we can define

$$T_g(f) := \int_{\mathbb{R}^N} f g\, d\mathbf{x}, \quad f \in \mathbb{H}_a^{1,p}\left(\mathbb{R}^N\right). \tag{145}$$

Let's prove that T_g is well-defined. Given $f \in \mathbb{H}_a^{1,p}\left(\mathbb{R}^N\right)$, by Definition 132 we have that $f \in L^p\left(\mathbb{R}^N\right)$ and there exists a cube $Q_0 \subset \mathbb{R}^N$ such that $\operatorname{supp} f \subseteq Q_0$. Hence,

$$\int_{\mathbb{R}^N} f g\, d\mathbf{x} = \int_{Q_0} f g\, d\mathbf{x},$$

which is well-defined since $g \in L_{\mathrm{loc}}^{p'}\left(\mathbb{R}^N\right)$ by Theorem 119. Note that since $\int_{\mathbb{R}^N} f\, d\boldsymbol{x} = 0$, we have that $T_g = T_{g+c}$ for any constant $c \in \mathbb{C}$.

The following theorem has been proved by Fefferman.

Theorem 140 (Duality of $\mathbb{H}^{1,p}$ and BMO) *Let $1 < p \leq \infty$. If $g \in \mathrm{BMO}\left(\mathbb{R}^N\right)$ then the linear functional T_g defined in* (145) *has a unique bounded extension to $\mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$ and furthermore*

$$C_1 \|g\|_{\mathrm{BMO}} \leq \|T_g\|_{(\mathbb{H}^{1,p})'} \leq C_2 \|g\|_{\mathrm{BMO}}$$

for some constants $C_1, C_2 > 0$ depending only N and p. Conversely, if $1 < p < \infty$ and if $T : \mathbb{H}^{1,p}\left(\mathbb{R}^N\right) \to \mathbb{R}$ is a continuous linear functional, then there exists a unique $g \in \mathrm{BMO}\left(\mathbb{R}^N\right)$ such that $T = T_g$ with

$$C_3 \|T\|_{(\mathbb{H}^{1,p})'} \leq \|g\|_{\mathrm{BMO}} \leq C_4 \|T\|_{(\mathbb{H}^{1,p})'},$$

for some constants $C_3, C_4 > 0$ depending only N and p.

Proof. Step 1: Let $1 < p \leq \infty$ and let $g \in \mathrm{BMO}\left(\mathbb{R}^N\right)$. Assume first that $g \in L^\infty\left(\mathbb{R}^N\right)$. Given $f \in \mathbb{H}^{1,p}\left(\mathbb{R}^N\right)$, by the decomposition theorem, there exist a sequence $\{f_n\} \subset L^p\left(\mathbb{R}^N\right)$ of atoms and $\{c_n\} \subset \mathbb{C}$ such that

$$f = \sum_{n=1}^{\infty} c_n f_n, \quad \sum_{n=1}^{\infty} |c_n| < \infty.$$

By Theorem 135(i), $f \in L^1\left(\mathbb{R}^N\right)$ and since $g \in L^\infty\left(\mathbb{R}^N\right)$, we have that $T_g(f)$ is well-defined with

$$T_g(f) = \int_{\mathbb{R}^N} f g\, d\mathbf{x} = \sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^N} f_n g\, d\mathbf{x}.$$

Since f_n is an atom, there exists a cube $Q_n \subset \mathbb{R}^N$ such that

$$\operatorname{supp} f_n \subseteq Q_n, \quad \|f_n\|_{L^p(\mathbb{R}^N)} \leq (\mathcal{L}^N(Q_n))^{1/p-1}$$

and

$$\int_{\mathbb{R}^N} f_n(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Hence,

$$\int_{\mathbb{R}^N} f_n g\, d\mathbf{x} = \int_{Q_n} f_n g\, d\mathbf{x} = \int_{Q_n} f_n(g - g_{Q_n})\, d\mathbf{x}.$$

In turn, by Hölder's inequality,

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f_n g\, d\mathbf{x}\right| = \left|\int_{Q_n} f_n(g - g_{Q_n})\, d\mathbf{x}\right| &\leq \|f_n\|_{L^p(\mathbb{R}^N)} \|g - g_{Q_n}\|_{L^{p'}(\mathbb{R}^N)} \\ &\leq (\mathcal{L}^N(Q_n))^{1/p'} \|g - g_{Q_n}\|_{L^{p'}(\mathbb{R}^N)} \leq c\|g\|_{\mathrm{BMO}},\end{aligned}$$

where we have used Remark 122.

It follows that

$$|T_g(f)| \leq c\|g\|_{\mathrm{BMO}} \sum_{n=1}^{\infty} |c_n|$$

and taking the infimum over all possible decompositions of f gives

$$|T_g(f)| \leq c\|g\|_{\mathrm{BMO}} \|f\|_{\mathbb{H}^{1,p}}.$$

Now if $g \in \mathrm{BMO}(\mathbb{R}^N)$, define

$$g_k(\boldsymbol{x}) := \begin{cases} g(\boldsymbol{x}) & \text{if } |g(\boldsymbol{x})| \leq k, \\ g(\boldsymbol{x})k/|g(\boldsymbol{x})| & \text{if } |g(\boldsymbol{x})| > k. \end{cases}$$

Then $g_k(\boldsymbol{x}) \to g(\boldsymbol{x})$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x}$. By the first part of the proof,

$$\left|\int_{\mathbb{R}^N} f g_k\, d\mathbf{x}\right| \leq \|g_k\|_{\mathrm{BMO}} \|f\|_{\mathbb{H}^{1,p}}$$

for all k and all $f \in \mathbb{H}^{1,p}(\mathbb{R}^N)$. In particular, if $f \in \mathbb{H}_a^{1,p}(\mathbb{R}^N)$, then since $f \in L^p(\mathbb{R}^N)$ and has compact support and $|g_k| \leq |g|$ which belongs to $L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$ by Theorem 119, it follows that we are in a position to apply the Lebesgue dominated convergence theorem to conclude that

$$\left|\int_{\mathbb{R}^N} f g\, d\mathbf{x}\right| \leq \lim_{k\to\infty} \|g\|_{\mathrm{BMO}} \|f\|_{\mathbb{H}^{1,p}} = \|g\|_{\mathrm{BMO}} \|f\|_{\mathbb{H}^{1,p}}$$

for all $f \in \mathbb{H}_a^{1,p}(\mathbb{R}^N)$. Hence, $T_g : \mathbb{H}_a^{1,p}(\mathbb{R}^N) \to \mathbb{C}$ is a bounded linear functional. Since $\mathbb{H}_a^{1,p}(\mathbb{R}^N)$ is dense in $\mathbb{H}^{1,p}(\mathbb{R}^N)$, the functional T_g admits a unique continuous extension to $\mathbb{H}^{1,p}(\mathbb{R}^N)$. Moreover,

$$\|T_g\|_{(\mathbb{H}^{1,p})'} \leq C_2 \|g\|_{\mathrm{BMO}}.$$

It remains to prove the inequality

$$C_1 \|g\|_{\text{BMO}} \leq \|T_g\|_{(\mathbb{H}^{1,p})'} .$$

We only prove this for $p = \infty$, since the case $1 < p < \infty$ follows from Step 2 below. Assume that g is real-valued. Fix a cube Q and consider the function

$$\varpi_g(s) := \mathcal{L}^N \left(\{ \boldsymbol{x} \in Q : \, g(\boldsymbol{x}) > s \} \right), \quad s \in \mathbb{R}.$$

Since $g \in L^1(Q)$, reasoning as in Proposition 1 and Exercise 2 we have that ϖ_g is decreasing, right-continuous,

$$\lim_{s\to\infty} \varpi_g(s) = 0, \quad \lim_{s\to-\infty} \varpi_g(s) = \mathcal{L}^N(Q),$$

and

$$(\varpi_g)_- (s) = \varpi_g (s) + \mathcal{L}^N \left(\{ \boldsymbol{x} \in Q : \, g(\boldsymbol{x}) = s \} \right).$$

Hence, there exists $s_Q \in \mathbb{R}$ such that either $\varpi_g(s_Q) = \frac{1}{2}\mathcal{L}^N(Q)$ or

$$\begin{aligned} \varpi_g(s_Q) &< \frac{1}{2}\mathcal{L}^N(Q) \leq (\varpi_g)_- (s_Q) \\ &= \varpi_g (s_Q) + \mathcal{L}^N \left(\{ \boldsymbol{x} \in Q : \, g(\boldsymbol{x}) = s_Q \} \right). \end{aligned}$$

In the second case, $\varpi_g(s_Q) < \frac{1}{2}\mathcal{L}^N(Q)$, we have that

$$\mathcal{L}^N \left(\{ \boldsymbol{x} \in Q : \, g(\boldsymbol{x}) = s_Q \} \right) \geq \frac{1}{2}\mathcal{L}^N(Q) - \varpi_g (s_Q) > 0$$

and thus we can find a measurable subset $E \subseteq \{ \boldsymbol{x} \in Q : \, g(\boldsymbol{x}) = s_Q \}$ such that

$$\mathcal{L}^N (E) = \frac{1}{2}\mathcal{L}^N(Q) - \varpi_g (s_Q) .$$

If $\varpi_g(s_Q) = \frac{1}{2}\mathcal{L}^N(Q)$, take E to be the empty set. Define

$$E_+ := \{ \boldsymbol{x} \in Q : \; g(\boldsymbol{x}) > s_Q \} \cup E, \quad E_- := Q \setminus E_+.$$

Then $\mathcal{L}^N (E_+) = \frac{1}{2}\mathcal{L}^N(Q)$ and, in turn, $\mathcal{L}^N (E_-) = \frac{1}{2}\mathcal{L}^N(Q)$. Define $f(\boldsymbol{x}) := 1$ if $\boldsymbol{x} \in E_+$, $f(\boldsymbol{x}) := -1$ if $\boldsymbol{x} \in E_-$, $f(\boldsymbol{x}) := 0$ if $\boldsymbol{x} \in \mathbb{R}^N \setminus Q$. Then

$$\int_{\mathbb{R}^n} f \; d\boldsymbol{x} = \int_Q f \; d\boldsymbol{x} = \mathcal{L}^N(E_+) - \mathcal{L}^N(E_-) = 0.$$

Moreover, since $|f| = 1$ in Q we have that $(\mathcal{L}^N(Q))^{-1} f$ is an ∞ atom, and so f belongs to $\mathbb{H}^{1,\infty} \left(\mathbb{R}^N \right)$ and

$$\|f\|_{\mathbb{H}^{1,\infty}(\mathbb{R}^N)} = \mathcal{L}^N(Q) \left\| (\mathcal{L}^N(Q))^{-1} f \right\|_{\mathbb{H}^{1,\infty}(\mathbb{R}^N)} \leq \mathcal{L}^N(Q).$$

On the other hand

$$\int_Q |g(\boldsymbol{x}) - s_Q| \, d\boldsymbol{x} = \int_{E_+} (g(\boldsymbol{x}) - s_Q) \, d\boldsymbol{x} - \int_{E_-} (g(\boldsymbol{x}) - s_Q) \, d\boldsymbol{x}$$
$$= \int_Q (g(\boldsymbol{x}) - s_Q) f(x) \, d\boldsymbol{x}$$
$$= \int_Q g(\boldsymbol{x}) f(x) \, d\boldsymbol{x} = T_g(f).$$

In turn,

$$\frac{1}{\mathcal{L}^N(Q)} \int_Q |g(\boldsymbol{x}) - s_Q| \, d\boldsymbol{x} = \frac{1}{\mathcal{L}^N(Q)} T_g(f) \le \frac{1}{\mathcal{L}^N(Q)} \|T_g\|_{(\mathbb{H}^{1,\infty})'} \|f\|_{\mathbb{H}^{1,\infty}(\mathbb{R}^N)}$$
$$\le \|T_g\|_{(\mathbb{H}^{1,\infty})'},$$

which, by Remark 122, implies that

$$\frac{1}{2} \|g\|_{\mathrm{BMO}} \le \|T_g\|_{(\mathbb{H}^{1,\infty})'}.$$

If $g = g_1 + \boldsymbol{i} g_2$ with g_1 and g_2 real-valued, we repeat the previous reasoning for g_1 and g_2 to find two real numbers s_Q^1 and s_Q^2 and two functions f_1 and f_2 . We then consider $f := f_1 + \boldsymbol{i} f_2 \in \mathbb{H}^{1,\infty}(\mathbb{R}^N; \mathbb{C})$ and $s_Q := s_Q^1 + \boldsymbol{i} s_Q^2$ and proceed as above. We omit the details. ■

Monday, March 30, 2015

Proof. Step 2: Conversely, let $1 < p < \infty$ and let $T : \mathbb{H}^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be a continuous linear functional. Fix $n \in \mathbb{N}$, let $Q_n := [-n, n]^N$ and consider the space

$$L^p_{Q_n} := \left\{ f \in L^p(\mathbb{R}^N) : \ \operatorname{supp} f \subseteq Q_n, \quad \int_{\mathbb{R}^N} f(\boldsymbol{x}) \, d\boldsymbol{x} = 0 \right\}.$$

By Theorem 137, if $f \in L^p_{Q_n}$, then $f \in \mathbb{H}^{1,p}(\mathbb{R}^N)$ with

$$\|f\|_{\mathbb{H}^{1,p}(\mathbb{R}^N)} \le \|f\|_{L^p(\mathbb{R}^N)} (\mathcal{L}^N(Q_n))^{1/p'}.$$

Hence,

$$|T(f)| \le \|T\|_{(\mathbb{H}^{1,p})'} \|f\|_{\mathbb{H}^{1,p}} \le \|T\|_{(\mathbb{H}^{1,p})'} (\mathcal{L}^N(Q_n))^{1/p'} \|f\|_{L^p}. \tag{146}$$

It follows that $T : L^p_n \to \mathbb{R}$ is a continuous linear functional. Since the dual of $L^p_{Q_n}$ can be identified with $L^{p'}(Q_n)/\mathbb{R}$, by the Riesz representation theorem there exists a unique (up to constants) function $g_n \in L^{p'}(Q_n)$ such that

$$T(f) = \int_{Q_n} f g_n \, d\boldsymbol{x} \tag{147}$$

for all $f \in L^p_{Q_n}$. To fix the constant, assume that take $h_n := g_n - \int_{Q_1} g_n \, d\boldsymbol{x}$, so that

$$\int_{Q_1} h_n \, d\boldsymbol{x} = 0. \tag{148}$$

Since $\int_{\mathbb{R}^N} f(\boldsymbol{x}) \, d\boldsymbol{x} = 0$ for all $f \in L^p_{Q_n}$, for every $c \in \mathbb{R}$ we have that

$$T(f) = \int_{Q_n} f h_n \, d\boldsymbol{x}$$

for all $f \in L^p_{Q_n}$. Hence, by (146) and (147),

$$\|h_n\|_{L^{p'}(Q_n)} \leq \|T\|_{(\mathbb{H}^{1,p})'} (\mathcal{L}^N(Q_n))^{1/p'}. \tag{149}$$

On the other hand, since $L^p_{Q_n} \subset L^p_{Q_{n+1}}$, we must have

$$T(f) = \int_{Q_{n+1}} f h_{n+1} \, d\boldsymbol{x} = \int_{Q_n} f h_n \, d\boldsymbol{x}$$

for all $f \in L^p_{Q_n}$, which by uniqueness, implies that $h_{n+1}(\boldsymbol{x}) = h_n(\boldsymbol{x}) + c_{n+1}$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in Q_n$ and for some constant $c_{n+1} \in \mathbb{C}$. In view of (148), $c_{n+1} = 0$. Hence, we can define a function $h \in L^{p'}_{\text{loc}}(\mathbb{R}^N)$ by $h(\boldsymbol{x}) := h_n(\boldsymbol{x})$ for $\boldsymbol{x} \in Q_n$. Note that by (149),

$$\|h\|_{L^{p'}(Q_n)} \leq \|T\|_{(\mathbb{H}^{1,p})'} (\mathcal{L}^N(Q_n))^{1/p'}. \tag{150}$$

We claim that h belongs to $\text{BMO}(\mathbb{R}^N)$. To see this, let Q be any cube. By repeating the same argument with Q in place of Q_n we can find a function $g_Q \in L^{p'}(Q)$ with such that

$$T(f) = \int_Q f g_Q \, d\boldsymbol{x}$$

for all $f \in L^p_Q$. Extend g_Q to be zero outside Q and define $h_Q := g_Q - \int_{Q_1} g_Q \, d\boldsymbol{x}$. Then as before $h_Q(\boldsymbol{x}) = h(\boldsymbol{x})$ for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in Q$ and

$$\|h\|_{L^{p'}(Q)} \leq \|T\|_{(\mathbb{H}^{1,p})'} (\mathcal{L}^N(Q))^{1/p'}. \tag{151}$$

Then by Hölder's inequality

$$\frac{1}{\mathcal{L}^N(Q)} \int_Q |h(\boldsymbol{x})| \, d\boldsymbol{x} \leq \frac{1}{\mathcal{L}^N(Q)} \|h\|_{L^{p'}(Q)} (\mathcal{L}^N(Q))^{1/p} \leq \|T\|_{(\mathbb{H}^{1,p})'}.$$

Hence, by Remark 112, $h \in \text{BMO}(\mathbb{R}^N)$ with

$$\frac{1}{2} |h|_{\text{BMO}} \leq \|T\|_{(\mathbb{H}^{1,p})'}.$$

We can now continue as in the proof of Step 1 to conclude that $T = T_h$. ■

We are finally ready to prove that all the spaces $\mathbb{H}^{1,p}(\mathbb{R}^N)$ coincide. The proof relies on the following theorem.

Theorem 141 *Let X, Y be Banach spaces and let $L : X \to Y$ be a bounded linear operator. Then $L(X)$ is closed in Y if and only if $L^*(Y')$ is closed in X', where $L^* : Y' \to X'$ is the transpose of L.*

Theorem 142 *For $1 < p < \infty$,*

$$\mathbb{H}^{1,p}(\mathbb{R}^N) = \mathbb{H}^{1,\infty}(\mathbb{R}^N)$$

and the dual of $\mathbb{H}^{1,\infty}(\mathbb{R}^N)$ can be identified with $\mathrm{BMO}(\mathbb{R}^N)$.

Proof. We apply the previous theorem with $X = \mathbb{H}^{1,\infty}(\mathbb{R}^N)$ and $Y = \mathbb{H}^{1,p}(\mathbb{R}^N)$ and $L(f) := f$ (see Remark 134). Then $L^* : (\mathbb{H}^{1,p}(\mathbb{R}^N))' \to (\mathbb{H}^{1,\infty}(\mathbb{R}^N))'$ is still given by the canonical inclusion, that is, $L^*(T)$ is simply T restricted to $\mathbb{H}^{1,\infty}(\mathbb{R}^N)$. By Theorem 140, for every $T \in (\mathbb{H}^{1,p}(\mathbb{R}^N))'$ there exists a unique (up to constants) function $g \in \mathrm{BMO}(\mathbb{R}^N)$ such that $T = T_g$ and

$$C_1 \|T\|_{(\mathbb{H}^{1,p})'} \le |g|_{\mathrm{BMO}} \le C_2 \|T\|_{(\mathbb{H}^{1,p})'} . \tag{152}$$

On the other hand, by the first part of Theorem 140, the functional $T = T_g : \mathbb{H}^{1,\infty}(\mathbb{R}^N) \to \mathbb{R}$ has the property that

$$C_3 |g|_{\mathrm{BMO}} \le \|T\|_{(\mathbb{H}^{1,\infty})'} \le C_4 |g|_{\mathrm{BMO}} . \tag{153}$$

Hence,

$$\|L^*(T)\|_{(\mathbb{H}^{1,\infty})'} = \|T\|_{(\mathbb{H}^{1,\infty})'} \ge C_3 |g|_{\mathrm{BMO}} \ge \frac{C_3}{C_1} \|T\|_{(\mathbb{H}^{1,p})'} .$$

This implies that $L^*((\mathbb{H}^{1,p}(\mathbb{R}^N))')$ is closed in $(\mathbb{H}^{1,\infty}(\mathbb{R}^N))'$. In view of Theorem 141, we have that $L(\mathbb{H}^{1,\infty}(\mathbb{R}^N)) = \mathbb{H}^{1,\infty}(\mathbb{R}^N)$ is closed in $\mathbb{H}^{1,p}(\mathbb{R}^N)$. To prove that they coincide, in view of the Hahn–Banach theorem, it is enough to show that any functional $T \in (\mathbb{H}^{1,p}(\mathbb{R}^N))'$ that vanishes on $\mathbb{H}^{1,\infty}(\mathbb{R}^N)$ also vanishes on $\mathbb{H}^{1,p}(\mathbb{R}^N)$. But reasoning as before, we have that $T = T_g$ for some $g \in \mathrm{BMO}(\mathbb{R}^N)$. Since T vanishes on $\mathbb{H}^{1,\infty}(\mathbb{R}^N)$ it follows from (153) that $g = 0$ (up to a constant) and so $|g|_{\mathrm{BMO}(\mathbb{R}^N)} = 0$. In turn, $T = 0$ on $\mathbb{H}^{1,p}(\mathbb{R}^N)$ by (152). ■

Wednesday, April 1, 2015

Next we prove a compactness result in $\mathbb{H}^1(\mathbb{R}^N)$.

