Monday, August 24, 2009

1 Metric Spaces

Definition 1 *A* metric *on a set X is a map $d : X \times X \to [0, \infty)$ such that*

(i) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$,

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(iii) $d(x, y) = 0$ *if and only if* $x = y$.

A *metric space* (X, d) is a set X endowed with a metric d. When there is no possibility of confusion, we abbreviate by saying that X is a metric space.

Example 2 *Here are some of the most important examples of metric spaces.*

(i) In $\mathbb{R}$ *we have* $d(x, y) := |x - y|$.

(ii) In $\mathbb{R}^N$, $N \geq 1$, *for* $x = (x_1, \ldots, x_N)$ *and* $y = (y_1, \ldots, y_N)$,

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + \ldots + (x_N - y_N)^2}.$$

(iii) Following what we do in $\mathbb{R}^N$, *given two metric spaces* (X, d_X) *and* (Y, d_Y), *we can construct a metric in* $X \times Y$, *taking*

$$d((x_1, y_1), (x_2, y_2)) := \sqrt{d_1(x_1, x_2)^2 + d_1(y_1, y_2)^2}.$$

(iv) Given a nonempty set X, *the metric*

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

is called the discrete metric.

(v) Consider the interval $[a, b] \subset \mathbb{R}$ *and take*

$$C([a, b]) := \{f : [a, b] \to \mathbb{R} : f \text{ continuous}\}$$

with the metric

$$d(f, g) := \max_{x \in [a,b]} |f(x) - g(x)|.$$

Note that here we are using the Weierstrass theorem. More generally, if $K \subset \mathbb{R}^N$ *is compact, then we can take*

$$C(K) := \{f : K \to \mathbb{R} : f \text{ continuous}\}$$

with the metric

$$d(f, g) := \max_{x \in K} |f(x) - g(x)|.$$

(vi) Consider the interval $[a,b] \subset \mathbb{R}$ *and take*

$$C([a,b]) := \{f : [a,b] \to \mathbb{R} : \ f \text{ continuous}\}$$

with

$$d(f,g) := \int_a^b |f(x) - g(x)| \, dx,$$

where the integral is the Riemann integral. Is this a metric?

If we take instead

$$\mathcal{R}([a,b]) := \{f : [a,b] \to \mathbb{R} : \ f \text{ bounded and Riemann integrable}\}$$

with

$$d(f,g) := \int_a^b |f(x) - g(x)| \, dx,$$

then do we have a metric? This is called a pseudometric. This will motivate quotient spaces.

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Example 3 *(vii) Consider* $C((0,1)) := \{f : (0,1) \to \mathbb{R} : f \text{ is continuous}\}$. *Consider* $K_n := \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$. *Then*

$$\bigcup_{n=1}^{\infty} K_n = (0,1).$$

Define

$$d(f,g) := \max_n \frac{1}{2^n} \frac{\max_{x\in K_n} |f(x) - g(x)|}{1 + \max_{x\in K_n} |f(x) - g(x)|}. \tag{1}$$

Then d *is a metric and* $d(f_n, f) \to 0 \to 0$ *if and only if* $f_n \to f$ *uniformly on compact sets.*

More generally if $\Omega \subset \mathbb{R}^N$ *is an open set, construct an increasing sequence of compact sets such* $\bigcup_{n=1}^{\infty} K_n = \Omega$ *and put on* $C(\Omega)$ *a metric as in (1).*

(viii) Given a set X *let*

$$B_b(X) = \ell^{\infty}(X) := \{f : X \to \mathbb{R} : f \text{ is bounded}\}$$

with

$$d_{\infty}(f,g) := \sup_{x\in X} |f(x) - g(x)|.$$

For the next example we need the notion of infinite sums.

Definition 4 *Given a set* X *and a function* $f : X \to [0,\infty]$ *the* infinite sum

$$\sum_{x\in X} f(x)$$

is defined as

$$\sum_{x\in X} f(x) := \sup\left\{\sum_{x\in Y} f(x) : Y \subset X,\, Y \text{ finite}\right\}.$$

Proposition 5 *Given a set* X *and a function* $f : X \to [0,\infty]$, *if*

$$\sum_{x\in X} f(x) < \infty,$$

then the set $\{x \in X : f(x) > 0\}$ *is countable, say,* $\{x_n\}_n$ *and*

$$\sum_{x\in X} f(x) = \sum_n f(x_n),$$

where the right-hand side is either a finite sum or a series. Moreover, f *does not take the value* ∞.

Proof. Define
$$M := \sum_{x\in X} f(x) < \infty.$$
For $k \in \mathbb{N}$ set $X_k := \left\{x \in X : f(x) > \frac{1}{k}\right\}$ and let Y be a finite subset of X_k. Then
$$\frac{1}{k}\text{number of elements of } Y \leq \sum_{x\in Y} f(x) \leq M,$$
which shows that Y cannot have more than $\lfloor kM \rfloor$ elements, where $\lfloor\cdot\rfloor$ is the integer part. In turn, X_k has a finite number of elements, and so
$$\{x \in X : f(x) > 0\} = \bigcup_{k=1}^{\infty} X_k$$
is countable. ■

Exercise 6 *Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Prove that the set of discontinuity points of f is countable.*

Exercise 7 *Given a nonempty set X and two functions $f, g : X \to [0,\infty]$.*

(i) Prove that
$$\sum_{x\in X} (f(x) + g(x)) \leq \sum_{x\in X} f(x) + \sum_{x\in X} g(x).$$

(ii) If $f \leq g$, then
$$\sum_{x\in X} f(x) \leq \sum_{x\in X} g(x).$$

Example 8 *Given a nonempty set X and $1 \leq p < \infty$, we define the space*
$$\ell^p(X) := \left\{ f : X \to [-\infty,\infty] : \sum_{x\in X} |f(x)|^p < \infty \right\}.$$
Consider the metric
$$d_p(f,g) := \left(\sum_{x\in X} |f(x) - g(x)|^p \right)^{\frac{1}{p}}.$$

In the particular case in which $X = \mathbb{N}$, then a function
$$f : \mathbb{N} \to [-\infty,\infty]$$
$$n \mapsto f(n) =: a_n$$
is just a sequence $\{a_n\}_n$ and so we have
$$\ell^p(\mathbb{N}) := \left\{ \{a_n\}_n \subset \mathbb{R} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}.$$

We usually write $\ell^p := \ell^p(\mathbb{N})$.

Given a number $1 \leq p \leq \infty$, the *Hölder conjugate exponent of* p is the number $1 \leq q \leq \infty$ defined as

$$q := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that, with an abuse of notation, we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the Hölder conjugate exponent of p will often be denoted by p'.

Proposition 9 (Young's inequality) *Let $1 < p < \infty$, and let q be its Hölder conjugate exponent. Then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

for all $a, b \geq 0$.

Proof. If $a = 0$ or $b = 0$, then there is nothing to prove. Thus, assume that $a, b > 0$. Since the function $t \in [0, \infty) \mapsto \ln t$ is concave and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\ln \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) \geq \frac{1}{p} \ln a^p + \frac{1}{q} \ln b^q = \ln ab,$$

that is

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab.$$

■

Theorem 10 (Hölder's inequality) *Let X be a set, let $1 \leq p \leq \infty$, and let q be its Hölder conjugate exponent. Given $f, g : X \to [-\infty, \infty]$, then*

$$\sum_{x \in X} |f(x) g(x)| \leq \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} \left(\sum_{y \in X} |g(y)|^q \right)^{\frac{1}{q}}$$

if $1 < p < \infty$, while

$$\sum_{x \in X} |f(x) g(x)| \leq \left(\sum_{x \in X} |f(x)| \right) \left(\sup_{y \in X} |g(y)| \right)$$

if $p = 1$. In particular, if $f \in \ell^p(X)$ and $g \in \ell^q(X)$ then $fg \in \ell^1(X)$.[1]

Proof. Assume that $1 < p < \infty$. If $\sum_{x \in X} |f(x)|^p = 0$ or $\sum_{y \in X} |g(y)|^q = 0$, then $f(x) g(x) = 0$ for all $x \in X$ and so there is nothing to prove. Thus assume that both sums are positive. If one of them is infinite, then the right-hand side is ∞ and so there is nothing to prove. Hence in what follows we consider the case in which both sums are finite belong to $(0, \infty)$.

If we apply Young's inequality with

$$a = \frac{|f(x)|}{\left(\sum_{y \in X} |f(y)|^p \right)^{\frac{1}{p}}} \quad \text{and} \quad b = \frac{|g(x)|}{\left(\sum_{y \in X} |g(y)|^q \right)^{\frac{1}{q}}},$$

we get

$$\frac{|f(x) g(x)|}{\left(\sum_{y \in X} |f(y)|^p \right)^{\frac{1}{p}} \left(\sum_{z \in X} |g(z)|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|f(x)|^p}{\sum_{y \in X} |f(y)|^p} + \frac{1}{q} \frac{|g(x)|^q}{\sum_{z \in X} |g(z)|^q}.$$

By Exercise 7, taking the sum on both sides, we obtain

$$\begin{aligned} \frac{\sum_{x \in X} |f(x) g(x)|}{\left(\sum_{y \in X} |f(y)|^p \right)^{\frac{1}{p}} \left(\sum_{z \in X} |g(z)|^q \right)^{\frac{1}{q}}} &\leq \frac{1}{p} \frac{\sum_{x \in X} |f(x)|^p}{\sum_{y \in X} |f(y)|^p} + \frac{1}{q} \frac{\sum_{x \in X} |g(x)|^q}{\sum_{z \in X} |g(z)|^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

[1] Here we define $0 \cdot \infty$ to be 0.

This gives the desired result for $1 < p < \infty$.

If $p = 1$ and $q = \infty$, then

$$|f(x)\, g(x)| \le |f(x)| \sup_{y \in X} |g(y)|,$$

and we can now sum both sides. ■

Theorem 11 (Minkowski's inequality) *Let X be a set, let $1 \le p < \infty$, and let $f, g : X \to [-\infty, \infty]$ be two functions. Then,*

$$\left(\sum_{x \in X} |f(x) + g(x)|^p \right)^{\frac{1}{p}} \le \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} + \left(\sum_{x \in X} |g(x)|^p \right)^{\frac{1}{p}}.$$

In particular, if $f, g \in \ell^p(X)$, then $f + g \in \ell^p(X)$.

Proof. If $\sum_{x \in X} |f(x)|^p = \infty$ or $\sum_{x \in X} |g(x)|^p = \infty$ then the right-hand side of Minkowski's inequality is ∞, and so there is nothing to prove. Thus assume that both sums are finite.

We consider first the case $1 < p < \infty$. By the convexity of the function $t \in [0, \infty) \mapsto t^p$, for any a, $b > 0$, we have

$$(a + b)^p = 2^p \left(\frac{a + b}{2} \right)^p \le \frac{2^p}{2} a^p + \frac{2^p}{2} b^p = 2^{p-1} (a^p + b^p),$$

and so by Exercise 7,

$$\sum_{x \in X} |f(x) + g(x)|^p \le \sum_{x \in X} (|f(x)| + |g(x)|)^p \le 2^{p-1} \left(\sum_{x \in X} (|f(x)|^p + |g(x)|^p) \right),$$

which shows that $f + g \in \ell^p(X)$. To prove Minkowski's inequality, we observe that

$$\begin{aligned} \sum_{x \in X} |f(x) + g(x)|^p &= \sum_{x \in X} |f(x) + g(x)| \, |f(x) + g(x)|^{p-1} \\ &\le \sum_{x \in X} |f(x)| \, |f(x) + g(x)|^{p-1} + \sum_{x \in X} |g(x)| \, |f(x) + g(x)|^{p-1}. \end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned} \sum_{x \in X} |f(x) + g(x)|^p &\le \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} \left(\sum_{x \in X} |f(x) + g(x)|^{(p-1)p'} \right)^{\frac{1}{p'}} \\ &\quad + \left(\sum_{x \in X} |g(x)|^p \right)^{\frac{1}{p}} \left(\sum_{x \in X} |f(x) + g(x)|^{(p-1)p'} \right)^{\frac{1}{p'}} \\ &\le \left(\left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} + \left(\sum_{x \in X} |g(x)|^p \right)^{\frac{1}{p}} \right) \left(\sum_{x \in X} |f(x) + g(x)|^p \right)^{\frac{1}{p'}}, \end{aligned}$$

where we have used the fact that $(p-1)\,p' = p$. If $\sum_{x\in X} |f(x)+g(x)|^p = 0$, then there is nothing to prove, thus assume that $\sum_{x\in X} |f(x)+g(x)|^p \in (0,\infty)$. Hence, we may divide both sides of the previous inequality by $\left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p'}}$ to obtain

$$\left(\sum_{x\in X} |f(x)+g(x)|^p\right)^{\frac{1}{p}} \leq \left(\sum_{x\in X} |f(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{x\in X} |g(x)|^p\right)^{\frac{1}{p}},$$

where we have used the fact that $\frac{1}{p}+\frac{1}{p'} = 1$.

If $p = 1$, Minkowski's inequality follows from the triangle inequality. ■

Monday, August 31, 2009

Minkowski's inequality shows that $\ell^p(X)$ is a metric space.

Corollary 12 *Let X be a nonempty set and let $1 \leq p < \infty$. Then $\ell^p(X)$ is a metric space.*

Proof. Define

$$\|f\|_p := \left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}}.$$

By Minkowski's inequality, we have that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, and so, since

$$d(f,g) = \|f-g\|_p,$$

it follows that

$$\begin{aligned} d(f,g) &= \|f-g\|_p = \|f \pm h - g\|_p \leq \|f-h\|_p + \|h-g\|_p \\ &= d(f,h) + d(h,g). \end{aligned}$$

■

Definition 13 *A* pseudometric *on a set X is a map $\rho : X \times X \to [0,\infty)$ such that*

(i) $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$ *for all* $x,y,z \in X$,

(ii) $\rho(x,y) = \rho(y,x)$ *for all* $x,y \in X$,

(iii) $\rho(x,x) = 0$ *for all* $x \in X$.

The pair (X,ρ) is called a *pseudometric* space.

Example 14 *(i) Consider the set*

$$\mathcal{R}([a,b]) := \{f : [a,b] \to \mathbb{R} : f \text{ is Riemann integrable}\}$$

and define

$$\rho(f,g) := \int_a^b |f(x) - g(x)|\, dx, \quad f,g \in \mathcal{R}([a,b]).$$

Then ρ is a pseudometric on $\mathcal{R}([a,b])$.

(ii) If X is a nonempty set and $f : X \to \mathbb{R}$ is any function, then the function

$$\rho_f(x,y) := |f(x) - f(y)|, \quad x,y \in X,$$

is a pseudometric on X.

(iii) *Given a nonempty set X, consider the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$ (see Example 123). Fix $x_0 \in X$ and define*

$$\rho_{x_0}(f, g) := |f(x_0) - g(x_0)|, \quad f, g \in \mathbb{R}^X.$$

Then ρ_{x_0} is a pseudometric on $\mathbb{R}^X$.

(iv) *Given $1 \leq p < \infty$ consider the set*

$$\mathcal{L}^p([a, b]) := \left\{ f : [a, b] \to \mathbb{R} : f \text{ is Lebesgue measurable and } \int_a^b |f(x)|^p \, dx < \infty \right\}$$

and define

$$\rho_p(f, g) := \left(\int_a^b |f(x) - g(x)|^p \, dx \right)^{\frac{1}{p}}, \quad f, g \in \mathcal{L}^p([a, b]).$$

Then ρ_p is a pseudometric on $\mathcal{L}^p([a, b])$.

Given a nonempty set X, a set $\mathcal{R} \subset X \times X$ is called an equivalence relation if

(i) (Reflexivity) $(x, x) \in \mathcal{R}$ for every $x \in X$.

(ii) (Symmetry) For all $x, y \in X$, if $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$.

(iii) (Transitivity) For all $x, y, z \in X$, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

We write $x \sim y$ if $(x, y) \in \mathcal{R}$. Given $x \in X$, the *equivalence class* determined by x is given by

$$[x] := \{z \in X : x \sim z\}.$$

We define the *quotient space*

$$Y = X \diagup \sim := \{[x] : x \in X\}$$

Given a pseudometric space, we can define an equivalence relation $\sim$ on X. Given $x, y \in X$, we say that $x \sim y$ if $\rho(x, y) = 0$. Consider the quotient space $Y = X \diagup$. In Y we can define

$$d([x], [y]) := \rho(x, y), \quad [x], [y] \in Y. \tag{2}$$

Exercise 15 *Let (X, ρ) be a pseudometric space.*

(i) *Prove that the function d defined in (2) is well-defined.*

(ii) *Prove that d is a metric.*

1.1 Topological Properties of Metric Spaces

Let (X, d) be a metric space. Given $r > 0$ and $x_0 \in X$, the *ball* of center x_0 and radius r is the set

$$B(x_0, r) := \{x \in X : d(x_0, x) < r\}.$$

Given a metric space (X, d) a subset $U \subset X$ is *open* if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$.

Proposition 16 *Let (X, d) be a metric space. Then for every $x_0 \in X$ and $r > 0$ the ball $B(x_0, r)$ is open.*

Proof. Given $x \in B(x_0, r)$, we have that $r_1 := r - d(x_0, x) > 0$. Let $y \in B(x, r_1)$. Then

$$d(x_0, y) \leq d(x_0, x) + d(x, y) < d(x_0, x) + r_1 = d(x_0, x) + r - d(x_0, x) = r,$$

which shows that $B(x, r_1) \subset B(x_0, r)$. Hence, $B(x_0, r)$ is open. ∎

The main properties of open sets are given in the next proposition.

Proposition 17 *Let (X, d) be a metric space. Then*

(i) $\emptyset$ and X are open.

(ii) If $U_i \subset X$, $i = 1, \ldots, n$, is a finite family of open sets of X, then $U_1 \cap \cdots \cap U_n$ is open.

(iii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ is an arbitrary collection of open sets of X, then $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open.

Proof. To prove (ii), let $x \in U_1 \cap \cdots \cap U_M$. Then $x \in U_i$ for every $i = 1, \ldots, n$, and since U_i is open, there exists $r_i > 0$ such that $B(x, r_i) \subset U_i$. Take $r := \min\{r_1, \ldots, r_n\} > 0$. Then

$$B(x, r) \subset U_1 \cap \cdots \cap U_n,$$

which shows that $U_1 \cap \cdots \cap U_n$ is open.

To prove (iii), let $x \in U := \bigcup_{\alpha \in \Lambda} U_\alpha$. Then there is $\alpha \in \Lambda$ such that $x \in U_\alpha$ and since U_α is open, there exists $r > 0$ such that $B(x, r) \subset U_\alpha \subset U$. This shows that U is open. ∎

Properties (i)–(iii) will be used to define topological spaces.

Wednesday, September 2, 2009

Remark 18 *The intersection of infinitely many open sets is not open in general. Take* $X = \mathbb{R}$ *and* $U_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$ *for* $n \in \mathbb{N}$*. Then*

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

which is not open.

The following characterization of open sets is a corollary of the previous propositions.

Corollary 19 *Let* (X, d) *be a metric space. A nonempty set* $U \subset X$ *is open if and only if it is given by the union of balls.*

Proof. By property (iii) of Proposition 17 and by Proposition 16 the arbitrary union of balls is open. Conversely, if U is open, then for every $x \in U$ there is a ball $B(x, r_x) \subset U$ and so

$$U = \bigcup_{x \in U} B(x, r_x).$$

■

Exercise 20 *Prove that in* $\mathbb{R}^N$ *(with the Euclidean metric) every nonempty open set can be written as union of countably many balls.*

Example 21 *If we take a different metric in* $\mathbb{R}^N$*, then the previous exercise is no longer true. Indeed, consider the discrete metric in* $\mathbb{R}^N$*,*

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Then

$$B(x, r) = \begin{cases} \{x\} & \text{if } 0 < r \leq 1, \\ \mathbb{R}^N & \text{if } r > 1. \end{cases}$$

Consider the open set $U = \mathbb{R}^N \setminus \{0\}$*. Then by the previous corollary, we can write*

$$U = \bigcup_{\alpha \in \Lambda} B(x_\alpha, r_\alpha).$$

Since $\mathbb{R}^N \setminus \{0\} \neq \mathbb{R}^N$*, necessarily* $r_\alpha \leq 1$ *for all* $\alpha \in \Lambda$*, and so*

$$U = \bigcup_{\alpha \in \Lambda} \{x_\alpha\}.$$

Thus, the number Λ *of balls is uncountable.*

Given a metric space (X,d) and a set $E \subset X$, a point $x \in E$ is called an *interior point* of E if there exists $r > 0$ such that $B(x,r) \subset E$. The *interior* E° of a set $E \subset X$ is the union of all its interior points.

The proof of following proposition is left as an exercise.

Proposition 22 *Let (X,d) be a metric space and let $E \subset X$. Then*

(i) *E° is an open subset of E,*

(ii) *E° is given by the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,*

(iii) *E is open if and only if $E = E^\circ$,*

(iv) *$(E^\circ)^\circ = E^\circ$.*

Exercise 23 *Let (X,d) be a metric space.*

(i) *Prove that if $E_1, \ldots, E_n$ are subsets of X, then*

$$(E_1)^\circ \cap \cdots \cap (E_n)^\circ = (E_1 \cap \cdots \cap E_n)^\circ ,$$
$$(E_1)^\circ \cup \cdots \cup (E_n)^\circ \subset (E_1 \cup \cdots \cup E_n)^\circ .$$

(ii) *Show that in general $(E_1)^\circ \cup \cdots \cup (E_n)^\circ \neq (E_1 \cup \cdots \cup E_n)^\circ$.*

(iii) *Let $\{E_\alpha\}_{\alpha \in \Lambda}$ be an arbitrary collection of sets of X. What is the relation, if any, between $\bigcap_{\alpha\in\Lambda} (U_\alpha)^\circ$ and $\left(\bigcap_{\alpha\in\Lambda} U_\alpha\right)^\circ$? And between $\bigcup_{\alpha\in\Lambda} (U_\alpha)^\circ$ and $\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right)^\circ$?*

(iii) *Think of some nontrivial condition on $\{E_\alpha\}_{\alpha\in\Lambda}$ that guarantees*

$$\bigcap_{\alpha\in\Lambda} (U_\alpha)^\circ = \left(\bigcap_{\alpha\in\Lambda} U_\alpha\right)^\circ .$$

Given a metric space (X,d), a subset $C \subset X$ is *closed* if its complement $X \setminus C$.