Theorem 143 (Compactness) *Let $\{f_m\}$ be a bounded sequence in $\mathbb{H}^1(\mathbb{R}^N)$. Then there exist a subsequence $\{f_{m_i}\}$ and $f \in \mathbb{H}^1(\mathbb{R}^N)$ such that*

$$\lim_{i\to\infty} \int_{\mathbb{R}^N} f_{m_i}(\boldsymbol{x}) g(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\mathbb{R}^N} f(\boldsymbol{x}) g(\boldsymbol{x}) \, d\boldsymbol{x} \tag{154}$$

for all $g \in C_c\left(\mathbb{R}^N\right)$.

We begin with some preliminary lemmas.

Lemma 144 *There exists a sequence of cubes $Q^*_{l,k}$, $l \in \mathbb{N}$, $k \in \mathbb{Z}$, such that for every $k \in \mathbb{Z}$ each cube $Q^*_{l,k}$ has side-length 3^{k+1} and intersects at most $(12)^N$ cubes $Q^*_{i,k}$. Moreover every $f \in \mathbb{H}^1(\mathbb{R}^N)$ can be written as*

$$f = \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{l,k} f_{l,k},$$

where $f_{l,k}$ are ∞-atoms supported in $Q^*_{l,k}$, and

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} |c_{l,k}| \leq 3^{N+1} \|f\|_{\mathbb{H}^1(\mathbb{R}^N)} .$$

Proof. Given any cube Q with side-length $r > 0$ let $3Q$ be the cube with the same center and side-length $3r$. Note that $3Q$ contains all cubes with side-length r and which intersect Q. For each $k \in \mathbb{Z}$ decompose $\mathbb{R}^N$ into a sequence of cubes $\{Q_{l,k}\}_{l \in \mathbb{N}}$ with side-length 3^k and pairwise disjoint interiors. Define $Q^*_{l,k} := 3Q_{l,k}$.

Step 1: Given $f \in \mathbb{H}^1(\mathbb{R}^N)$, consider an admissible decomposition $f = \sum_{n=1}^{\infty} c_n f_n$ where each ∞-atom has support in some cube $Q^*_{l,k}$, $l \in \mathbb{N}$, $k \in \mathbb{Z}$. Let $\|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)}$ be the infimum of $\sum_{n=1}^{\infty} |c_n|$ over all such decompositions. We claim that

$$\|f\|_{\mathbb{H}^1(\mathbb{R}^N)} \leq \|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)} \leq 3^N \|f\|_{\mathbb{H}^1(\mathbb{R}^N)} . \tag{155}$$

Indeed, consider an arbitrary decomposition $f = \sum_{n=1}^{\infty} c_n f_n$. Then for each n there exists a cube a cube $Q_n \subset \mathbb{R}^N$ such that

$$\operatorname{supp} f_n \subseteq Q_n, \quad \|f\|_{L^\infty(\mathbb{R}^N)} \leq (\mathcal{L}^N(Q_n))^{-1}.$$

If r_n is the side-length of Q_n, then we can find an integer k_n such that

$$3^{k_n - 1} < r_n \leq 3^{k_n}.$$

Since the cubes $\{Q_{l,k_n}\}_{l \in \mathbb{N}}$ cover $\mathbb{R}^N$ there exists a cube Q_{l_n,k_n} which intersects Q_n. Then $\operatorname{supp} f_n \subseteq Q_n \subseteq Q^*_{l_n,k_n}$ and

$$\|f_n\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{\mathcal{L}^N(Q_n)} = \frac{1}{r_n^N} \leq \frac{1}{(3^{k_n - 1})^N} = \frac{3^N}{\mathcal{L}^N(Q^*_{l_n,k_n})}.$$

Hence, $\frac{1}{3^N} f_n$ is an ∞-atom, and so writing

$$f = \sum_{n=1}^{\infty} 3^N c_n \left(\frac{1}{3^N} f_n \right),$$

we have an admissible decomposition for $\|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)}$, so that

$$\|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)} \leq 3^N \sum_{n=1}^{\infty} |c_n| .$$

Taking the infimum over all such decomposition gives

$$\|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)} \leq 3^N \|f\|_{\mathbb{H}^1(\mathbb{R}^N)} .$$

Step 2: Given $f \in \mathbb{H}^1(\mathbb{R}^N)$, we can find an admissible decomposition $f = \sum_{n=1}^{\infty} c_n f_n$ for $\|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)}$ such that

$$\sum_{l=1}^{\infty} |c_n| \leq 3 \|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)} . \tag{156}$$

Let $k = 0$ and consider all atoms f_n supported in $Q^*_{1,0}$. The sum of terms $c_n f_n$ of all such atoms can be written as $c_{1,1} f_{1,1}$, where $f_{1,1}$ is an atom supported on $Q^*_{1,0}$. Moreover the side-length of $Q^*_{1,0}$ is 3. Now consider all remaining atoms supported in $Q^*_{2,0}$ and write the sum of all such atoms as $c_{2,0} f_{2,0}$, where $f_{2,0}$ is an atom supported on $Q^*_{2,0}$. Continue inductively until all cubes $Q^*_{l,0}$ have been exhausted.

Next consider $k = -1$ and consider all atoms f_n supported in $Q^*_{1,-1}$. Write the sum of all such atoms as $c_{1,-1} f_{1,-1}$, where $f_{1,-1}$ is an atom supported on $Q^*_{1,-1}$. Note that the side-length of $Q^*_{1,-1}$ is 1. Continue inductively until all cubes $Q^*_{l,-1}$ have been exhausted. Next consider $k = 1$ and cubes $Q^*_{1,1}$, and so on.

Then $c_{l,k} f_{l,k}$ is a sum of the type $g_{l,k} := \sum_n c_n f_n$ over all n such that f_n has supporting cube $Q^*_{l,k}$ of side-length 3^{k+1}. Then

$$\frac{g_{l,k}}{\|g_{l,k}\|_{L^\infty(\mathbb{R}^N)} (3^{k+1})^N}$$

is an ∞-atom supported in $Q^*_{l,k}$. Define

$$c_{l,k} := \|g_{l,k}\|_{L^\infty(\mathbb{R}^N)} (3^{k+1})^N$$

Then

$$\begin{aligned} |c_{l,k}| = \|g_{l,k}\|_{L^\infty(\mathbb{R}^N)} (3^{k+1})^N &= \left\| \sum_n c_n f_n \right\|_{L^\infty(\mathbb{R}^N)} (3^{k+1})^N \\ \leq \sum_n \frac{|c_n|}{\mathcal{L}^N(Q^*_{l,k})} (3^{k+1})^N &= \sum_n |c_n|. \end{aligned}$$

This shows that

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} |c_{l,k}| \leq \sum_{l=1}^{\infty} |c_n| \leq 3 \|f\|^*_{\mathbb{H}^1(\mathbb{R}^N)} \leq 3^{N+1} \|f\|_{\mathbb{H}^1(\mathbb{R}^N)} ,$$

where we have used (155) and (156). ■

Lemma 145 *Given a sequence of nonnegative numbers* $c_j^{(m)}$, $j, m \in \mathbb{N}$, *such that*

$$\sum_{j=1}^{\infty} c_j^{(m)} \leq 1 \quad \textit{for every } m,$$

there exists a strictly increasing sequence $m_1 < m_2 < \cdots < m_i < \cdots$ *such that there exists*

$$\lim_{i \to \infty} c_j^{(m_i)} := c_j \quad \textit{for every } j,$$

and

$$\sum_{j=1}^{\infty} c_j \leq 1.$$

The proof is left as an exercise.

Friday, April 3, 2015

Next we turn to the proof of Theorem 143.

Proof of Theorem 143. Without loss of generality assume that $\|f_m\|_{\mathbb{H}^1} \leq 1/3^{N+1}$. Write f_m as in the previous lemma, that is,

$$f_m = \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{l,k}^{(m)} f_{l,k}^{(m)},$$

where $f_{l,k}^{(m)}$ are ∞-atoms supported in $Q_{l,k}^*$, and

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} |c_{l,k}^{(m)}| \leq 3^{N+1} \|f_m\|_{\mathbb{H}^1(\mathbb{R}^N)}.$$

Without loss of generality we may assume that $c_{l,k}^{(m)} \geq 0$. Then

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{l,k}^{(m)} \leq 3^{N+1} \|f_m\|_{\mathbb{H}^1(\mathbb{R}^N)} \leq 1.$$

By Lemma 145, up to a subsequence, not relabeled, we can assume that for every l and k there exists

$$\lim_{m \to \infty} c_{l,k}^{(m)} =: c_{l,k}$$

and that

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{l,k} \leq 1. \tag{157}$$

Moreover, for each fixed l and k all the atoms $f_{l,k}^{(m)}$ are supported in $Q_{l,k}^*$ and are uniformly bounded in $L^\infty(Q_{l,k}^*)$. Hence, there exists a subsequence (depending on l and k) of $\{f_{l,k}^{(m)}\}_m$ that converges weakly star to some function $f_{l,k}$ in

$L^\infty(Q^*_{l,k})$. Using a diagonal argument, we can find a sunsequence $\{f^{(m_i)}_{l,k}\}_{m_i}$ of $\{f^{(m)}_{l,k}\}_m$ such that $f^{(m_i)}_{l,k} \overset{*}{\rightharpoonup} f_{l,k}$ in $L^\infty(Q^*_{l,k})$ for every l and k, that is,

$$\lim_{i\to\infty} \int_{Q^*_{l,k}} f^{(m_i)}_{l,k}(\boldsymbol{x})g(\boldsymbol{x})\, d\boldsymbol{x} = \int_{Q^*_{l,k}} f_{l,k}(\boldsymbol{x})g(\boldsymbol{x})\, d\boldsymbol{x}$$

for all $g \in L^1(Q^*_{l,k})$. Define

$$f := \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} c_{l,k} f_{l,k}.$$

Then $f \in \mathbb{H}^1(\mathbb{R}^N)$, since $f_{l,k}$ is still an ∞-atom and (157) holds. It remains to show that (154) holds. Let $g \in C_c\left(\mathbb{R}^N\right)$. Write

$$\begin{aligned}
\int_{\mathbb{R}^N} f_{m_i} g\, d\boldsymbol{x} &= \sum_{k=-\infty}^{\infty} \sum_{l=1}^{\infty} c^{(m_i)}_{l,k} \int_{\mathbb{R}^N} f^{(m_i)}_{l,k} g\, d\boldsymbol{x} \\
&= \left(\sum_{k=-L}^{L} + \sum_{k<-L} + \sum_{k>L} \right) \left(\sum_{l=1}^{\infty} c^{(m_i)}_{l,k} \int_{\mathbb{R}^N} f^{(m_i)}_{l,k} g\, d\boldsymbol{x} \right) \\
&=: I_i + II_i + III_i.
\end{aligned}$$

We also decompose $\int_{\mathbb{R}^N} fg\, d\boldsymbol{x}$ in the same way,

$$\int_{\mathbb{R}^N} fg\, d\boldsymbol{x} = I + II + III.$$

Since $f^{(m_i)}_{l,k}$ is an ∞-atom supported in $Q^*_{l,k}$ given $\varepsilon > 0$ we have

$$\begin{aligned}
|III_i| &\le \sum_{k>L} \sum_{l=1}^{\infty} c^{(m_i)}_{l,k} \left\| f^{(m_i)}_{l,k} \right\|_{L^\infty} \|g\|_{L^1} \\
&\le \|g\|_{L^1} \sum_{k>L} \sum_{l=1}^{\infty} \frac{c^{(m_i)}_{l,k}}{\mathcal{L}^N(Q^*_{l,k})} = \|g\|_{L^1} \sum_{k>L} \sum_{l=1}^{\infty} \frac{c^{(m_i)}_{l,k}}{3^{(k+1)N}} \\
&\le \frac{1}{3^{(L+1)N}} \sum_{k>L} \sum_{l=1}^{\infty} c^{(m_i)}_{l,k} \le \frac{2}{3^{(L+1)N}} \le \varepsilon
\end{aligned}$$

for all L sufficiently large. Note that a similar estimate holds for $|III|$. On the other hand, since g is uniformly continuous, there exists $\delta > 0$ such that

$$|g(\boldsymbol{x}) - g(\boldsymbol{y})| \le \varepsilon$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ with $|\boldsymbol{x} - \boldsymbol{y}| \le \delta$. Write $Q^*_{l,k} = Q^*_{l,k}(\boldsymbol{x}_{l,k}, 3^{k+1})$. If L is sufficiently large, we have that $\sqrt{N} 3^{k+1} \le \delta$ for all $k \le -L$. Hence, since

$\int_{\mathbb{R}^N} f_{l,k}^{(m_i)}\, d\boldsymbol{x} = 0$,

$$|II_i| = \left| \sum_{k<-L} \sum_{l=1}^{\infty} c_{l,k}^{(m_i)} \int_{Q_{l,k}^*} (f_{l,k}^{(m_i)})(\boldsymbol{x})(g(\boldsymbol{x}) - g(\boldsymbol{x}_{l,k}))\, d\boldsymbol{x} \right|$$
$$\leq \sum_{k<-L} \sum_{l=1}^{\infty} c_{l,k}^{(m_i)} \left\| f_{l,k}^{(m_i)} \right\|_{L^\infty} \mathcal{L}^N(Q_{l,k}^*)\varepsilon \leq \sum_{k<-L} \sum_{l=1}^{\infty} c_{l,k}^{(m_i)} \varepsilon \leq \varepsilon.$$

Similarly, we have that $|II| \leq \varepsilon$.

Finally, since g has compact support and the cubes $Q_{l,k}^*$ for $k = -L, \ldots, L$ have side-length which varies from 3^{-L+1} to 3^{L+1} and each cube $Q_{l,k}^*$ intersects at most $(12)^N$ cubes $Q_{i,k}^*$, we have that there are only finitely many cubes $Q_{l,k}^*$ that intersect the support of g and this number is independent of m_i. Let I_L be the set of pairs (i, k) corresponding to these cubes. Then for $\boldsymbol{x} \in \operatorname{supp} g$,

$$\sum_{(l,k)\in I_L} c_{l,k}^{(m_i)} |f_{l,k}^{(m_i)}(\boldsymbol{x}) g(\boldsymbol{x})| \leq \sum_{(l,k)\in I_L} c_{l,k}^{(m_i)} \left\| f_{l,k}^{(m_i)} \right\|_{L^\infty} \|g\|_{L^\infty}$$
$$\leq \sum_{(l,k)\in I_L} \frac{c_{l,k}^{(m_i)}}{\mathcal{L}^N(Q_{l,k}^*)} \|g\|_{L^\infty}$$
$$\leq M_L \|g\|_{L^\infty} \sum_{(l,k)\in I_L} c_{l,k}^{(m_i)} \leq M_L \|g\|_{L^\infty}.$$

Thus we can apply the Lebesgue dominated convergence theorem to conclude that there exists

$$\lim_{i\to\infty} I_i = \lim_{i\to\infty} \int_{\mathbb{R}^N} \sum_{(l,k)\in I_L} c_{l,k}^{(m_i)} f_{l,k}^{(m_i)} g\, d\boldsymbol{x} = \int_{\mathbb{R}^N} \sum_{(l,k)\in I_L} c_{l,k} f_{l,k} g\, d\boldsymbol{x}$$
$$= \sum_{k=-L}^{L} \sum_{l=1}^{\infty} c_{l,k} \int_{\mathbb{R}^N} f_{l,k} g\, d\boldsymbol{x} = I.$$

Thus for all i sufficiently large we have that $|I_i - I| \leq \varepsilon$, which concludes the proof. ■

Using the previus result we can show that $\mathbb{H}^1(\mathbb{R}^N)$ is the dual of the space of functions with *vanishing mean oscillation* $\mathrm{VMO}(\mathbb{R}^N)$, which is the closure of $C_c^\infty(\mathbb{R}^N)$ in $\mathrm{BMO}(\mathbb{R}^N)$, that is,

$$\mathrm{VMO}(\mathbb{R}^N) = \overline{C_c^\infty(\mathbb{R}^N)}^{\mathrm{BMO}(\mathbb{R}^N)}.$$

To explain the name, given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for every $r > 0$ define

$$\mathcal{M}_r(f) := \sup \frac{1}{\mathcal{L}^N(Q)} \int_Q |f(\boldsymbol{x}) - f_Q|\, d\boldsymbol{x} < \infty,$$

where the supremum is taken over all cubes Q of side-length less than or equal than r. Note that the function $r \mapsto \mathcal{M}_r(f)$ is increasing and that f belongs to $\mathrm{BMO}(\mathbb{R}^N)$ if and only if

$$\lim_{r\to\infty} \mathcal{M}_r(f) < \infty.$$

Define

$$\mathcal{M}_0(f) := \lim_{r\to 0^+} \mathcal{M}_r(f).$$

It can be shown that if f belongs to $\mathrm{VMO}(\mathbb{R}^N)$ then $\mathcal{M}_0(f) = 0$.

Remark 146 *In some books the space* $\mathrm{VMO}(\mathbb{R}^N)$ *is defined as the subspace of functions in* $\mathrm{BMO}(\mathbb{R}^N)$ *such that* $\mathcal{M}_0(f) = 0$*, while* $\overline{C_c^\infty(\mathbb{R}^N)}^{\mathrm{BMO}(\mathbb{R}^N)}$ *is denoted* $\mathrm{VMO}_0(\mathbb{R}^N)$*. These two spaces do not coincide.*

Theorem 147 *The dual of* $\mathrm{VMO}(\mathbb{R}^N)$ *may be identified with* $\mathbb{H}^1(\mathbb{R}^N)$*. To be precise every linear continuous functional* $L : \mathrm{VMO}(\mathbb{R}^N) \to \mathbb{C}$ *has the form*

$$L(f) = \int_{\mathbb{R}^N} f(\boldsymbol{x})g(\boldsymbol{x})\, d\boldsymbol{x}$$

for all $f \in C_c\left(\mathbb{R}^N\right)$ *and for some* $g \in \mathbb{H}^1(\mathbb{R}^N)$*. Moreover,*

$$C_1 \|L\|_{(\mathrm{VMO}(\mathbb{R}^N))'} \le \|g\|_{\mathbb{H}^1(\mathbb{R}^N)} \le C_2 \|L\|_{(\mathrm{VMO}(\mathbb{R}^N))'}.$$

The proof makes use of the following theorem. Let $(X, \|\cdot\|)$ be a normed space and consider the linear operator mapping

$$J : (X, \|\cdot\|) \to (X'', \|\cdot\|_{X''})$$

defined by

$$J(x)(T) := T(x), \quad T \in X'.$$

As a corollary of Goldstein's theorem we have:

Theorem 148 *Let* X *be a Banach space. Then*

$$\|J(x)\|_{X''} = \|x\| \quad \text{for all } x \in X, \tag{158}$$

so that J *is injective and continuous. Moreover,*

$$J : (X, \sigma(X, X')) \to (J(X), \sigma(X'', X'))$$

is a homeomorphism and $J(X)$ *is dense in* X'' *with respect to the weak star topology* $\sigma(X'', X')$ *in* X''*.*

Monday, April 6, 2015

We now turn to the proof of Theorem 147.

Proof of Theorem 147. Let $L : \mathrm{VMO}(\mathbb{R}^N) \to \mathbb{C}$ be a linear continuous functional. By the Banach–Alaoglu theorem the space

$$Y := \left\{ T \in \left(\mathrm{VMO}(\mathbb{R}^N)\right)' : \ \|T\|_{(\mathrm{VMO}(\mathbb{R}^N))'} \leq \|L\|_{(\mathrm{VMO}(\mathbb{R}^N))'} \right\}$$

is compact with respect to the weak star topology $\sigma\left(\left(\mathrm{VMO}(\mathbb{R}^N)\right)', \mathrm{VMO}(\mathbb{R}^N)\right)$. Moreover, since $\mathrm{VMO}(\mathbb{R}^N)$ is separable, the weak star topology on Y is metrizable.

Since $(\mathbb{H}^1(\mathbb{R}^N))' \simeq \mathrm{BMO}(\mathbb{R}^N)$, in view of the previous theorem $J(\mathbb{H}^1(\mathbb{R}^N))$ is dense in $\left(\mathrm{BMO}(\mathbb{R}^N)\right)'$ with respect to the $\sigma\left(\left(\mathrm{BMO}(\mathbb{R}^N)\right)', \mathrm{BMO}(\mathbb{R}^N)\right)$ topology. Given, $T_1 \in \left(\mathrm{VMO}(\mathbb{R}^N)\right)'$, by the Hahn–Banach theorem T_1 can be extended to a linear continuous functional $T_2 : \left(\mathrm{BMO}(\mathbb{R}^N)\right)'$ with the same norm. Since $J(\mathbb{H}^1(\mathbb{R}^N))$ is dense in $\left(\mathrm{BMO}(\mathbb{R}^N)\right)'$ with respect to the $\sigma\left(\left(\mathrm{BMO}(\mathbb{R}^N)\right)', \mathrm{BMO}(\mathbb{R}^N)\right)$ topology, for every $\varepsilon > 0$ and for every $g_1, \ldots, g_n \in \mathrm{BMO}(\mathbb{R}^N)$ there exists $f_\varepsilon \in J(\mathbb{H}^1(\mathbb{R}^N))$ such that

$$|J(f_\varepsilon)(g_i) - T_2(g_i)| < \varepsilon$$

for all $i = 1, \ldots, n$. In particular, if $g_1, \ldots, g_n \in \mathrm{VMO}(\mathbb{R}^N)$, then

$$|J(f_\varepsilon)(g_i) - T_1(g_i)| < \varepsilon$$

for all $i = 1, \ldots, n$. Thus, if for every $f \in \mathbb{H}^1(\mathbb{R}^N)$ we restrict $J(f)$ to $\mathrm{VMO}(\mathbb{R}^N)$, it follows that $J(\mathbb{H}^1(\mathbb{R}^N))$ is dense in $\left(\mathrm{VMO}(\mathbb{R}^N)\right)'$ with respect to the $\sigma\left(\left(\mathrm{VMO}(\mathbb{R}^N)\right)', \mathrm{VMO}(\mathbb{R}^N)\right)$ topology. In turn $J(\mathbb{H}^1(\mathbb{R}^N)) \cap Y$ is dense in Y with respect to $\sigma\left(\left(\mathrm{VMO}(\mathbb{R}^N)\right)', \mathrm{VMO}(\mathbb{R}^N)\right)$.

Since $L \in Y$ there exists a sequence $\{f_m\} \subset \mathbb{H}^1(\mathbb{R}^N)$ such that $J(f_m) \in Y$ and $J(f_m) \to L$ with respect to $\sigma\left(\left(\mathrm{VMO}(\mathbb{R}^N)\right)', \mathrm{VMO}(\mathbb{R}^N)\right)$, that is,

$$J(f_m)(g) \to L(g)$$

for every $g \in \mathrm{VMO}(\mathbb{R}^N)$.

By the Banach–Steinhaus theorem it follows that the sequence $\{f_m\}$ is bounded in $\mathbb{H}^1(\mathbb{R}^N)$. This require to show that

$$\sup \left\{ \left| \int_{\mathbb{R}^N} fg \ d\boldsymbol{x} \right| : \ g \in C_c(\mathbb{R}^N), \ \|g\|_{\mathrm{BMO}(\mathbb{R}^N)} \leq 1 \right\}$$

defines an equivalent norm in $\mathbb{H}^1(\mathbb{R}^N)$ (Exercise). In view of Theorem 143, up to a subsequence,

$$\lim_{i \to \infty} \int_{\mathbb{R}^N} f_m(\boldsymbol{x}) g(\boldsymbol{x}) \ d\boldsymbol{x} = \int_{\mathbb{R}^N} f(\boldsymbol{x}) g(\boldsymbol{x}) \ d\boldsymbol{x}$$

for all $g \in C_c\left(\mathbb{R}^N\right)$. But this implies that $L = f$. ■

7.3 Interpolation $p = 1$

In what follows $L_c^\infty(\mathbb{R}^N)$ is the space of all functions in $L^\infty(\mathbb{R}^N)$ that vanish outside a compact set (depending on the function).

Theorem 149 *Let T be a sublinear operator which is bounded from $L^\infty(\mathbb{R}^N)$ into $\mathrm{BMO}(\mathbb{R}^N)$ and from $\mathbb{H}^1(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$. Then T can be extendend to an operator of strong type (p,p) for every $1 < p < \infty$.*

Proof. Let $f \in L_c^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} f \, d\boldsymbol{x} = 0$. Then $f \in \mathbb{H}^1(\mathbb{R}^N)$ and so $T(f) \in L^1(\mathbb{R}^N)$. In turn, $\mathrm{M}^d(T(f))$ is of weak type $(1,1)$. It follows that for all $1 < p < \infty$, $\varphi := \inf\{1, \mathrm{M}^d(T(f))\} \in L^p(\mathbb{R}^N)$. Indeed, by Theorem 4,

$$\begin{aligned}
\int_{\mathbb{R}^N} |\varphi|^p \, d\boldsymbol{x} &= p \int_0^\infty t^{p-1} \mathcal{L}^N \left(\left\{ \boldsymbol{x} \in \mathbb{R}^N : \varphi(\boldsymbol{x}) > t \right\} \right) \, dt \\
&= p \int_0^1 t^{p-1} \mathcal{L}^N \left(\left\{ \boldsymbol{x} \in \mathbb{R}^N : \varphi(\boldsymbol{x}) > t \right\} \right) \, dt \\
&\leq p \int_0^1 t^{p-1} \mathcal{L}^N (\{ \boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^d(T(f)(\boldsymbol{x}) > t \}) \, dt \\
&\leq pc \, \|T(f)\|_{L^1} \int_0^1 t^{p-2} \, dt < \infty.
\end{aligned}$$

Hence by the Fefferman–Stein Theorem and Exercise 128,

$$\left\| \mathrm{M}^d(T(f)) \right\|_{L^p} \leq c \left\| \mathrm{M}^\#(T(f)) \right\|_{L^p}. \tag{159}$$

Given $t > 0$ we apply Theorem 97 to the function $|f|^p \in L^1(\mathbb{R}^N)$ and the number t^p to find a countable family $\{Q_n\}$ of open mutually disjoint cubes such that

$$|f(\boldsymbol{x})|^p \leq t^p \quad \text{for } \mathcal{L}^N \text{ a.e. } \boldsymbol{x} \in \mathbb{R}^N \setminus \bigcup_{n=1}^\infty \overline{Q_n}, \tag{160}$$

and for every $n \in \mathbb{N}$,

$$t^p < (|f|^p)_{Q_n} \leq 2^N t^p. \tag{161}$$

Write

$$f = g + h,$$

where

$$g(\boldsymbol{x}) := \begin{cases} f_{Q_n} & \text{if } \boldsymbol{x} \in \overline{Q_n} \text{ for some } n, \\ f(\boldsymbol{x}) & \text{otherwise.} \end{cases} \qquad h(\boldsymbol{x}) := f(\boldsymbol{x}) - g(\boldsymbol{x}). \tag{162}$$

Note that in view of (162)-(161),

$$\|g\|_{L^\infty} \leq 2^{N/p} t.$$

Since T maps $L^\infty(\mathbb{R}^N)$ into $\mathrm{BMO}(\mathbb{R}^N)$, we have that

$$|T(g)|_{\mathrm{BMO}} \leq c_1 \|g\|_{L^\infty} \leq c_1 2^{N/p} t.$$

By Remark 124,

$$\left\|\mathrm{M}^{\#}(T(g))\right\|_{L^{\infty}} = |T(g)|_{\mathrm{BMO}} \leq c_1 2^{N/p} t,$$

where $\mathrm{M}^{\#}$ is the sharp maximal operator defined in (137). Let Consider a constant $c_0 > c_1 2^{N/p}$. Then $c_0 t > \left\|\mathrm{M}^{\#}(T(g))\right\|_{L^{\infty}}$. Hence, by the sublinearity of $\mathrm{M}^{\#}(T)$,

$$\begin{aligned} &\{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(f)\,(\boldsymbol{x}) > (c_0+1)t\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(g)\,(\boldsymbol{x}) > c_0 t\} \\ &\cup \{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(h)\,(\boldsymbol{x}) > t\} = \{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(h))\,(\boldsymbol{x}) > t\}. \end{aligned} \tag{163}$$

To estimate the latter set, define $h_n := (f - f_{Q_n})\chi_{Q_n}$. Then h_n has average zero and is bounded. Hence, up to a multiplicative constant we have that h_n is an p-atom and by Hölder's inequality and (161),

$$\begin{aligned} \|h_n\|_{L^p(\mathbb{R}^N)} &= \|h_n\|_{L^p(Q_n)} \leq \|f\|_{L^p(Q_n)} + |f_{Q_n}|(\mathcal{L}^N(Q_n))^{1/p} \\ &\leq 2(\mathcal{L}^N(Q_n))^{1/p}[(|f|^p)_{Q_n}]^{1/p} \leq 2^{1+N/p} t (\mathcal{L}^N(Q_n))^{1/p}. \end{aligned}$$

It follows from Definitions 132 and 133 that

$$\|h_n\|_{\mathbb{H}^{1,p}(\mathbb{R}^N)} \leq 2^{1+N/p} t \mathcal{L}^N(Q_n)$$

and in turn,

$$\begin{aligned} \|h\|_{\mathbb{H}^{1,p}(\mathbb{R}^N)} &\leq 2^{1+N/p} t \sum_{n=1}^{\infty} \mathcal{L}^N(Q_n) \leq 2^{1+N/p} t \sum_{n=1}^{\infty} \frac{1}{t^p} \int_{Q_n} |f|^p \, d\boldsymbol{x} \\ &\leq 2^{1+N/p} \frac{1}{t^{p-1}} \|f\|_{L^p(\mathbb{R}^N)}^p, \end{aligned} \tag{164}$$

where we used (161).

In view of Remark 125, $\mathrm{M}^{\#}(T(h)) \leq 2\,\mathrm{N}^{nc}(T(h))$, and so by (161), (164), and the facts that N^{nc} is of weak type $(1,1)$ (see Remark 41), and that T is bounded from from $\mathbb{H}^1(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$,

$$\begin{aligned} \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(f))\,(\boldsymbol{x}) > (c_0+1)t)\}) &\leq \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{M}^{\#}(T(h))\,(\boldsymbol{x}) > t\}) \\ &\leq \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \mathrm{N}^{nc}(T(h))\,(\boldsymbol{x}) > t/2\}) \\ &\leq \frac{c}{t} \int_{\mathbb{R}^N} |T(h)| \, d\boldsymbol{x} \leq \frac{c}{t} \|h\|_{\mathbb{H}^{1,p}(\mathbb{R}^N)} \\ &\leq \frac{c}{t^p} \|f\|_{L^p(\mathbb{R}^N)}^p. \end{aligned}$$

This shows that the sublinear operator $\mathrm{M}^{\#}(T)$ is of weak type (p,p) for every $1 < p < \infty$. It follows by the Marcinkiewicz Interpolation Theorem that $\mathrm{M}^{\#}(T)$ is of strong type (p,p) for every $1 < p < \infty$. In turn, by (159), $\mathrm{M}^d(T)$ is of strong type (p,p). Since $|T(f)(\boldsymbol{x})| \leq \mathrm{M}^d(T(f))(\boldsymbol{x})$ at every Lebesgue point $\boldsymbol{x}$ of $T(f)$, we finally conclude that T is of strong type (p,p). ■

Wednesday, April 8, 2015

Next we prove an interpolation result for singular integrals.

Theorem 150 *Let K be a standard kernel and let φ be as in (92). If T_{φ_ε} satisfies (100), then there exists a constant $c(N) > 0$ such that*

$$\|T_{\varphi_\varepsilon}(f)\|_{L^1} \leq c \max\{C_\varphi, C_1\} \|f\|_{\mathbb{H}^1}$$

for all $\varepsilon > 0$ and for all $f \in \mathbb{H}^1(\mathbb{R}^N)$. Moroever, if K satisfies (115) then the same estimate holds for the operator T given in (121).

Proof. Let f be a 2-atom with support contained in a cube $Q = Q(\boldsymbol{x}_0, r)$ and let $mQ := Q(\boldsymbol{x}_0, mr)$ where as usual $m := \lceil\sqrt{N} + 1\rceil$. Then

$$\int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} \leq \int_{mQ} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} + \int_{\mathbb{R}^N \setminus mQ} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x}.$$

By Holder's inequality, (100), and (142),

$$\begin{aligned}
\int_{mQ} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} &\leq (\mathcal{L}^N(mQ))^{1/2} \left(\int_{mQ} |T_{\varphi_\varepsilon}(f)|^2 \, d\boldsymbol{x} \right)^{1/2} \\
&\leq m^{N/2} (\mathcal{L}^N(Q))^{1/2} C_1 \left(\int_{\mathbb{R}^N} |f|^2 \, d\boldsymbol{x} \right)^{1/2} \\
&\leq m^{N/2} (\mathcal{L}^N(Q))^{1/2} C_1 (\mathcal{L}^N(Q))^{-1/2} = m^{N/2} C_1.
\end{aligned}$$

Since the support of f is contained in Q and $\int_Q f \, d\boldsymbol{x} = 0$, for $\boldsymbol{x} \in \mathbb{R}^N \setminus mQ$ write

$$\begin{aligned}
T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) &= \int_Q K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, d\boldsymbol{y} \\
&= \int_Q (K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{x}_0)) f(\boldsymbol{y}) \, d\boldsymbol{y}.
\end{aligned}$$

As in (107) we have that for $\boldsymbol{x} \in \mathbb{R}^N \setminus mQ$ and $\boldsymbol{y} \in Q$, $|\boldsymbol{y} - \boldsymbol{x}_0| \leq \frac{1}{2}|\boldsymbol{y} - \boldsymbol{x}|$, and so by Proposition 95,

$$|K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{y}) - K_{\varphi_\varepsilon}(\boldsymbol{x}, \boldsymbol{x}_0)| \leq \frac{c C_\varphi r^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_0|^{N+\alpha}}.$$

In turn,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus mQ} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} &\leq \int_Q |f(\boldsymbol{y})| \, d\boldsymbol{y} \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}_0, r)} \frac{c C_\varphi r^\alpha}{|\boldsymbol{x} - \boldsymbol{x}_0|^{N+\alpha}} d\boldsymbol{x} \\
&= c C_\varphi \int_Q |f(\boldsymbol{y})| \, d\boldsymbol{y} \\
&\leq c C_\varphi (\mathcal{L}^N(Q))^{1/2} \left(\int_Q |f|^2 \, d\boldsymbol{y} \right)^{1/2} \\
&\leq c C_\varphi (\mathcal{L}^N(Q))^{1/2} (\mathcal{L}^N(Q))^{-1/2} = c C_\varphi,
\end{aligned}$$

where we have used Holder's inequality and (142). Thus, we have proved that

$$\int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} \le c(C_1 + C_\varphi) \tag{165}$$

for every 2-atom f.

Next, given $f \in \mathbb{H}^{1,2}\left(\mathbb{R}^N\right)$, let

$$f = \sum_{n=1}^{\infty} c_n f_n$$

where $\{f_n\}$ are 2-atoms and $\{c_n\} \subset \mathbb{C}$ are such that $\sum_{n=1}^{\infty} |c_n| < \infty$. Since $\mathbb{H}^{1,2}\left(\mathbb{R}^N\right) = \mathbb{H}^1\left(\mathbb{R}^N\right) \subset L^1\left(\mathbb{R}^N\right)$ and T_{φ_ε} is of weak type $(1,1)$ by Theorem 100, for $t > 0$, we have

$$\begin{aligned}
&\mathcal{L}^N(\{|T_{\varphi_\varepsilon}(f) - \textstyle\sum_{n=1}^{\infty} c_n T_{\varphi_\varepsilon}(f_n)| > t\}) \le \mathcal{L}^N(\{|T_{\varphi_\varepsilon}(f - \textstyle\sum_{n=1}^{k} c_n f_n)| > t/2\}) \\
&\quad + \mathcal{L}^N(\{|\textstyle\sum_{n=k+1}^{\infty} c_n T_{\varphi_\varepsilon}(f_n)| > t/2\}) \le c \max\{C_\varphi, C_1\} t^{-1} \left\| f - \textstyle\sum_{n=1}^{k} c_n f_n \right\|_{L^1} \\
&\qquad + 2t^{-1} \left\| \textstyle\sum_{n=k+1}^{\infty} c_n T_{\varphi_\varepsilon}(f_n) \right\|_{L^1} \le c \max\{C_\varphi, C_1\} t^{-1} \| f - \textstyle\sum_{n=1}^{k} c_n f_n \|_{\mathbb{H}^1} \\
&\qquad + 2t^{-1} \textstyle\sum_{n=k+1}^{\infty} |c_n| \, \|T_{\varphi_\varepsilon}(f_n)\|_{L^1} \le c \max\{C_\varphi, C_1\} t^{-1} \left\| \textstyle\sum_{n=k+1}^{\infty} c_n f_n \right\|_{\mathbb{H}^1} \\
&\qquad + 2c(C_1 + C_\varphi) t^{-1} \textstyle\sum_{n=k+1}^{\infty} |c_n| \le c \max\{C_\varphi, C_1\} t^{-1} \textstyle\sum_{n=k+1}^{\infty} |c_n|,
\end{aligned}$$

where we used Definition 133, Theorem 135(i) and (165). Letting $k \to \infty$, shows that

$$\mathcal{L}^N(\{|T_{\varphi_\varepsilon}(f) - \textstyle\sum_{n=1}^{\infty} c_n T_{\varphi_\varepsilon}(f_n)| > t\}) = 0$$

for all $t > 0$, which implies that

$$T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) = \sum_{n=1}^{\infty} c_n T_{\varphi_\varepsilon}(f_n)(\boldsymbol{x})$$

for $\mathcal{L}^N$ a.e. $\boldsymbol{x} \in \mathbb{R}^N$. In turn,

$$\begin{aligned}
\int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} &\le \sum_{n=1}^{\infty} |c_n| \int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f_n)| \, d\boldsymbol{x} \\
&\le c(C_1 + C_\varphi) \sum_{n=1}^{\infty} |c_n|
\end{aligned}$$

by (165). Taking the infimum over all possible decompositions of f gives

$$\int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f)| \, d\boldsymbol{x} \le c(C_1 + C_\varphi) \|f\|_{\mathbb{H}^1} . \tag{166}$$

Next we prove that the same estimate holds for the operator T. Let $f \in \mathbb{H}_a^{1,2}\left(\mathbb{R}^N\right)$. Then $f \in L^2(\mathbb{R}^N)$ and has compact support. It follows by Theorem 105, that there exists

$$\lim_{\varepsilon \to 0^+} T_\varepsilon(f)(\boldsymbol{x}) = T(f)(\boldsymbol{x})$$

for $\mathcal{L}^N$ a.e $\boldsymbol{x} \in \mathbb{R}^N$ and that the operator $\|T(f)\|_{L^2} \leq C\|f\|_{L^2}$. Moreover, since

$$|T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) - T_\varepsilon(f)(\boldsymbol{x})| \leq C(N)\,\mathrm{M}(f)(\boldsymbol{x})$$

by Theorem **??**, reasoning exactly as in the proof of Theorem 105, with $T^\#$ replaced by M, we have that

$$\lim_{\varepsilon \to 0^+} T_{\varphi_\varepsilon}(f)(\boldsymbol{x}) = T(f)(\boldsymbol{x})$$

for $\mathcal{L}^N$ a.e $\boldsymbol{x} \in \mathbb{R}^N$. It follows by Fatou's lemma and (166) that

$$\int_{\mathbb{R}^N} |T(f)|\, d\boldsymbol{x} \leq \liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} |T_{\varphi_\varepsilon}(f)|\, d\boldsymbol{x} \leq c(C_1 + C_\varphi)\|f\|_{\mathbb{H}^1}.$$

Since $\mathbb{H}^{1,2}_a(\mathbb{R}^N)$ is dense in $\mathbb{H}^1(\mathbb{R}^N)$, we can extend $T : \mathbb{H}^{1,2}_a(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ to a bounded linear operator from $\mathbb{H}^1(\mathbb{R}^N)$ into $L^1(\mathbb{R}^N)$. ■

8 The Crux of the Matter: The $T(1)$ Theorem

Let K be a standard kernel and let φ be as in (92). In Theorem 106 we have seen that (100) and (140) hold in the very special case of homogeneous kernels of convolution type. It remains to understand when (100) and (140) hold for all the other type of kernels.