The main properties of closed sets are given in the next proposition.

Proposition 24 *Let (X,d) be a metric space. Then*

(i) *$\emptyset$ and X are closed.*

(ii) *If $C_i \subset X$, $i = 1, \ldots, n$, is a finite family of closed sets of X, then $C_1 \cup \cdots \cap U_n$ is closed.*

(iii) *If $\{C_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of closed sets of X, then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is closed.*

The proof follows from Proposition 17 and De Morgan's laws. If $\{E_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of subsets of a set X, then *De Morgan's laws* are

$$X \setminus \left(\bigcup_{\alpha\in\Lambda} E_\alpha\right) = \bigcap_{\alpha\in\Lambda} (X \setminus E_\alpha),$$

$$X \setminus \left(\bigcap_{\alpha\in\Lambda} E_\alpha\right) = \bigcup_{\alpha\in\Lambda} (X \setminus E_\alpha).$$

Given a metric space (X, d), a sequence $\{x_n\} \subset X$ *converges (strongly)* to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Exercise 25 (Uniqueness of the limit) *Given a metric space (X, d), prove that if $\{x_n\} \subset X$ converges to x and y in X, then $x = y$.*

Given a metric space (X, d), a subset $C \subset X$ is *sequentially closed* if for every sequence $\{x_n\} \subset C$ such that $\{x_n\}$ converges to some $x \in X$, then x belongs to C.

Proposition 26 *Let (X, d) be a metric space and let $C \subset X$. Then C is closed if and only if C is sequentially closed.*

Proof. Step 1: Assume that C is closed and let $\{x_n\} \subset C$ be such that $\{x_n\}$ converges to some $x \in X$. We need to show that x belongs to C. If not, then $x \in X \setminus C$. Since $X \setminus C$ is open, there exists $r > 0$ such that $B(x, r) \subset X \setminus C$. But then, taking $\varepsilon = r$ there exists $n_r \in \mathbb{N}$ such that $d(x_n, x) < r$ for all $n \geq n_r$, which implies that $x_n \in B(x, r) \subset X \setminus C$ for all $n \geq n_r$. This contradicts the fact that $\{x_n\} \subset C$.
Step 2: Assume that C is sequentially closed. We need to show that $X \setminus C$ is open. Let $x \in X \setminus C$. We claim that there exists $r > 0$ such that $B(x, r) \subset X \setminus C$. If not, then for every $r > 0$ we can find $y \in C$ such that $y \in B(x, r)$. Taking $r = \frac{1}{n}$ we can find $x_n \in C$ such that $d(x_n, x) < \frac{1}{n} \to 0$, which shows that $\{x_n\}$ converges to x. Since C is sequentially closed, it follows that $x \in C$, which is a contradiction. ■

Friday, September 4, 2009

Given a metric space (X, d) and a set $E \subset X$, the closure of E, denoted $\overline{E}$, is the intersection of all closed sets that contain E; in other words, the closure of E is the smallest (with respect to inclusion) closed set that contains E. It follows by Proposition 24 that $\overline{E}$ is closed.

The proof of following proposition is left as an exercise.

Proposition 27 *Let (X, d) be a metric space and let $C \subset X$. Then C is closed if and only if $C = \overline{C}$.*

Exercise 28 *Let (X, d) be a metric space, let $x_0 \in X$, and let $r > 0$.*

(i) Prove that $\overline{B(x_0, r)} \subset \{x \in X : d(x_0, x) \leq r\}$.

(ii) Prove that in general it may happen that

$$\overline{B(x_0, r)} \neq \{x \in X : d(x_0, x) \leq r\}.$$

Proposition 29 *Let (X, d) be a metric space, let $E \subset X$, and let $x \in X$. Then $x \in \overline{E}$ if and only if $B(x, r) \cap E \neq \emptyset$ for every $r > 0$.*

Proof. Let $x \in \overline{E}$ and assume by contradiction that there exists $r > 0$ such that $B(x, r) \cap E = \emptyset$. Since $B(x, r)$ is open and $B(x, r) \cap E = \emptyset$, it follows that $X \setminus B(x, r)$ is closed and contains E. By the definition of $\overline{E}$ we have that $\overline{E} \subset X \setminus B(x, r)$, which contradicts the fact that $x \in \overline{E}$.

Conversely, let $x \in X$ and assume that $B(x, r) \cap E \neq \emptyset$ for every $r > 0$. We claim that $x \in \overline{E}$. Indeed, if not, then $x \in X \setminus \overline{E}$, which is open. Thus, there exists $B(x, r) \subset X \setminus \overline{E}$, which contradicts the fact that $B(x, r) \cap E \neq \emptyset$. ■

The previous proposition leads us to the definition of accumulation points.

Given a metric space (X, d) and a set $E \subset X$, a point $x \in X$ is an *accumulation*, or *limit*, point of E if for every $r > 0$ the ball $B(x, r)$ contains at least one point of E different from x. The set of all accumulation points of E is denoted $\operatorname{acc} E$.

Remark 30 *Note take if $x \in X$ is an accumulation point of E, then by taking $r = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a sequence $\{x_n\} \subset E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $d(x_n, x) < \frac{1}{n} \to 0$. Thus $\{x_n\}$ converges to x. Conversely, if there exists a sequence $\{x_n\} \subset E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $d(x_n, x) \to 0$, then x is an accumulation point of E.*

It turns out that the closure of a set is given by the set and all its accumulations points.

Proposition 31 *Let (X, d) be a metric space and let $E \subset X$. Then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, a set $C \subset X$ is closed if and only if C contains all its accumulation points.

Proof. Let $x \in \overline{E}$ and assume by contradiction that $x \notin E \cup \operatorname{acc} E$. Since $x \notin \operatorname{acc} E$, then there exists a ball $B(x, r)$ that contains no other point of E other than x, but since $x \notin E$, it follows that $B(x, r) \subset X \setminus E$. This contradicts Proposition 29.

Conversely, let $x \in E \cup \operatorname{acc} E$. If $x \in E$, then since $E \subset \overline{E}$, there is nothing to prove. If $x \in \operatorname{acc} E$, then the result follows from Proposition 29. ∎

Exercise 32 *Let (X, d) be a metric space.*

(i) Prove that if $E_1, \ldots, E_n$ are subsets of X, then

$$\overline{E_1} \cap \cdots \cap \overline{E_n} \supset \overline{E_1 \cap \cdots \cap E_n},$$
$$\overline{E_1} \cup \cdots \cup \overline{E_n} = \overline{E_1 \cup \cdots \cup E_n}.$$

(ii) Show that in general $\overline{E_1} \cap \cdots \cap \overline{E_n} \neq \overline{E_1 \cap \cdots \cap E_n}$.

(iii) Let $\{E_\alpha\}_{\alpha \in \Lambda}$ be an arbitrary collection of sets of X. What is the relation, if any, between between $\bigcap_{\alpha \in \Lambda} \overline{U_\alpha}$ and $\overline{\bigcap_{\alpha \in \Lambda} U_\alpha}$? And between $\bigcup_{\alpha \in \Lambda} \overline{U_\alpha}$ and $\overline{\bigcup_{\alpha \in \Lambda} U_\alpha}$?

(iii) Think of some nontrivial condition on $\{E_\alpha\}_{\alpha \in \Lambda}$ that guarantees

$$\bigcup_{\alpha \in \Lambda} \overline{U_\alpha} = \overline{\bigcup_{\alpha \in \Lambda} U_\alpha}.$$

Wednesday, September 9, 2009

Definition 33 *Given a metric space* (X, d)*, a set* $E \subset X$ *is said to be* dense *if* $\overline{E} = X$*. The metric space* (X, d) *is* separable *if there exists a sequence* $\{x_n\} \subset X$ *that is dense in* X*.*

Example 34 *We discuss separability of some of the examples introduced before.*

(i) $\mathbb{R}^N$ *is separable, since* $\mathbb{Q}^N$ *is dense in* $\mathbb{R}^N$*.*

(ii) Given a nonempty set X *with discrete metric*

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

X *is separable if and only if* X *is countable. Why?*

(iii) Using uniform continuity, one can show that piecewise affine functions are dense in $C([a, b])$*. By approximating a piecewise affine function with one with rational slopes and endpoints, it follow that* $C([a, b])$ *is separable.*

(iv) $\ell^\infty = \ell^\infty(\mathbb{N})$ *is not separable (exercise).*

(v) $\ell^p = \ell^p(\mathbb{N})$*,* $1 \leq p < \infty$*, is separable. Take*

$$E = \{x = (r_1, \ldots, r_n, 0, \ldots) : r_i \in \mathbb{Q},\, i = 1, \ldots, n,\, n \in \mathbb{N}\}.$$

If $x = \{x_n\} \in \ell^p$ *and* $\varepsilon > 0$*, let* $n_\varepsilon \in \mathbb{N}$ *be so large that*

$$\sum_{n=n_\varepsilon+1}^{\infty} |x_n|^p \leq \frac{\varepsilon^p}{2}.$$

Using the density of $\mathbb{Q}$ *in* $\mathbb{R}$ *we may find* $r_1, \ldots, r_{n_\varepsilon}$ *such that*

$$\sum_{n=1}^{n_\varepsilon} |x_n - r_n|^p \leq \frac{\varepsilon^p}{2}.$$

Then $y = (r_1, \ldots, r_{n_\varepsilon}, 0, \ldots)$ *belongs to* E *and*

$$d_p(x, y) = \left(\sum_{n=1}^{n_\varepsilon} |x_n - r_n|^p + \sum_{n=n_\varepsilon+1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \leq \left(\frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} \right)^{\frac{1}{p}} = \varepsilon.$$

(vi) If X *is uncountable, then* $\ell^p(X)$*,* $1 \leq p < \infty$*, is not separable (exercise).*

1.2 Completeness

A *Cauchy sequence* in a metric space is a sequence $\{x_n\} \subset X$ such that

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

A set E in a metric space X is *bounded* (*in the metric sense*) if it is contained in some ball.

Exercise 35 *Let (X, d) be a metric space and let $\{x_n\} \subset X$.*

(i) Prove that if $\{x_n\}$ converges to some $x \in X$, then $\{x_n\}$ is a Cauchy sequence.

(ii) Prove that if $\{x_n\}$ is a Cauchy sequence, then the set $E := \{x_n : n \in \mathbb{N}\}$ is bounded.

Exercise 36 *Let (X, d) be a metric space and let $\{x_n\} \subset X$.*

(i) Prove that if $\{x_n\}$ is a Cauchy sequence and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to some $x \in X$, then $\{x_n\}$ converges to x.

(ii) Prove that if there exists $x \in X$ such that for every *subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a further subsequence $\left\{x_{n_{k_j}}\right\}$ that converges to x, then $\{x_n\}$ converges to x.*

A metric space X is said to be *complete* if every Cauchy sequence is convergent.

Proposition 37 *Let (X,d) be a complete metric space and let $C \subset X$. Then C is closed if and only if (C,d) is a complete metric space.*

Proof. Assume that C is closed and let $\{x_n\} \subset C$ be a Cauchy sequence. Then $\{x_n\}$ is a Cauchy sequence in X, and since X is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. Using Proposition 26 we have that C is sequentially closed, and so $x \in C$, which proves that (C,d) complete. Conversely, assume that (C,d) is a complete metric space. To prove that C is closed, again by Proposition 26 it is enough to show that C is sequentially closed. Let $\{x_n\} \subset C$ be a sequence converging to some $x \in X$. Then by the previous exercise, $\{x_n\}$ is a Cauchy sequence in X, and in turn also in C. Since (C,d) is complete, it follows that $\{x_n\}$ converges to some $y \in C$. By the uniqueness of the limit, we have that $x = y \in C$, which shows that C is sequentially closed. ■

Example 38 *We discuss completeness of some of the examples introduced before.*

(i) $\mathbb{R}^N$ *is complete.*

(ii) Given a nonempty set X with the discrete metric, then X is complete. Why?

(iii) Given a nonempty set X, $\ell^p(X)$, $1 \le p < \infty$, is complete. To see this, let $\{f_n\} \subset \ell^\infty(X)$ be a Cauchy sequence. Let $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ so large that

$$\left(\sum_{x\in X} |f_n(x) - f_m(x)|^p\right)^{\frac{1}{p}} = d_p(f_n, f_m) \le \varepsilon$$

for all $n, m \ge n_\varepsilon$. Since for every fixed $x \in X$,

$$|f_n(x) - f_m(x)| \le \left(\sum_{y\in X} |f_n(y) - f_m(y)|^p\right)^{\frac{1}{p}} \le \varepsilon$$

for all $n, m \ge n_\varepsilon$, this implies that the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ and so there exists

$$\lim_{n\to\infty} f_n(x) = f(x) \in \mathbb{R}.$$

Fix a finite *subset $Y \subset X$. Since*

$$\sum_{y\in Y} |f_n(y) - f_m(y)|^p \le \varepsilon^p$$

for all $n, m \geq n_\varepsilon$, *letting* $n \to \infty$ *gives*

$$\sum_{y \in Y} |f(y) - f_m(y)|^p \leq \varepsilon^p$$

for all $m \geq n_\varepsilon$. *This holds for every finite subset* $Y \subset X$. *Hence, taking the supremum over all finite subsets* $Y \subset X$ *gives*

$$\sum_{y \in X} |f(y) - f_m(y)|^p = [d_p(f, f_m)]^p \leq \varepsilon^p$$

for all $m \geq n_\varepsilon$; *that is* $d_p(f, f_m) \to 0$ *as* $m \to \infty$. *Moreover, by Minkowski's inequality,*

$$\left(\sum_{x \in X} |f(x)|^p \right)^{\frac{1}{p}} = \left(\sum_{x \in X} |f(x) \pm f_{n_\varepsilon}(x)|^p \right)^{\frac{1}{p}} \leq \left(\sum_{x \in X} |f(x) - f_{n_\varepsilon}(x)|^p \right)^{\frac{1}{p}}$$

$$+ \left(\sum_{x \in X} |f_{n_\varepsilon}(x)|^p \right)^{\frac{1}{p}} \leq \varepsilon + \left(\sum_{x \in X} |f_{n_\varepsilon}(x)|^p \right)^{\frac{1}{p}} < \infty,$$

which implies that $f \in \ell^p(X)$.

(iv) Given a nonempty set X, *the space* $\ell^\infty(X)$ *is complete. To see this, let* $\{f_n\} \subset \ell^\infty(X)$ *be a Cauchy sequence. Let* $\varepsilon > 0$ *and find* $n_\varepsilon \in \mathbb{N}$ *so large that*

$$\sup_{x \in X} |f_n(x) - f_m(x)| = d_\infty(f_n, f_m) \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$. *This implies that for every fixed* $x \in X$, *the sequence of real numbers* $\{f_n(x)\}$ *is a Cauchy sequence in* $\mathbb{R}$ *and so there exists*

$$\lim_{n \to \infty} f_n(x) = f(x) \in \mathbb{R}.$$

Since

$$|f_n(x) - f_m(x)| \leq \varepsilon$$

for all $n, m \geq n_\varepsilon$, *letting* $n \to \infty$ *gives*

$$|f(x) - f_m(x)| \leq \varepsilon$$

for all $m \geq n_\varepsilon$. *This holds for every* $x \in X$. *Hence, taking the supremum over all* $x \in X$ *gives*

$$\sup_{x \in X} |f(x) - f_m(x)| = d_\infty(f, f_m) \leq \varepsilon$$

for all $m \geq n_\varepsilon$; *that is* $d_\infty(f, f_m) \to 0$ *as* $m \to \infty$. *Moreover, for every* $x \in X$,

$$|f(x)| \leq |f(x) \pm f_{n_\varepsilon}(x)| \leq \sup_{x \in X} |f(x) - f_{n_\varepsilon}(x)| + \sup_{x \in X} |f_{n_\varepsilon}(x)| \leq \varepsilon + \sup_{x \in X} |f_{n_\varepsilon}(x)|,$$

which implies that $f \in \ell^\infty(X)$.

(v) *To prove that* $C([a,b])$ *is complete, it is enough to show that* $C([a,b])$ *is a sequentially closed subset of the complete metric space* $\ell^{\infty}([a,b])$ *(see Propositions 26 and 37). Let* $\{f_n\} \subset C([a,b])$ *be such that*

$$d_{\infty}(f, f_n) \to 0$$

as $n \to \infty$ *for some function* $f \in \ell^{\infty}([a,b])$*. We claim that* f *is continuous. Fix* $x_0 \in [a,b]$ *and* $\varepsilon > 0$*. Then there exists* $n_{\varepsilon} \in \mathbb{N}$ *such that*

$$\sup_{x \in X} |f(x) - f_n(x)| = d_{\infty}(f, f_n) \leq \varepsilon$$

for all $n \geq n_{\varepsilon}$*. Since* $f_{n_{\varepsilon}}$ *is continuous at* x_0 *there exists* $\delta > 0$ *such that*

$$|f_{n_{\varepsilon}}(x) - f_{n_{\varepsilon}}(x_0)| \leq \varepsilon$$

for all $x \in [a,b]$ *with* $|x - x_0| \leq \delta$*. Hence, for all* $x \in [a,b]$ *with* $|x - x_0| \leq \delta$*,*

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{n_{\varepsilon}}(x)| + |f_{n_{\varepsilon}}(x) - f_{n_{\varepsilon}}(x_0)| + |f_{n_{\varepsilon}}(x_0) - f(x_0)| \\ &\leq \varepsilon + \varepsilon + \varepsilon, \end{aligned}$$

which shows that f *is continuous.*

(vi) *The space* $C([a,b])$ *with the metric*

$$d(f,g) := \int_a^b |f(x) - g(x)| \, dx,$$

where the integral is the Riemann integral, is not complete (Exercise).

(vii) *Prove that* $C((0,1))$ *is complete.*

Monday, September 14, 2009

Example 39 *The following example shows that if a sequence of functions converges uniformly on every compact interval of an open interval, then it does not necessarily converge uniformly in the open interval. For $n \in \mathbb{N}$ consider the function*

$$f_n(x) = x^n, \quad x \in \mathbb{R}.$$

Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & \text{if } x > 1, \\ 1 & \text{if } x = 1, \\ 0 & \text{if } -1 < x < 1, \\ \text{does not exist} & \text{if } x \leq -1. \end{cases}$$

Thus the sequence of functions $\{f_n\}$ converges pointwise in the set $E = (-1, 1]$ to the discontinuous function

$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } -1 < x < 1. \end{cases}$$

Since the functions f_n are continuous and f is discontinuous, there cannot be uniform convergence in $(-1, 1]$. Consider now the interval $[-a, a]$, where $0 < a < 1$. Then

$$\sup_{x\in[-a,a]} |f_n(x) - f(x)| = \sup_{x\in[-a,a]} |x^n - 0| = \sup_{x\in[-a,a]} |x|^n = a^n \to 0$$

as $n \to \infty$, since $0 < a < 1$. Hence, $\{f_n\}$ converges uniformly to 0 in every $[-a, a] \subset (-1, 1)$. Note that

$$\sup_{x\in(-1,1)} |f_n(x) - f(x)| = \sup_{x\in(-1,1)} |x^n - 0| = \sup_{x\in(-1,1)} |x|^n = 1 \nrightarrow 0,$$

so there is no uniform convergence in $(-1, 1)$.

Theorem 40 (Baire category theorem) *Let (X, d) be a complete metric space. Then the intersection of a countable family of open dense sets in X is still dense in X.*

Proof. Let $\{U_n\} \subset X$ be a countable family of dense open sets. We consider here only the case in which the family $\{U_n\}$ is infinite. The case in which the family is finite is simpler and is left as an exercise. Let

$$E := \bigcap_{n=1}^{\infty} U_n.$$

We claim that $\overline{E} = X$. Fix $x_0 \in X$. We claim that $x_0 \in \overline{E}$. To see this, in view of Proposition 29, it is enough to show that for

$$B(x_0, r) \cap E \neq \emptyset$$

for every $r > 0$. Fix $r > 0$. Since U_1 is dense, $x_0 \in X = \overline{U_1}$, and so by Proposition 29, the open set $B(x_0, r) \cap U_1$ is nonempty. Let $x_1 \in B(x_0, r) \cap U_1$. Since $B(x_0, r) \cap U_1$ is open, there exists $0 < r_1 < 1$ such that

$$B(x_1, 2r_1) \subset B(x_0, r) \cap U_1. \tag{3}$$

Inductively, assume that $x_n \in X$ and $0 < r_n < \frac{1}{n}$ have been chosen. Since U_{n+1} is dense, $x_n \in X = \overline{U_{n+1}}$, and so by Proposition 29, the open set $B(x_n, r_n) \cap U_{n+1}$ is nonempty, and so there exist $x_{n+1} \in X$ and $0 < r_{n+1} < \frac{1}{n+1}$ such that

$$B(x_{n+1}, 2r_{n+1}) \subset B(x_n, r_n) \cap U_{n+1}. \tag{4}$$

By induction we can construct two sequences $\{x_n\}$ and $\{r_n\}$ such that (4) holds and $0 < r_n < \frac{1}{n}$ for all $n \geq 1$. Note that, by construction (see (3) and (4)), for every $n \in \mathbb{N}$,

$$B(x_{n+1}, 2r_{n+1}) \subset B(x_n, r_n) \subset B(x_n, 2r_n) \subset \cdots \subset B(x_1, 2r_1) \subset B(x_0, r) \cap U_1. \tag{5}$$

Hence, if $n, m > k$, then $x_n, x_m \in B(x_k, r_k)$, so that

$$d(x_n, x_m) \leq d(x_n, x_k) + d(x_k, x_m) < r_k + r_k < \frac{2}{k}.$$

By letting $k \to \infty$, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. We claim that $x \in B(x_0, r) \cap U_\ell$ for every $\ell \in \mathbb{N}$. Indeed, fix $k \in \mathbb{N}$. Then for all $n > k$ we have that $x_n \in B(x_k, r_k)$, and so,

$$d(x_k, x) \leq d(x_k, x_n) + d(x_n, x) < r_k + d(x_n, x).$$

Letting $n \to \infty$ we conclude that $d(x_k, x) \leq r_k < 2r_k$. It follows that $x \in B(x_k, 2r_k) \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all $k \in \mathbb{N}$ (where we set $r_0 := r$). In turn, by (3) and (4), we have that $x \in B(x_0, r) \cap E$ holds and the proof is complete. ■

Definition 41 *Let (X, d) be a metric space. A set $E \subset X$ is called*

(i) nowhere dense *if the interior of its closure is empty.*

(ii) meager *if it can be written as a countable union of nowhere dense sets.*

Note that if U is open and dense, then its complement is closed and nowhere dense. Hence, we have the following.

Corollary 42 *Let (X, d) be a nonempty complete metric space. If*

$$X = \bigcup_{n=1} C_n,$$

where C_n is closed for every $n \in \mathbb{N}$. Then at least one C_n has nonempty interior. In particular, every complete nonempty metric space is not meager.

Proof. If every C_n has empty interior, then C_n is nowhere dense. Hence, $U_n := X \setminus C_n$ is open and dense. By De Morgan's laws,

$$X = \bigcup_{n=1} C_n = \bigcup_{n=1} (X \setminus U_n) = X \setminus \bigcap_{n=1} U_n,$$

which implies that $\bigcap_{n=1} U_n$ is empty. This contradicts Baire's theorem. ■

Theorem 43 *There exists a continuous $f : [0,1] \to \mathbb{R}$ that is nowhere monotone.*

Proof. Let I be a closed interval of $[0,1]$ and let

$$\mathcal{I}_I := \{f \in C([0,1]) : f \text{ is increasing in } I\},$$
$$\mathcal{D}_I := \{f \in C([0,1]) : f \text{ is decreasing in } I\}$$

The sets $\mathcal{I}_I$ and $\mathcal{D}_I$ are closed. Define

$$\mathcal{M}_I := \mathcal{I}_I \cup \mathcal{D}_I.$$

Then $\mathcal{M}_I$ is closed. Moreover, $\mathcal{M}_I$ has empty interior (why?). Consider the sequence $\{I_n\}_n$ of closed intervals $I_1 = \left[0, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, 1\right]$, $I_3 = \left[0, \frac{1}{3}\right]$, $I_4 = \left[\frac{1}{3}, \frac{2}{3}\right]$, $I_5 = \left[\frac{2}{3}, 1\right]$, $I_6 = \left[0, \frac{1}{4}\right]$, etc... and let $\mathcal{M}_n := \mathcal{M}_{I_n}$. Then

$$\mathcal{M} := \bigcup_{n=1}^{\infty} \mathcal{M}_n$$

is a meager set. By the previous corollary

$$C([0,1]) \neq \bigcup_{n=1}^{\infty} \mathcal{M}_n.$$

Any function $f \in C([0,1]) \setminus (\bigcup_{n=1}^{\infty} \mathcal{M}_n)$ is nowhere monotone. ■

1.3 Completion of a Metric Space

In this subsection we show that every metric space can be completed.

Definition 44 *Given two metric spaces* (X, d_X) *and* (Y, d_Y)*, a function* $f : X \to Y$ *is called an* isometry *if*

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$*.*

Note that f is one-to-one (injective). If f is also onto, then the metric spaces (X, d_X) and (Y, d_Y) are called *isometric*.

Exercise 45 *Let* (Y, d) *be a metric space and let* $E \subset X$ *be a dense set with the property that every Cauchy sequence* $\{y_n\} \subset E$ *converges to some point in* Y*. Prove that* Y *is complete.*

Theorem 46 *Given a metric space* (X, d_X)*, there exists a complete metric space* (Y, d_Y) *and an isometry* $f : X \to Y$ *such that* $f(X)$ *is dense in* Y*. The space* Y *is unique up to isometries, that is, if* $(Y', d_{Y'})$ *is a complete metric space having a dense subset isometric to* X*, then* Y *and* Y' *are isometric.*

Proof. [2]Let $\mathcal{Y} := \{\{x_n\} \subset X : \{x_n\}$ is a Cauchy sequence$\}$. Given two Cauchy sequences $\{x_n\}, \{z_n\} \subset X$, we say that they are *equivalent*, and we write $\{x_n\} \sim \{z_n\}$, if

$$\lim_{n\to\infty} d_X(x_n, z_n) = 0.$$

It can be checked that $\sim$ is an equivalence relations; that is,

(i) (Reflexivity) $\{x_n\} \sim \{x_n\}$ for every Cauchy sequence $\{x_n\} \subset X$.

(ii) (Symmetry) For all Cauchy sequences $\{x_n\}, \{z_n\} \subset X$, if $\{x_n\} \sim \{z_n\}$, then $\{z_n\} \sim \{x_n\}$.

(iii) (Transitivity) For all Cauchy sequences $\{x_n\}, \{z_n\}, \{w_n\} \subset X$, if $\{x_n\} \sim \{z_n\}$ and $\{z_n\} \sim \{w_n\}$, then $\{x_n\} \sim \{w_n\}$.

Given a Cauchy sequence $\{x_n\} \subset X$ the equivalence class determined by $\{x_n\}$ is given by

$$[\{x_n\}] := \{\{z_n\} \in \mathcal{Y} : \{x_n\} \sim \{z_n\}\}.$$

We define

$$Y := \{[\{x_n\}] : \{x_n\} \in \mathcal{Y}\}$$

and $d_Y : Y \times Y \to [0, \infty)$ as

$$d_Y([\{x_n\}], [\{z_n\}]) := \lim_{n\to\infty} d_X(x_n, z_n).$$

Step 1: We claim that d_Y is well-defined. We begin by showing that the limit exists in $\mathbb{R}$. By the triangle inequality applied twice,

$$d_X(x_n, z_n) \le d_X(x_n, x_m) + d_X(x_m, z_m) + d_X(z_m, z_n),$$

[2]I did not prove this in class, so I will not ask for the proof in the written exams, but please read it because it is important.

and so (interchanging the roles of x_n and y_n)

$$|d_X(x_n, z_n) - d_X(x_m, z_m)| \leq d_X(x_n, x_m) + d_X(z_m, z_n).$$

Letting $n, m \to \infty$ and using the fact that $\{x_n\}, \{z_n\}$ are Cauchy sequences in X gives

$$\lim_{n,m\to\infty} |d_X(x_n, z_n) - d_X(x_m, z_m)| = 0,$$

which shows that $\{d_X(x_n, z_n)\}$ is a Cauchy sequences in $\mathbb{R}$. Thus, there exists $\lim_{n\to\infty} d_X(x_n, z_n) \in [0, \infty)$.

It remains to show that this limit does not depend on the choice of the representatives in the equivalence class. Thus, let $\{x'_n\}, \{y'_n\} \in \mathcal{Y}$ be such that $\{x_n\} \sim \{x'_n\}$ and $\{z_n\} \sim \{z'_n\}$, so that

$$\lim_{n\to\infty} d_X(x_n, x'_n) = \lim_{n\to\infty} d_X(z_n, z'_n) = 0.$$

By the triangle inequality,

$$d_X(x_n, z_n) \leq d_X(x_n, x'_n) + d_X(x'_n, z'_n) + d_X(z'_n, z_n),$$

and so (interchanging the roles of x_n and y_n)

$$|d_X(x_n, z_n) - d_X(x'_n, z'_n)| \leq d_X(x_n, x'_n) + d_X(z'_n, z_n).$$

Letting $n \to \infty$ and using the fact that $\{d_X(x_n, z_n)\}$ is a convergent sequence, we obtain

$$\lim_{n\to\infty} d_X(x_n, z_n) = \lim_{n\to\infty} d_X(x'_n, z'_n),$$

which shows that d_Y is well-defined.

Step 2: Next we prove that d_Y is a metric. Given Cauchy sequences $\{x_n\}, \{z_n\}, \{w_n\} \subset X$, we have that

$$0 = d_Y([\{x_n\}], [\{z_n\}]) = \lim_{n\to\infty} d_X(x_n, z_n) \quad \text{if and only if } \{x_n\} \sim \{z_n\},$$

which is equivalent to say that $[\{x_n\}] = [\{z_n\}]$.

Since d_X is a metric,

$$d_Y([\{x_n\}], [\{z_n\}]) = \lim_{n\to\infty} d_X(x_n, z_n) = \lim_{n\to\infty} d_X(z_n, x_n) = d_Y([\{z_n\}], [\{x_n\}]).$$

Finally, since $d_X(x_n, z_n) \leq d_X(x_n, w_n) + d_X(w_n, z_n)$, we have

$$\lim_{n\to\infty} d_X(x_n, z_n) \leq \lim_{n\to\infty} d_X(x_n, w_n) + \lim_{n\to\infty} d_X(w_n, z_n);$$

that is,

$$d_Y([\{x_n\}], [\{z_n\}]) \leq d_Y([\{x_n\}], [\{w_n\}]) + d_Y([\{w_n\}], [\{z_n\}]).$$

Step 3: We construct an isometry $f : X \to Y$. For every $x \in X$ define $\widehat{x} := [\{x_n\}]$, where $x_n := x$ for all $n \in \mathbb{N}$. Define

$$\begin{aligned} f : X &\to Y \\ x &\mapsto \widehat{x} \end{aligned}$$

We claim that f is an isometry. To see this, note that

$$\begin{aligned} d_Y(f(x), f(z)) &= d_Y(\widehat{x}, \widehat{z}) = d_Y([\{x, \ldots, x, \ldots\}], [\{z, \ldots, z, \ldots\}]) \\ &= \lim_{n\to\infty} d_X(x, z) = d_X(x, z). \end{aligned}$$

Step 4: We prove that $f(X)$ is dense in Y. Let $[\{x_n\}] \in Y$ and let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_X(x_n, x_m) \le \varepsilon$$

for all $n, m \in \mathbb{N}$ with $n, m \ge n_\varepsilon$. Consider $f(x_{n_\varepsilon}) = \widehat{x_{n_\varepsilon}} \in Y$. Then

$$d_Y([\{x_n\}], \widehat{x_{n_\varepsilon}}) = \lim_{n\to\infty} d_X(x_n, x_{n_\varepsilon}) \le \varepsilon,$$

which proves that $f(X)$ is dense in Y.

Step 5: We prove that Y is a complete metric space. In view of the previous exercise and Step 4, it suffices to show that every Cauchy sequence $\{y_k\}$ contained in $f(X)$ converges in Y. Let $\{y_k\}$ be a Cauchy sequence in $f(X)$. Then for each $k \in \mathbb{N}$ we have that

$$y_k = \widehat{x_k} = [\{x_k, \ldots, x_k, \ldots\}]$$

for some $x_k \in X$. By Step 3, we have that

$$\lim_{l,k\to\infty} d_X(x_l, x_k) = \lim_{l,k\to\infty} d_Y(\widehat{x_l}, \widehat{x_k}) = 0$$

and so the sequence $\{x_n\} \subset X$ is a Cauchy sequence. Hence, $[\{x_n\}] \in Y$. We claim that $\{y_k\}$ converges to $[\{x_n\}]$. Fix $\varepsilon > 0$. Since $\{y_k = \widehat{x_k}\}$ is a Cauchy sequence, there exists $k_\varepsilon \in \mathbb{N}$ such that

$$d_Y(\widehat{x_l}, \widehat{x_k}) = d_X(x_l, x_k) \le \varepsilon$$

for all $l, k \in \mathbb{N}$ with $l, k \ge k_\varepsilon$. Then for every $k \ge k_\varepsilon$,

$$d_Y([\{x_n\}], \widehat{x_k}) = \lim_{n\to\infty} d_X(x_n, x_k) \le \varepsilon,$$

which proves that $\{y_k\}$ converges to $[\{x_n\}]$.

Step 6: (sketch) Finally, we prove uniqueness. Assume that $(Y', d_{Y'})$ is a complete metric space having a dense subset X' isometric to X. Given $y', w' \in Y'$ by the density of X' in Y', we can find two sequences $\{y'_n\}, \{w'_n\} \subset X'$ such that $d_{Y'}(y'_n, y') \to 0$ and $d_{Y'}(w'_n, w') \to 0$. In particular, $\{y'_n\}, \{w'_n\}$ are

Cauchy sequences in X'. Since X' is isometric to X and X is isometric to $f(X)$, there exist $\{y_n\}, \{w_n\} \subset X$ corresponding to $\{y'_n\}, \{w'_n\}$ such that

$$\begin{aligned} d_Y(f(y_n), f(y_m)) = d_X(y_n, y_m) = d_{Y'}(y'_n, y'_m), \\ d_Y(f(w_n), f(w_m)) = d_X(w_n, w_m) = d_{Y'}(w'_n, w'_m), \\ d_Y(f(y_n), f(w_n)) = d_X(y_n, w_n) = d_{Y'}(y'_n, w'_n), \end{aligned}$$

which implies that $\{f(y_n)\}, \{f(w_n)\}$ are Cauchy sequences in Y and so, by the completeness of Y, they converge to some $y, w \in Y$, respectively. Define $g(y') := y$, $g(w') := w$.

By the triangle inequality,

$$\begin{aligned} |d_{Y'}(y', z') - d_{Y'}(y'_n, w'_n)| \le d_{Y'}(y'_n, y') + d_{Y'}(w'_n, w'), \\ |d_Y(y, z) - d_Y(f(y_n), f(w_n))| \le d_Y(f(y_n), y) + d_{Y'}(f(w_n), w) \end{aligned}$$

and so

$$d_{Y'}(y', z') = \lim_{n\to\infty} d_{Y'}(y'_n, w'_n) = \lim_{n\to\infty} d_Y(f(y_n), f(w_n)) = d_Y(y, z).$$

Thus, Y and Y' are isometric. ■

Example 47 *If we take* $\mathbb{Q}$ *with the metric* $d(x, y) := |x - y|$ *and we complete it, we obtain* $\mathbb{R}$.

Given a metric space (X, d) and a function $f : X \to \mathbb{R}$, the *support* of f is the closed set

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

Exercise 48 *Consider an open interval* $I \subset \mathbb{R}$ *and consider the space*

$$C_c(I) := \{f : I \to \mathbb{R} \text{ continuous, } \operatorname{supp} f \text{ is a compact set of } I\}$$

with the metric

$$d(f, g) := \max_{x\in I} |f(x) - g(x)|.$$

Prove that $C_c(I)$ *is not complete and find its completion.*

Friday, September 18, 2009

1.4 Limits

Definition 49 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$*, where* $E \subset X$*. Given* $x_0 \in \operatorname{acc} E$*, if there exists* $y \in Y$ *with the property that for every* $\varepsilon > 0$ *there exists* $\delta = \delta(x_0, \varepsilon) > 0$ *such that*

$$d_Y(f(x), y) < \varepsilon$$

for all $x \in E$ *with* $0 < d_X(x, x_0) < \delta$*, we write*

$$y = \lim_{x \to x_0} f(x)$$

that y *is the* limit *of* f *as* x *approaches* x_0*.*

Note that x_0 need not belong to E.

Proposition 50 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$*, where* $E \subset X$*. Given* $x_0 \in \operatorname{acc} E$ *and* $y_0 \in Y$ *the following are equivalent.*

(i) $y_0 = \lim_{x \to x_0} f(x)$*.*

(ii) $f(x_n) \to y_0$ *for every sequence* $\{x_n\} \subset E \setminus \{x_0\}$ *converging to* x_0*.*

Proof. Assume that there exists $y_0 = \lim_{x \to x_0} f(x)$ and let $\{x_n\} \subset E \setminus \{x_0\}$ converge to x_0. Fix $\varepsilon > 0$ and find $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$d_Y(f(x), y_0) < \varepsilon$$

for all $x \in E$ with $d_X(x, x_0) < \delta$. Since $x_n \to x_0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta$ for all $n \geq n_\varepsilon$ and so

$$d_Y(f(x_n), y_0) < \varepsilon$$

for all $n \geq n_\varepsilon$, which shows that $f(x_n) \to y_0$.

Conversely, assume that (ii) holds and assume by contradiction that (i) does not hold. Then there exists $\varepsilon > 0$ such that for every δ there exists $x \in E \setminus \{x_0\}$ with $d_X(x, x_0) < \delta$ such that

$$d_Y(f(x), y_0) \geq \varepsilon.$$

Take $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$ and find $x_n \in E \setminus \{x_0\}$ with $d_X(x_n, x_0) < \frac{1}{n}$ such that

$$d_Y(f(x_n), y_0) \geq \varepsilon.$$

This contradicts (ii). ■

In the special case in which $Y = \mathbb{R}$ or $Y = [-\infty, \infty]$ all the standard theorems about the sum, product, quotient of limits continue to hold with the standard modifications. We omit the details.

1.5 Continuity

Definition 51 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a function $f : E \to Y$, where $E \subset X$. Given $x_0 \in E$, the function f is said to be* continuous at x_0 *if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x_0) > 0$ such that*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

for all $x \in E$ with $d_X(x, x_0) < \delta$. The function f is said to be continuous *if it is continuous at every point of E.*

Remark 52 *Note that if $x_0 \in E$ is an* isolated point *of E; that is, if there exists $r > 0$ such that $B_X(x_0, r) \cap E = \{x_0\}$, then f is continuous at x_0 (take $\delta = r$). Thus, it is enough to check the continuity of a function $f : E \to Y$ at points $x_0 \in E \cap \operatorname{acc} E$, in which case continuity at x_0 is equivalent to saying that there exists the limit*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The space of all continuous functions $f : X \to Y$ is denoted $C(X; Y)$. If $Y = \mathbb{R}$ we write $C(X)$. The space of all continuous bounded functions $f : X \to Y$ is denoted $C_b(X; Y)$. It is a metric space with the distance

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

Proposition 53 *Consider two metric spaces (X, d_X) and (Y, d_Y) with X nonempty. Then the metric space $(C_b(X; Y), d_\infty)$ is complete if and only if (Y, d_Y) is complete.*

Proof. If (Y, d_Y) is complete, then reasoning as in Example 38, one can show that $(C_b(X; Y), d_\infty)$ is a closed subset of the space of all bounded functions $f : X \to Y$. We omit the details.

Conversely, assume that $(C_b(X; Y), d_\infty)$ is complete and let $\{y_n\} \subset Y$ be a Cauchy sequence. Define the functions

$$f_n(x) = y_n, \quad x \in X.$$

Then f_n is bounded and continuous, and so it belongs to $C_b(X; Y)$. Moreover,

$$d_\infty(f_n, f_m) = \sup_{x \in X} d_Y(f_n(x), f_m(x)) = d_Y(y_n, y_m) \to 0$$

as $n, m \to \infty$, and so $\{f_n\}$ is a Cauchy sequence in $C_b(X; Y)$. Since $(C_b(X; Y), d_\infty)$ is complete, there exists $f \in C_b(X; Y)$ such that $f_n \to f$ in $C_b(X; Y)$. Fix $x_0 \in X$. Then

$$0 \le d_Y(y_n, f(x_0)) = d_Y(f_n(x_0), f(x_0)) \le \sup_{x \in X} d_Y(f_n(x), f(x)) = d_\infty(f_n, f) \to 0$$

as $n \to \infty$. Hence, $y_n \to f(x_0)$ in Y, which shows that Y is complete. ∎

The proof of the next proposition follows from Proposition 50.

Proposition 54 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$*, where* $E \subset X$*. Given* $x_0 \in E \cap \operatorname{acc} E$*, the following are equivalent.*

(i) f *is continuous at* x_0*.*

(ii) f *is* sequentially continuous *at* x_0*; that is,* $f(x_n) \to f(x_0)$ *for every sequence* $\{x_n\} \subset E$ *that converges that to* x_0*.*

Exercise 55 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : X \to Y$*. Prove that the following are equivalent.*

(i) f *is continuous.*

(ii) $f^{-1}(U)$ *is open for every open set* $U \subset Y$*.*

(iii) $f^{-1}(C)$ *is closed for every closed set* $C \subset Y$*.*

Next we introduce the notion of uniform continuity.

Definition 56 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$*, where* $E \subset X$*. The function* f *is said to be* uniformly continuous *if for every* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$d_Y(f(x), f(x_0)) < \varepsilon$$

for all $x, x_0 \in E$ *with* $d_X(x, x_0) < \delta$*.*

Remark 57 *To negate uniform continuity it is enough to fins two sequences* $\{x_n\}, \{z_n\} \subset E$ *such that*

$$\lim_{n\to\infty} d_X(x_n, z_n) = 0$$

and $d_Y(f(x_n), f(z_n)) \nrightarrow 0$ *(so either the limit does not exist or it exists but it is not zero).*

Example 58 *The function* $f(x) = x$, $x \in \mathbb{R}$, *is uniformly continuous, while the function* $g(x) = x^2$, $x \in \mathbb{R}$, *is not. To see this, take* $\varepsilon = \delta$ *for the function* f. *To prove that* g *is not uniformly continuous, consider the two sequences* $x_n = n + \frac{1}{n}$ *and* $z_n = n$. *Then* $x_n - z_n = \frac{1}{n} \to 0$, *while*

$$f(x_n) - f(z_n) = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n} \to 2 \neq 0,$$

which implies that g *is not uniformly continuous, by the previous remark.*

Simple examples of uniformly continuous functions are Lipschitz and Hölder's continuous functions.

Definition 59 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$, *where* $E \subset X$.

(i) The function f *is said to be* Lipschitz continuous *if there exists* $L > 0$ *such that*

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$

for all $x_1, x_2 \in E$. *The number*

$$\mathrm{Lip}(f; E) := \sup_{x_1, x_2 \in E,\, x_1, \neq x_2} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq L$$

is called the Lipschitz constant *of* f. *It is also denoted* $\mathrm{Lip}\, f$. *The function* f *is called a* contraction *if* $\mathrm{Lip}\, f < 1$.

(ii) The function f *is said to be* Hölder continuous with exponent $\alpha \in (0, 1)$ *if there exists* $L > 0$ *such that*

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in E$.

Remark 60 *Consider two metric spaces* (X, d_X) *and* (Y, d_Y) *and a function* $f : E \to Y$, *where* $E \subset X$.