The following fundamental theorem was proved by David and Journé in 1984.

Theorem 151 ($T(1)$ Theorem) *Let K be a standard kernel and let φ be as in (92). Then (140) holds if and only if there exists a constant $C > 0$ such that*

(i) $T_{\varphi_{\varepsilon,R}}(1) \in \mathrm{BMO}(\mathbb{R}^N)$, *with*

$$|T_{\varphi_{\varepsilon,R}}(1)|_{\mathrm{BMO}} \leq C$$

for all $\varepsilon > 0$ and $R \geq 1$,

(ii) $T^*_{\varphi_{\varepsilon,R}}(1) \in \mathrm{BMO}(\mathbb{R}^N)$, *with*

$$\left|T^*_{\varphi_{\varepsilon,R}}(1)\right|_{\mathrm{BMO}} \leq C$$

for all $\varepsilon > 0$ and $R > 1$,

(iii) $T_{\varphi_{\varepsilon,R}}$ *is* weakly bounded*, that is,*

$$\left|\int_{\mathbb{R}^N} T_{\varphi_{\varepsilon,R}}(\chi_Q)(\boldsymbol{x})\chi_Q(\boldsymbol{x})\, d\boldsymbol{x}\right| \leq C\mathcal{L}^N(Q)$$

for every cube Q.

Here $T^*_{\varphi_{\varepsilon,R}}$ is the adjoint operator of $T_{\varphi_{\varepsilon,R}}$ and is defined by

$$T^*_{\varphi_{\varepsilon,R}}(f)(\boldsymbol{x}) = \int_{\mathbb{R}^N} K_{\varphi_{\varepsilon,R}}(\boldsymbol{y},\boldsymbol{x}) f(\boldsymbol{y})\, d\boldsymbol{y}.$$

Remark 152 *In the definition of weak boundedness one can use balls instead of cubes (exercise).*

The following is a a local version of the $T(1)$ theorem.

Theorem 153 (Local $T(1)$ Theorem) *Let K be a standard kernel and let φ be as in (92). Then (140) holds if and only if there exists a constant $C > 0$ such that*

$$\int_Q \left|T_{\varphi_{\varepsilon,R}}(\chi_Q)(\boldsymbol{x})\right|^2 d\boldsymbol{x} \le C\mathcal{L}^N(Q), \quad \int_Q \left|T^*_{\varphi_{\varepsilon,R}}(\chi_Q)(\boldsymbol{x})\right|^2 d\boldsymbol{x} \le C\mathcal{L}^N(Q)$$

for all cubes Q.

The $T(1)$ theorem was extended to the case when the function 1 is replaced by two bounded accretive functions, one for T and one for T^*, by David, Journé, and Semmes in 1985, who proved the $T(b)$ theorem. A function $b : \mathbb{R}^N \to \mathbb{C}$ is *accretive* if there exists a constant $\delta > 0$ such that

$$\operatorname{Re} b(\boldsymbol{x}) \ge \delta$$

for all $\boldsymbol{x} \in \mathbb{R}^N$.

The extension of the $T(1)$ and the $T(b)$ theorems to non-doubling measures is due to Nazarov, Treil, Volberg in 2003.

Exercise 154 *Let K be a standard kernel and let $T_1 : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $T_2 : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be two bounded linear operators such that*

$$T_i(f)(\boldsymbol{x}) = \int_{\mathbb{R}^N} K(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}$$

for all $f \in L^2(\mathbb{R}^N)$ which vanish outside of a compact set $K_f \subset \mathbb{R}^N$ and for all $\boldsymbol{x} \in \mathbb{R}^N \setminus K_f$, $i = 1, 2$. Prove that there exists a function $g \in L^\infty(\mathbb{R}^N)$ such that $T_1(f) - T_2(f) = gf$ for all $f \in L^2(\mathbb{R}^N)$.

Friday, April 10, 2015

9 Carleson Measures

Given a function $f \in \mathcal{S}(\mathbb{R}^N)$, consider the function

$$u(\boldsymbol{x},t) := c_N \int_{\mathbb{R}^N} \frac{t f(\boldsymbol{y})}{(|\boldsymbol{x}-\boldsymbol{y}|^2 + t^2)^{(N+1)/2}}\, d\boldsymbol{y} = (f * P(\cdot, y))(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \mathbb{R}^{N+1}_+,$$

where P is the *Poisson's kernel*

$$P(\boldsymbol{x},t) := c_N \frac{t}{(|\boldsymbol{x}|^2+t^2)^{(N+1)/2}}, \quad (\boldsymbol{x},t)\in\mathbb{R}^{N+1}_+$$

and

$$c_N := \frac{\Gamma((N+1)/2)}{\pi^{(N+1)/2}}.$$

It can be shown that u is the harmonic extension of f in $\mathbb{R}^{N+1}_+$, that is, u is a solution of the Dirichlet problem

$$\begin{cases} \Delta u(\boldsymbol{x},t) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ u(\boldsymbol{x},t) = f(\boldsymbol{x}) & \text{in } \mathbb{R}^N. \end{cases}$$

Write $u = u_f$. Note that by Hölder's inequality the function u_f is well-defined for every $f\in L^p(\mathbb{R}^N)$, $1<p<\infty$, since

$$\begin{aligned} |u_f(\boldsymbol{x},t)| &\le c_N \|f\|_{L^p}\, t\left(\int_{\mathbb{R}^N} \frac{1}{(|\boldsymbol{x}-\boldsymbol{y}|^2+t^2)^{p'(N+1)/2}}\,d\boldsymbol{y}\right)^{1/p'} \\ &= c_N\|f\|_{L^p}\, t\left(\alpha_N\int_0^\infty \frac{r^{N-1}}{(r^2+t^2)^{p'(N+1)/2}}\,dr\right)^{1/p'} < \infty, \end{aligned}$$

since $p'(N+1)>N$.

Carleson was interested in characterizing the Radon measures $\mu:\mathcal{B}(\mathbb{R}^{N+1}_+)\to[0,\infty)$ for which

$$\int_{\mathbb{R}^{N+1}_+} |u_f(\boldsymbol{x},t)|^2\,d\mu(\boldsymbol{x},t) \le C_\mu \int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2\,d\boldsymbol{x} \tag{167}$$

for all $f\in L^2(\mathbb{R}^N)$. To derive a necessary condition, take $f=\chi_{2Q}$, where $Q=Q(\boldsymbol{x}_0,r)$. If $\boldsymbol{x}\in Q$ and $0<t<r$, then $B(\boldsymbol{x},r)\subset 2Q$. Indeed, if $\boldsymbol{y}\in B(\boldsymbol{x},r)$, then $\|\boldsymbol{x}-\boldsymbol{y}\|<r$ and so $|x_i-y_i|<r$, but since $|x_i-x_{0,i}|<r$, it follows that $|y_i-x_{0,i}|<2r$. Hence,

$$\begin{aligned} u_f(\boldsymbol{x},t) &= c_N\int_{2Q}\frac{t}{(|\boldsymbol{x}-\boldsymbol{y}|^2+t^2)^{(N+1)/2}}\,d\boldsymbol{y} \\ &\ge c_N\int_{B(\boldsymbol{x},r)}\frac{t}{(|\boldsymbol{x}-\boldsymbol{y}|^2+t^2)^{(N+1)/2}}\,d\boldsymbol{y} \\ &= c_N\int_{B(\boldsymbol{0},r/t)}\frac{1}{(|\boldsymbol{\xi}|^2+1)^{(N+1)/2}}\,d\boldsymbol{\xi} \\ &\ge c_N\int_{B(\boldsymbol{0},1)}\frac{1}{(|\boldsymbol{\xi}|^2+1)^{(N+1)/2}}\,d\boldsymbol{\xi} = c_0 > 0, \end{aligned}$$

where we made the change of variables $\boldsymbol{\xi}:=(\boldsymbol{x}-\boldsymbol{y})/t$ and used the fact that $t<r$. Hence, $u_f(\boldsymbol{x},t)\ge c_0>0$ for all $(\boldsymbol{x},t)\in Q\times(0,r)$, and so if (167) holds, then a necessary condition is that

$$c_0^2\mu(Q\times(0,r))\le\int_{\mathbb{R}_+^{N+1}}|u_f(\boldsymbol{x},t)|^2\,d\mu(\boldsymbol{x},t)\le C_\mu\int_{\mathbb{R}^N}|f(\boldsymbol{x})|^2\,d\boldsymbol{x}$$
$$=C_\mu 2^N r^N.$$

We will see that this condition is also sufficient.

Definition 155 *A measure $\mu:\mathcal{B}(\mathbb{R}_+^{N+1})\to[0,\infty]$ is a* Carleson measure *if there exists a constant $C>0$ such that*

$$\mu(Q(\mathbf{x},r)\times(0,r])\le Cr^N \tag{168}$$

for all $\mathbf{x}\in\mathbb{R}^N$ and $r>0$. The best constant C for which (168) holds is called Carleson norm *and is denoted C_μ.*

Example 156 *The Lebesgue measure $\mathcal{L}^{N+1}$ is not a Carleson measure. The Lebesgue measure $\mathcal{L}^N$ on an hyperplane $\{(\mathbf{x},t_0):\ \mathbf{x}\in\mathbb{R}^N\}$ of $\mathbb{R}_+^{N+1}$ is a Carleson measure. Given a function $g=g(t)$, the measure $\mu=g(t)\,d\boldsymbol{x}dt$ is a Carleson measure if and only if $g\in L^1(\mathbb{R}_+)$ (exercise). In particular, $\mu=d\boldsymbol{x}\frac{dt}{t}$ is not a Carleson measure.*

Theorem 157 (Whitney) *Given an open set $\Omega\subset\mathbb{R}^N$, there exists a countable family $\mathcal{F}=\left\{\overline{Q(\boldsymbol{x}_n,r_n)}\right\}_n$ of closed dyadic cubes such that*

- *(i)* $\Omega=\bigcup_n\overline{Q(\boldsymbol{x}_n,r_n)}$,
- *(ii)* $Q(\boldsymbol{x}_n,r_n)\cap Q(\boldsymbol{x}_m,r_m)=\emptyset$ *for all* $n\ne m$,
- *(iii)* $\sqrt{N}r_n\le\operatorname{dist}\left(\overline{Q(\boldsymbol{x}_n,r_n)},\partial\Omega\right)\le4\sqrt{N}r_n$,[1]
- *(iv) if* $\overline{Q(\boldsymbol{x}_n,r_n)}$ *and* $\overline{Q(\boldsymbol{x}_m,r_m)}$ *touch, then* $\frac14r_n\le r_m\le4r_n$,
- *(v) for every fixed cube* $\overline{Q(\boldsymbol{x}_n,r_n)}$ *in* $\mathcal{F}$ *there are at most* $(12)^N$ *cubes in* $\mathcal{F}$ *that touch* $\overline{Q(\boldsymbol{x}_n,r_n)}$,
- *(vi) for every fixed* $0<\varepsilon<\frac14$ *and for every* $\boldsymbol{x}\in\Omega$ *there exist at most* $(12)^N$ *cubes* $\overline{Q(\boldsymbol{x}_n,(1+\varepsilon)r_n)}$ *that contain* $\boldsymbol{x}$.

Proof. Let $\mathcal{G}_0:=\left\{\overline{Q(\boldsymbol{x},1)}:\ \boldsymbol{x}\in\mathbb{Z}^N\right\}$. The family $\mathcal{G}_0$ leads to a family $\{\mathcal{G}_k\}_{k\in\mathbb{Z}}$ of collections of cubes with the property that each cube in the family $\mathcal{G}_k$ gives rise to 2^N cubes in the family $\mathcal{G}_{k+1}$ by bisecting the sides. The cubes in the family $\mathcal{G}_k$ have side length $\frac{1}{2^k}$ and, in turn, diameter $\frac{\sqrt{N}}{2^k}$.

[1] Note that $\sqrt{N}r_n=\operatorname{diam}\overline{Q(x_n,r_n)}$.

For $k \in \mathbb{Z}$ define

$$\Omega_k := \left\{ \boldsymbol{x} \in \Omega : \frac{\sqrt{N}}{2^{k-1}} < \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \leq \frac{\sqrt{N}}{2^{k-2}} \right\}.$$

Then

$$\Omega = \bigcup_{k \in \mathbb{Z}} \Omega_k.$$

Consider the family

$$\mathcal{F}_0 := \bigcup_{k \in \mathbb{Z}} \left\{ \overline{Q} \in \mathcal{G}_k : \overline{Q} \cap \Omega_k \neq \emptyset \right\}.$$

Since each family $\mathcal{G}_k$ is a partition of $\mathbb{R}^N$, we have that

$$\Omega \subset \bigcup_{\overline{Q} \in \mathcal{F}_0} \overline{Q}. \tag{169}$$

Next we claim that for every $\overline{Q} \in \mathcal{F}_0$,

$$\operatorname{diam} \overline{Q} < \operatorname{dist}\left(\overline{Q}, \partial\Omega\right) \leq 4 \operatorname{diam} \overline{Q}. \tag{170}$$

Indeed, if $\overline{Q} \in \mathcal{F}_0$, then $\overline{Q} \in \mathcal{G}_k$ for some $k \in \mathbb{Z}$, so that $\operatorname{diam} \overline{Q} = \frac{\sqrt{N}}{2^k}$. Moreover, since $\overline{Q} \cap \Omega_k \neq \emptyset$, there exists $\boldsymbol{x} \in \overline{Q} \cap \Omega_k$. Thus, also by the definition of Ω_k,

$$\operatorname{dist}\left(\overline{Q}, \partial\Omega\right) \leq \operatorname{dist}(\boldsymbol{x}, \partial\Omega) \leq 4 \frac{\sqrt{N}}{2^k}.$$

On the other hand,

$$\operatorname{dist}\left(\overline{Q}, \partial\Omega\right) \geq \operatorname{dist}(\boldsymbol{x}, \partial\Omega) - \operatorname{diam} \overline{Q} > \frac{\sqrt{N}}{2^{k-1}} - \frac{\sqrt{N}}{2^k} = \frac{\sqrt{N}}{2^k},$$

which proves (170).

Note that the first inequality in (170) shows, in particular, that every $\overline{Q} \in \mathcal{F}_0$ is contained in Ω, so that also by (169),

$$\Omega = \bigcup_{\overline{Q} \in \mathcal{F}_0} \overline{Q}.$$

Thus, properties (i) and (iii) are satisfied. To obtain (ii), we construct an appropriate subfamily of $\mathcal{F}_0$. We begin by observing that if $\overline{Q}_1 \in \mathcal{G}_{k_1}$ and $\overline{Q}_2 \in \mathcal{G}_{k_2}$ intersect, with $k_1 < k_2$, then, necessarily, $\overline{Q}_1 \supset \overline{Q}_2$.

Start from any cube $\overline{Q} \in \mathcal{F}_0$ and consider the maximal cube $\overline{Q}'$ (with respect to inclusion) in $\mathcal{F}_0$ that contains $\overline{Q}$. Such a cube exists, since, by (170), the diameter of any cube in $\mathcal{F}_0$ containing $\overline{Q}$ cannot exceed $4 \operatorname{diam} \overline{Q}$. Note that by the previous observation there is only one such maximal cube $\overline{Q}'$. Let $\mathcal{F}$ be the subfamily of maximal cubes in $\mathcal{F}_0$. Then (i)–(iii) hold.

To prove (iv), assume that $\overline{Q}_1, \overline{Q}_2 \in \mathcal{F}$ touch. Then

$$\operatorname{diam} \overline{Q}_2 \leq \operatorname{dist}\left(\overline{Q}_2, \partial\Omega\right) \leq \operatorname{dist}\left(\overline{Q}_1, \partial\Omega\right) + \operatorname{diam} \overline{Q}_1 \leq 5 \operatorname{diam} \overline{Q_1},$$

where we have used (170). On the other hand, since $\operatorname{diam} \overline{Q}_2 = 2^k \operatorname{diam} \overline{Q}_1$ for some integer $k \in \mathbb{Z}$, then, necessarily, $\operatorname{diam} \overline{Q}_2 \leq 4 \operatorname{diam} \overline{Q}_1$. By reversing the roles of $\overline{Q}_1$ and $\overline{Q}_2$, we obtain (iv).

Next we show that (v) holds. Fix a cube $\overline{Q} \in \mathcal{F}$ and let $k \in \mathbb{Z}$ be such that $\overline{Q} \in \mathcal{G}_k$. In the family $\mathcal{G}_k$ there are only $3^N - 1$ cubes that touch $\overline{Q}$. Each cube in $\mathcal{G}_k$ can contain at most 4^N cubes of $\mathcal{F}$ with diameter greater than or equal to $\frac{1}{4} \operatorname{diam} \overline{Q}$. Hence (v) follows from (iv).

Finally, we prove (vi). For every $\boldsymbol{x} \in \Omega$ let $\overline{Q} \in \mathcal{F}$ be such that $\boldsymbol{x} \in \overline{Q}$. We claim that if $\overline{Q(\boldsymbol{x}_n, r_n)} \in \mathcal{F}$, then $\overline{Q(\boldsymbol{x}_n, (1+\varepsilon) r_n)}$ intersects $\overline{Q}$ only if $\overline{Q(\boldsymbol{x}_n, r_n)}$ touches $\overline{Q}$. Indeed, consider the union of all the cubes in $\mathcal{F}$ that touch $\overline{Q}$. By (iv), the diameter of each of these cubes is greater than or equal to $\frac{1}{4} \operatorname{diam} \overline{Q}$. Since $0 < \varepsilon < \frac{1}{4}$, the union of these cubes contains $\overline{Q(\boldsymbol{x}_n, (1+\varepsilon) r_n)}$. By the maximality of the family $\mathcal{F}$, it follows that $\overline{Q(\boldsymbol{x}_n, r_n)}$ must be one of these cubes and the claim is proved. Property (vi) now follows from (v). ■

The family $\mathcal{F}$ is called a *Whitney decomposition* of Ω.