(i) If f is Lipschitz continuous with Lipschitz constant L, then to see that it is uniformly continuous, given $\varepsilon > 0$, it is enough to take $\delta = \frac{\varepsilon}{L}$. The function $f(x) = \sqrt{x}$, $x \in [0,1]$, is uniformly continuous, but not Lipschitz. Indeed, in the case $X = Y = \mathbb{R}$, if f is Lipschitz, then

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le L$$

for all $x, y \in E$, $x \neq y$. In particular, if f is differentiable at x, then letting $y \to x$ in the previous inequality gives,

$$|f'(x)| \le L.$$

Hence, the derivative is bounded (where it exists). For $f(x) = \sqrt{x}$, $x \in [0,1]$, we have that $f'(x) = \frac{1}{2\sqrt{x}}$ for $x \in (0,1)$, which is not bounded. Similarly, if $X = \mathbb{R}^N$ and $Y = \mathbb{R}$, then if f is Lipschitz and admits a partial derivative $\frac{\partial f}{\partial x_i}$ at some point x, then $\left|\frac{\partial f}{\partial x_i}(x)\right| \le L$.

(ii) If f is Hölder continuous with exponent $\alpha \in (0,1)$ and constant $L > 0$, then to see that it is uniformly continuous, given $\varepsilon > 0$, it is enough to take $\delta = \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$. Indeed, if $x_1, x_2 \in E$ with $d_X(x_1, x_2) < \left(\frac{\varepsilon}{L}\right)^{\frac{1}{\alpha}}$, then

$$d_Y(f(x_1), f(x_2)) \le L(d_X(x_1, x_2))^a < L\left(\frac{\varepsilon}{L}\right)^{\frac{\alpha}{\alpha}} = \varepsilon.$$

The Weierstrass nowhere differentiable function is an example of a uniformly continuous function that is not Hölder continuous of any $\alpha \in (0,1)$.

(iii) If f is Lipschitz continuous with Lipschitz constant $L > 0$ and if E is bounded, then f is Hölder continuous of any exponent $\alpha \in (0,1)$. To see this, let $E \subset B_X(x_0, r)$. Then for all $x_1, x_2 \in E$, we have

$$\begin{aligned} d_Y(f(x_1), f(x_2)) &\le L d_X(x_1, x_2) = L(d_X(x_1, x_2))^a (d_X(x_1, x_2))^{1-a} \\ &\le L(d_X(x_1, x_2))^a (2r)^{1-a}, \end{aligned}$$

where in the last inequality we have used the fact that $d_X(x_1, x_2) \le d_X(x_1, x_0) + d_X(x_0, x_2) < r + r$, since $E \subset B_X(x_0, r)$. If E is unbounded, then this is no longer true. Indeed, the function $f(x) = x$, $x \in \mathbb{R}$, cannot be Hölder continuous of any exponent $\alpha \in (0,1)$. To see this, take $x_1 = x > 0$ and $x_2 = 0$, then we cannot have an inequality of the type

$$x = |f(x) - f(0)| \le Lx^a,$$

because as $x \to \infty$, x goes faster than x^a.

Let (X, d) be a metric space. If $x \in X$ and $E \subset X$, the *distance* of x from the set E is defined by

$$\operatorname{dist}(x, E) := \inf\{d(x, y) : y \in E\},$$

while the distance between two sets E_1, $E_2 \subset X$ is defined by

$$\operatorname{dist}(E_1, E_2) := \inf \{d(x, y) : x \in E_1, y \in E_2\}.$$

Exercise 61 *Let (X, d) be a metric space and let $E \subset X$ be a nonempty set.*

(i) Fix $x_0 \in X$. Prove that the function

$$x \in X \mapsto d(x, x_0)$$

is Lipschitz continuous with Lipschitz constant one.

(ii) Prove that the distance function

$$x \in X \mapsto \operatorname{dist}(x, E)$$

is Lipschitz continuous with Lipschitz constant one.

(ii) Characterize the points $x \in X$ such that $\operatorname{dist}(x, E) = 0$.

Proposition 62 *Let (X, d_X) be a metric spaces, let (Y, d_Y) be a complete metric spaces, and let $f : E \to Y$ be uniformly continuous, where $E \subset X$. Then f can be extended uniquely to a uniformly continuous function $g : \overline{E} \to Y$.*

Proof. Step 1: We begin by showing that if $\{x_n\} \subset E$ is a Cauchy sequence, then $\{f(x_n)\}$ is a Cauchy sequence in Y. Fix $\varepsilon > 0$. By the uniform continuity of f there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_Y(f(x'), f(x'')) < \varepsilon \tag{6}$$

for all $x', x'' \in E$ with $d_X(x', x'') < \delta$. Since $\{x_n\} \subset X$ is a Cauchy sequence, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d_X(x_n, x_m) < \delta$$

for all $n, m \geq n_\varepsilon$, and so

$$d_Y(f(x_n), f(x_m)) < \varepsilon \tag{7}$$

for all $n, m \geq n_\varepsilon$, which shows that $\{f(x_n)\}$ is a Cauchy sequence in Y.
Step 2: For $x \in E$ define $g(x) := f(x)$. Fix $x \in \overline{E} \setminus E$. By Remark 30 there exists a sequence $\{x_n\} \subset E$ with $x_n \neq x$ for all $n \in \mathbb{N}$ such that $\{x_n\}$ converges to x. In particular, $\{x_n\}$ is a Cauchy sequence, and so by the previous step $\{f(x_n)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{f(x_n)\}$ converges to some element $y \in Y$. Note that for any continuous extension $h : \overline{E} \to Y$ of f to $\overline{E}$ we must have $h(x) = y$ (this will show uniqueness). Thus, we define $g(x) := y$. To make sure that g is well-defined, we need to verify that $g(x)$ does not depend on the particular sequence $\{x_n\}$ converging to x. Thus, let $\{z_n\} \subset E$ be another sequence converging to x. Then by the triangle inequality we have

$$d_X(x_n, z_n) \leq d_X(x_n, x) + d_X(x, z_n) < \frac{\delta}{2} + \frac{\delta}{2}$$

for all $n \in \mathbb{N}$ sufficiently large, say $n \geq n_1$. Hence, by (6),

$$d_Y(f(x_n), f(z_n)) < \varepsilon$$

for all $n \geq n_1$. Then, since $\{f(x_n)\}$ converges to y,

$$d_Y(y, f(z_n)) \leq d_Y(y, f(x_n)) + d_Y(f(x_n), f(z_n)) \leq \varepsilon + \varepsilon$$

for all sufficiently large, which shows that $\{f(x_n)\}$ converges to y. Thus, $g(x)$ is well-defined.
Step 3: It remains to show that g is uniformly continuous. Let $x', x'' \in \overline{E}$ be such that $d_X(x', x'') < \delta$ and consider two sequences $\{x'_n\}, \{x''_n\} \subset E$ converging to x' and x'', respectively. Then for all n sufficiently large we have that

$$d_X(x'_n, x''_n) \leq d_X(x'_n, x') + d_X(x', x'') + d_X(x'', x''_n) < \delta,$$

and so by (6),

$$d_Y(f(x'_n), f(x''_n)) < \varepsilon$$

for all n sufficiently large. Letting $n \to \infty$, we obtain

$$d_Y(g(x'), g(x'')) \leq \varepsilon,$$

which shows that g is uniformly continuous. ■

Theorem 63 (Banach's contraction principle) *Let (X,d) be a nonempty complete metric space and let $f : X \to X$ be a contraction. Then f has a unique fixed point; that is, there is a unique $x \in X$ such that $f(x) = x$.*

1.6 Application to Ordinary Differential Equations

An important application of Banach's contraction principle is the existence of solutions of ODE. Consider the initial value problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(t_0) &= u_0. \end{aligned}$$

Here, $I \subset \mathbb{R}$ is an open interval, $t_0 \in I$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume that f satisfies the following Lipschitz condition

$$|f(t, z_1) - f(t, z_2)| \leq L |z_1 - z_2|$$

for all $t \in I$, $z_1, z_2 \in \mathbb{R}$. Then we can prove short time existence of solutions. Consider the space $X = C([t_0, t_0 + T])$, where T will be chosen later and consider the operator

$$F : C([t_0, t_0 + T]) \to C([t_0, t_0 + T])$$

given by

$$F(g)(t) = u_0 + \int_{t_0}^{t} f(s, g(s))\, ds$$

for $g \in C([t_0, t_0 + T])$ and $t \in [t_0, t_0 + T]$. It is clear that F is well-defined, since the function on the right-hand side is continuous. Let's prove that F is a contraction. Take $g_1, g_2 \in C([t_0, t_0 + T])$. Then

$$\begin{aligned} |F(g_1)(t) - F(g_2)(t)| &= \left| \int_{t_0}^{t} [f(s, g_1(s)) - f(s, g_2(s))]\, ds \right| \\ &\leq \int_{t_0}^{t} |f(s, g_1(s)) - f(s, g_2(s))|\, ds \\ &\leq L \int_{t_0}^{t} |g_1(s) - g_2(s)|\, ds \leq L \max_{y \in [t_0, t_0+T]} |g_1(y) - g_2(y)| \int_{t_0}^{t} ds \\ &\leq LT d_\infty(g_1, g_2), \end{aligned}$$

and so taking the maximum over all $t \in [t_0, t_0 + T]$, we get

$$d_\infty(F(g_1), F(g_2)) \leq LT d_\infty(g_1, g_2).$$

If we take T so small that $LT < 1$ and $t_0 + T \in I$, then F is a contraction. By Banach's contraction principle there exists a unique fixed point $u \in C([t_0, t_0 + T])$, that is,

$$u(t) = F(u)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))\, ds$$

for all $t \in [t_0, t_0 + T]$. Since u is continuous, the right-hand side is of class C^1, and so u is actually of class C^1. By differentiating both sides, we get that u is a solution of the ODE. Moreover, $u(t_0) = u_0$. Since any other solution of the initial value problem is a fixed point of F, we have uniqueness.

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1.7 Continuity, continued

Next we prove Banach's contraction theorem.
Proof. Step 1: Let's first prove uniqueness. Assume that x_1 and x_2 are fixed points of f. Then

$$d_X(x_1, x_2) = d_X(f(x_1), f(x_2)) \leq L d_X(x_1, x_2),$$

which implies that

$$(1 - L) d_X(x_1, x_2) \leq 0.$$

Since $L < 1$, we have that $d_X(x_1, x_2) = 0$, and so $x_1 = x_2$.
Step 2: To prove existence, fix $x_0 \in X$ and define inductively

$$x_1 := f(x_0), \quad x_{n+1} := f(x_n).$$

We claim that $\{x_n\}$ is a Cauchy sequence. Indeed, note that

$$d_X(x_1, x_2) = d_X(f(x_0), f(x_1)) \leq L d_X(x_0, x_1)$$

and by induction

$$d_X(x_n, x_{n+1}) = d_X(f(x_{n-1}), f(x_n)) \leq L^n d_X(x_0, x_1).$$

Hence, for every $m, n \in \mathbb{N}$, by the triangle inequality

$$\begin{aligned} d_X(x_n, x_{n+m}) &\leq \sum_{i=n}^{n+m-1} d_X(x_i, x_{i+1}) \leq d_X(x_0, x_1) \sum_{i=n}^{n+m-1} L^i \\ &\leq d_X(x_0, x_1) \sum_{i=n}^{\infty} L^i = d_X(x_0, x_1) \frac{L^n}{1 - L}. \end{aligned}$$

Letting $n \to \infty$, we have that $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists $x \in X$ such that $\{x_n\}$ converges to x. But by the continuity of f,

$$x \leftarrow x_{n+1} = f(x_n) \to f(x),$$

which shows that $f(x) = x$. ■

Another important fixed point theorem is the following.

Theorem 64 (Brouwer's fixed point theorem) *Let $K \subset \mathbb{R}^N$ be a nonempty compact convex set and let $f : K \to K$ be a continuous function. Then there exists $x \in K$ such that $f(x) = x$.*

In the next two subsections we prove that a continuous function preserves two important notions: connectedness and compactness.

1.8 Connectedness

Definition 65 *Let (X, d) be a metric space.*

(i) A set $E \subset X$ is disconnected *if there exist two nonempty disjoint open sets $U, V \subset X$ such that*

$$E \subset U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset.$$

(ii) A set $E \subset X$ is connected *if it is not disconnected.*

Next we show that continuous functions preserve connectedness.

Proposition 66 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a continuous function $f : X \to Y$. Then $f(E)$ is connected for every connected set $E \subset X$.*

Proof. Let $E \subset X$ be a connected set and assume by contradiction that $f(E)$ is disconnected. Then there exist two nonempty disjoint open sets $U, V \subset X$ such that

$$f(E) \subset U \cup V, \quad f(E) \cap U \neq \emptyset, \quad f(E) \cap V \neq \emptyset.$$

By continuity, $f^{-1}(U)$ and $f^{-1}(V)$ are open,

$$E \subset f^{-1}(U) \cup f^{-1}(V), \quad E \cap f^{-1}(U) \neq \emptyset, \quad E \cap f^{-1}(V) \neq \emptyset,$$

which shows that E is disconnected. ■

Definition 67 *A set $C \subset \mathbb{R}^N$ is convex if for every $x, y \in C$, the segment of endpoints x and y, precisely,*

$$\{tx + (1-t)y : t \in [0,1]\},$$

is contained in C.

Theorem 68 *A set $C \subset \mathbb{R}$ is connected if and only if it is convex.*

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Proof. Step 1: Assume that C is convex. We claim that C is connected. If not, then there exist two nonempty disjoint open sets $U, V \subset \mathbb{R}$ such that

$$C \subset U \cup V, \quad C \cap U \neq \emptyset, \quad C \cap V \neq \emptyset.$$

Let $x \in C \cap U$ and $y \in C \cap V$. Without loss of generality, we may assume that $x < y$. By convexity, the interval $[x, y]$ is contained in C. Let

$$z := \sup (U \cap [x, y]).$$

Then $x \leq z \leq y$. Since $x \in U$ and $y \in V$ and U and V are open, we can find $\delta > 0$ such that $x + \delta < z < y - \delta$. We will show that $z \notin U \cup V$. Indeed, if $z \in U$, then, since since U is open, we can find $r > 0$ such that $(z - r, z + r) \subset U$ and taking $r < y - z$, we have that $[z, z + r) \subset U \cap [x, y]$, which contradicts the definition of z. Hence, $z \notin U$ but since

$$z \in [x, y] \subset C \subset U \cup V,$$

this implies that $z \in V$. Since V is open, we can find $r_1 > 0$ such that $(z - r_1, z + r_1) \subset V$ and taking $r_1 < z - x$, we have that $(z - r, z] \subset V \cap [x, y]$, which contradicts the definition of z. This shows that C is connected.

Step 2: Assume that C is connected, let $x, y \in C$, with, say, $x < y$. We claim that the interval $[x, y]$ is contained in C. If not, then there exists $x < z < y$ such that $z \notin C$. Define

$$U := (-\infty, z), \quad V := (z, \infty).$$

Then U and V are open, disjoint, both intersect C and their union covers C. This contradicts the fact that C is connected. ■

We now introduce another notion of connectedness, which is simpler to verify.

Definition 69 *Given a metric space (X, d), a* continuous path*, or* curve*, is a continuous function $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval. The set $\gamma(I) \subset X$ is called the* range *of the path. If $I = [a, b]$, the points $\gamma(a)$ and $\gamma(b)$ are called* endpoints *of the path. A set $E \subset X$ is called* pathwise connected *if for all $x, y \in E$ there exists a continuous path with endpoints x, and y and range contained in E.*

Proposition 70 *Let (X, d) be a metric space and let $E \subset X$ be pathwise connected. Then E is connected.*

Proof. We claim that E is connected. If not, then there exist two nonempty disjoint open sets $U, V \subset X$ such that

$$E \subset U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset.$$

Let $x \in C \cap U$ and $y \in C \cap V$. By hypothesis there exists a continuous path $\gamma : [a, b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma([a, b]) \subset E$. By Proposition 66 and Theorem 68, we have that $\gamma([a, b])$ is connected. On the other hand,

$$\gamma([a, b]) \subset E \subset U \cup V, \quad x \in \gamma([a, b]) \cap U, \quad y \in \gamma([a, b]) \cap V,$$

which is a contradiction. ■

Proposition 71 *Let (X, d) be a metric space and let $E \subset X$ be a connected set. Then $\overline{E}$ is connected.*

Proof. If not, then there exist two nonempty disjoint open sets $U, V \subset X$ such that

$$\overline{E} \subset U \cup V, \quad \overline{E} \cap U \neq \emptyset, \quad \overline{E} \cap V \neq \emptyset.$$

By Proposition 29, it follows that $E \cap U \neq \emptyset$ and $E \cap V \neq \emptyset$, which contradicts the fact that E is connected. ■

Wednesday, September 30, 2009

The next example and exercise show that in $\mathbb{R}^N$ a connected set may fail to be pathwise connected, unless the set is open.

Example 72 *Let $E \subset \mathbb{R}^2$ be the set given by*

$$E_1 = \left\{(x,y) \in \mathbb{R}^2 : x = 0, -1 \le y \le 1\right\},$$
$$E_2 = \left\{(x,y) \in \mathbb{R}^2 : x > 0, y = \sin\frac{1}{x}\right\},$$
$$E = E_1 \cup E_2.$$

The set E_2 is connected since it is the image of $(0,\infty)$ through the continuous function

$$g : (0,\infty) \to \mathbb{R}^2$$
$$x \mapsto \left(x, \sin\frac{1}{x}\right)$$

and since $E = \overline{E_2}$, it follows by Proposition 71 that E is connected. We claim that E is not pathwise connected. Indeed, assume by contradiction that there exists a continuous curve $\gamma : [0,1] \to E$ such that $\gamma(0) = \left(\frac{1}{\pi}, 0\right)$ and $\gamma(1) = (0,0)$. Let

$$t_0 = \inf\{t \in [0,1] : \gamma(t) \in E_1\}.$$

Then $\gamma([0,t_0])$ contains at most one point of E_1, while $\overline{\gamma([0,t_0])}$ contains E_1. Indeed, $\gamma([0,t_0])$ contains all points $\left(x, \sin\frac{1}{x}\right)$ with $0 < x < \frac{1}{\pi}$. Indeed, if there existed $0 < x_0 < \frac{1}{\pi}$ such that $\gamma([0,t_0])$ $\left(x_0, \sin\frac{1}{x_0}\right) \notin \gamma([0,t_0])$, then the open sets $(-\infty, x_0) \times \mathbb{R}$ and $(x_0, \infty) \times \mathbb{R}$ would disconnect. In particular, $\gamma([0,t_0])$ is not closed, which contradicts the fact that a continuous function sends compact sets into compact sets (we will prove this later).

Definition 73 *In the Euclidean space $\mathbb{R}^N$, a* polygonal path *is a continuous path $\gamma : [a,b] \to \mathbb{R}^N$ for which there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ with the property that $\gamma : [x_{i-1}, x_i] \to \mathbb{R}^N$ is affine for all $i = 1, \ldots, n$, that is,*

$$\gamma(t) = c_i + t d_i \quad \textit{for } t \in [x_{i-1}, x_i],$$

for some $c_i, d_i \in \mathbb{R}^N$.

Exercise 74 *Let $O \subset \mathbb{R}^N$ be open and connected and let $x_0 \in O$.*

(i) Prove that the set

$$U := \{x \in O : \text{ there exists a polygonal path with endpoints } x \text{ and } x_0 \text{ and range contained in } O\}$$

is open and nonempty.

(ii) Prove that the set

$$V := \{x \in O : \text{ there does not exist a polygonal path with endpoints } x \text{ and } x_0 \text{ and range contained in } O\}$$

is open.

(iii) Prove that O is pathwise connected.

Exercise 75 *Prove that the set $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected.*

Exercise 76 *Let (X, d) be a metric space and let $E_1, E_2 \subset X$ be two connected sets. Prove that if there exists $x \in E_1 \cap \overline{E_2}$, then $E_1 \cup E_2$ is connected.*

Next we show that if a set is not connected, we can decompose it uniquely into a disjoint union of maximal connected subsets.

Proposition 77 *Let (X, d) be a metric space and let $E \subset X$. Assume that*

$$E = \bigcup_{\alpha \in \Lambda} E_\alpha,$$

where each E_α is a connected set. If $\bigcap_{\alpha \in \Lambda} E_\alpha$ is nonempty, then E is connected.

Proof. We claim that E is connected. If not, then there exist two nonempty disjoint open sets $U, V \subset X$ such that

$$E \subset U \cup V, \quad E \cap U \neq \emptyset, \quad E \cap V \neq \emptyset.$$

Since each E_α is connected, we must have that either $E_\alpha \subset U$ or $E_\alpha \subset V$. On the other hand, if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta$ is nonempty, while $U \cap V$ is empty. Thus, all E_α either belong to U or to V. This contradicts the fact that $E \cap U \neq \emptyset$ and that $E \cap V \neq \emptyset$. ■

Let (X, d) be a metric space and let $E \subset X$. For every $x \in E$, let E_x be the union of all the connected subsets of E that contain x. Note that E_x is nonempty, since $\{x\}$ is a connected subset. In view of the previous proposition, the set E_x is connected. Moreover, if $x, y \in E$ and $x \neq y$, then either $E_x \cap E_y = \emptyset$ or $E_x = E_y$. Indeed, if not, then again by the previous proposition the set $E_x \cup E_y$ would be connected, contained in E, and would contain x and y, which would contradict the definition of E_x and of E_y. Thus, we can partition E into a disjoint union of maximal connected subsets, called the *connected components* of E.

Proposition 78 *Let (X, d) be a metric space and let $C \subset X$ be a closed set. Then the connected components of C are closed.*

Proof. Let C_α be a connected component of C. Then $C_\alpha \subset \overline{C_\alpha} \subset \overline{C} = C$. By Proposition 71, $\overline{C_\alpha}$ is connected, and so by the maximality of C_α, $\overline{C_\alpha} = C_\alpha$, i.e., C_α is closed. ■

Exercise 79 *Prove that if $U \subset \mathbb{R}^N$ is open, then the connected components of U are open. Is this still true for open subsets of arbitrary metric spaces?*

Friday, October 02, 2009

1.9 Compactness

Definition 80 *Let (X, d) be a metric space.*

(i) A set $K \subset X$ is compact *if for every open cover of K, i.e., for every collection $\{U_\alpha\}$ of open sets such that $\bigcup_\alpha U_\alpha \supset K$, there exists a finite subcover (i.e., a finite subcollection of $\{U_\alpha\}$ whose union still contains K).*

(ii) A set $K \subset X$ is sequentially compact *if for every sequence $\{x_n\} \subset K$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$.*

(iii) A set $K \subset X$ is totally bounded *if for every $\varepsilon > 0$ there exist $x_1, \ldots, x_m \in K$ such that*

$$K \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon).$$

(iv) A set $E \subset X$ is relatively compact *(or* precompact*) if its closure $\overline{E}$ is compact.*

Example 81 *Let (X, d) be a metric space. A finite set $K \subset X$ is compact, sequentially compact, and totally bounded.*

The following theorem is one of the main results of this subsection.

Theorem 82 *Let (X, d) be a metric space and let $K \subset X$. The the following are equivalent.*

(i) K is sequentially compact.

(ii) K is complete and totally bounded.

(iii) K is compact.

We begin with a preliminary result.

Lemma 83 *Let (X, d) be a metric space and let $K \subset X$ be compact. Then K is closed.*

Proof. It is enough to show that $X \setminus K$ is open. Fix $x \in X \setminus K$. For every $y \in K$ consider $U_y := B(y, r_y)$ and $V_y := B(x, r_y)$, where $r := \frac{d(x,y)}{4}$. Then $\{U_y\}_{y \in K}$ is an open cover of K, and so there exist $y_1, \ldots, y_m \in K$ such that

$$K \subset \bigcup_{i=1}^{m} U_{y_i} =: U.$$

Let $V := \bigcap_{i=1}^{m} V_{y_i}$. Then V is open and does not intersect U. In particular, $V \subset X \setminus K$. This shows that every point of $X \setminus K$ is an interior point, and so $X \setminus K$ is open. ■

We now turn to the proof of Theorem 82.

Proof of Theorem 82. (i)⇒(ii) Assume that K is sequentially compact. We claim that (K, d) is complete. To see this, let $\{x_n\} \subset K$ be a Cauchy sequence. Since $K \subset X$ is sequentially compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Since $\{x_n\} \subset K$ is Cauchy sequence, for every fixed $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$d(x_n, x_m) \leq \frac{\varepsilon}{2} \tag{8}$$

for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$. On the other hand, since $x_{n_k} \to x$ as $k \to \infty$, there exists exists $k_\varepsilon \in \mathbb{N}$ such that

$$d(x_{n_k}, x) \leq \frac{\varepsilon}{2} \tag{9}$$

for all $k \in \mathbb{N}$ with $k \geq k_\varepsilon$. Fix $k \in \mathbb{N}$ so large that $n_k \geq \max\{n_\varepsilon, n_{k_\varepsilon}\}$. Then for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ we have that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

which implies that $d(x_n, x) \to 0$ as $n \to \infty$. Hence, (K, d) is complete.

Next we show that K is totally bounded. Assume by contradiction that K is not totally bounded. Then there exists $\varepsilon_0 > 0$ such that K cannot be covered by a finite number of balls of radius ε_0. Fix $x_1 \in K$. Then there exists $x_2 \in K$ such that $d(x_1, x_2) \geq \varepsilon_0$ (otherwise $B(x_1, \varepsilon_0)$ would cover K). Similarly, we can find $x_3 \in K$ such that $d(x_1, x_3) \geq \varepsilon_0$ and $d(x_2, x_3) \geq \varepsilon_0$ (otherwise $B(x_1, \varepsilon_0)$ and $B(x_2, \varepsilon_0)$ would cover K). Inductively, construct a sequence $\{x_n\} \subset K$ such that $d(x_n, x_m) \geq \varepsilon_0$ for all $n, m \in \mathbb{N}$ with $n \neq m$. The sequence $\{x_n\}$ cannot have a convergent subsequence, which contradicts the fact that K is sequentially compact.

(ii)⇒(i) Assume that K is complete and totally bounded. Let $\{x_n\} \subset K$. We want to prove that a subsequence of $\{x_n\}$ is A Cauchy sequence and then use the completeness of K. For every $k \in \mathbb{N}$ let $\mathcal{B}_k$ be a finite cover of K with balls of radius $\frac{1}{2^k}$ and centers in K. Since $\mathcal{B}_1$ covers K there exists a ball $B_1 \in \mathcal{B}_1$ such that $x_n \in B_1$ for infinitely many $n \in \mathbb{N}$. Since $\mathcal{B}_1$ covers $K \cap B_1$ there exists a ball $B_2 \in \mathcal{B}_2$ such that $x_n \in B_1 \cap B_2$ for infinitely many $n \in \mathbb{N}$. Inductively, for every $k \in \mathbb{N}$ we may find a ball $B_k \in \mathcal{B}_k$ such that $x_n \in B_1 \cap \cdots \cap B_k$ for infinitely many $n \in \mathbb{N}$.

Let $n_1 \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $x_n \in B_1$, let $n_2 \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $n > n_1$ and $x_n \in B_1 \cap B_2$. Inductively, for every $k \in \mathbb{N}$ let $n_k \in \mathbb{N}$ be the first of the integers $n \in \mathbb{N}$ such that $n > n_{k-1}$ and $x_n \in B_1 \cap \cdots \cap B_k$. We claim that the subsequence $\{x_{n_k}\}$ is a Cauchy sequence. Indeed, if $k, \ell \in \mathbb{N}$ with $k, \ell \geq m$, then $x_{n_k}, x_{n_\ell} \in B_m$, and so

$$d(x_{n_k}, x_{n_\ell}) \leq \frac{1}{2^m} \to 0$$

as $m \to \infty$. Thus, $\{x_{n_k}\}$ is a Cauchy sequence and since K is complete, $\{x_{n_k}\}$ converges to a point in K. ■

Proof. (ii)⇒(iii) Assume that K is complete and totally bounded. Let $\{U_\alpha\}$ be a collection of open sets such that $\bigcup_\alpha U_\alpha \supset K$. As in the previous part, for every $k \in \mathbb{N}$ let $\mathcal{B}_k$ be a finite cover of K with balls of radius $\frac{1}{2^k}$ and centers in K. We want to prove that there exists $\overline{k} \in \mathbb{N}$ such that every ball in $\mathcal{B}_{\overline{k}}$ is contained in some U_α. Note that this would conclude the proof. Indeed, for every $B \in \mathcal{B}_{\overline{k}}$ fix one U_α containing B. Since $\mathcal{B}_{\overline{k}}$ is a finite family and covers K, the subcover of $\{U_\alpha\}$ just constructed has the same properties.

To find $\overline{k}$, assume by contradiction that for every $k \in \mathbb{N}$ there exists a ball $B\left(x_k, \frac{1}{2^k}\right) \in \mathcal{B}_k$ that is not contained in any U_α. Since $\{x_k\} \subset K$, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $x \in K$ such that $x_{k_j} \to x$ as $j \to \infty$. Since $\bigcup_\alpha U_\alpha \supset K$, there exists a such that $x \in U_\alpha$. Since U_α is an open set, there exists $r > 0$ such that $B(x, r) \subset U_\alpha$. Let $j \in \mathbb{N}$ be so large that $d\left(x_{k_j}, x\right) < \frac{r}{2}$ and $\frac{1}{2^{k_j}} < \frac{r}{2}$. Then

$$B\left(x_{k_j}, \frac{1}{2^{k_j}}\right) \subset B(x, r) \subset U_\alpha,$$

which is a contradiction.

(iii)⇒(ii) Assume that K is compact. By the previous lemma, K is closed, and so sequentially closed by Proposition 26. We claim that K is sequentially compact. To see this, assume by contradiction that there exists a sequence $\{x_n\} \subset K$ that has no subsequence converging in K. Then for every $m \in \mathbb{N}$ the number of $n \in \mathbb{N}$ such that $x_n = x_m$ is finite (otherwise, if $x_n = x_m$ for infinitely many $n \in \mathbb{N}$, then this would be a convergent subsequence). Moreover, the set $C := \{x_n : n \in \mathbb{N}\}$ has no accumulation points. Indeed, if C had an accumulation point, then since K is sequentially closed, there would a subsequence of $\{x_n\}$ converging to K. Since C has no accumulation point, it follows, in particular, that C is closed. Similarly, for every $m \in \mathbb{N}$ the sets $C_m := \{x_n : n \in \mathbb{N}, n \geq m\}$ are closed. Moreover, $C_{m+1} \subset C_m$ and by what we said before,

$$\bigcap_{m=1}^{\infty} C_m = \emptyset. \tag{10}$$

For every $m \in \mathbb{N}$ the set $U_m := X \setminus C_m$ is open, $U_{m+1} \supset U_m$ and by (10) and De Morgan's laws

$$\bigcup_{m=1}^{\infty} U_m = \bigcup_{m=1}^{\infty} (X \setminus C_m) = X \setminus \left(\bigcap_{m=1}^{\infty} C_m\right) = X.$$

In particular, $\{U_m\}_m$ is an open cover of K. By compactness, it follows that there $\overline{m} \in \mathbb{N}$ such that

$$K \subset \bigcup_{m=1}^{\overline{m}} U_m = U_{\overline{m}} = X \setminus C_{\overline{m}},$$

which implies that $K \cap C_{\overline{m}} = \emptyset$. This is a contradiction, since $C_{\overline{m}}$ is nonempty and contained in K. ■

Exercise 84 *Let* $\overline{\mathbb{R}} := [-\infty, \infty]$ *be the extended real line. Prove that there is a metric* d *on* $\overline{\mathbb{R}}$ *that makes* $\overline{\mathbb{R}}$ *compact.*

Exercise 85 *Let* (X, d) *be a compact metric space. Prove that* X *is separable and complete.*

Exercise 86 *Let* (X, d) *be a metric space and let*

$$Y := \{C \subset X : \ C \text{ is closed and bounded}\}.$$

Given $C_1, C_2 \in Y$, *define*

$$d_Y(C_1, C_2) := \sup\{\operatorname{dist}(x, C_2) : \ x \in C_1\} + \sup\{\operatorname{dist}(y, C_1) : \ y \in C_2\}.$$

Prove that (Y, d_Y) *is a metric space.*

Proposition 87 *Let* (X, d) *be a metric space and let* $K \subset X$ *be compact. Then*

(i) K *is closed and bounded (in the metric sense),*

(ii) if $C \subset K$ *is closed, then* C *is compact.*

Proof. (i) Let K be compact. We have already proved in Lemma 83 that K is closed. To prove that K is bounded, assume by contradiction that it is not. Fix $x_0 \in K$. Then we may construct a sequence $\{x_n\} \subset K$ such that $d(x_n, x_0) \to \infty$ as $n \to \infty$. On the other hand, since K is sequentially compact by Theorem 82, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Then

$$d(x_{n_k}, x_0) \le d(x_{n_k}, x) + d(x, x_0) \le M$$

for all $k \in \mathbb{N}$ and for some $M \in \mathbb{R}$, which is a contradiction.

(ii) Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of C. Then $\{U_\alpha\}_{\alpha \in \Lambda} \cup \{X \setminus C\}$ is an open cover of K. By the compactness of K there exist $U_{\alpha_1}, \dots, U_{\alpha_n}$ such that

$$U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \cup (X \setminus C) \supset K \supset C.$$

Then

$$U_{\alpha_1} \cup \dots \cup U_{\alpha_n} \supset C,$$

which shows that C is compact. ■

Exercise 88 *This exercise shows that in* $\mathbb{R}^N$ *a closed and bounded set is compact.*

(i) Prove that an interval $[a, b] \subset \mathbb{R}$ *is compact.*

(ii) Prove that the Cartesian product of two compact metric space is compact.

(iii) Prove that a rectangle $[a_1, b_1] \times \dots \times [a_N, b_N] \subset \mathbb{R}^N$ *is compact.*

Definition 89 *A family* $\{E_\alpha\}_{\alpha \in \Lambda}$ *of subsets of a set* X *has the* finite intersection property *if every finite subfamily has nonempty intersection.*

A decreasing sequence of nonempty sets has the finite intersection property.

Exercise 90 *Let (X, d) be a metric space. Prove that a set $K \subset X$ is compact if and only if for every family $\{C_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of K with the finite intersection property,*

$$\bigcap_{\alpha\in\Lambda} C_\alpha \neq \emptyset.$$

The importance of compactness is explained by the following results.

Theorem 91 (Weierstrass) *Let (X, d) be a metric space, let $K \subset X$ be compact, and let $f : X \to \mathbb{R}$ be a continuous function. Then there exists $x_0, x_1 \in K$ such that*

$$f(x_0) = \min_{x\in K} f(x), \quad f(x_1) = \max_{x\in K} f(x).$$

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First proof. This proof only uses compactness. Let

$$t := \inf_{x \in K} f(x).$$

If the infimum is not attained, then for every $x \in K$ we may find $t < t_x < f(x)$. Then the family of open sets

$$U_x := \{y \in X : f(y) > t_x\}, \quad x \in K,$$

is an open cover for the compact set K, and so we may find a finite cover $U_{x_1}, \ldots, U_{x_l}$ of the set K. But then for all $x \in K$,

$$f(x) \geq \min_{i=1,\ldots,l} t_{x_i} > t = \inf_{w \in K} f(w),$$

which contradicts the definition of t.

The proof for the existence the maximum is similar. ■

Remark 92 *Note that to prove the existence of a minimum we only used a weaker form of continuity, namely that the set* $\{y \in X : f(y) > t\}$ *is open for all* $t \in \mathbb{R}$. *A function satisfying this property is called* lower semicontinuous.

Second proof. This proof uses sequential compactness. Construct a sequence of real numbers $t < t_n$ such that $t_n \to t$ as $n \to \infty$ (if $t \in \mathbb{R}$ we can take $t_n = t + \frac{1}{n}$, while if $t = -\infty$, take $t_n = -n$). By the definition of infimum, for every $n \in \mathbb{N}$ we may find $x_n \in K$ such that

$$t < f(x_n) < t_n.$$

Letting $n \to \infty$, by the squeezing theorem, we get

$$\lim_{n \to \infty} f(x_n) = t. \tag{11}$$

Since $\{x_n\} \subset K$, and K is sequentially compact by Theorem 82, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in K$ such that $x_{n_k} \to x$ as $k \to \infty$. Using the continuity of f and (11), we get

$$t = \lim_{n \to \infty} f(x_n) = \lim_{k \to \infty} f(x_{n_k}) = f(x),$$

which shows that the infimum is a minimum. ■

The sequence $\{x_n\}$ constructed in the previous proof is called a *minimizing sequence.*

Remark 93 *Note that to prove the existence of a minimum in the second proof we only used a weaker form of continuity, namely that the set*

$$\liminf_{j \to \infty} f(x_j) \geq f(x)$$

for all sequences $\{x_j\}$ *converging to* $x \in \mathbb{R}$. *A function satisfying this property is called* sequentially lower semicontinuous.

In view of the previous theorem if (X, d) is a compact metric space, then $C(X)$ is a metric space with the distance

$$d_\infty(f, g) := \max_{x \in X} |f(x) - g(x)|.$$

If the metric space (X, d) is not compact, then we cannot use the distance d_∞. However, we can consider a smaller space, precisely

$$C_c(X) := \{f \in C(X) : \operatorname{supp} f \text{ is a compact set}\}$$

with the metric d_∞.

Exercise 94 *Let (X, d) be a metric space. Characterize the completion of $C_c(X)$. It is denoted $C_0(X)$ and is called* the space of functions vanishing at infinity*.*

Remark 95 *A typical application of the Weierstrass theorem is the following. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a continuous function. Assume that f is bounded from below, so that*

$$\ell = \inf_{x \in X} f(x) > -\infty$$

and that for a given $x_0 \in X$,

$$\lim_{d(x, x_0) \to \infty} f(x) = \infty.$$

We would like to know if f has a minimum. By the definition of limit, we can find $R > 0$ such that $f(x) > \ell$ for all $x \in X$ such that $d(x, x_0) \geq R$. Thus,

$$\ell = \inf_{x \in X} f(x) = \inf_{x \in \overline{B(x_0, R)}} f(x).$$

If $X = \mathbb{R}^N$ with the euclidean distance, then we know that $\overline{B(x_0, R)}$ is compact and so by the Weierstrass theorem, f has a minimum in $\overline{B(x_0, R)}$ and we are done. Unfortunately, for all reasonable infinite-dimensional metric spaces $\overline{B(x_0, R)}$ is neither compact, nor sequentially compact. We will see in functional analysis that the way around this is to put a weaker topology on the metric space in such a way that $\overline{B(x_0, R)}$ becomes compact.

Theorem 96 *Let (X, d_X) and (Y, d_Y) be two metric spaces, let $K \subset X$ be compact, and let $f : K \to Y$ be a continuous function. Then f is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. By continuity for every $x \in K$ there exists $\delta_x = \delta_x(\varepsilon) > 0$ such that

$$d_Y(f(x), f(z)) < \varepsilon \tag{12}$$

for all $z \in K$ with $d_X(x,z) < \delta_x$. The family $\left\{B\left(x, \frac{\delta_x}{2}\right)\right\}_{x\in K}$ is an open cover for the compact set K, and so we may find a finite cover $x_1, \ldots, x_m$ such that

$$K \subset \bigcup_{i=1}^{m} B\left(x_i, \frac{\delta_{x_i}}{2}\right).$$

Let

$$\delta := \min_{i=1,\ldots,m} \frac{\delta_{x_i}}{2} > 0.$$

Let $x, z \in K$ be such that $d_X(x,z) < \delta$. Since $K \subset \bigcup_{i=1}^{m} B\left(x_i, \frac{\delta_{x_i}}{2}\right)$, there exists we may find x_i such that $x \in B\left(x_i, \frac{\delta_{x_i}}{2}\right)$, and so by the triangle inequality

$$d_X(z, x_i) \leq d_X(z,x) + d_X(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i},$$

which shows that $z \in B(x_i, \delta_{x_i})$. Hence, by (12),

$$d_Y(f(x), f(z)) \leq d_Y(f(x), f(x_i)) + d_Y(f(x_i), f(z)) < 2\varepsilon.$$

■

Next we show that continuous functions preserve compactness.

Proposition 97 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a continuous function $f : X \to Y$. Then $f(K)$ is compact for every compact set $K \subset X$.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(K)$. By continuity, $f^{-1}(U_\alpha)$ is open for every $\alpha \in \Lambda$, and so $\left\{f^{-1}(U_\alpha)\right\}_{\alpha\in\Lambda}$ is an open cover of K. Since K is compact, we may find $U_{\alpha_1}, \ldots, U_{\alpha_l}$ such that $\left\{f^{-1}(U_{\alpha_i})\right\}_{i=1}^{l}$ cover K. In turn, $U_{\alpha_1}, \ldots, U_{\alpha_l}$ cover $f(K)$. Indeed, if $y \in f(K)$, then there exists $x \in K$ such that $f(x) = y$. Let $i = 1, \ldots, m$ be such that $x \in f^{-1}(U_{\alpha_i})$. Then $y = f(x) \in U_{\alpha_i}$. ■

Definition 98 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a bijective function $f : X \to Y$. Then f is a* homeomorphism *if f and f^{-1} are continuous.*

Proposition 99 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a bijective function $f : X \to Y$. If X is compact and f is continuous, then f^{-1} is continuous. In particular, f is a homeomorphism.*

Proof. For every closed set $C \subset X$, we have that C is compact by Proposition 87, and so $f(C)$ is compact by Proposition 97. Again by Proposition 87 we have that $f(C)$ is closed. Let $g := f^{-1}$. We have shown that $g^{-1}(C)$ is closed for every closed set $C \subset X$. Thus, by Exercise 55, g is continuous. ■

Note that without compactness the result is false.

Example 100 *Let $X = (0,1) \cup [2,3]$ and define*

$$f(x) = \begin{cases} x & \text{if } x \in (0,1), \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$

Then $f : (0,1) \cup [2,3] \to (0,2]$ is continuous, bijective, but the inverse function is discontinuous. The problem here is the fact that $(0,1) \cup [2,3]$ is not compact nor connected.

1.10 The Ascoli–Arzela Theorem

Next we show that in an arbitrary metric space, closed and bounded sets are not compact.

Example 101 *Let $X := C([0,1])$. The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0,1]$$

is bounded in $C([0,1])$, but no subsequence converges uniformly to a continuous function. This shows that $\overline{B_X(0,1)}$ is closed and bounded but not compact. Hence, Bolzano–Weierstrass theorem fails for infinite dimensional metric spaces.

Definition 102 *Let (X, d_X) and (Y, d_Y) be metric spaces. A family $\mathcal{F}$ of functions $f : X \to Y$ is said to be* equicontinuous at a point $x_0 \in X$ *if for every $\varepsilon > 0$ there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that*

$$d_Y(f(x), f(x_0)) \le \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x \in X$ with $d(x, x_0) \leq \delta$. The family $\mathcal{F}$ of functions $f : X \to Y$ is said to be (uniformly) equicontinuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \leq \delta$.

Definition 103 *Let (X, d) be a metric space. A family $\mathcal{F}$ of functions $f : X \to \mathbb{R}$ is said to be* pointwise bounded *if for every $x \in X$ there exists $M_x > 0$ such that*

$$|f(x)| \leq M_x$$

for all $f \in \mathcal{F}$.

Example 104 *The sequence of functions*

$$f_n(x) = x^n, \quad x \in [0, 1],$$

is pointwise bounded but not equicontinuous at $x = 1$. To see this, fix $0 < \varepsilon < 1$. We want to find $\delta > 0$ such that $1 - x^n \leq \varepsilon$ for all $1 - \delta \leq x < 1$. We have $(1 - \varepsilon)^{1/n} \leq x$. So for each n the best δ is $1 - \delta_n = (1 - \varepsilon)^{1/n}$, that is, $\delta_n = 1 - (1 - \varepsilon)^{1/n} \to 0$ as $n \to \infty$. Hence, no δ works for all n.

Example 105 *Consider two metric spaces (X, d_X) and (Y, d_Y) and a family $\mathcal{F}$ of functions from X into Y. If there exist $\alpha \in (0, 1]$ if there exists $L > 0$ such that*

$$d_Y(f(x_1), f(x_2)) \leq L(d_X(x_1, x_2))^a$$

for all $x_1, x_2 \in X$ and for all $f \in \mathcal{F}$, then the family $\mathcal{F}$ is equicontinuous. The sequence of functions

$$f_n(x) = \frac{x^n}{n}, \quad x \in [0, 1],$$

is pointwise bounded and equicontinuous at $x = 1$. Indeed,

$$f_n'(x) = x^{n-1}, \quad x \in [0, 1],$$

so that $\max_{x \in [0,1]} \left|x^{n-1}\right| = 1$, which shows that the sequence $\{f_n\}$ is equi-Lipschitz (take $L = 1$). Hence, it is equicontinuous.