Monday, April 13, 2015

Given a measurable function $u : \mathbb{R}^{N+1}_+ \to \mathbb{C}$, the *non-tangential maximal function* of u is the function defined by

$$\mathrm{N}(u)(\boldsymbol{x}) := \sup_{(\boldsymbol{y},t):\ |\boldsymbol{x}-\boldsymbol{y}|<t} |u(\boldsymbol{y},t)|, \quad \boldsymbol{x} \in \mathbb{R}^N. \tag{171}$$

Theorem 158 *Let μ be a Carleson measure and let $u : \mathbb{R}^{N+1}_+ \to \mathbb{C}$ be a continuous function. Then for every $1 < p < \infty$,*

$$\int_{\mathbb{R}^{N+1}_+} |u_f(\boldsymbol{x}, t)|^p \, d\mu(\boldsymbol{x}, t) \leq C(N) C_\mu \int_{\mathbb{R}^N} (\mathrm{N}(u)(\boldsymbol{x}))^p \, d\boldsymbol{x}.$$

Proof. Step 1: We claim that for every $s > 0$,

$$\mu(\{(\boldsymbol{x}, t) \in \mathbb{R}^{N+1}_+ : \ |u(\boldsymbol{x}, t)| > s\}) \leq C(N) C_\mu \mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{N}(u)(\boldsymbol{x}) > s\}).$$

Consider the set $U_s := \{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{N}(u)(\boldsymbol{x}) > s\}$. Since u is continuous, the set U_s is open. If $U_s = \mathbb{R}^N$, then there is nothing to prove, so assume that $U_s \subset \mathbb{R}^N$ and apply Whitney's decomposition theorem to find a family of dyadic cubes $Q_n = Q(\boldsymbol{x}_n, r_n)$ as in the statement. Let $(\boldsymbol{x}, t) \in \mathbb{R}^{N+1}_+$ be such that $|u(\boldsymbol{x}, t)| > s$, then $\boldsymbol{x} \in U_s$, and so $\boldsymbol{x} \in \overline{Q_n}$ for some n. By part (ii) in Theorem 157,

$$\sqrt{N} r_n \leq \operatorname{dist}\left(\overline{Q_n}, \partial U_s\right) \leq 4\sqrt{N} r_n$$

and so there exist $\boldsymbol{y}_n \in \partial U_s$ and $\boldsymbol{\xi}_n \in \overline{Q_n}$ such that

$$\sqrt{N} r_n \leq \operatorname{dist}\left(\overline{Q_n}, \partial U_s\right) = |\boldsymbol{\xi}_n - \boldsymbol{y}_n| \leq 4\sqrt{N} r_n.$$

In turn,

$$\sqrt{N}r_n \leq |\boldsymbol{\xi}_n-\boldsymbol{y}_n| \leq |\boldsymbol{x}-\boldsymbol{y}_n| \leq |\boldsymbol{x}-\boldsymbol{\xi}_n|+|\boldsymbol{\xi}_n-\boldsymbol{y}_n| \leq \sqrt{N}r_n+4\sqrt{N}r_n = 5\sqrt{N}r_n.$$

Since $\boldsymbol{y}_n \notin U_s$, and $|u(\boldsymbol{x},t)| > s$, necessarily, $t \leq |\boldsymbol{x}-\boldsymbol{y}_n| \leq 5\sqrt{N}r_n$. Hence, $(\boldsymbol{x},t) \in \overline{Q_n} \times (0, 5\sqrt{N}r_n]$, and so

$$\begin{aligned}
\mu(\{(\boldsymbol{x},t) \in \mathbb{R}^{N+1}_+ : \ |u(\boldsymbol{x},t)| > s\}) &\leq \mu\left(\bigcup_n \overline{Q_n} \times (0, 5\sqrt{N}r_n]\right) \\
&\leq \sum_n \mu\left(\overline{Q_n} \times (0, 5\sqrt{N}r_n]\right) \leq \sum_n \mu\left(\ell Q_n \times (0, \ell r_n]\right) \\
&\leq C_\mu \sum_n \ell^N r_n^N = C_\mu \ell^N \sum_n \mathcal{L}^N(Q_n) = C_\mu \ell^N \mathcal{L}^N(U_s),
\end{aligned}$$

where $\ell := \left\lceil 5\sqrt{N}r_n \right\rceil + 1$ and we have used the fact that μ is a Carleson measure and that the cubes Q_n are pairwise disjoint and their closure covers U_s (see parts (i) and (ii) in Theorem 157).
Step 2: By Theorem 4 and and the previous step,

$$\begin{aligned}
\int_{\mathbb{R}^{N+1}_+} |u(\boldsymbol{x},t)|^p \, d\mu(\boldsymbol{x},t) &= p\int_0^\infty s^{p-1}\mu\left(\left\{(\boldsymbol{x},t) \in \mathbb{R}^{N+1}_+ : \, u(\boldsymbol{x},t) > s\right\}\right) \, ds \\
&\leq pC(N)C_\mu \int_0^\infty s^{p-1}\mathcal{L}^N(\{\boldsymbol{x} \in \mathbb{R}^N : \ \mathrm{N}(u)(\boldsymbol{x}) > s\}) \, ds \\
&= C(N)C_\mu \int_{\mathbb{R}^N} (\mathrm{N}(u)(\boldsymbol{x}))^p \, d\boldsymbol{x},
\end{aligned}$$

which concludes the proof. ■

We are now ready to prove that condition (167) is also sufficient for μ to be a Carleson measure.

Theorem 159 *Let $\mu : \mathcal{B}(\mathbb{R}^{N+1}_+) \to [0,\infty]$ be a measure. Then μ is a Carleson measure if and only if*

$$\int_{\mathbb{R}^{N+1}_+} |u_f(\boldsymbol{x},t)|^p \, d\mu(\boldsymbol{x},t) \leq C(N,p)C_\mu \int_{\mathbb{R}^N} |f(\boldsymbol{x})|^p \, d\boldsymbol{x}$$

for all $f \in L^p(\mathbb{R}^N)$ and $1 < p < \infty$.

Proof. We have already proved that if (167) holds, then μ is a Calreson measure. Assume now that μ is a Carelson measure. We claim that for $f \in L^p(\mathbb{R}^N)$ and $1 < p < \infty$,

$$\mathrm{N}(u_f)(\boldsymbol{x}_0) \leq C(N)\,\mathrm{M}(f)(\boldsymbol{x}_0)$$

for every $\boldsymbol{x}_0 \in \mathbb{R}^N$. To see this, let $(\boldsymbol{x}, t) \in \mathbb{R}^{N+1}_+$ be such that $|\boldsymbol{x}_0 - \boldsymbol{x}| < t$ and write

$$\begin{aligned} u_f(\boldsymbol{x}, t) &= c_N \int_{\mathbb{R}^N} \frac{t f(\boldsymbol{y})}{(|\boldsymbol{x} - \boldsymbol{y}|^2 + t^2)^{(N+1)/2}} \, d\boldsymbol{y} \\ &= c_N \int_{B(\boldsymbol{x},t)} \frac{t f(\boldsymbol{y})}{(|\boldsymbol{x} - \boldsymbol{y}|^2 + t^2)^{(N+1)/2}} \, d\boldsymbol{y} \\ &\quad + c_N \sum_{n=1}^{\infty} \int_{B(\boldsymbol{x},2^n t)\setminus B(\boldsymbol{x},2^{n-1}t)} \frac{t f(\boldsymbol{y})}{(|\boldsymbol{x} - \boldsymbol{y}|^2 + t^2)^{(N+1)/2}} \, d\boldsymbol{y} \\ &= c_N I_0 + c_N \sum_{n=1}^{\infty} I_n. \end{aligned}$$

Since $|\boldsymbol{x}_0 - \boldsymbol{x}| < t$, if $|\boldsymbol{y} - \boldsymbol{x}| < t$ then $|\boldsymbol{x}_0 - \boldsymbol{y}| < 2t$ and so

$$|I_0| \le \int_{B(\boldsymbol{x}_0,2t)} \frac{t|f(\boldsymbol{y})|}{(|\boldsymbol{x} - \boldsymbol{y}|^2 + t^2)^{(N+1)/2}} \, d\boldsymbol{y} \le \frac{1}{t^N} \int_{B(\boldsymbol{x}_0,2t)} |f(\boldsymbol{y})| \, d\boldsymbol{y} \le 2^N \alpha_N \, \mathrm{M}(f)(\boldsymbol{x}_0).$$

On the other hand, since $|\boldsymbol{x}_0 - \boldsymbol{x}| < t$, if $2^{n-1}t \le |\boldsymbol{y} - \boldsymbol{x}| < 2^n t$, we have that $|\boldsymbol{x}_0 - \boldsymbol{y}| < 2^{n+1}t$ and so

$$\begin{aligned} |I_n| &\le \frac{t}{(2^{n-1}t)^{N+1}} \int_{B(\boldsymbol{x}_0,2^{n+1}t)} |f(\boldsymbol{y})| \, d\boldsymbol{y} \\ &= \frac{4^N}{2^{n-1}} \frac{1}{(2^{n+1}t)^N} \int_{B(\boldsymbol{x}_0,2^{n+1}t)} |f(\boldsymbol{y})| \, d\boldsymbol{y} \le \frac{4^N}{2^{n-1}} \alpha_N \, \mathrm{M}(f)(\boldsymbol{x}_0). \end{aligned}$$

It follows that

$$u_f(\boldsymbol{x}, t) \le c_N 2^N \alpha_N \, \mathrm{M}(f)(\boldsymbol{x}_0) + c_N 4^N \alpha_N \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \, \mathrm{M}(f)(\boldsymbol{x}_0)$$

for all $(\boldsymbol{x}, t) \in \mathbb{R}^{N+1}_+$ with $|\boldsymbol{x}_0 - \boldsymbol{x}| < t$. Taking the supremum over all such $(\boldsymbol{x}, t)$ gives the desired inequality.

It now follows from Theorems 33 and 158,

$$\begin{aligned} \int_{\mathbb{R}^{N+1}_+} |u_f(\boldsymbol{x}, t)|^p \, d\mu(\boldsymbol{x}, t) &\le C(N) C_\mu \int_{\mathbb{R}^N} (\mathrm{N}(u)\,(\boldsymbol{x}))^p \, d\boldsymbol{x} \\ &\le C(N) C_\mu \int_{\mathbb{R}^N} |\mathrm{M}(f)|^p \, d\boldsymbol{x} \le C(N,p) C_\mu \int_{\mathbb{R}^N} |f|^p \, d\boldsymbol{x}, \end{aligned}$$

which concludes the proof. ■

Corollary 160 *Let* $\mu : \mathcal{B}(\mathbb{R}^{N+1}_+) \to [0, \infty]$ *be a measure and let* $\varphi \in C^\infty_c(\mathbb{R}^N)$ *be supported on the unit ball. Then*

$$\int_{\mathbb{R}^{N+1}_+} |(f * \varphi_t)(\boldsymbol{x})|^p \, d\mu(\boldsymbol{x}, t) \le C(N, p) \, \|\varphi\|^p_\infty \, C_\mu \int_{\mathbb{R}^N} |f(\boldsymbol{x})|^p \, d\boldsymbol{x}$$

for all $f \in L^p(\mathbb{R}^N)$ *and* $1 < p < \infty$, *where*

$$\varphi_t(\boldsymbol{x}) := \frac{1}{t^N}\varphi\left(\frac{\boldsymbol{x}}{t}\right)$$

Proof. By Theorem 158,

$$\int_{\mathbb{R}^{N+1}_+} |(f * \varphi_t)(\boldsymbol{x})|^p \, d\mu(\boldsymbol{x}, t) \leq C(N) C_\mu \int_{\mathbb{R}^N} (\mathrm{N}(f * \varphi_t)(\boldsymbol{x}))^p \, d\boldsymbol{x}.$$

On the other hand, by (171),

$$\begin{aligned}
\mathrm{N}(f * \varphi_t)(\boldsymbol{x}) &= \sup_{(\boldsymbol{y},t):\ |\boldsymbol{x}-\boldsymbol{y}|<t} |(f * \varphi_t)(\boldsymbol{y})| \\
&= \sup_{(\boldsymbol{y},t):\ |\boldsymbol{x}-\boldsymbol{y}|<t} \frac{1}{t^N} \left| \int_{\mathbb{R}^N} \varphi\left(\frac{\mathbf{y}-\xi}{t}\right) f(\xi) \, d\xi \right| \\
&= \sup_{(\boldsymbol{y},t):\ |\boldsymbol{x}-\boldsymbol{y}|<t} \frac{1}{t^N} \left| \int_{B(\mathbf{y},t)} \varphi\left(\frac{\mathbf{y}-\xi}{t}\right) f(\xi) \, d\xi \right| \\
&\leq \|\varphi\|_\infty \sup_{(\boldsymbol{y},t):\ \boldsymbol{x} \in B(\boldsymbol{y},t)} \frac{1}{t^N} \int_{B(\mathbf{y},t)} |f(\xi)| \, d\xi \\
&= \|\varphi\|_\infty \mathrm{M}^{nc}(f)(\boldsymbol{x}).
\end{aligned}$$

The result now follows from Theorem 40. ■

Wednesday, April 15, 2015

One of the most important examples of Carleson measure is given by the following theorem.

Theorem 161 *Let* $\psi \in \mathcal{S}(\mathbb{R}^N)$ *be such that*

$$\int_{\mathbb{R}^N} \psi(\mathbf{x}) \, d\mathbf{x} = 0$$

and let $f \in \mathrm{BMO}\left(\mathbb{R}^N\right)$. *Then the measure*

$$\mu(E) := \int_E |(f * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t}, \quad E \in \mathcal{B}(\mathbb{R}^{N+1}_+),$$

is a Carleson measure, with

$$C_\mu \leq C(N, \psi) \, |f|^2_{\mathrm{BMO}}.$$

Here, as usual,

$$\psi_t(\boldsymbol{x}) := \frac{1}{t^N}\psi\left(\frac{\boldsymbol{x}}{t}\right).$$

Lemma 162 *The operator*

$$T(g)(\boldsymbol{x}) := \left(\int_0^\infty |(g * \psi_t)(\boldsymbol{x})|^2 \, \frac{dt}{t}\right)^{1/2}, \quad g \in L^2(\mathbb{R}^N),$$

is bounded in $L^2(\mathbb{R}^N)$.

Proof. By the Fubini and the Plancherel theorems,

$$\int_{\mathbb{R}^N} (T(g)(\boldsymbol{x}))^2 \; d\boldsymbol{x} = \int_0^\infty \int_{\mathbb{R}^N} |(g * \psi_t)(\boldsymbol{x})|^2 \; d\boldsymbol{x} \frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^N} |\widehat{(g * \psi_t)}(\boldsymbol{x})|^2 \; d\boldsymbol{x} \frac{dt}{t}$$
$$= \int_0^\infty \int_{\mathbb{R}^N} |\widehat{g}(\boldsymbol{x})\widehat{\psi_t}(\boldsymbol{x})|^2 \; d\boldsymbol{x} \frac{dt}{t} = \int_{\mathbb{R}^N} |\widehat{g}(\boldsymbol{x})|^2 \int_0^\infty |\widehat{\psi_t}(\boldsymbol{x})|^2 \; \frac{dt}{t} d\boldsymbol{x}.$$

Since

$$\widehat{\psi_t}(\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \psi_t(\mathbf{y}) \; d\boldsymbol{y} = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{y}} \frac{1}{t^N} \psi\left(\frac{\boldsymbol{y}}{t}\right) \; d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^N} e^{-2\pi i t \boldsymbol{x}\cdot\boldsymbol{\xi}} \psi\left(\boldsymbol{\xi}\right) \; d\boldsymbol{\xi} = \widehat{\psi}(t\boldsymbol{x})$$

where we have made the change of variables $\boldsymbol{\xi} = \frac{\boldsymbol{y}}{t}$. Hence,

$$\int_0^\infty |\widehat{\psi_t}(\boldsymbol{x})|^2 \; \frac{dt}{t} = \int_0^\infty |\widehat{\psi}(t\boldsymbol{x})|^2 \; \frac{dt}{t} = \int_0^\infty \left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right|^2 \; \frac{ds}{s},$$

with $s = |\boldsymbol{x}|t$. In view of Theorem 59, $\widehat{\psi}$ belongs to $\mathcal{S}(\mathbb{R}^N)$. Moreover, since $\int_{\mathbb{R}^N} \psi(\mathbf{x}) \; d\mathbf{x} = 0$, we have that $\widehat{\psi}(\mathbf{0}) = 0$. Hence, for $s \in (0,1)$, by the mean value theorem,

$$\left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right| = \left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) - \widehat{\psi}(\mathbf{0}) \right| \leq s \max_{B(\mathbf{0},1)} \left| \nabla \widehat{\psi} \right| \leq s \left\| \widehat{\psi} \right\|_{0,1},$$

while, for $s \geq 1$,

$$\left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right| \leq \frac{1}{s} \left\| \widehat{\psi} \right\|_{1,0}.$$

Thus,

$$\int_0^\infty \left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right|^2 \; \frac{ds}{s} = \int_0^1 \left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right|^2 \; \frac{ds}{s} + \int_1^\infty \left| \widehat{\psi}\left(s\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \right|^2 \; \frac{ds}{s}$$
$$\leq \left\| \widehat{\psi} \right\|_{0,1} + \left\| \widehat{\psi} \right\|_{1,0} = C_\psi.$$

This completes the proof. ■

Remark 163 *Given $0 < \varepsilon < R$, consider the operator*

$$T_{\varepsilon,R}(g)(\boldsymbol{x}) := \left(\int_\varepsilon^R |(g * \psi_t)(\boldsymbol{x})|^2 \; \frac{dt}{t} \right)^{1/2}, \quad g \in L^2(\mathbb{R}^N).$$

Then reasoning as in the previous proof we have that

$$\int_\varepsilon^R |\widehat{\psi_t}(\boldsymbol{x})|^2 \; \frac{dt}{t} \leq \int_0^\infty |\widehat{\psi_t}(\boldsymbol{x})|^2 \; \frac{dt}{t} \leq C_\psi.$$

Hence, by the Lebesgue dominated convergence theorem

$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}^N}(T_{\varepsilon,R}(g)(\boldsymbol{x}))^2\ d\boldsymbol{x} = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}^N}|\widehat{g}(\boldsymbol{x})|^2\int_{\varepsilon}^{R}|\widehat{\psi_t}(\boldsymbol{x})|^2\ \frac{dt}{t}d\boldsymbol{x}.$$
$$= \int_{\mathbb{R}^N}|\widehat{g}(\boldsymbol{x})|^2\int_0^\infty|\widehat{\psi_t}(\boldsymbol{x})|^2\ \frac{dt}{t}d\boldsymbol{x}$$
$$\leq C_\psi\left\|g\right\|_{L^2}^2.$$

Lemma 164 *Let* $f\in \mathrm{BMO}\left(\mathbb{R}^N\right)$ *and let* $Q=Q(\mathbf{x}_0,r)$ *be a cube. Then for every* $n\geq 1$,

$$\int_{\mathbb{R}^N\setminus mQ}\frac{|f(\boldsymbol{x})-f_{2Q}|}{|\boldsymbol{x}_0-\boldsymbol{x}|^{N+1}}\ d\mathbf{x}\leq\frac{C(N)}{r}\left|f\right|_{\mathrm{BMO}},$$

where $mQ:=Q(\mathbf{x}_0,mr)$ *and* $m:=2^{\lceil\sqrt{N}+1\rceil}$.

Proof. Write

$$\int_{\mathbb{R}^N\setminus mQ}\frac{|f(\boldsymbol{x})-f_{mQ}|}{|\boldsymbol{x}_0-\boldsymbol{x}|^{N+1}}\ d\mathbf{x}\leq\sum_{n=1}^\infty\int_{2^{n+1}Q\setminus 2^nQ}\frac{|f(\boldsymbol{x})-f_{mQ}|}{|\boldsymbol{x}_0-\boldsymbol{x}|^{N+1}}\ d\boldsymbol{x}$$
$$\leq\sum_{n=1}^\infty\frac{1}{2^{n(N+1)}r^{N+1}}\int_{2^{n+1}Q}|f(\boldsymbol{x})-f_{mQ}|\ d\boldsymbol{x}$$

where we used the fact that for $\boldsymbol{y}\in 2^{n+1}Q\setminus 2^nQ$, we have $|\boldsymbol{x}_0-\boldsymbol{x}|\geq 2^n r$. Then

$$\int_{2^{n+1}Q}|f(\boldsymbol{x})-f_{mQ}|\ d\boldsymbol{x}\leq\int_{2^{n+1}Q}|f(\boldsymbol{x})-f_{2^{n+1}Q}|\ d\boldsymbol{x}+\sum_{k=1}^n|f_{2^{k+1}Q}-f_{2^kQ}|2^{(n+1)N}r^N$$
$$\leq 2^{(n+1)N}r^N\left|f\right|_{\mathrm{BMO}}+\sum_{k=1}^n|f_{2^{k+1}Q}-f_{2^kQ}|2^{(n+1)N}r^N.$$

On the other hand,

$$|f_{2^{k+1}Q}-f_{2^kQ}|\leq\frac{1}{\mathcal{L}^N(2^kQ)}\int_{2^kQ}|f(\boldsymbol{x})-f_{2^{k+1}Q}|\ d\boldsymbol{x}\leq 2^N\left|f\right|_{\mathrm{BMO}},$$

and so

$$\sum_{k=1}^n|f_{2^{k+1}Q}-f_{2^kQ}|2^{(n+1)N}r^N\leq n2^{(n+1)N}r^N2^N\left|f\right|_{\mathrm{BMO}}.$$

In turn,

$$\int_{\mathbb{R}^N\setminus mQ}\frac{|f(\boldsymbol{x})-f_{mQ}|}{|\boldsymbol{x}_0-\boldsymbol{x}|^{N+1}}\ d\mathbf{x}\leq\frac{1}{r}\sum_{n=1}^\infty\frac{1}{2^{n(N+1)}}\left(2^{(n+1)N}+n2^{(n+1)N}2^N\right)\left|f\right|_{\mathrm{BMO}}$$
$$\leq\frac{2^{2N+1}}{r}\left|f\right|_{\mathrm{BMO}}\sum_{n=1}^\infty\frac{n+1}{2^n},$$

which concludes the proof. ■

We now turn to the proof of Theorem 161.