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Theorem 106 (Ascoli–Arzelà) *Let (X, d) be a separable metric space and let $\mathcal{F} \subset C(X)$ be a family of functions. Assume that $\mathcal{F}$ is pointwise bounded and equicontinuous. Then every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of X to a continuous function $g : X \to \mathbb{R}$.*

Proof. Without loss of generality, we may assume that $\mathcal{F}$ has infinite many elements, otherwise there is nothing to prove. Since X is separable, there exists a countable set $E \subset X$ such that $X = \overline{E}$.
Step 1: Let $\mathcal{G} \subset \mathcal{F}$ be an infinite set. We claim that $\mathcal{G}$ contains a sequence $\{f_n\}$ such that the limit $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$ for all $x \in E$. The proof makes use of the *Cantor diagonal argument.* Write $E = \{x_k\}_k$. Since the set

$$\{f(x_1) : f \in \mathcal{G}\}$$

is bounded in $\mathbb{R}$, by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,1}\}_n \subset \mathcal{G}$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,1}(x_1) = \ell_1 \in \mathbb{R}.$$

Since the set

$$\{f_{n,1}(x_2) : n \in \mathbb{N}\}$$

is bounded in $\mathbb{R}$, again by the Bolzano–Weierstrass theorem we can find a sequence $\{f_{n,2}\}_n \subset \{f_{n,1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,2}(x_2) = \ell_2 \in \mathbb{R}.$$

By induction for every $k \in \mathbb{N}$, $k > 1$, we can find a subsequence $\{f_{n,k}\}_n \subset \{f_{n,k-1}\}_n$ for which there exists the limit

$$\lim_{n\to\infty} f_{n,k}(x_k) = \ell_k \in \mathbb{R}.$$

We now consider the diagonal elements of the infinite matrix, that is, the sequence $\{f_{n,n}\}_n$. For every fixed $x_k \in E$ we have that the sequence $\{f_{n,n}(x_k)\}_{n=k}^{\infty}$ is a subsequence of $\{f_{n,k}(x_k)\}_n$, and thus it converges to ℓ_k as $n \to \infty$. This completes the proof of the claim. Set $f_n := f_{n,n}$ and define $g : E \to \mathbb{R}$ by

$$g(x) := \lim_{n\to\infty} f_n(x) \in \mathbb{R}, \quad x \in E. \tag{13}$$

Step 2: Let $K \subset X$ be compact and fix $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon \tag{14}$$

for all $f \in \mathcal{F}$ and for all $x, y \in X$ with $d(x, y) \le \delta$. Since K is compact, we may cover it with a finite number of balls $B\left(y_1, \frac{\delta}{2}\right), \ldots, B\left(y_M, \frac{\delta}{2}\right)$. Since E is dense, for every $i = 1, \ldots, M$ there exists $z_i \in B\left(y_i, \frac{\delta}{2}\right) \cap E$. Using (13), we have that there exists an integer $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(z_i) - f_m(z_i)| \leq \varepsilon \tag{15}$$

for all $i = 1, \ldots, M$ and for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$. Fix $x \in K$. Then x belongs to $B\left(y_i, \frac{\delta}{2}\right)$ for some i. In particular,

$$d(x, z_i) \leq d(x, y_i) + d(y, z_i) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Using (14) and (15), we have that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(z_i)| + |f_n(z_i) - f_m(z_i)| + |f_m(z_i) - f_m(x)| \leq \varepsilon + \varepsilon + \varepsilon$$

for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$, which shows that the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, there exists

$$g(x) := \lim_{n \to \infty} f_n(x) \in \mathbb{R}.$$

Moreover, since

$$|f_n(x) - f_m(x)| \leq 3\varepsilon$$

for all $x \in K$ and all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$, letting $m \to \infty$, we conclude that

$$|f_n(x) - g(x)| \leq 3\varepsilon$$

for all $x \in K$ and all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, or, equivalently,

$$\sup_{x \in K} |f_n(x) - g(x)| \leq 3\varepsilon$$

for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, which shows that $\{f_n\}$ converges to g uniformly on K. In turn, g restricted to K is continuous.
Step 3: Is g defined everywhere? Yes, for every $x \in X$, take K to be the singleton $\{x\}$. Is g continuous? Yes, this follows from (14). ■

Corollary 107 *Let (X, d) be a compact metric space. Then $\mathcal{F} \subset C(X)$ is relatively compact if and only if is bounded and equicontinuous. In particular, $\mathcal{F} \subset C(X)$ is compact if and only if is closed, bounded, and equicontinuous.*

Proof. If $\mathcal{F}$ is bounded and equicontinuous, then by the previous theorem it follows that the closure of $\mathcal{F}$ is sequentially compact, and so by Theorem 82 the closure of $\mathcal{F}$ is compact. Conversely, assume that $\mathcal{F} \subset C(X)$ is relatively compact. Then by Proposition 87, the closure of $\mathcal{F}$ is bounded. It remains to show that $\mathcal{F}$ is equicontinuous. Assume, by contradiction, that this is not the case. Then there exist $\varepsilon > 0$, $\{f_n\} \subset \mathcal{F}$, and $\{x_n\}, \{y_n\} \subset X$ such that

$$|f_n(x_n) - f_n(y_n)| > \varepsilon$$

and $d(x_n, y_n) \leq \frac{1}{n}$. Since X is compact (and so sequentially compact), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0 \in X$ such that $d(x_{n_k}, x) \to 0$ as $k \to \infty$. In turn, since $\{f_{n_k}\} \subset \mathcal{F}$, again by Theorem 82, there exist a subsequence $\left\{f_{n_{k_j}}\right\}$ of $\{f_{n_k}\}$ and $f_0 \in C(X)$ such that $d_{C(X)}\left(f_{n_{k_j}}, f_0\right) \to 0$ as $j \to \infty$. By your homework, $f_{n_{k_j}}\left(x_{n_{k_j}}\right) \to f(x)$ and $f_{n_{k_j}}\left(y_{n_{k_j}}\right) \to f(x)$. Hence,

$$\varepsilon < \left|f_{n_{k_j}}\left(x_{n_{k_j}}\right) - f_{n_{k_j}}\left(y_{n_{k_j}}\right)\right| \to |f(x) - f(x)| = 0,$$

which is a contradiction. ■

Theorem 108 *Let $1 \leq p < \infty$. Then $\mathcal{F} \subset L^p(\mathbb{R}^N)$ is relatively compact if and only if $\mathcal{F}$ is bounded in $L^p(\mathbb{R}^N)$ and for every $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that*

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p \, dx \leq \varepsilon$$

for all $f \in \mathcal{F}$ and $h \in \mathbb{R}^N$ with $|h| < \delta$ and

$$\int_{\mathbb{R}^N \setminus B(0,R)} |f(x)|^p \, dx \leq \varepsilon$$

for all $f \in \mathcal{F}$.

1.11 The Stone–Weierstrass Theorem

Another important application of compactness is given by the Stone–Weierstrass theorem. We begin with some results that are of interest in themselves.

Theorem 109 (Dini) *Let (X, d) be a compact metric space and let $\{f_n\} \subset C(X)$ be a sequence of functions such that*

$$f_n(x) \leq f_{n+1}(x) \tag{16}$$

for every $x \in X$ and for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} f_n(x) =: f(x) \in \mathbb{R}. \tag{17}$$

If $f : X \to \mathbb{R}$ is continuous, then the convergence is uniform.

Proof. Fix $\varepsilon > 0$. By (17) for every $x \in X$ there exists $n_x \in \mathbb{N}$ such that

$$0 \le f(x) - f_n(x) \le \varepsilon$$

for all $n \ge n_x$. Since the functions f and f_{n_x} are continuous at x, there exists $B(x, r_x)$ such that

$$0 \le f(y) - f_{n_x}(y) \le 3\varepsilon$$

for all $y \in B(x, r_x)$. By (16), we have that

$$0 \le f(y) - f_n(y) \le 3\varepsilon \tag{18}$$

for all $y \in B(x, r_x)$ and for all $n \ge n_x$. By the compactness of X we may find a finite number of balls that cover X, say $B(x_1, r_{x_1})$, $\ldots$, $B(x_m, r_{x_m})$. Let $n_\varepsilon := \max\{n_{x_1}, \ldots, n_{x_m}\}$. Then for all $y \in X$ and $n \ge n_\varepsilon$ we have that y is contained in one of the balls $B(x_i, r_{x_i})$, $i = 1, \ldots, m$, and so (18) holds. In turn,

$$\max_{y\in X} |f(y) - f_n(y)| \le 3\varepsilon$$

for all $n \ge n_\varepsilon$, which gives the desired result. ∎

Example 110 *If f is not assumed to be continuous, then the result is false. Take*

$$f_n(x) = -x^n, \quad x \in [0,1].$$

Then $f_n \le f_{n+1}$ but the convergence is not uniform.

Similarly, if X is not compact, then the result fails. Take $X = \mathbb{R}$ and

$$f_n(x) = \begin{cases} 1 & \text{if } x \le n, \\ 1 + n - x & \text{if } n < x < n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \le f_{n+1}$, $f_n(x) \to f(x) = 1$, which is a continuous function, but there is no uniform convergence.

Exercise 111 *Let $f(x) := \sqrt{x}$, $x \in [0,1]$. For $n \in \mathbb{N}$ and $x \in [0,1]$ define recursively*

$$p_0(x) := 0, \quad p_n(x) := p_{n-1}(x) + \frac{1}{2}\left[x - p_{n-1}^2(x)\right].$$

(i) Prove that each p_n is a polynomial.

(ii) Prove that $|p_n(x)| \le 1$ and for all $n \in \mathbb{N}$ and $x \in [0,1]$.

(iii) Prove that $\{p_n\}$ *converges uniformly to* f *in* $[0,1]$.

Theorem 112 (Stone) *Let* (X,d) *be a compact metric space and let* $\mathcal{F} \subset C(X)$ *be a family of functions such that*

(i) $\mathcal{F}$ *separates points; that is, if* $x, y \in X$ *with* $x \neq y$, *then there exists* $f \in \mathcal{F}$ *such that* $f(x) \neq f(y)$,

(ii) $\mathcal{F}$ *contains the constant functions,*

(iii) $\mathcal{F}$ *is an algebra; that is, if* $f, g \in \mathcal{F}$ *and* $t \in \mathbb{R}$, *then* $f+g$, fg, *and* tf *belong to* $\mathcal{F}$.

Then $\mathcal{F}$ *is dense in* $C(X)$.

Proof. Step 1: We claim that $\overline{\mathcal{F}}$ satisfies properties (i)-(iii). We only need to prove property (iii). Given $f, g \in \overline{\mathcal{F}}$ and $t \in \mathbb{R}$, by Remark 30 there exist $\{f_n\}, \{g_n\} \subset \mathcal{F}$ such that $d_\infty(f_n, f) \to 0$ and $d_\infty(g_n, f) \to 0$. By property (iii), $f_n + g_n$, $f_n g_n$, and tf_n belong to $\mathcal{F}$. Since $d_\infty(f_n + g_n, f+g) \to 0$, $d_\infty(f_n g_n, fg) \to 0$, and $d_\infty(tf_n, tf) \to 0$ (exercise), it follows again by Remark 30, that $f+g$, fg, and tf belong to $\overline{\mathcal{F}}$. It remains to show that $\overline{\mathcal{F}}. = C(X)$. ■

Proof. Step 2: We prove that if f belongs to $\overline{\mathcal{F}}$, then so does $|f|$. Since X is compact, by the Weierstrass theorem f is bounded by some constant $M > 0$. Define

$$g(x) := \frac{|f(x)|}{M}, \quad x \in X.$$

Then $g(x) \in [0,1]$. In view of (iii), it suffices to show that g belongs to $\overline{\mathcal{F}}$. By the previous exercise there exists a sequence of polynomials $\{p_n\}$ that converges uniformly in $[0,1]$ to the function $h(t) := \sqrt{t}$, $t \in [0,1]$. Define

$$g_n(x) := p_n\left(\left(\frac{f(x)}{M}\right)^2\right), \quad x \in X.$$

Then g_n converges uniformly in X to the function $\sqrt{\left(\frac{f}{M}\right)^2} = g$. Since $\overline{\mathcal{F}}$ is an algebra, we have that $g_n \in \mathcal{F}$. Hence, using the fact that $\overline{\mathcal{F}}$ is closed, it follows that g belongs to $\overline{\mathcal{F}}$.

Step 3: We prove that if f, g belong to $\overline{\mathcal{F}}$, then so do $\max\{f,g\}$ and $\min\{f,g\}$. It is enough to observe that

$$\max\{f,g\} = \frac{1}{2}[f + g + |f - g|],$$
$$\min\{f,g\} = \frac{1}{2}[f + g - |f - g|].$$

Step 4: We prove that if $x, y \in X$ with $x \neq y$ and $\alpha, \beta \in \mathbb{R}$, then there exists $g \in \overline{\mathcal{F}}$ such that $g(x) = \alpha$ and $g(y) = \beta$. To see this, use property (i) to find $f \in \overline{\mathcal{F}}$ such that $f(x) \neq f(y)$ and define

$$g(z) := \frac{\alpha(f(z) - f(y)) - \beta(f(x) - f(z))}{(f(x) - f(y))}, \quad z \in X.$$

Step 5: We are now ready to prove that $\overline{\mathcal{F}} = C(X)$. Let $f \in C(X)$ and $\varepsilon > 0$. By the previous step, for every $x, y \in X$ there exists a function $g_{x,y} \in \overline{\mathcal{F}}$ such that $g_{x,y}(x) = f(x)$ and $g_{x,y}(y) = f(y)$. Define

$$U_{x,y} := \{z \in X : g_{x,y}(z) < f(z) + \varepsilon\},$$
$$V_{x,y} := \{z \in X : g_{x,y}(z) > f(z) - \varepsilon\}.$$

By the continuity of $g_{x,y}$ and f we have that $U_{x,y}$ and $V_{x,y}$ are open sets containing x and y. Since $\{U_{x,y}\}_{x \in X}$ is an open cover of X, it follows by compactness that there exist $x_1, \ldots, x_{m_y} \in X$ such that

$$\bigcup_{i=1}^{m_y} U_{x_i,y} = X. \tag{19}$$

Define

$$g_y := \min\left\{g_{x_1,y}, \ldots, g_{x_{m_y},y}\right\}.$$

Then g_y belongs to $\overline{\mathcal{F}}$ by Step 3 and by (19) and the definition of $U_{x_i,y}$ and $V_{x_i,y}$,

$$g_y(z) < f(z) + \varepsilon \text{ for all } z \in X, \tag{20}$$

$$g_y(z) > f(z) - \varepsilon \text{ for all } z \in V_y := \bigcap_{n=1}^{m_y} V_{x_i,y}. \tag{21}$$

Since V_y is open and contains y, the family $\{V_y\}_{y\in X}$ is an open cover of X. Again by compactness, there exist $y_1, \ldots, y_n \in X$ such that

$$\bigcup_{i=1}^{n} V_{y_i} = X.$$

Define

$$g := \max\{g_{y_1}, \ldots, g_{y_n}\}.$$

Then g belongs to $\overline{\mathcal{F}}$ by Step 3 and by (20) and (21),

$$f(z) - \varepsilon < g(z) < f(z) + \varepsilon \text{ for all } z \in X.$$

Hence $\max_{z\in X}|f(z) - g(z)| \le \varepsilon$. Since $g \in \overline{\mathcal{F}}$, we may find $h \in \mathcal{F}$ such that $\max_{z\in X}|h(z) - g(z)| \le \varepsilon$, and thus, by the triangle inequality, $\max_{z\in X}|f(z) - h(z)| \le 2\varepsilon$. This concludes the proof. ■

Exercise 113 (Weierstrass) *Let $K \subset \mathbb{R}^N$ be a compact set. Prove that every continuous function $f : K \to \mathbb{R}$ is the uniform limit in K of a sequence of polynomials.*

Corollary 114 *Let (X, d) be a compact metric space. Then $C(X)$ is separable.*

Proof. Since X is separable by Exercise 85, there exists a sequence $\{x_n\} \subset X$ such that $\overline{\{x_n\}} = X$. For every n define

$$f_n(x) := d(x, x_n), \quad x \in X.$$

Then f_n is continuous. We claim that $\{f_n\}$ separates points. Indeed, assume the contrary. Then there exist $x, y \in X$ such that $f_n(x) = f_n(y)$ for every $n \in \mathbb{N}$. By density we may find a subsequence $\{x_{n_k}\} \subset X$ of $\{x_n\}$ such that $x_{n_k} \to x$. Hence,

$$d(y, x_{n_k}) = f_{n_k}(y) = f_{n_k}(x) = d(x, x_{n_k}) \to 0$$

as $k \to \infty$. Thus, $x_{n_k} \to y$. By the uniqueness of limits, it follows that $x = y$. This proves the claim.

Define $f_0 := 1$ and for every $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}_0$ define

$$f_{n_1,\ldots,n_k}(x) := f_{n_1}(x) \cdots f_{n_k}(x), \quad x \in X.$$

Consider the family $\mathcal{F}$ given by all finite linear combinations of functions of the form $f_{n_1,\ldots,n_k}$. Then $\mathcal{F}$ satisfies the hypotheses of Stone's theorem, and so $\mathcal{F}$ is dense in $C(X)$. On the other hand, the family $\mathcal{F}'$ given by all finite rational linear combinations of functions of the form $f_{n_1,\ldots,n_k}$ is countable. For every $C(X)$ and $\varepsilon > 0$ we may find $g \in \mathcal{F}$ such that

$$d_\infty(f, g) \leq \varepsilon.$$

Since g is a finite linear combinations of functions of the form $f_{n_1,\ldots,n_k}$, using the density of the rationals in the real, we may find $h \in\in \mathcal{F}$ such that

$$d_\infty(h, g) \leq \varepsilon.$$

This shows that $\mathcal{F}'$ is dense in $C(X)$ and, in turn, that $C(X)$ is separable. ■

Exercise 115 *Prove that $C_b(\mathbb{R})$ is not separable.*

Wednesday, October 21, 2009

2 Topological Spaces

Definition 116 *Let X be a nonempty set. A collection $\tau \subset \mathcal{P}(X)$ is a* topology *if the following hold.*

(i) $\emptyset, X \in \tau$.

(ii) If $U_i \in \tau$ *for* $i = 1, \ldots, M$*, then* $U_1 \cap \ldots \cap U_M \in \tau$.

(iii) If $\{U_\alpha\}_{\alpha \in \Lambda}$ *is an arbitrary collection of elements of* τ*, then* $\bigcup_{\alpha \in \Lambda} U_\alpha \in \tau$.

The pair (X, τ) is called a *topological space* and the elements of τ are *open sets*. For simplicity, we often apply the term topological space only to X.

Example 117 *(i) Given a nonempty set X, the smallest topology consists of $\{\emptyset, X\}$, while the largest topology contains all subsets as open sets.*

(ii) Given a metric space (X, d) the family of open sets is a topology.

(iii) Given a topological space (X, τ_X) and an onto function $f : X \to Y$, the family of sets

$$\tau_Y := \left\{E \subset Y : f^{-1}(E) \in \tau_X\right\}$$

is a topology on Y. It is called the quotient topology *(relative to f and τ_X). We now consider an important special case. Given a nonempty set X, let $\sim$ be an equivalence relation. We define*

$$Y = X \diagup \sim := \{[x] : x \in X\}$$

and consider the projection of X onto Y

$$\begin{aligned} P : X &\to Y \\ x &\mapsto [x] \end{aligned}$$

If X is a topological space with topology τ, we can consider in Y the quotient topology (relative to P and τ), precisely,

$$\tau_Y := \left\{E \subset Y : P^{-1}(E) \in \tau_X\right\}.$$

Note that

$$P^{-1}(E) = \{x \in X : [x] \in E\} = \bigcup_{[x] \in E} [x],$$

that $P^{-1}(E)$ is is given by the union of the equivalence classes belonging to E. Thus, an open set in the quotient topology is a collection of equivalence classes whose union is an open set of X.

As Example 117(i) shows, a set X can have more than one topology. Note that if τ is any topology on the set X, then the inclusions

$$\{\emptyset, X\} \subset \tau \subset \mathcal{P}(X)$$

hold. If τ_1 and τ_2 are two topologies on X, we say that τ_1 is *weaker*, or *coarser*, than τ_1 if $\tau_1 \subset \tau_2$.

Remark 118 *Given a family of topologies $\{\tau_\alpha\}_{\alpha\in\Lambda}$ on a set X, the family of sets*

$$\bigcap_{\alpha\in\Lambda} \tau_\alpha := \{U \subset X : U \in \tau_\alpha \text{ for every } \alpha \in \Lambda\}$$

is still a topology on X, while in general $\bigcup_{\alpha\in\Lambda} \tau_\alpha$ is not.

In view of the previous remark, we can construct topologies starting from any family of subsets of X.

Proposition 119 *Let X be a set and let $\mathcal{F}$ be a family of subsets of X. Then there exists a unique, smallest topology τ containing $\mathcal{F}$. Moreover, τ consists of X, $\emptyset$, finite intersections of elements of $\mathcal{F}$ and arbitrary unions of finite intersections of elements of $\mathcal{F}$.*

The family $\mathcal{F}$ is called a *subbase* for τ and τ is said to be *generated* by $\mathcal{F}$.
Proof. Let $\{\tau_\alpha\}_{\alpha\in\Lambda}$ be the family of all topologies that contain $\mathcal{F}$. Note that this family is nonempty since $\mathcal{P}(X)$ is one such topology. Then

$$\tau := \bigcap_{\alpha\in\Lambda} \tau_\alpha$$

is still a topology, contains $\mathcal{F}$, and it is the smallest such topology. Moreover, it is unique in view of its definition. By the properties of a topology, we have that τ contains X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$ and all arbitrary unions of finite intersections of elements of $\mathcal{F}$. On the other hand, let τ' be the family that consists of X, $\emptyset$, all finite intersections of elements of $\mathcal{F}$ and arbitrary unions of finite intersections of elements of $\mathcal{F}$. Then τ' is a topology. ■

Example 120 *(i) In $\mathbb{R}$ we can consider the family $\mathcal{F} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$. The smallest topology containing $\mathcal{F}$ is the standard one.*

(ii) Given two topological spaces (X, τ_X) and (Y, τ_Y), by considering the family $\mathcal{F} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$, the smallest topology containing $\mathcal{F}$ on $X \times Y$ is called the product topology*.*

Given a point $x \in X$, a *neighborhood*[3] of x is an open set $U \in \tau$ that contains x. Given a set $E \subset X$, a *neighborhood* of E is an open set $U \in \tau$ that contains E.

[3] In some texts the definition of neighborhood is different.

Given a topological space (X, τ) and a set $E \subset X$, a point $x \in E$ is called an *interior point* of E if there exists a neighborhood U of x such that $U \subset E$. The *interior* E° of a set $E \subset X$ is the union of all its interior points.

The proof of following proposition is left as an exercise.

Proposition 121 *Let (X, τ) be a topological space and let $E \subset X$. Then*

(i) *E° is an open subset of E,*

(ii) *E° is given by the union of all open subsets contained in E; that is, E° is the largest (in the sense of union) open set contained in E,*

(iii) *E is open if and only if $E = E^\circ$,*

(iv) *$(E^\circ)^\circ = E^\circ$.*

We now introduce the notion of a base for a topology. Let (X, τ) be a topological space. A family β of open sets of X is a *base* for the topology τ if every open set $U \in \tau$ may be written as the union of elements of β.

Friday, October 23, 2009

Given a topological space (X,τ) and a point $x\in X$, a family β_x of neighborhoods of x is a *local base at* x if every neighborhood of x contains an element of β_x.

Proposition 122 *Let X be a nonempty set and let $\beta\subset\mathcal{P}(X)$ be a family of sets. Then β is a base for a topology τ if and only if*

(i) it contains the empty set;

(ii) for every $x\in X$ there exists $B\in\beta$ such that $x\in B$,

(iii) for every $B_1,B_2\in\beta$ with $B_1\cap B_2\neq\emptyset$ and for every $x\in B_1\cap B_2$ there exists $B_3\in\beta$ such that $x\in B_3$ and $B_3\subset B_1\cap B_2$.

Proof. Assume that τ is a topology and that β is a base for τ. Then every open set is a union of sets of β. In particular, the empty set and X can be written as union of sets of β, and so (i) and (ii) hold. To prove (iii), note that if $B_1,B_2\in\beta$, then $B_1\cap B_2$ is open, and so it can be written as union of elements of β, say

$$B_1\cap B_2=\bigcup_\gamma B_\gamma$$

Hence, if $x\in B_1\cap B_2$, then there is a B_γ such that $x\in B_\gamma\subset B_1\cap B_2$.

Conversely, let $\beta=\{B_\alpha\}_{\alpha\in\Lambda}$ be a family of sets satisfying (i)-(iii) and let τ be given by arbitrary unions of elements of β. We need to show that τ is a topology. By property (i), we have that the empty set belongs to τ, while by property (ii) we have that

$$X=\bigcup_{\alpha\in\Lambda}B_\alpha,$$

and so $X\in\tau$.

If $U_i\in\tau$ for $i=1,\dots,M$, then we can write

$$U_i=\bigcup_{\alpha\in\Lambda_i}B_\alpha$$

for some $\Lambda_i\subset\Lambda$. Then

$$U_1\cap\ldots\cap U_M=\bigcap_{i=1}^M\bigcup_{\alpha\in\Lambda_i}B_\alpha.$$

If $x\in U_1\cap\ldots\cap U_M$, then there exist $\alpha_i\in\Lambda_i$ such that $x\in B_{\alpha_1}\cap\ldots\cap B_{\alpha_M}$. By property (iii) and an induction argument we may find $B_x\in\beta$ such that $x\in B_x$ and $B_x\subset B_{\alpha_1}\cap\ldots\cap B_{\alpha_M}$. Hence

$$U_1\cap\ldots\cap U_M=\bigcup_{x\in U_1\cap\ldots\cap U_{Mi}}B_x\in\tau.$$

Finally, given an arbitrary collection $\{U_\gamma\}_{\gamma\in\Xi}$ of elements of τ, since each U_γ is a union of elements of β, we have that $\bigcup_{\gamma\in\Xi} U_\gamma$ is a union of elements of β, and so it belongs to τ.

Thus, τ is a topology. The fact that β is a base for τ follows from the definition of τ. ■

Example 123 *We give some examples of bases.*

(i) Given a metric space (X, d), consider the family

$$\beta = \{B(x, r) : x \in X,\, r > 0\}.$$

Then properties (i) and (ii) are satisfied, and so by the previous proposition we obtain a topological space (X, τ) by taking as open sets arbitrary unions of open balls.

(ii) Given a pseudometric space (X, ρ), we can construct a topology τ on X by taking for every $x \in X$ the local base of all pseudoballs centered at x and of radius $r > 0$,

$$\{y \in X : \rho(x, y) < r\}.$$

(iii) Given a nonempty set X, we recall that $\mathbb{R}^X$ denotes the space of all functions $f : X \to \mathbb{R}$. We construct a local base for a topology. For every $f \in \mathbb{R}^X$ let $r > 0$ and let $Y \subset X$ be a finite subset. Consider

$$B(f; r; Y) := \left\{g \in \mathbb{R}^X : |g(x) - f(x)| < r \text{ for all } x \in Y\right\}.$$

The family $\beta = \left\{B(f; r; Y) : f \in \mathbb{R}^X,\, r > 0,\, Y \subset X \text{ finite}\right\}$ is a base for a topology τ on $\mathbb{R}^X$. See Example 14.

(iv) Given a set X and a family $\mathcal{F}$ of subsets of X, let τ be the smallest topology τ containing $\mathcal{F}$. Then a basis for τ is given by finite intersections of elements of $\mathcal{F}$.

Exercise 124 *Let (X, ρ) be a pseudometric space. Let $Y = X/\sim$ be as in Exercise 15.*

(i) Prove that a set $U \subset X$ is open (with respect to the topology τ_X) if and only if the set $P(U)$ is open (with respect to the metric d).

(ii) Prove that the topology induced by the metric d is the quotient topology.

Exercise 125 *For every $(x, y) \in \mathbb{R}^2$ consider the family of rectangles $[x, x + r) \times [y, y + t]$, where $r, t > 0$. Consider the family*

$$\beta = \left\{[x, x + r) \times [y, y + t] : (x, y) \in \mathbb{R}^2,\, r, t > 0\right\}.$$

Prove that β is a base for a topology τ on $\mathbb{R}^2$ that is not the canonical topology on $\mathbb{R}^2$.

Exercise 126 *Given a nonempty set X, consider two metrics $d_1 : X \times X \to [0, \infty)$ and $d_2 : X \times X \to [0, \infty)$. What are the relations, if any, among the following properties?*

(i) d_1 and d_2 are equivalent*, that is, for every sequence $\{x_n\} \subset X$ and every $x \in X$,*

$$\lim_{n\to\infty} d_1(x_n, x) = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} d_2(x_n, x) = 0.$$

(ii) For every $x \in X$ and $r > 0$ there exist $r_1, r_2 > 0$ such that

$$B_{d_2}(x, r_1) \subset B_{d_1}(x, r) \subset B_{d_2}(x, r_2).$$

(iii) d_1 and d_2 generate the same topology.

A set $C \subset X$ is *closed* if its complement $X \setminus C$ is open. The next proposition follows from De Morgan's law.

Proposition 127 *Let (X, τ) be a topological space. Then*

(i) $\emptyset$ and X are closed.

(ii) If $C_i \subset X$, $i = 1, \ldots, n$, is a finite family of closed sets of X, then $C_1 \cup \cdots \cap U_n$ is closed.

(iii) If $\{C_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary collection of closed sets of X, then $\bigcap_{\alpha\in\Lambda} C_\alpha$ is closed.

Let (X, τ) be a topological space. The *closure* $\overline{E}$ *of a set* $E \subset X$ is the smallest closed set that contains E. A subset E of a topological space X is said to be *dense* if its closure is the entire space, i.e., $\overline{E} = X$. We say that a topological space is *separable* if it contains a countable dense subset.

Proposition 128 *Let (X, τ) be a topological space and let $E \subset X$. Then $x \in \overline{E}$ if and only if $E \cap U$ is nonempty for every neighborhood U of x.*

Proposition 129 *Let (X, τ) be a topological space and let $\{E_\alpha\}_\alpha$ be an arbitrary family of subsets of X. Then*

$$\bigcup_\alpha \overline{U_\alpha} \subset \overline{\bigcup_\alpha U_\alpha}$$

and equality holds if the family $\{E_\alpha\}_\alpha$ is finite.

The proof of following proposition is left as an exercise.

Proposition 130 *Let (X, τ) be a topological space and let $C \subset X$. Then C is closed if and only if $C = \overline{C}$.*

Monday, October 26, 2009

A point $x_0 \in X$ is an *accumulation point* for a set $E \subset X$ if for every open set U that contains x_0 there exists $x \in E \cap U$, with $x \neq x_0$. The set of accumulation points of E is denoted $\operatorname{acc} E$. Exactly as in the case of metric spaces we have the following proposition.

Proposition 131 *Let (X, τ) be a topological space and let $E \subset X$. Then*

$$\overline{E} = E \cup \operatorname{acc} E.$$

In particular, a set $C \subset X$ is closed if and only if C contains all its accumulation points.

2.1 Inadequacy of Sequences

A topological space (X, τ) is a *Hausdorff space* if for any $x, y \in X$ with $x \neq y$ we may find two disjoint neighborhoods of x and y.

Proposition 132 *Let (X, d) be a metric space and let τ be the topology determined by d. Then (X, τ) is a Hausdorff space.*

Proof. If $x \neq y$, then $B\left(x, \frac{d(x,y)}{2}\right)$ and $B\left(y, \frac{d(x,y)}{2}\right)$ are disjoint neighborhoods of x and y, respectively. ■

Given a topological space (X, τ) and a sequence $\{x_n\}$, we say that $\{x_n\}$ *converges* to a point $x \in X$ if for every neighborhood U of x we have that $x_n \in U$ for all n sufficiently large. Note that unless the space is Hausdorff, the limit may not be unique.

Proposition 133 (Uniqueness of limits) *Let (X, τ) be a Hausdorff space. If $\{x_n\}$ converges to x and to y, then $x = y$.*

Proof. Assume that $x \neq y$. Then there exist two disjoint neighborhoods U and V of x and y, respectively. Since $\{x_n\}$ converges to x we have that $x_n \in U$ for all n sufficiently large. This implies that $x_n \notin V$ for all n sufficiently large, and so $\{x_n\}$ cannot converge to y. ■

Given a topological space (X, τ), a subset $C \subset X$ is *sequentially closed* if for every sequence $\{x_n\} \subset C$ such that $\{x_n\}$ converges to some $x \in X$, then x belongs to C.

Proposition 134 *Let (X, τ) be a topological space and let $C \subset X$ be a closed set. Then C is sequentially closed.*

Proof. Assume that C is closed and let $\{x_n\} \subset C$ be such that $\{x_n\}$ converges to some $x \in X$. We need to show that x belongs to C. If not, then $x \in X \setminus C$. Since $X \setminus C$ is open, there exists a neighborhood U of x such that $U \subset X \setminus C$. But then $x_n \in U$ for all n sufficiently large, which implies that $x_n \in X \setminus C$ for all n large. This contradicts the fact that $\{x_n\} \subset C$. ■

Given a topological space (X, τ) and a set $E \subset X$, the *sequential closure* of E is the set

$$\overline{E}^{\text{seq}} := \{x \in X : \text{ there exists } \{x_n\} \subset E \text{ converging to } x\}.$$

Proposition 135 *Let (X, τ) be a topological space and let $E \subset X$ set. Then*

$$E \subset \overline{E}^{\text{seq}} \subset \overline{E}.$$

Proof. Since $\overline{E}$ is closed, by the previous proposition it is sequentially closed. Hence, $\overline{E}^{\text{seq}} \subset \overline{E}$. ■

The next example shows that strict inclusion in $\overline{E}^{\text{seq}} \subset \overline{E}$ is possible.

Exercise 136 *Let $\left(\mathbb{R}^{[0,1]}, \tau\right)$ be as in Example 123 and consider the subset E of $\mathbb{R}^{[0,1]}$ that consists of all functions f that take value zero at a finite number of $x \in [0, 1]$ and that take value 1 otherwise.*

(i) Prove that the function $f_0 \equiv 0$ belongs to $\overline{E}$.

(ii) Prove that $f_0 \notin \overline{E}^{\text{seq}}$.

(iii) Prove that there is no metric on $\mathbb{R}^{[0,1]}$ compatible with the topology τ.

Definition 137 *Let (X, τ) be a topological space.*

(i) The space X satisfies the first axiom of countability *if every $x \in X$ admits a countable local base.*

(ii) The space X satisfies the second axiom of countability *if it has a countable base.*

Example 138 *Consider an uncountable set X with the discrete topology τ. Then X satisfies the first axiom of countability, since for every $x \in X$, the singleton $\{x\}$ is a local base. However, X does not have a countable base, since singletons belong to τ and are uncountable.*

Example 139 *A metric space (X, d) satisfies the satisfies the first axiom of countability (take as a local base at $x \in X$ the family of balls $\left\{B\left(x, \frac{1}{n}\right)\right\}_{n \in \mathbb{N}}$, but not necessarily the second (see the next exercise).*

Exercise 140 *Let (X, τ) be a topological space satisfying the second axiom of countability. Prove that X is separable.*

Proposition 141 *Let (X, τ) be a topological space satisfying the first axiom of countability and let $E \subset X$. Then*

$$\overline{E}^{\text{seq}} = \overline{E}.$$

Proof. In view of Proposition 135, it remains to show that $\overline{E}^{\text{seq}} \supset \overline{E}$. Let $x \in \overline{E}$ and let $\{B_n\}_n$ be a local base at x. By replacing B_n with $B_1 \cap \cdots \cap B_n$, we may assume that $\{B_n\}_n$ is a decreasing sequence. By Proposition 128 we have that $E \cap B_n$ is nonempty, and so there exists $x_n \in E \cap B_n$. We claim that the sequence $\{x_n\}$ converges to x. Indeed, let U be a neighborhood of x. Since $\{B_n\}_n$ is a local base at x, there exists $\overline{n} \in \mathbb{N}$ such that $B_{\overline{n}} \subset U$. Using the fact that $\{B_n\}_n$ is a decreasing sequence, we have that

$$x_n \in B_n \subset B_{\overline{n}} \subset U$$

for all $n \geq \overline{n}$, which shows that $\{x_n\}$ converges to x. ■

2.2 Limit, Limsup, and Liminf

Definition 142 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : E \to Y$, where $E \subset Y$. Given $x_0 \in \operatorname{acc} E$, if there exists $y_0 \in Y$ with the property that for every neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 such that*

$$f(x) \in V$$

for all $x \in U \cap (E \setminus \{x_0\})$, we write

$$y_0 = \lim_{x \to x_0} f(x)$$

and we say that y_0 is the limit *of f as x approaches x_0.*

Note that x_0 need not belong to E.

Remark 143 *In applications it is enough to take V in a local base of y and U in a local base of x_0.*

Proposition 144 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces with Y a Hausdorff space and let $f : E \to Y$, where $E \subset Y$. Given $x_0 \in \operatorname{acc} E$, if there exist $\lim_{x \to x_0} f(x) = y_1$ and $\lim_{x \to x_0} f(x) = y_2$, then $y_1 = y_2$.*

Proof. Assume by contradiction that there $y_1 \neq y_2$. Then there exist two disjoint neighborhoods V_1 and V_2 of y_1 and y_2, respectively. In turn, by the definition of limit, there exist two neighborhoods U_1 and U_2 of x_0 such that $f(x) \in V_1$ for all $x \in U_1 \cap (E \setminus \{x_0\})$ and $f(x) \in V_2$ for all $x \in U_2 \cap (E \setminus \{x_0\})$. Since $x_0 \in \operatorname{acc} E$, it follows by Proposition 128 that there exists $x \in U_1 \cap U_2 \cap (E \setminus \{x_0\})$. But then $f(x) \in V_1 \cap V_2$, which contradicts the fact that V_1 and V_2 are disjoint. ■

We have seen that for metric spaces, when working with limits, it is enough to work with sequences. For topological spaces, this is no longer true, although we have one implication.

Proposition 145 *Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in \operatorname{acc} E$, if there exists $\lim_{x \to x_0} f(x) = y_0$, then $f(x_n) \to y_0$ for every sequence $\{x_n\} \subset E \setminus \{x_0\}$ that converges that to x_0.*

Proof. Let $\{x_n\} \subset E \setminus \{x_0\}$ converge to x_0. Given a neighborhood $V \subset Y$ of y_0, there exists a neighborhood $U \subset X$ of x_0 such that

$$f(x) \in V$$

for all $x \in U \cap (E \setminus \{x_0\})$. Since $x_n \to x_0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_\varepsilon$ and so

$$f(x_n) \in V$$

for all $n \geq n_\varepsilon$, which shows that $f(x_n) \to f(x_0)$. ■

The converse is false in general.

Proposition 146 *Let (X, τ_X) be a topological space satisfying the first axiom of countability, let (Y, τ_Y) be a topological space, and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in E \cap \operatorname{acc} E$, if there exists $y_0 \in Y$ such that $f(x_n) \to y_0$ for every sequence $\{x_n\} \subset E \setminus \{x_0\}$ that converges that to x_0, then there exists $\lim_{x \to x_0} f(x) = y_0$.*

Proof. We claim that $\lim_{x \to x_0} f(x) = y_0$. If not, then there exists a neighborhood $V \subset Y$ of y_0 such that for every neighborhood $U \subset X$ of x_0 there exists $x \in U \cap (E \setminus \{x_0\})$ such that $f(x) \notin V$. Since (X, τ) satisfies the first axiom of countability, there exists a countable local base $\{B_n\}_n$ at x_0. By selecting a subsequence, we may assume that $B_{n+1} \subset B_n$ for every $n \in \mathbb{N}$. Then for every n we may find $x_n \in B_n \cap (E \setminus \{x_0\})$ such that $f(x_n) \notin V$. Since $\{B_n\}_n$ is a decreasing local base at x_0, the sequence $\{x_n\}$ converges to x_0. By hypothesis, $f(x_n) \to y_0$, and so $f(x_n) \in V$ for all n sufficiently large, which is a contradiction. ■

In the special case in which $Y = \mathbb{R}$ or $Y = [-\infty, \infty]$ all the standard theorems about the sum, product, quotient of limits continue to hold with the standard modifications. We omit the details.

Definition 147 *Let (X, τ) be a topological space, let $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and let $x_0 \in \operatorname{acc} E$. The* limit inferior *of f as x tends to x_0 is defined as*

$$\liminf_{x \to x_0} f(x) := \sup_{U \in \tau(x_0)} \inf_{x \in U \cap (E \setminus \{x_0\})} f(x),$$

while the limit superior *of f as x tends to x_0 is defined as*

$$\limsup_{x \to x_0} f(x) := \inf_{U \in \tau(x_0)} \sup_{x \in U \cap (E \setminus \{x_0\})} f(x),$$

where $\tau(x_0)$ stands for the collection of all neighborhoods of x_0.[4]

Remark 148 *If U and V are two neighborhoods of x_0 with $U \subset V$, then*

$$\inf_{x \in V \cap (E \setminus \{x_0\})} f(x) \leq \inf_{x \in U \cap (E \setminus \{x_0\})} f(x),$$

and since we are interested in the supremum over all neighborhoods of x_0, we can neglect V. Thus, we can focus on "small" neighborhoods of x_0 (this is the analogous of $\varepsilon > 0$ small in the standard definition of limits). In particular, we could replace $\tau(x_0)$ with a local base at x_0 in the definition of $\liminf_{x \to x_0}$. A similar reasoning works for $\limsup_{x \to x_0}$. This is why in a metric space we only consider balls.

[4] In several books, $\sup_{A \in \tau(x_0)} \inf_{x \in A \cap E} f(x)$ is used as a definition for the limit inferior $\liminf_{x \to x_0} f(x)$. The definition we use here is in accordance with the definition of limit, in which the value of the function at the point x_0 plays no role. In particular, with our definition we recover the fact that the limit exists at x_0 if and only if the limit inferior and superior coincide, while with the other definition one would get that the limit inferior and superior at x_0 coincide if and only if the function f is continuous at f_0.