Proof of Theorem 161. Fix a cube $Q = Q(\mathbf{x}_0, r)$ and let $mQ := Q(\boldsymbol{x}_0, mr)$, where $m := 2^{\lceil \sqrt{N}+1 \rceil}$. Since $\int_{\mathbb{R}^N} \psi(\mathbf{x})\, d\mathbf{x} = 0$, for any constant c, we have that $c * \psi_t = 0$. Hence,

$$\begin{aligned}\mu(Q(\mathbf{x}_0, r) \times (0, r)) &= \int_0^r \int_{Q(\mathbf{x}_0,r)} |(f * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t} \\ &= \int_0^r \int_{Q(\mathbf{x}_0,r)} |((f - f_{mQ}) * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t}.\end{aligned}$$

Write $f - f_{mQ} = (f - f_{mQ})\chi_{mQ} + (f - f_{mQ})\chi_{\mathbb{R}^N \setminus mQ} =: g + h$. Then

$$\begin{aligned}\mu(Q(\mathbf{x}_0, r) \times (0, r)) \le{}& 2\int_0^r \int_{Q(\mathbf{x}_0,r)} |(g * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t} \\ &+ 2\int_0^r \int_{Q(\mathbf{x}_0,r)} |(h * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t} =: I + II.\end{aligned}$$

Since $g \in L^2(\mathbb{R}^N)$ by Corollary 121, it follows from Lemma 162 that

$$I \le C \int_{\mathbb{R}^N} g^2 \, d\mathbf{x} = C \int_{mQ} |f - f_{mQ}|^2 \, d\mathbf{x} \le C \, |f|^2_{\mathrm{BMO}} \, \mathcal{L}^N(Q).$$

On the other hand, since $\psi \in \mathcal{S}(\mathbb{R}^N)$, we have that $(1 + |\boldsymbol{x}|^{N+1}) \, |\psi(\boldsymbol{x})| \le \|\psi\|_{N+1,0}$, and so, for $\mathbf{x} \in Q$,

$$\begin{aligned}|(h * \psi_t)(\boldsymbol{x})| &\le \int_{\mathbb{R}^N \setminus mQ} \frac{1}{t^N} \left| \psi\left(\frac{\boldsymbol{x} - \boldsymbol{y}}{t} \right) \right| |h(\boldsymbol{y})| \, d\boldsymbol{y} \\ &\le C \int_{\mathbb{R}^N \setminus mQ} \frac{1}{t^N (1 + |(\boldsymbol{x} - \boldsymbol{y})/t|^{N+1}} |h(\boldsymbol{y})| \, d\boldsymbol{y} \\ &\le C \int_{\mathbb{R}^N \setminus mQ} \frac{t}{t^{N+1} + |\boldsymbol{x} - \boldsymbol{y}|^{N+1}} |h(\boldsymbol{y})| \, d\boldsymbol{y} \\ &\le Ct \int_{\mathbb{R}^N \setminus mQ} \frac{|f - f_{mQ}|}{|\boldsymbol{x}_0 - \boldsymbol{y}|^{N+1}} \, d\boldsymbol{y},\end{aligned}$$

where we used the fact that for $\boldsymbol{y} \in \mathbb{R}^N \setminus mQ$ and $\boldsymbol{x} \in Q$, we have $|\boldsymbol{y} - \boldsymbol{x}_0| \le \frac{1}{2}|\boldsymbol{y} - \boldsymbol{x}|$. It follows by Lemma 164 that

$$|(h * \psi_t)(\boldsymbol{x})| \le C \frac{t}{r} \, |f|_{\mathrm{BMO}} \, .$$

In turn,

$$\begin{aligned}II &= 2\int_0^r \int_{Q(\mathbf{x}_0,r)} |(g * \psi_t)(\boldsymbol{x})|^2 \, d\mathbf{x} \frac{dt}{t} \\ &\le 2C \int_0^r \int_{Q(\mathbf{x}_0,r)} \frac{t^2}{r^2} \, |f|^2_{\mathrm{BMO}} \, d\mathbf{x} \frac{dt}{t} = C \, |f|^2_{\mathrm{BMO}} \, \mathcal{L}^N(Q),\end{aligned}$$

which concludes the proof. ■

Friday, April 17, 2015

Carnival

Monday, April 20, 2015

10 Paraproducts

Given $\varphi, \psi \in C_c^\infty(\mathbb{R}^N)$ be radial functions supported on the unit ball, with $\varphi \geq 0$ and

$$\int_{\mathbb{R}^N} \varphi(\boldsymbol{x})\, d\boldsymbol{x} = 1, \quad \int_{\mathbb{R}^N} \psi(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

For $t > 0$ consider

$$\varphi_t(\boldsymbol{x}) := \frac{1}{t^N}\varphi\left(\frac{\boldsymbol{x}}{t}\right), \quad \psi_t(\boldsymbol{x}) := \frac{1}{t^N}\psi\left(\frac{\boldsymbol{x}}{t}\right).$$

Given two functions f and b, the *paraproduct* of f and b is defined formally as

$$\Pi(f,b) := c\int_0^\infty \psi_t * ((\psi_t * b)(\varphi_t * f))\, \frac{dt}{t},$$

where $c > 0$. We will fix b and consider the operator $\Pi_b := \Pi(\cdot, b)$. Then formally,

$$\begin{aligned}
\Pi_b(f)(\mathbf{x}) &= c\int_0^\infty \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})(\varphi_t * f)(\mathbf{z})\, d\mathbf{z}\frac{dt}{t} \\
&= c\int_0^\infty \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z}) \int_{\mathbb{R}^N} \varphi_t(\mathbf{z}-\mathbf{y})f(\mathbf{y})\, d\mathbf{y}d\mathbf{z}\frac{dt}{t} \\
&= \int_{\mathbb{R}^N}\left(c\int_0^\infty \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{y})\, d\mathbf{z}\frac{dt}{t}\right) f(\mathbf{y})\, d\mathbf{y}.
\end{aligned}$$

Define

$$K_b(\mathbf{x},\mathbf{y}) := c\int_0^\infty \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{y})\, d\mathbf{z}\frac{dt}{t}.$$

Theorem 165 *Given $b \in \mathrm{BMO}\left(\mathbb{R}^N\right)$, the kernel K_b is a standard kernel and the corresponding singular operator Π_b is bounded in L^2 and satisfies*

$$\Pi_b(1) = b, \quad \Pi_b^*(1) = 0. \tag{172}$$

Proof. Step 1: Define

$$K_{1,t}(\mathbf{x},\mathbf{y}) := \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{y})\, d\mathbf{z}.$$

For every fixed $\mathbf{z}$ consider the cube $Q = Q(\mathbf{z}, 2t)$. Since $\int_{\mathbb{R}^N} \psi_t(\boldsymbol{x})\, d\boldsymbol{x} = 0$ and $\operatorname{supp}\psi \subseteq B(\mathbf{0}, 1)$, we have that

$$\begin{aligned}|(\psi_t * b)(\mathbf{z})| &= \left|\int_{\mathbb{R}^N} \psi_t(\mathbf{z}-\xi)\,(b(\xi)-b_{2Q})\ d\xi\right| \\ &= \left|\frac{1}{t^N}\int_{2Q} \psi\left(\frac{\mathbf{z}-\xi}{t}\right)(b(\xi)-b_{2Q})\ d\xi\right| \le 2^N \|\psi\|_\infty |b|_{\mathrm{BMO}}\,,\end{aligned}$$

and so

$$\begin{aligned}|K_{1,t}(\mathbf{x},\mathbf{y})| &\le 2^N \|\psi\|_\infty |b|_{\mathrm{BMO}} \int_{\mathbb{R}^N} |\psi_t(\mathbf{x}-\mathbf{z})|\varphi_t(\mathbf{z}-\mathbf{y})\ d\mathbf{z} \\ &\le \frac{1}{t^N} 2^N \|\psi\|_\infty^2 |b|_{\mathrm{BMO}} \|\varphi\|_{L^1}\,.\end{aligned}$$

Since $\operatorname{supp}\psi_t(\mathbf{x}-\cdot) \subseteq B(\boldsymbol{x}, t)$ and $\operatorname{supp}\varphi_t(\cdot-\mathbf{y}) \subseteq B(\boldsymbol{y}, t)$, if $|\boldsymbol{x}-\boldsymbol{y}| > 2t$, we have that $K_{1,t}(\mathbf{x},\mathbf{y}) = 0$. It follows that

$$\begin{aligned}|K_b(\mathbf{x},\mathbf{y})| &= \left|c\int_0^\infty K_{1,t}(\mathbf{x},\mathbf{y})\ \frac{dt}{t}\right| \le c2^N \|\psi\|_\infty^2 |b|_{\mathrm{BMO}} \|\varphi\|_{L^1} \int_{2/|\boldsymbol{x}-\boldsymbol{y}|}^\infty \frac{1}{t^N}\ \frac{dt}{t} \\ &= \frac{c2^N}{N} \|\psi\|_\infty^2 |b|_{\mathrm{BMO}} \|\varphi\|_{L^1} \frac{2^N}{|\boldsymbol{x}-\boldsymbol{y}|^N}.\end{aligned}$$

Since

$$\frac{\partial K_b}{\partial x_i}(\mathbf{x},\mathbf{y}) = c\int_0^\infty \int_{\mathbb{R}^N} \frac{1}{t^{N+1}} \frac{\partial \psi}{\partial x_i}\left(\frac{\mathbf{z}-\xi}{t}\right)(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{y})\ d\mathbf{z}\frac{dt}{t},$$

reasoning as before we conclude that

$$\left|\frac{\partial K_b}{\partial x_i}(\mathbf{x},\mathbf{y})\right| \le \frac{c2^N}{N+1} \|\psi\|_\infty \left\|\frac{\partial \psi}{\partial x_i}\right\|_\infty |b|_{\mathrm{BMO}} \|\varphi\|_{L^1} \frac{2^{N+1}}{|\boldsymbol{x}-\boldsymbol{y}|^{N+1}}.$$

A similar estimate can be obtained for $\frac{\partial K_b}{\partial y_i}$. It now follows from Example 94 that K_b is a standard kernel.

Step 2: To prove that the operator is bounded in $L^2(\mathbb{R}^N)$, we need to truncate the kernel. Define

$$K_{b,\varepsilon,R}(\mathbf{x},\mathbf{y}) := c\int_\varepsilon^R \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{y})\ d\mathbf{z}\frac{dt}{t}$$

and

$$\begin{aligned}\Pi_{b,\varepsilon,R}(f)(\mathbf{x}) &:= \int_{\mathbb{R}^N} K_{b,\varepsilon,R}(\mathbf{x},\mathbf{y})f(\mathbf{y})\ d\mathbf{y} \\ &= c\int_\varepsilon^R \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})(\varphi_t * f)(\mathbf{z})\ d\mathbf{z}\frac{dt}{t}\end{aligned}$$

In view of Step 1 we have that

$$|K_{b,\varepsilon,R}(\mathbf{x},\mathbf{y})| \leq \frac{C}{\varepsilon^N}$$

for all $|\boldsymbol{x}-\boldsymbol{y}| \leq \varepsilon$ and $K_{b,\varepsilon,R}(\mathbf{x},\mathbf{y}) = 0$ for all $|\boldsymbol{x}-\boldsymbol{y}| > 2R$. Thus, $\Pi_{b,\varepsilon,R}(f)$ is well-defined for $f \in L^2(\mathbb{R}^N)$.

Let $f, g \in L^2(\mathbb{R}^N)$. Then by Fubini's theorem and the fact that ψ is radial, so that $\psi(-\mathbf{x}) = \psi(\mathbf{x})$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\, d\mathbf{x} \\
&= c\int_\varepsilon^R \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} g(\boldsymbol{x})\psi_t(\mathbf{x}-\mathbf{z})\, d\mathbf{x}\right)(\psi_t * b)(\mathbf{z})(\varphi_t * f)(\mathbf{z})\, d\mathbf{z}\frac{dt}{t} \\
&= c\int_\varepsilon^R \int_{\mathbb{R}^N} (\psi_t * g)(\mathbf{z})(\psi_t * b)(\mathbf{z})(\varphi_t * f)(\mathbf{z})\, d\mathbf{z}\frac{dt}{t}.
\end{aligned}$$

It follows by Hölder's inequality that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\, d\mathbf{x}\right| &\leq \left(\int_\varepsilon^R \int_{\mathbb{R}^N} |(\psi_t * g)(\mathbf{z})|^2\, d\mathbf{z}\frac{dt}{t}\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^{N+1}_+} |(\psi_t * b)(\mathbf{z})|^2 |(\varphi_t * f)(\mathbf{z})|^2\, d\mathbf{z}\frac{dt}{t}\right)^{1/2}.
\end{aligned}$$

By Lemma 162,

$$\begin{aligned}
&\left(\int_\varepsilon^R \int_{\mathbb{R}^N} |(\psi_t * g)(\mathbf{z})|^2\, d\mathbf{z}\frac{dt}{t}\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}^{N+1}_+} |(\psi_t * g)(\mathbf{z})|^2\, d\mathbf{z}\frac{dt}{t}\right)^{1/2} \leq C_\psi \|g\|_{L^2}.
\end{aligned}$$

On the other hand, by Theorem 161, the measure

$$\mu(E) := \int_E |(b * \psi_t)(\boldsymbol{z})|^2\, d\mathbf{z}\frac{dt}{t}, \quad E \in \mathcal{B}(\mathbb{R}^{N+1}_+),$$

is a Carleson measure with

$$C_\mu \leq C(N,\psi)\, |b|^2_{\mathrm{BMO}}.$$

Hence, by Corollary 160,

$$\begin{aligned}
\left(\int_{\mathbb{R}^{N+1}_+} |(\psi_t * b)(\mathbf{z})|^2 |(\varphi_t * f)(\mathbf{z})|^2\, d\mathbf{z}\frac{dt}{t}\right)^{1/2} &\leq C(N,\varphi,\psi) C_\mu^{1/2} \|f\|_{L^2} \\
&\leq C(N,\varphi,\psi)\, |b|_{\mathrm{BMO}} \|f\|_{L^2}.
\end{aligned}$$

Thus we have proved that

$$\left|\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\ d\mathbf{x}\right| \leq C(N,\varphi,\psi)\,|b|_{\mathrm{BMO}}\,\|f\|_{L^2}\,\|g\|_{L^2}\,.$$

Taking the supremum over all $g \in L^2(\mathbb{R}^N)$ gives

$$\|\Pi_{b,\varepsilon,R}(f)\|_{L^2} \leq C(N,\varphi,\psi)\,|b|_{\mathrm{BMO}}\,\|f\|_{L^2}\,.$$

Moreover, in view of Remark 163,

$$\lim_{\varepsilon\to0^+}\lim_{R\to\infty}\int_{(0,\infty)\setminus(\varepsilon,R)}\int_{\mathbb{R}^N}|(\psi_t * g)(\mathbf{z})|^2\ d\mathbf{z}\frac{dt}{t} = 0$$

and so there exists

$$\lim_{\varepsilon\to0^+}\lim_{R\to\infty}\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\ d\mathbf{x}$$

and

$$\left|\lim_{\varepsilon\to0^+}\lim_{R\to\infty}\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\ d\mathbf{x}\right| \leq C(N,\varphi,\psi)\,|b|_{\mathrm{BMO}}\,\|f\|_{L^2}\,\|g\|_{L^2}\,.$$

Thus we can define the continuous linear operator

$$L_{b,f} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$$

given by

$$L_{b,f}(g) := \lim_{\varepsilon\to0^+}\lim_{R\to\infty}\int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_{b,\varepsilon,R}(f)(\mathbf{x})\ d\mathbf{x}, \quad g \in L^2(\mathbb{R}^N).$$

It follows by the Riesz representation theorem that there exists a unique function $\Pi_b(f) \in L^2(\mathbb{R}^N)$ such that

$$L_{b,f}(g) = \int_{\mathbb{R}^N} g(\boldsymbol{x})\Pi_b(f)(\mathbf{x})\ d\mathbf{x}, \quad g \in L^2(\mathbb{R}^N),$$

with

$$\|\Pi_b(f)\|_{L^2} \leq C(N,\varphi,\psi)\,|b|_{\mathrm{BMO}}\,\|f\|_{L^2}\,.$$

■

Wedneday, April 22, 2015

Proof. Step 3: Finally, we give a formal proof of (172). For $f \in L^2(\mathbb{R}^N)$ we have

$$K_b^*(\mathbf{x},\mathbf{y}) = K_b(\mathbf{y},\mathbf{x}) = c\int_0^\infty\int_{\mathbb{R}^N}\psi_t(\mathbf{y}-\mathbf{z})(\psi_t * b)(\mathbf{z})\varphi_t(\mathbf{z}-\mathbf{x})\ d\mathbf{z}\frac{dt}{t}$$

and

$$\Pi_b^*(f)(\mathbf{x}) = \int_{\mathbb{R}^N} K_b^*(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$
$$= c\int_0^\infty \int_{\mathbb{R}^N} \varphi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})(\psi_t * f)(\mathbf{z})\, d\mathbf{z}\frac{dt}{t}$$

for for $\mathcal{L}^N$ a.e. $\mathbf{x}\in\mathbb{R}^N$.

Hence, formally, taking $f=1$, it follows that $(\psi_t * 1)(\mathbf{z}) = 0$, since $\int_{\mathbb{R}^N}\psi_t(\boldsymbol{x})\,d\boldsymbol{x} = 0$. This shows that $\Pi_{b,\varepsilon,R}^*(1)(\mathbf{x}) = 0$ for all $\mathbf{x}\in\mathbb{R}^N$.

On the other hand, since $(\varphi_t * 1)(\mathbf{z}) = 1$ because $\int_{\mathbb{R}^N}\varphi_t(\boldsymbol{x})\; d\boldsymbol{x} = 1$, again formally,

$$\Pi_b(1)(\mathbf{x}) = c\int_0^\infty \int_{\mathbb{R}^N} \psi_t(\mathbf{x}-\mathbf{z})(\psi_t * b)(\mathbf{z})\; d\mathbf{z}\frac{dt}{t}$$
$$= c\int_0^\infty (\psi_t * (\psi_t * b))(\mathbf{x})\; \frac{dt}{t}.$$

So we need

$$c\int_0^\infty (\psi_t * (\psi_t * b))(\mathbf{x})\; \frac{dt}{t} = b(\mathbf{x}).$$

Let's consider the Fourier transform. Again formally we have

$$\widehat{\Pi_b(1)}(\boldsymbol{y}) = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{y}}\Pi_b(1)(\mathbf{x})\; d\boldsymbol{x}$$
$$= c\int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{y}} \int_0^\infty (\psi_t * (\psi_t * b))(\mathbf{x})\; \frac{dt}{t} d\boldsymbol{x}$$
$$= c\int_0^\infty \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{y}} (\psi_t * (\psi_t * b))(\mathbf{x})\; d\boldsymbol{x}\frac{dt}{t}$$
$$= c\int_0^\infty (\psi_t \widehat{* (\psi_t} * b))(\boldsymbol{y})\; \frac{dt}{t} = c\int_0^\infty \widehat{\psi_t}(\boldsymbol{y})(\widehat{\psi_t * b)})(\boldsymbol{y})\; \frac{dt}{t}$$
$$= c\int_0^\infty |\widehat{\psi_t}(\boldsymbol{y})|^2 \widehat{b}(\boldsymbol{y})\; \frac{dt}{t} = c\widehat{b}(\boldsymbol{y})\int_0^\infty |\widehat{\psi_t}(\boldsymbol{y})|^2\; \frac{dt}{t}.$$

So we need

$$c\int_0^\infty |\widehat{\psi_t}(\boldsymbol{y})|^2\; \frac{dt}{t} = 1.$$

Observe that

$$\widehat{\psi_t}(\boldsymbol{y}) = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{x}}\psi_t(\mathbf{x})\; d\boldsymbol{x} = \int_{\mathbb{R}^N} e^{-2\pi i \boldsymbol{y}\cdot\boldsymbol{x}}\frac{1}{t^N}\psi\left(\frac{\boldsymbol{x}}{t}\right)\; d\boldsymbol{x}$$
$$= \int_{\mathbb{R}^N} e^{-2\pi i t\boldsymbol{y}\cdot\boldsymbol{\xi}}\psi\left(\boldsymbol{\xi}\right)\; d\boldsymbol{\xi} = \widehat{\psi}(t\boldsymbol{y}).$$

Since ψ is radial, we have that $\widehat{\psi}$ is also radial, so that $\widehat{\psi}(t\boldsymbol{y}) = g(t|\boldsymbol{y}|)$. Hence,

$$c\int_0^\infty |\widehat{\psi_t}(\boldsymbol{y})|^2\; \frac{dt}{t} = c\int_0^\infty |g(s)|^2\; \frac{ds}{s} = 1$$

by the appropriate choce of c and where we have made the change of variables $s = t|\boldsymbol{y}|$. ■

Remark 166 *The rigororous proof of Step 3 is significantly more complicated.*

11 Proof of the $T(1)$ Theorem

Theorem 167 (Krein) *Let H be a Hilbert space with inner product $(\cdot,\cdot)_H : H \times H \to \mathbb{C}$ and let Y be a normed space for which the inclusion $i : Y \hookrightarrow H$ is well-defined, continuous, and with dense range. Let $T_1 : Y \to Y$ and $T_2 : Y \to Y$ be two linear bounded operators such that*

$$(T_1(x), y)_H = (x, T_2(y))_H \tag{173}$$

for all $x, y \in Y$. Then both T_1 and T_2 can be extended to bounded linear operators from H to H with

$$\|T_1\|_{\mathcal{L}(H,H)} \leq \|T_1\|_{\mathcal{L}(Y,Y)}^{1/2} \|T_2\|_{\mathcal{L}(Y,Y)}^{1/2}, \quad \|T_2\|_{\mathcal{L}(H,H)} \leq \|T_1\|_{\mathcal{L}(Y,Y)}^{1/2} \|T_2\|_{\mathcal{L}(Y,Y)}^{1/2}.$$

Proof. For $y \in Y$, by (173),

$$\begin{aligned} \|T_1(y)\|_H^2 &= (T_1(y), T_1(y))_H = (y, T_2(T_1(y)))_H \\ &\leq \|y\|_H \|T_2(T_1(y))\|_H . \end{aligned}$$

Therefore

$$\|T_1(y)\|_H \leq \|y\|_H^{1/2} \|T_2(T_1(y))\|_H^{1/2}.$$

Hence, if we can extend the operator $L := T_2 \circ T_1$ as a bounded linear operator from H to H, then by the previous inequality we can also extend T_1 (and, similarly, T_2).

Note that by the properties of the inner product and by (173),

$$\begin{aligned} (L(x), y)_H &= (T_2(T_1(x)), y)_H = \overline{(y, T_2(T_1(x)))_H} \\ &= \overline{(T_1(y), T_1(x))_H} = (T_1(x), T_1(y))_H \\ &= (x, T_2(T_1(y)))_H = (x, L(y))_H. \end{aligned}$$

Hence, repeating the previous argument with T_1 and T_2 both replaced by L we get

$$\|L(y)\|_H \leq \|y\|_H^{1/2} \left\|L^2(y)\right\|_H^{1/2}.$$

By iterating this inequality 2^n times we get

$$\|L(y)\|_H \leq \|y\|_H^{2^{-1}+\cdots+2^{-n}} \left\|L^{2^n}(y)\right\|_H^{2^{-n}}.$$

Now using the fact that the inclusion $i : Y \hookrightarrow H$ continuous, for every $x \in Y$ we have that

$$\|z\|_H = \|i(z)\|_H \leq \|i\|_{\mathcal{L}(Y,H)} \|z\|_Y .$$

Hence, taking $z = L^{2^n}(y)$, we obtain

$$\begin{aligned}
\|L(y)\|_H &\le \|y\|_H^{2^{-1}+\cdots+2^{-n}} \|i\|_{\mathcal{L}(Y,H)}^{2^{-n}} \left\|L^{2^n}(y)\right\|_Y^{2^{-n}} \\
&\le \|y\|_H^{2^{-1}+\cdots+2^{-n}} \|i\|_{\mathcal{L}(Y,H)}^{2^{-n}} \left\|L^{2^n}\right\|_{\mathcal{L}(Y,Y)}^{2^{-n}} \|y\|_Y^{2^{-n}} \\
&\le \|y\|_H^{2^{-1}+\cdots+2^{-n}} \|i\|_{\mathcal{L}(Y,H)}^{2^{-n}} \|L\|_{\mathcal{L}(Y,Y)} \|y\|_Y^{2^{-n}},
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\left\|L^{2^n}\right\|_{\mathcal{L}(Y,Y)} &= \sup_{y\neq 0} \frac{\left\|L^{2^n}(z)\right\|_Y}{\|z\|_Y} \le \|L\|_{\mathcal{L}(Y,Y)} \sup_{y\neq 0} \frac{\left\|L^{2^n-1}(z)\right\|_Y}{\|z\|_Y} \\
&\le \cdots \le \|L\|_{\mathcal{L}(Y,Y)}^{2^n}.
\end{aligned}$$

Letting $n \to \infty$ gives

$$\|L(y)\|_H \le \|y\|_H \|L\|_{\mathcal{L}(Y,Y)}$$

for all $y \in Y$. By density of $i(Y)$ in H, we can extend L to H in a continuous way. ■

Friday, April 24, 2015

We now turn to the proof of the $T(1)$ theorem.

Proof of Theorem 151. By Theorem 165 there exist two operators $L_{\varepsilon,R}$ and $M_{\varepsilon,R}$ such that

$$\begin{aligned}
L_{\varepsilon,R}(1) &= T_{\varphi_{\varepsilon,R}}(1), \quad L_{\varepsilon,R}^*(1) = 0, \\
M_{\varepsilon,R}(1) &= T_{\varphi_{\varepsilon,R}}^*(1), \quad M_{\varepsilon,R}^*(1) = 0.
\end{aligned}$$

Define

$$\widetilde{T}_{\varepsilon,R} := T_{\varphi_{\varepsilon,R}} - L_{\varepsilon,R} - M_{\varepsilon,R}^*.$$

Then

$$\widetilde{T}_{\varepsilon,R}(1) = \widetilde{T}_{\varepsilon,R}^*(1) = 0. \tag{174}$$

Let $\widetilde{K}_{\varepsilon,R}$ be the standard kernel associated to $\widetilde{T}_{\varepsilon,R}$. Note that $\widetilde{K}_{\varepsilon,R}$ still satisfies the standard estimates but it is not truncated anymore.

Step 1: We claim that there exists a constant $C > 0$ such that

$$\left|\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x})\right| \le C$$

for all $\varepsilon, R, \delta > 0$ and for all $\boldsymbol{x} \in \mathbb{R}^N$.

By (174) applied twice we can write

$$\begin{aligned}
\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x}) &= \widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x}) + \widetilde{T}_{\varepsilon,R}(1)(\boldsymbol{z}) \\
&= \widetilde{T}_{\varepsilon,R}\left(\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x}) + \widetilde{T}_{\varepsilon,R}(1)(\boldsymbol{z}) \\
&= \widetilde{T}_{\varepsilon,R}\left(\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x}) - \widetilde{T}_{\varepsilon,R}\left(\chi_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\right)(\boldsymbol{z}) + \widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{z}).
\end{aligned}$$

Now average in $\boldsymbol{z}$ over $B(\boldsymbol{x}, \delta/2)$ to get

$$\left|\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{x})\right| \leq \frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\int_{B(\boldsymbol{x},\delta/2)}\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\left|\widetilde{K}_{\varepsilon,R}(\boldsymbol{z},\boldsymbol{y})-\widetilde{K}_{\varepsilon,R}(\boldsymbol{x},\boldsymbol{y})\right|\,d\boldsymbol{y}d\boldsymbol{z}$$
$$+\frac{1}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\left|\int_{B(\boldsymbol{x},\delta/2)}\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{z})d\boldsymbol{z}\right|$$
$$\leq \frac{C}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\int_{B(\boldsymbol{x},\delta/2)}\int_{\mathbb{R}^N\setminus B(\boldsymbol{x},\delta)}\frac{|\boldsymbol{z}-\boldsymbol{x}|^\alpha}{|\boldsymbol{x}-\boldsymbol{y}|^{N+\alpha}}\,d\boldsymbol{y}d\boldsymbol{z}$$
$$+\frac{C}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\left(\int_{B(\boldsymbol{x},\delta)}\left|\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},\delta)}\right)(\boldsymbol{z})\right|^2 d\boldsymbol{z}\right)^{1/2}\left(\mathcal{L}^N(B(\boldsymbol{x},\delta/2))\right)^{1/2}$$

where we used (87) and Hölder's inequality. Using spherical coordinates the first term on the right-hand size equals to

$$\frac{C\alpha_N}{\mathcal{L}^N(B(\boldsymbol{x},\delta/2))}\int_{B(\boldsymbol{x},\delta/2)}|\boldsymbol{z}-\boldsymbol{x}|^\alpha\,d\boldsymbol{z}\int_\delta^\infty\frac{r^{N-1}}{r^{N+\alpha}}\,dr\leq Cr^\alpha,$$

while for the second term we use Theorem 153.

Step 2: Let $\varphi\in C_c^\infty(B(\boldsymbol{0},2))$ and let $M_\varphi(f):=\varphi f$ be the multiplication operator. We claim that the operator

$$M_\varphi\circ\widetilde{T}_{\varepsilon,R}\circ M_\varphi : C^{0,\beta}(B(\boldsymbol{0},2))\to C^{0,\beta}(B(\boldsymbol{0},2))$$

is bounded, where $0<\beta<\alpha$. For $\boldsymbol{x}\in B(\boldsymbol{0},2)$, we have that $B(\boldsymbol{0},2)\subseteq B(\boldsymbol{x},4)$. Extend φf to be zero outside $B(\boldsymbol{0},2)$. Then we can write

$$\widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{x})=\int_{\mathbb{R}^N}\widetilde{K}_{\varepsilon,R}(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}=\int_{B(\boldsymbol{x},4)}\widetilde{K}_{\varepsilon,R}(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}$$
$$=\int_{B(\boldsymbol{x},4)}\widetilde{K}_{\varepsilon,R}(\boldsymbol{x},\boldsymbol{y})[\varphi(\boldsymbol{y})f(\boldsymbol{y})-\varphi(\boldsymbol{x})f(\boldsymbol{x})]\,d\boldsymbol{y}+\varphi(\boldsymbol{x})f(\boldsymbol{x})\widetilde{T}_{\varepsilon,R}\left(\chi_{B(\boldsymbol{x},4)}\right)(\boldsymbol{x}).$$

By the fact that φf is Hölder's continuous, Step 1, and (85),

$$|\widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{x})|\leq C\,|\varphi f|_{C^{0,\beta}}\int_{B(\boldsymbol{x},4)}\frac{|\boldsymbol{x}-\boldsymbol{y}|^\beta}{|\boldsymbol{x}-\boldsymbol{y}|^N}\,d\boldsymbol{y}+C\,\|\varphi f\|_\infty$$
$$=C\,|\varphi f|_{C^{0,\beta}}\,\alpha_N\int_0^4\frac{r^{N-1+\beta}}{r^N}\,dr+C\,\|\varphi f\|_\infty.$$

Hence,

$$\left\|\widetilde{T}_{\varepsilon,R}(\varphi f)\right\|_\infty\leq C\,\|\varphi f\|_{0,\beta}\leq C\,\|\varphi\|_{0,\beta}\,\|f\|_{0,\beta}. \tag{175}$$

■

Monday, April 27, 2015

Proof. On the other hand, for all $\boldsymbol{x},\boldsymbol{z}\in B(\boldsymbol{0},2)$ let $\boldsymbol{h}:=\boldsymbol{z}-\boldsymbol{x}$ so that $\boldsymbol{y}=\boldsymbol{x}+\boldsymbol{h}$. Using (174) we have

$$\left|\widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{x})-\widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{x}+\boldsymbol{h})\right|\leq I+II+III+IV,$$

where

$$I := \left| \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \left(\widetilde{K}_{\varepsilon,R}(\boldsymbol{x}+\boldsymbol{h}, \boldsymbol{y}) - \widetilde{K}_{\varepsilon,R}(\boldsymbol{x}, \boldsymbol{y}) \right) \left((\varphi f)(\boldsymbol{y}) - (\varphi f)(\boldsymbol{x}) \right) \, d\boldsymbol{y} \right|,$$

$$II := \left| \int_{B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \widetilde{K}_{\varepsilon,R}(\boldsymbol{x}, \boldsymbol{y}) \left((\varphi f)(\boldsymbol{y}) - (\varphi f)(\boldsymbol{x}) \right) \, d\boldsymbol{y} \right|,$$

$$III := \left| \int_{B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \widetilde{K}_{\varepsilon,R}(\boldsymbol{x}+\boldsymbol{h}, \boldsymbol{y}) \left((\varphi f)(\boldsymbol{y}) - (\varphi f)(\boldsymbol{x}+\boldsymbol{h}) \right) \, d\boldsymbol{y} \right|$$

$$IV := \left| \left((\varphi f)(\boldsymbol{x}+\boldsymbol{h}) - (\varphi f)(\boldsymbol{x}) \right) \int_{B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \widetilde{K}_{\varepsilon,R}(\boldsymbol{x}+\boldsymbol{h}, \boldsymbol{y}) \, d\boldsymbol{y} \right|$$

By the fact that φf is Hölder's continuous and (87),

$$\begin{aligned} I &\le C \left| \varphi f \right|_{C^{0,\beta}} |\boldsymbol{h}|^\alpha \int_{\mathbb{R}^N \setminus B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \frac{|\boldsymbol{x}-\boldsymbol{y}|^\beta}{|\boldsymbol{x}-\boldsymbol{y}|^{N+\alpha}} \, d\boldsymbol{y} \\ &= C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N |\boldsymbol{h}|^\alpha \int_{2|\boldsymbol{h}|}^{\infty} \frac{r^{N-1+\beta}}{r^{N+\alpha}} \, dr = C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N \frac{|\boldsymbol{h}|^\alpha (2|\boldsymbol{h}|)^{\beta-\alpha}}{\alpha-\beta}. \end{aligned}$$

Similarly, by the fact that φf is Hölder's continuous and (85),

$$\begin{aligned} II &\le C \left| \varphi f \right|_{C^{0,\beta}} \int_{B(\boldsymbol{x}, 2|\boldsymbol{h}|)} \frac{|\boldsymbol{x}-\boldsymbol{y}|^\beta}{|\boldsymbol{x}-\boldsymbol{y}|^{N}} \, d\boldsymbol{y} \\ &= C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N \int_0^{2|\boldsymbol{h}|} \frac{r^{N-1+\beta}}{r^{N}} \, dr = C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N \frac{(2|\boldsymbol{h}|)^\beta}{\beta}. \end{aligned}$$

Since $B(\boldsymbol{x}, 2|\boldsymbol{h}|) \subseteq B(\boldsymbol{x}+\boldsymbol{h}, 3|\boldsymbol{h}|)$, by the fact that φf is Hölder's continuous, (85) and (87),

$$\begin{aligned} III &\le C \left| \varphi f \right|_{C^{0,\beta}} \int_{B(\boldsymbol{x}+\boldsymbol{h}, 3|\boldsymbol{h}|)} \frac{|\boldsymbol{x}+\boldsymbol{h}-\boldsymbol{y}|^\beta}{|\boldsymbol{x}+\boldsymbol{h}-\boldsymbol{y}|^{N}} \, d\boldsymbol{y} \\ &= C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N \int_0^{3|\boldsymbol{h}|} \frac{r^{N-1+\beta}}{r^{N}} \, dr = C \left| \varphi f \right|_{C^{0,\beta}} \alpha_N \frac{(3|\boldsymbol{h}|)^\beta}{\beta}. \end{aligned}$$

Finally, since $B(\boldsymbol{x}, 2|\boldsymbol{h}|) \setminus B(\boldsymbol{x}+\boldsymbol{h}, |\boldsymbol{h}|) \subset B(\boldsymbol{x}+\boldsymbol{h}, 3|\boldsymbol{h}|) \setminus B(\boldsymbol{x}+\boldsymbol{h}, |\boldsymbol{h}|)$ and $B(\boldsymbol{x}+\boldsymbol{h}, |\boldsymbol{h}|) \subset B(\boldsymbol{x}, 2|\boldsymbol{h}|)$ by Step 1, the fact that φf is Hölder's continuous,

(85) and (87),

$$
\begin{aligned}
III &\leq |(\varphi f)(\boldsymbol{x}+\boldsymbol{h}) - (\varphi f)(\boldsymbol{x})| \left| \int_{B(\boldsymbol{x}+\boldsymbol{h},|\boldsymbol{h}|)} \widetilde{K}_{\varepsilon,R}(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{y})\, d\boldsymbol{y} \right| \\
&\quad + |(\varphi f)(\boldsymbol{x}+\boldsymbol{h}) - (\varphi f)(\boldsymbol{x})| \int_{B(\boldsymbol{x}+\boldsymbol{h},3|\boldsymbol{h}|)\setminus B(\boldsymbol{x}+\boldsymbol{h},|\boldsymbol{h}|)} |\widetilde{K}_{\varepsilon,R}(\boldsymbol{x}+\boldsymbol{h},\boldsymbol{y})|\, d\boldsymbol{y} \\
&\leq C\, |\varphi f|_{C^{0,\beta}}\, |\boldsymbol{h}|^\beta \left(1 + \int_{B(\boldsymbol{x}+\boldsymbol{h},3|\boldsymbol{h}|)\setminus B(\boldsymbol{x}+\boldsymbol{h},|\boldsymbol{h}|)} \frac{1}{|\boldsymbol{x}+\boldsymbol{h}-\boldsymbol{y}|^N}\, d\boldsymbol{y} \right) \\
&\leq C\, |\varphi f|_{C^{0,\beta}}\, |\boldsymbol{h}|^\beta \left(1 + \alpha_N \frac{|3\boldsymbol{h}|^N}{|\boldsymbol{h}|^N} \right).
\end{aligned}
$$

In conclusion we have shown that

$$\left| \widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{x}) - \widetilde{T}_{\varepsilon,R}(\varphi f)(\boldsymbol{y}) \right| \leq C\, |\varphi f|_{C^{0,\beta}}\, |\boldsymbol{x}-\boldsymbol{y}|^\beta,$$

which together with (175) implies that

$$\left\| \widetilde{T}_{\varepsilon,R}(\varphi f) \right\|_{0,\beta} \leq C\, \|\varphi f\|_{0,\beta}$$

In turn,

$$\left\| \varphi \widetilde{T}_{\varepsilon,R}(\varphi f) \right\|_{0,\beta} \leq C\, \|\varphi\|_{0,\beta}\, \|\varphi f\|_{0,\beta} \leq C\, \|\varphi\|_{0,\beta}\, \|f\|_{0,\beta},$$

which proves the claim.
Step 3: By applying Krein's theorem to the operator $M_\varphi \circ \widetilde{T}_{\varepsilon,R} \circ M_\varphi$ and its adjugate with $Y := C^{0,\beta}(B(\mathbf{0},2))$ and $H := L^2(B(\mathbf{0},2))$, we get that both $M_\varphi \circ \widetilde{T}_{\varepsilon,R} \circ M_\varphi$ and its adjugate can be extended to linear bounded operators from $L^2(B(\mathbf{0},2))$ to $L^2(B(\mathbf{0},2))$.

Next assume that $\varphi = 1$ in $B(\mathbf{0},1)$. Given $f \in L^2(B(\mathbf{0},1))$, extend f to be zero in $B(\mathbf{0},2) \setminus B(\mathbf{0},1)$. Then

$$
\begin{aligned}
\left\| \widetilde{T}_{\varepsilon,R}\left(f\chi_{B(\mathbf{0},1)}\right) \right\|_{L^2(B(\mathbf{0},1))} &= \left\| \widetilde{T}_{\varepsilon,R}(\varphi f) \right\|_{L^2(B(\mathbf{0},1))} \leq \left\| \varphi \widetilde{T}_{\varepsilon,R}(\varphi f) \right\|_{L^2(B(\mathbf{0},2))} \\
&\leq C\, \|f\|_{L^2(B(\mathbf{0},2))} = C\, \|f\|_{L^2(B(\mathbf{0},1))}.
\end{aligned}
$$

By rescaling, we can show that

$$\left\| \widetilde{T}_{\varepsilon,R}\left(f\chi_{B(\mathbf{0},n)}\right) \right\|_{L^2(B(\mathbf{0},n))} \leq C\, \|f\|_{L^2(B(\mathbf{0},n))}$$

for all $f \in L^2(\mathbb{R}^N)$, where the constant $C > 0$ does not depend on n. Recall that

$$\widetilde{T}_{\varepsilon,R} := T_{\varphi_{\varepsilon,R}} - L_{\varepsilon,R} - M_{\varepsilon,R}$$

and that by Theorem 165, $L_{\varepsilon,R}$ and $M_{\varepsilon,R}$ are bounded in L^2. It follows that

$$\left\| T_{\varphi_{\varepsilon,R}}\left(f\chi_{B(\mathbf{0},n)}\right) \right\|_{L^2(B(\mathbf{0},n))} \leq C\, \|f\|_{L^2(B(\mathbf{0},n))}.$$

Since $K_{\varphi_{\varepsilon,R}}(x-y)=0$ if either $|\boldsymbol{x}-\boldsymbol{y}|\le\varepsilon$ or $|\boldsymbol{x}-\boldsymbol{y}|\ge 2R$, we can apply the Lebesgue dominated convergence theorem and Fatou's lemma to conclude that

$$\left\|T_{\varphi_{\varepsilon,R}}(f)\right\|_{L^2(\mathbb{R}^N)}\le C\,\|f\|_{L^2(\mathbb{R}^N)}\,.$$

This concludes the proof. ■

Wednesday, April 29, 2015

12 The Cauchy Integral, Continued

[?]

We recall that given a simple closed oriented rectifiable curve γ in $\mathbb{C}$ or a simple oriented locally rectifiable curve γ through infinity and a measurable function $f:\gamma\to\mathbb{C}$, then the Cauchy integral of f is given by

$$\mathcal{C}(f)(\boldsymbol{z}):=\lim_{\varepsilon\to 0^+}\mathcal{C}_\varepsilon(f)(\boldsymbol{z}),\quad \boldsymbol{z}\in\gamma,$$

where

$$\mathcal{C}_\varepsilon(f)(\boldsymbol{z}):=\frac{1}{2\pi\boldsymbol{i}}\int_{\gamma\setminus B(\boldsymbol{z},\varepsilon)}\frac{f(\boldsymbol{\zeta})}{\boldsymbol{\zeta}-\boldsymbol{z}}\,d\boldsymbol{\zeta},\quad \boldsymbol{z}\in\gamma.$$

Consider a curve γ parametrized by $\boldsymbol{\zeta}(x)=x+\boldsymbol{i}g(x)$, where $g:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function (the general case can be obtained using a partition of unity). Then

$$\mathcal{C}_\varepsilon(f)(\boldsymbol{z}(x))=\frac{1}{2\pi\boldsymbol{i}}\int_{\{t:\ |\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)|>\varepsilon\}}\frac{f(\boldsymbol{\zeta}(t))\boldsymbol{\zeta}'(t)}{\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)}\,dt.$$

Since $\boldsymbol{\zeta}'(t)=1+\boldsymbol{i}g'(x)$ is bounded, by changing the truncation we consider the slightly different truncated operator

$$\mathcal{C}_\varepsilon(f)(x)=\int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}\frac{f(t)}{\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)}\,dt,\quad f\in L^2(\mathbb{R}).$$

We will prove the following result:

Theorem 168 *There exists $C_1>0$ depending only on $\|g'\|_\infty$ such that*

$$\|\mathcal{C}_\varepsilon(f)\|_{L^2(\mathbb{R})}\le C_1\,\|f\|_{L^2(\mathbb{R})}$$

for all $f\in L^2(\mathbb{R})$ and $\varepsilon>0$.

We begin with some preliminary results.

Lemma 169 *Let $\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3$ be three distinct points in $\mathbb{C}$, then*

$$\sum_\sigma\frac{1}{\boldsymbol{z}_{\sigma(2)}-\boldsymbol{z}_{\sigma(1)}}\frac{1}{\overline{\boldsymbol{z}_{\sigma(3)}-\boldsymbol{z}_{\sigma(1)}}}=\left(\frac{4A(\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3)}{|\boldsymbol{z}_2-\boldsymbol{z}_1||\boldsymbol{z}_3-\boldsymbol{z}_1||\boldsymbol{z}_3-\boldsymbol{z}_2|}\right)^2,$$

where the sum is done over all six permutations of the set $\{1,2,3\}$ and $A(\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3)$ is the area of the triangle of vertices $\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3$.

Proof. Exercise ■

Remark 170 *If* $\boldsymbol{z}_i = x_i + \boldsymbol{i}y_i$*,* $i = 1, 2, 3$*, and* $x_i \neq x_j$ *for* $i \neq j$*, then*

$$A(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3) = \frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|,$$

and so

$$\begin{aligned}\frac{4A(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3)}{|\boldsymbol{z}_2 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_2|} &\leq \frac{2|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|}{|\boldsymbol{z}_2 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_2|} \\ &= 2\frac{|x_3 - x_1||x_2 - x_1|\left|\frac{y_3 - y_1}{x_3 - x_1} - \frac{y_2 - y_1}{x_2 - x_1}\right|}{|\boldsymbol{z}_2 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_1||\boldsymbol{z}_3 - \boldsymbol{z}_2|} \\ &\leq 2\frac{\left|\frac{y_3 - y_1}{x_3 - x_1} - \frac{y_2 - y_1}{x_2 - x_1}\right|}{|x_3 - x_2|}.\end{aligned}$$

We begin with some preliminary results.

Lemma 171 *Let* $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ *with* $\varphi' \in L^2(\mathbb{R})$*. Then*

$$\iiint_{\mathbb{R}^3} \left(\frac{\frac{\varphi(y) - \varphi(x)}{y - x} - \frac{\varphi(t) - \varphi(x)}{t - x}}{t - y} \right)^2 dxdydt = c \left\|\varphi'\right\|^2_{L^2}$$

for some constant $c > 0$ *independent of* φ*.*

Proof. Consider the change of variables $h = y - x$ and $k = t - x$ so that $dh = dy$ and $dk = dt$. Then also by Tonelli's theorem the left-hand side equals to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\frac{\varphi(x+h) - \varphi(x)}{h} - \frac{\varphi(x+k) - \varphi(x)}{k}}{h - k} \right)^2 dxdhdk.$$

We now use the Plancherel theorem in the variable x. We need to compute the Fourier transform of a difference quotient. Let $\psi(x) := \frac{\varphi(x+h) - \varphi(x)}{h}$. Then we have

$$\begin{aligned}\widehat{\psi}(\xi) &= \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi x} \psi(x)\ dx = \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi x} \frac{\varphi(x+h) - \varphi(x)}{h}\ dx \\ &= \frac{1}{h} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi x} \varphi(x+h)\ dx - \frac{1}{h} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi x} \varphi(x)\ dx \\ &= \frac{1}{h} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi (s-h)} \varphi(s)\ ds - \frac{1}{h} \int_{\mathbb{R}} e^{-2\pi \boldsymbol{i}\xi x} \varphi(x)\ dx = \frac{e^{2\pi \boldsymbol{i}\xi h} - 1}{h} \widehat{\varphi}(\xi),\end{aligned}$$

where we made the change of variables $s = x + h$. Hence by the Plancherel theorem

$$\int_{\mathbb{R}} \left(\frac{\varphi(x+h) - \varphi(x)}{h} - \frac{\varphi(x+k) - \varphi(x)}{k} \right)^2 \, dx$$
$$= \int_{\mathbb{R}} \left| \frac{e^{2\pi \boldsymbol{i} \xi h} - 1}{h} - \frac{e^{2\pi \boldsymbol{i} \xi k} - 1}{h} \right|^2 |\widehat{\varphi}(\xi)|^2 \, d\xi$$
$$= \int_{\mathbb{R}} \left| \frac{e^{2\pi \boldsymbol{i} \xi h} - 1}{\xi h} - \frac{e^{2\pi \boldsymbol{i} \xi k} - 1}{\xi h} \right|^2 |\xi \widehat{\varphi}(\xi)|^2 \, d\xi$$
$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \left| \frac{e^{2\pi \boldsymbol{i} \xi h} - 1}{\xi h} - \frac{e^{2\pi \boldsymbol{i} \xi k} - 1}{\xi h} \right|^2 |\widehat{\varphi'}(\xi)|^2 \, d\xi,$$

where we used the used the fact that by integrating by parts, we have that

$$\begin{aligned} \xi \widehat{\varphi}(\xi) &= \int_{\mathbb{R}} e^{-2\pi \xi y \boldsymbol{i}} \xi \varphi(y) \, dy \\ &= \frac{-1}{2\pi \boldsymbol{i}} \int_{\mathbb{R}} \frac{d}{dy} \left(e^{-2\pi \xi y \boldsymbol{i}} \right) \varphi(y) \, dy \\ &= \frac{1}{2\pi \boldsymbol{i}} \int_{\mathbb{R}} e^{-2\pi \xi y \boldsymbol{i}} \varphi'(y) \, dy = \frac{1}{2\pi \boldsymbol{i}} \widehat{\varphi'}(\xi). \end{aligned}$$

In turn, again by Tonelli's theorem,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{\frac{\varphi(x+h)-\varphi(x)}{h} - \frac{\varphi(x+k)-\varphi(x)}{k}}{h-k} \right)^2 \, dx dh dk$$
$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\frac{e^{2\pi \boldsymbol{i} \xi h}-1}{\xi h} - \frac{e^{2\pi \boldsymbol{i} \xi k}-1}{\xi h}}{h-k} \right|^2 |\widehat{\varphi'}(\xi)|^2 \, d\xi dh dk$$
$$= \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\frac{e^{2\pi \boldsymbol{i} u}-1}{u} - \frac{e^{2\pi \boldsymbol{i} v}-1}{v}}{u-v} \right|^2 |\widehat{\varphi'}(\xi)|^2 \, d\xi du dv$$
$$= \frac{1}{4\pi^2} \|\varphi'\|_{L^2}^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\frac{e^{2\pi \boldsymbol{i} u}-1}{u} - \frac{e^{2\pi \boldsymbol{i} v}-1}{v}}{u-v} \right|^2 \, du dv,$$

where $u = \xi h$ and $v = \xi k$, so that $du = \xi dh$ and $dv = \xi dk$. It remains to show that the double integral is finite. Define

$$E(u) = \frac{e^{2\pi \boldsymbol{i} u} - 1}{\boldsymbol{i} u}.$$

Then by the Plancherel theorem and the previous computation on the Fourier transform of a difference quotient we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{\frac{e^{2\pi i u}-1}{u}-\frac{e^{2\pi i v}-1}{v}}{u-v}\right|^2\,dudv=\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{E(u)-E(v)}{u-v}\right|^2\,dudv$$
$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{E(v+t)-E(v)}{t}\right|^2\,dvdt$$
$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{e^{2\pi i\xi t}-1}{t}\right|^2|\widehat{E}(\xi)|^2\,d\xi dt$$

where $t=u-v$.

To find $\widehat{E}$ it is simpler to compute the Fourier transform of $\chi_{[a,b]}$. We have

$$\widehat{\chi_{[a,b]}}(y)=\int_a^b e^{-2\pi i yx}\,dx=-\left[\frac{e^{-2\pi i yx}}{2\pi i y}\right]_{x=a}^{x=b}$$
$$=-\frac{e^{-2\pi i yb}}{2\pi i y}+\frac{e^{-2\pi i ya}}{2\pi i y}.$$

In particular, $\widehat{\chi_{[0,1]}}(y)=-\frac{e^{-2\pi i y}}{2\pi i y}+\frac{1}{2\pi i y}=\frac{1}{2\pi}E(-y)$ and so by the Fourier inversion theorem,

$$\widehat{E}(\xi)=2\pi\chi_{[0,1]}(\xi).$$

Hence, the double integral equals to

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\frac{e^{2\pi i\xi t}-1}{t}\right|^2|\widehat{E}(\xi)|^2\,d\xi dt=2\pi\int_{\mathbb{R}}\int_0^1\left|\frac{e^{2\pi i\xi t}-1}{t}\right|^2\,d\xi dt$$
$$=2\pi\int_{-1}^1\int_0^1\left|\frac{e^{2\pi i\xi t}-1}{t}\right|^2\,d\xi dt+2\pi\int_{\mathbb{R}\setminus[-1,1]}\int_0^1\left|\frac{e^{2\pi i\xi t}-1}{t}\right|^2\,d\xi dt$$
$$\leq 2\pi\int_{-1}^1\int_0^1\left|2\pi i\xi e^{2\pi i\xi s_t}\right|^2\,d\xi dt+8\pi\int_{\mathbb{R}\setminus[-1,1]}\frac{1}{t^2}dt<\infty,$$

where we used the mean value theorem. ■

Friday, May 1, 2015

Proof. Step 1: The main part of the proof consists in obtaining a good expression for

$$\int_I|\mathcal{C}_\varepsilon(\chi_I)(x)|^2\,dx,$$

where $I\subset\mathbb{R}$ is a bounded interval. We have

$$\int_I|\mathcal{C}_\varepsilon(\chi_I)(x)|^2\,dx=\int_I\mathcal{C}_\varepsilon(\chi_I)(x)\overline{\mathcal{C}_\varepsilon(\chi_I)(x)}\,dx$$
$$=\iiint_{T_\varepsilon}\frac{1}{\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)}\frac{1}{\overline{\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)}}\,dxdydt,$$

where

$$T_\varepsilon:=\{(x,y,t)\in I^3:\ |y-x|>\varepsilon,\ |t-x|>\varepsilon\}.$$

Now the triple integral is not symmetric either in the domain or in the kernel. Let

$$S_\varepsilon := \{(x,y,t) \in I^3 : \ |y-x| > \varepsilon, \ |t-x| > \varepsilon, \ |t-y| > \varepsilon\}.$$

We claim that

$$\int_I |\mathcal{C}_\varepsilon(\chi_I)(x)|^2 \, dx = \iiint_{S_\varepsilon} \frac{1}{\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)} \frac{1}{\overline{\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)}} \, dxdydt + O(\mathcal{L}^1(I)). \quad (176)$$

To prove (83) let

$$U_{\varepsilon,1} := \{(x,y,t) \in I^3 : \ |y-x| > \varepsilon, \ |t-x| > 2\varepsilon, \ |t-y| \le \varepsilon\},$$
$$U_{\varepsilon,2} := \{(x,y,t) \in I^3 : \ |y-x| > \varepsilon, \ \varepsilon < |t-x| \le 2\varepsilon, \ |t-y| \le \varepsilon\}.$$

On $U_{\varepsilon,1}$ we have that

$$\begin{aligned} |\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)| &\ge |y-x| \ge |t-x| - |t-y| \\ &= \frac{1}{2}|t-x| + \frac{1}{2}|t-x| - |t-y| \ge \frac{1}{2}|t-x| + \varepsilon - \varepsilon \end{aligned}$$

and $|\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)| \ge |t-x|$, and so,

$$\begin{aligned} &\iiint_{U_{\varepsilon,1}} \frac{1}{|\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)|} \frac{1}{|\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)|} \, dxdydt \le \int_I \left(\int_{I\setminus[x-2\varepsilon,x+2\varepsilon]} \left(\int_{t-\varepsilon}^{t+\varepsilon} \frac{2}{(t-x)^2} \, dy \right) dt \right) \, dx \\ &= \int_I \left(\int_{I\setminus[x-2\varepsilon,x+2\varepsilon]} \frac{4\varepsilon}{(t-x)^2} \, dt \right) \, dx = \int_I \left(\left[-\frac{4\varepsilon}{(t-x)} \right]_{t=x+2\varepsilon}^{t=b} + \left[-\frac{4\varepsilon}{(t-x)} \right]_{t=a}^{t=x-2\varepsilon} \right) \, dx \\ &\le \int_I \left(0 + \frac{4\varepsilon}{2\varepsilon} + \frac{4\varepsilon}{2\varepsilon} \right) \, dx = 4\mathcal{L}^1(I), \end{aligned}$$

while on $U_{\varepsilon,2}$ we have that $|\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)| \ge |y-x| > \varepsilon$ and $|\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)| \ge |t-x| > \varepsilon$, and so,

$$\iiint_{U_{\varepsilon,2}} \frac{1}{|\boldsymbol{\zeta}(y)-\boldsymbol{\zeta}(x)|} \frac{1}{|\boldsymbol{\zeta}(t)-\boldsymbol{\zeta}(x)|} \, dxdydt \le \frac{1}{\varepsilon^2} \int_I \left(\int_{t-2\varepsilon}^{t+2\varepsilon} dx \int_{t-\varepsilon}^{t+\varepsilon} dy \right) \, dt \le \frac{8\varepsilon^2}{\varepsilon^2} \mathcal{L}^1(I).$$

Since

$$\iiint_{T_\varepsilon} = \iiint_{S_\varepsilon} + \iiint_{T_\varepsilon \setminus S_\varepsilon}$$

and $T_\varepsilon \setminus S_\varepsilon \subset U_{\varepsilon,1} \cup U_{\varepsilon,2}$, (83) holds.

Step 2: Next we symmetrize the kernel by permuting the position of all three variables in all possible ways. We can write

$$6\int_I |\mathcal{C}_\varepsilon(\chi_I)(x)|^2 \, dx = \iiint_{S_\varepsilon} \sum_\sigma \frac{1}{\boldsymbol{\zeta}(x_{\sigma(2)})-\boldsymbol{\zeta}(x_{\sigma(1)})} \frac{1}{\overline{\boldsymbol{\zeta}(x_{\sigma(3)})-\boldsymbol{\zeta}(x_{\sigma(1)})}} \, dx_1dx_2dx_3 + O(\mathcal{L}^1(I)),$$

where the sum is done over all six permutations of the set $\{1,2,3\}$. Using Lemma 169 and Remark 170 we get

$$
\begin{aligned}
&6\int_I |\mathcal{C}_\varepsilon(\chi_I)(x)|^2\,dx\\
&=\iiint_{S_\varepsilon}\left(\frac{4A(\boldsymbol{\zeta}(x_1),\boldsymbol{\zeta}(x_2),\boldsymbol{\zeta}(x_3))}{|\boldsymbol{\zeta}(x_2)-\boldsymbol{\zeta}(x_1)||\boldsymbol{\zeta}(x_3)-\boldsymbol{\zeta}(x_1)||\boldsymbol{\zeta}(x_3)-\boldsymbol{\zeta}(x_2)|}\right)^2 dx_1dx_2dx_3+O(\mathcal{L}^1(I))\\
&\le\iiint_{I^3}\left|\frac{\frac{g(x_3)-g(x_1)}{x_3-x_1}-\frac{g(x_2)-g(x_1)}{x_2-x_1}}{x_3-x_2}\right|^2 dx_1dx_2dx_3+O(\mathcal{L}^1(I)).
\end{aligned}
$$

Let $I=[a,b]$ and define the functions

$$p(x):=g(a)+c(x-a),\quad \varphi(x):=(g(x)-p(x))\chi_I(x),$$

where

$$c:=\frac{1}{b-a}\int_a^b g'(x)\,dx=\frac{g(b)-g(a)}{b-a}.$$

Note that $p(a)=g(a)$ and $p(b)=g(a)$, so $\varphi\in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ and $\varphi'\in L^2(\mathbb{R})$. Moreover, since

$$\frac{p(x_3)-p(x_1)}{x_3-x_1}-\frac{p(x_2)-p(x_1)}{x_2-x_1}=c-c=0,$$

we have that

$$
\begin{aligned}
&\iiint_{I^3}\left|\frac{\frac{g(x_3)-g(x_1)}{x_3-x_1}-\frac{g(x_2)-g(x_1)}{x_2-x_1}}{x_3-x_2}\right|^2 dx_1dx_2dx_3\\
&\le\iiint_{\mathbb{R}^3}\left|\frac{\frac{\varphi(x_3)-\varphi(x_1)}{x_3-x_1}-\frac{\varphi(x_2)-\varphi(x_1)}{x_2-x_1}}{x_3-x_2}\right|^2 dx_1dx_2dx_3=c_0\left\|\varphi'\right\|^2_{L^2}\\
&\le c_0(\operatorname{Lip} g)^2\mathcal{L}^1(I)
\end{aligned}
$$

where we have used Lemma 171.

In conclusion we have shown that

$$\int_I |\mathcal{C}_\varepsilon(\chi_I)(x)|^2\,dx\le c_0(\operatorname{Lip} g)^2\mathcal{L}^1(I). \tag{177}$$

We are now in a position to apply the local $T(1)$ theorem. ■

Remark 172 *One can actually avoid the $T(1)$ theorem by extending the argument of the previous proof to show that T maps L^∞ into* $\mathrm{BMO}(\mathbb{R})$ *and* $\mathbb{H}^1(\mathbb{R})$ *into* $L^1(\mathbb{R})$.

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