Theorem 149 *Let* (X,τ) *be a topological space, let* $f : E \to \mathbb{R}$*, where* $E \subset \mathbb{R}$*, and let* $x_0 \in \operatorname{acc} E$*. Then*

$$\liminf_{x\to x_0} f(x) \le \limsup_{x\to x_0} f(x). \tag{22}$$

Moreover there exists $\lim_{x\to x_0} f(x)$ *if and only if equality holds in (22), and in this case the limit coincides with the common value in (22).*

Proof. Let U and V be two neighborhoods of x_0. Then

$$\begin{aligned}\inf_{x\in U\cap(E\setminus\{x_0\})} f(x) &\le \inf_{x\in U\cap V\cap(E\setminus\{x_0\})} f(x) \\ &\le \sup_{x\in U\cap V\cap(E\setminus\{x_0\})} f(x) \le \sup_{x\in V\cap(E\setminus\{x_0\})} f(x).\end{aligned}$$

Taking the supremum over all $U \in \tau(x_0)$ gives

$$\liminf_{x\to x_0} f(x) \le \sup_{x\in V\cap(E\setminus\{x_0\})} f(x).$$

Taking the infimum over all $V \in \tau(x_0)$ gives (22). ■

Proof. To prove the second part of the theorem assume that there exists $\lim_{x\to x_0} f(x) = \ell$. I will consider only the case $\ell \in \mathbb{R}$ and leave the cases $\ell = \infty$ and $\ell = -\infty$ as an exercise. By the definition of limit, for every $\varepsilon > 0$ there exists a neighborhood U_ε of x_0 such that

$$\ell - \varepsilon \le f(x) \le \ell + \varepsilon$$

for all $x \in U_\varepsilon \cap (E \setminus \{x_0\})$. Hence,

$$\ell - \varepsilon \le \inf_{x\in U_\varepsilon \cap (E\setminus\{x_0\})} f(x) \le \sup_{U\in\tau(x_0)} \inf_{x\in U\cap(E\setminus\{x_0\})} f(x) = \liminf_{x\to x_0} f(x)$$

$$\limsup_{x\to x_0} f(x) \le \inf_{U\in\tau(x_0)} \sup_{x\in U\cap(E\setminus\{x_0\})} f(x) \le \sup_{x\in U_\varepsilon\cap(E\setminus\{x_0\})} f(x) \le \ell + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = \ell.$$

Conversely, assume that

$$\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = L$$

for some $L \in [-\infty, \infty]$. Again we consider the case $L \in \mathbb{R}$ and leave the cases $L = \infty$ and $L = -\infty$ as an exercise. Fix $\varepsilon > 0$. By the definition of $\liminf_{x\to x_0} f(x)$ and $\limsup_{x\to x_0} f(x)$ there exist two neighborhoods U_ε and V_ε of x_0 such that

$$L - \varepsilon \le \inf_{x\in U_\varepsilon\cap(E\setminus\{x_0\})} f(x),$$

$$\sup_{x\in V_\varepsilon\cap(E\setminus\{x_0\})} f(x) \le L + \varepsilon.$$

Taking $U := U_\varepsilon \cap V_\varepsilon$, we have that for all $x \in U \cap (E \setminus \{x_0\})$,

$$L - \varepsilon \le f(x) \le L + \varepsilon,$$

which shows that there exists $\lim_{x\to x_0} f(x) = L$. ■

Exercise 150 *Let (X, τ) be a topological space, let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$, where $E \subset \mathbb{R}$, and let $x_0 \in \operatorname{acc} E$. Assume that one of the two functions is bounded. Prove that*

$$\begin{aligned}\liminf_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x) &\le \liminf_{x\to x_0} (f(x) + g(x)) \le \limsup_{x\to x_0} f(x) + \liminf_{x\to x_0} g(x)\\ &\le \limsup_{x\to x_0} (f(x) + g(x)) \le \limsup_{x\to x_0} f(x) + \limsup_{x\to x_0} g(x)\end{aligned}$$

and that in general all inequalities may be strict. Prove that if there exists $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$, *then*

$$\liminf_{x \to x_0} f(x) + \liminf_{x \to x_0} g(x) = \liminf_{x \to x_0} (f(x) + g(x)),$$
$$\limsup_{x \to x_0} (f(x) + g(x)) = \limsup_{x \to x_0} f(x) + \limsup_{x \to x_0} g(x).$$

The proof of the following corollary is left as an exercise.

Corollary 151 (Cauchy) *Let* (X, τ) *be a topological space, let* $f : E \to \mathbb{R}$, *where* $E \subset \mathbb{R}$, *and let* $x_0 \in \operatorname{acc} E$. *Then a necessary and sufficient condition for* $\lim_{x \to x_0} f(x)$ *to exist in* $\mathbb{R}$ *is that for every* $\varepsilon > 0$ *there exists a neighborhood* U_ε *of* x_0 *such that*

$$|f(x_1) - f(x_2)| \le \varepsilon$$

for all $x_1, x_2 \in U_\varepsilon \cap (E \setminus \{x_0\})$.

2.3 Continuity

Definition 152 *Let* (X, τ_X) *and* (Y, τ_Y) *be two topological spaces, and let* $f : E \to Y$, *where* $E \subset Y$. *Given* $x_0 \in E$, *the function* f *is said to be* continuous at x_0 *if for every neighborhood* $V \subset Y$ *of* $f(x_0)$, *there exists a neighborhood* $U \subset X$ *of* x_0 *such that*

$$f(x) \in V$$

for all $x \in U \cap E$. *The function* f *is said to be* continuous *if it is continuous at every point of* E.

Remark 153 *Note that if* $x_0 \in E$ *is an* isolated point *of* E*; that is, if there exists a neighborhood* $U \subset X$ *of* x_0 *such that* $U_0 \cap E = \{x_0\}$, *then* f *is continuous at* x_0 *(take* $U = U_0$*). Thus, it is enough to check the continuity of a function* $f : E \to Y$ *at points* $x_0 \in E \cap \operatorname{acc} E$, *in which case continuity at* x_0 *is equivalent to saying that there exists the limit*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

The space of all continuous functions $f : X \to Y$ is denoted by $C(X; Y)$. If $Y = \mathbb{R}$ we write $C(X)$. If Y is a metric space, the space of all continuous bounded functions $f : X \to Y$ is denoted $C_b(X, Y)$. It is a metric space with the distance

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

The proof of the next two propositions follows from Propositions 145 and 146.

Proposition 154 *Let* (X, τ_X) *and* (Y, τ_Y) *be two topological spaces and let* $f : E \to Y$, *where* $E \subset X$. *Given* $x_0 \in E \cap \operatorname{acc} E$, *if* f *is continuous at* x_0, *then* f *is* sequentially continuous *at* x_0*; that is,* $f(x_n) \to f(x_0)$ *for every sequence* $\{x_n\} \subset E$ *that converges that to* x_0.

Proposition 155 *Let (X, τ_X) be a topological space satisfying the first axiom of countability, let (Y, τ_Y) be a topological space, and let $f : E \to Y$, where $E \subset X$. Given $x_0 \in E \cap \operatorname{acc} E$, if f is sequentially continuous at x_0, then f is continuous at x_0.*

Exercise 156 *Consider two topological spaces (X, τ_X) and (Y, τ_Y) and a function $f : X \to Y$. Prove that the following are equivalent.*

(i) *f is continuous.*

(ii) *$f^{-1}(U)$ is open for every open set $U \subset Y$.*

(iii) *$f^{-1}(C)$ is closed for every closed set $C \subset Y$.*

(iv) *$f^{-1}(B)$ is open for every set $B \subset Y$ in a base (or subbase) of τ_Y.*

Exercise 157 *Consider three topological spaces (X, τ_X), (Y, τ_Y), and (Z, τ_Z) and two continuous functions $f : X \to Y$ and $g : Y \to Z$. Prove that $g \circ f : X \to Z$ is continuous.*

Definition 158 *Given two topological spaces (X, τ_X) and (Y, τ_Y), a function $f : X \to Y$ is said to be* homeomorphism *if f is one-to-one, onto, and continuous and $f^{-1} : Y \to X$ is continuous. In this case, the topological spaces (X, τ_X) and (Y, τ_Y) are said to be* homeomorphic.

Given a topological space (X, τ), a *topological property* is a property that if possessed by X, is possessed by all spaces homeomorphic to X.

Definition 159 *Given two topological spaces (X, τ_X) and (Y, τ_Y), a function $f : X \to Y$ is said* open *if $f(U)$ is open for every open set $U \subset X$. Similarly, $f : X \to Y$ is said* closed *if $f(C)$ is closed for every closed set $C \subset X$.*

Thus, if $f : X \to Y$ is f is one-to-one and onto, then f is a homeomorphism if and only if f is continuous and open (or continuous and closed).

2.4 Lower Semicontinuity

Definition 160 *Let (X, τ) be a topological space and let $f : E \to \mathbb{R}$ be a function, where $E \subset X$.*

(i) *The function f is said to be* lower semicontinuous *at a point $x_0 \in E$ if either x_0 is an isolated point or $x_0 \in \operatorname{acc} E$ and*

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

The function f is said to be lower semicontinuous *if it is lower semicontinuous at every point of E.*

(ii) *The function f is said to be* upper semicontinuous *at a point $x_0 \in E$ if $-f$ is lower semicontinuous at x_0. The f is said to be* upper semicontinuous *if it is upper semicontinuous at every point of E.*

We recall $x_0 \in E$ is an *isolated point* of E if there exists a neighborhood $U \subset X$ of x_0 such that $U_0 \cap E = \{x_0\}$.

Example 161 *Given the function*

$$f(x) = \left\{ \begin{array}{ll} \sin \frac{1}{x} & \text{if } x \neq 0, \\ a & \text{if } x = 0, \end{array} \right\}$$

we have that

$$\liminf_{x \to 0} f(x) = -1 \quad \textit{and} \quad \limsup_{x \to 0} f(x) = 1.$$

Thus f is lower semicontinuous at 0 if and only if $a \leq -1$, f is upper semicontinuous at 0 if and only if $a \geq 1$, while if $-1 < a < 1$, then f is neither lower nor upper semicontinuous at 0.

The following characterizations of lower semicontinuity will be of use in the sequel. We recall the definition of epigraph and of limit inferior.

Definition 162 *Let (X, τ) be a topological space and let $f : E \to \mathbb{R}$, where $E \subset \mathbb{R}$. The* epigraph *of f is the set*

$$\operatorname{epi} f := \{(x, t) \in E \times \mathbb{R} : f(x) \leq t\}.$$

Proposition 163 *Let X be a topological space and let $f : X \to \mathbb{R}$. Then the following are equivalent:*

(i) *$\{x \in X : f(x) \leq t\}$ is closed for every $t \in \mathbb{R}$;*

(ii) *$\operatorname{epi} f$ is closed;*

(iii) *f is lower semicontinuous.*

Proof. Step 1: We prove that (i) is equivalent to (ii). Assume that (i) holds and let

$$D := (X \times \mathbb{R}) \setminus \operatorname{epi} f = \{(x, t) \in X \times \mathbb{R} : f(x) > t\}. \tag{23}$$

Fix $(x_0, t_0) \in D$ and let $0 < \varepsilon < f(x_0) - t_0$. Then the set

$$U := f^{-1}((t_0 + \varepsilon, \infty])$$

is open and contains x_0, and so $U \times (t_0 - \varepsilon, t_0 + \varepsilon)$ is a neighborhood of (x_0, t_0). If $(x, t) \in U \times (t_0 - \varepsilon, t_0 + \varepsilon)$ then $f(x) > t_0 + \varepsilon > t$, which implies that $(x, t) \in D$. Hence D is open and its complement, $\operatorname{epi} f$, is closed.

Conversely, assume that $\operatorname{epi} f$ is closed (and so the set D defined in (23) is open), and for $t_0 \in \mathbb{R}$ consider the set

$$U := \{x \in X : f(x) > t_0\}.$$

If $x_0 \in U$ then for any fixed $0 < \varepsilon_0 < f(x_0) - t_0$ the pair $(x_0, t_0 + \varepsilon_0)$ belongs to the open set D, and so we may find a neighborhood $U_0 \subset X$ of x_0 and $0 < \varepsilon \leq \varepsilon_0$ such that

$$U_0 \times (t_0 + \varepsilon_0 - \varepsilon, t_0 + \varepsilon_0 + \varepsilon) \subset D.$$

In particular, if $x \in U_0$ then $f(x) > t_0 + \varepsilon_0 - \varepsilon > t_0$, and so $U_0 \subset U$, which shows that U is open and in turn proves (i)

Step 2: We show that (i) is equivalent to (iii). Let f satisfy (i) and assume by contradiction that there exists $x_0 \in X$ such that

$$f(x_0) > \liminf_{x \to x_0} f(x).$$

Fix $f(x_0) > t > \liminf_{x \to x_0} f(x)$. Then the open set

$$U_t := \{x \in X : f(x) > t\} \tag{24}$$

belongs to $\tau(x_0)$, and so

$$\liminf_{x \to x_0} f(x) = \sup_{U \in \tau(x_0)} \inf_{x \in U \setminus \{x_0\}} f(x) \geq \inf_{x \in U_t \setminus \{x_0\}} f(x) \geq t.$$

which is a contradiction.

Conversely, assume that (iii) holds and fix $t \in \mathbb{R}$. If the set U_t defined in (24) is nonempty, fix any x_0 that belongs to it. Since

$$\liminf_{x \to x_0} f(x) \geq f(x_0) > t,$$

by the definition of limit inferior there exists $U \in \tau(x_0)$ such that

$$\inf_{x \in U \setminus \{x_0\}} f(x) > t,$$

which, together with the fact that $f(x_0) > t$, implies that $U \subset U_t$. Hence (i) holds. ■

Monday, November 2, 2009

2.5 Compactness

Definition 164 *Let (X, τ) be a topological space.*

(i) A set $K \subset X$ is compact *if for every open cover of K, i.e., for every collection $\{U_\alpha\}$ of elements of τ such that $\bigcup_\alpha U_\alpha \supset K$, there exists a finite subcover (i.e., a finite subcollection of $\{U_\alpha\}$ whose union still contains K).*

(ii) A set $K \subset X$ is sequentially compact *if for every sequence $\{x_n\}$ of elements of K, there exists a subsequence converging to an element of K.*

(iii) A set $E \subset X$ is relatively compact *(or* precompact*) if its closure $\overline{E}$ is compact.*

Remark 165 *Consider a nonempty set X and let τ_1 and τ_2 be two topologies on X with $\tau_1 \subset \tau_2$. If $K \subset X$ is compact with respect to τ_2, then $K \subset X$ is compact with respect to τ_1. The less open sets we have, the easier it becomes for a set to be compact. This remark will be important in Functional Analysis and it is related to Remark 95.*

Proposition 166 *Let (X, τ) be a Hausdorff space, let $K \subset X$ be a compact set and let $x_0 \in X \setminus K$. Then there exist two disjoint open sets U and V such that U contains K and $x_0 \in V$.*

Proof. Since X is Hausdorff, for every $x \in K$ there is a neighborhood U_x of x and a neighborhood V_x of x_0 such that $U_x \cap V_x = \emptyset$. Since $\{U_x\}_{x \in X}$ is an open cover of K, by the compactness of K there exist $x_1, \ldots, x_m \in K$ such that

$$U := U_{x_1} \cup \cdots \cup U_{x_m} \supset K.$$

The open set

$$V := V_{x_1} \cap \cdots \cap V_{x_m}$$

is a neighborhood of x_0 and does not intersect U. ■

Proposition 167 *Let (X, τ) be a topological space and let $K \subset X$ be a compact set.*

(i) If $C \subset K$ is closed, then C is compact.

(ii) If X is a Hausdorff space, then K is closed.

Proof. The proof of (i) is the same as the proof of Proposition 87. It remains to prove (ii). By the previous proposition for every $x_0 \in X \setminus K$ there exists a neighborhood U of x_0 that does not intersect K. This shows that x_0 is an interior point of $X \setminus K$ and, in turn, that $X \setminus K$ is open. ■

Note that if the topology is not Hausdorff then compact sets may not be closed.

Example 168 *Given a nonempty set X endowed with the smallest topology, any nonempty set strictly contained in X is compact but not closed.*

We have seen that for metric spaces, compactness and sequential compactness are the same. For topological spaces, these two notions are not related.

Exercise 169 *Consider the space $[0,1]^{[0,1]}$ of all functions $f : [0,1] \to [0,1]$ with the topology introduced in Example 123. We will see later that this space is compact (see Tychonoff's theorem). Prove that $[0,1]^{[0,1]}$ is not sequentially compact.*

Wednesday, November 4, 2009

Exercise 170 *Let X be the family of all nonempty countable subsets of $\mathbb{R}$. Given $E, F \in X$, define*

$$\rho(E,F) := \begin{cases} \min\{1, \operatorname{dist}(E \setminus F, F)\} & \text{if } F \subsetneqq E, \\ 0 & \text{if } F = E, \\ 1 & \text{otherwise.} \end{cases}$$

(i) Prove that (X, ρ) is a pseudometric space.

(ii) Let $\{E_n\} \subset X$ be a sequence such that $E_n \supset \mathbb{Q}$ for every $n \in \mathbb{N}$. Prove that $\{E_n\}$ converges to

$$E := \bigcup_{n=1}^{\infty} E_n.$$

(iii) Prove that X is sequentially compact.

(iv) For every $E \in X$, prove that the set $\mathcal{P}(E) \setminus \{\emptyset\} \subset X$ is open in the topology determined by ρ.

(v) Prove that X is not compact.

Exercise 171 *Let X be the first uncountable ordinal with its well-ordering. Define*

$$\rho(x,y) := \begin{cases} 0 & \text{if } x \geq y, \\ 1 & \text{if } x < y. \end{cases}$$

(i) Prove that (X, ρ) is a pseudometric space.

(ii) Prove that X is sequentially compact. Hint: prove that every sequence contains a monotone subsequence.

(iii) Prove that X is not compact.

Proposition 172 *Let (X, τ_X) be a topological space and let $K \subset X$ be a compact set. Then every infinite set of K has an accumulation point.*

Proof. Let $E \subset K$ be an infinite set and assume by contradiction that E has no accumulation points. Then E is closed. Moreover, for every $x \in E$ there exists a neighborhood U_x of x such that $U_x \cap E = \{x\}$. The family of sets $\{U_x\}_{x \in E} \cup \{X \setminus E\}$ covers the compact set K. Hence, there exist $x_1, \ldots, x_m \in E$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} \cup (X \setminus E) \supset K \supset E.$$

Since $X \setminus E$ does not intersect E and each U_{x_i} intersects E only at the point x_i, it follows that E cannot have more than m elements, which is a contradiction. ■

Proposition 173 *Let (X, τ_X) be a topological space satisfying the first axiom of countability and let $K \subset X$ be closed and compact. Then K is sequentially compact. In particular, if X is also Hausdorff, then every compact set is sequentially compact.*

Proof. Consider a sequence $\{x_n\} \subset K$ and $E := \{x_n : n \in \mathbb{N}\}$. By the previous proposition, either the set E is finite, in which case one element is repeated infinitely many times (so we have a convergent subsequence) or it has an accumulation point x_0. Since (X, τ) satisfies the first axiom of countability, there exists a countable local base $\{B_k\}_k$ at x_0. By selecting a subsequence, we may assume that $B_{k+1} \subset B_k$ for every $k \in \mathbb{N}$. Since x_0 is ana accumulation point of E, for every k we may find $n_k \in \mathbb{N}$ such that $x_{n_k} \in B_k \cap E$. Since $\{B_k\}_k$ is a decreasing local base at x_0, the sequence $\{x_{n_k}\}$ converges to x_0. Since K is closed and $E \subset K$, it follows that $x_0 \in \operatorname{acc} E \subset \overline{E} \subset \overline{K} = K$, which shows that $x_0 \in X$. ■

We now show that Weierstrass's theorem continues to hold for lower semicontinuous functions. This fact alone explains the importance of this class of functions in minimization problems.

Let (X, τ) be a topological space, let $E \subset X$ and let $f : E \to \mathbb{R}$. The function f is said to be sequentially lower semicontinuous at some $x_0 \in E$ if

$$\liminf_{n \to \infty} f(x_n) \geq f(x_0)$$

for every sequence $\{x_n\} \subset E$ such that $x_n \to x_0$ as $n \to \infty$.

Theorem 174 (Weierstrass) *Let (X, τ) be a topological space, let $K \subset X$ be compact (respectively sequentially compact), and let $f : X \to \mathbb{R}$ be a lower semicontinuous (respectively sequentially lower semicontinuous, that is) function. Then there exists $x_0 \in K$ such that*

$$f(x_0) = \min_{x \in K} f(x).$$

Proof. The proof is the same of the ones we gave for Weierstrass theorem in metric spaces. ■

Theorem 175 *The Cartesian product of two compact topological spaces is a compact topological space.*

Proof. Let (X, τ_X) and (Y, τ_Y) be two compact topological spaces. Assume by contradiction that $X \times Y$ is not compact. Then there is an open cover $\mathcal{W}$ of $X \times Y$ with the property that no finite subfamily of $\mathcal{W}$ covers $X \times Y$.
Step 1: We claim that there is $x_0 \in X$ such that for every neighborhood U of x_0, no finite subfamily of $\mathcal{W}$ covers $U \times Y$. If this is not the case, then for all $x \in X$ there is a neighborhood U_x of x and a finite subfamily of $\mathcal{W}$ that covers $U_x \times Y$. Since $\{U_x\}_{x \in X}$ is an open cover of X, by the compactness of X there exist $x_1, \ldots, x_m$ such that

$$U_{x_1} \cup \cdots \cup U_{x_m} = X.$$

For each x_i, $i = 1, \ldots, m$, find a finite subfamily $\mathcal{W}_i$ of $\mathcal{W}$ that covers $U_{x_i} \times Y$. Then the subfamily

$$\{W : W \in \mathcal{W}_i \text{ for some } i = 1, \ldots, m\}$$

is finite and covers $X \times Y$, which is a contradiction. This proves the claim. ■

Proof. Step 2: We claim that there is $y_0 \in Y$ such that for every neighborhood $U \times V$ of (x_0, y_0), no finite subfamily of $\mathcal{W}$ covers $U \times V$. If this is not the case, then for all $y \in Y$ there is a neighborhood $U_y \times V_y$ of (x_0, y) and a finite subfamily of $\mathcal{W}$ that covers $U_y \times V_y$. Since $\{V_y\}_{y \in Y}$ is an open cover of Y, by the compactness of Y there exist $y_1, \ldots, y_\ell$ such that

$$V_{y_1} \cup \cdots \cup V_{y_\ell} = Y.$$

For each y_i, $i = 1, \ldots, \ell$, find a finite subfamily $\mathcal{W}_i'$ of $\mathcal{W}$ that covers $U_{y_i} \times V_{y_i}$. Then the subfamily

$$\{W : W \in \mathcal{W}_i' \text{ for some } i = 1, \ldots, m\}$$

is finite and covers $(U_{y_1} \cap \cdots \cap U_{y_\ell}) \times Y$, which contradicts Step 1. This proves the claim.

Step 3: Let $x_0 \in X$ and $y_0 \in Y$ be given as in Steps 1 and 2, respectively. Since $\mathcal{W}$ is an open cover of $X \times Y$, there exists $W \in \mathcal{W}$ such that $(x_0, y_0) \in W$. But then we can find neighborhoods U and V of x_0 and y_0 such that $U \times V \subset W$, which contradicts Step 2. This completes the proof. ■

If X is a topological space, we denote by $C_c(X)$ the space of all continuous functions $f : X \to \mathbb{R}$ whose support is compact.

2.6 Product Topology

Given a collection $\{X_\alpha\}_{\alpha \in \Lambda}$ of sets, we define the product of the collection as

$$\prod_{\alpha \in \Lambda} X_\alpha := \left\{ f : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha : f(\alpha) \in X_\alpha \text{ for every } a \in \Lambda \right\}.$$

Remark 176 *Note that, if the collection if finite, then $\prod_{\alpha \in \Lambda} X_\alpha$ reduces to the usual Cartesian product.*

On the other hand, if $X_\alpha = X$ for all $\alpha \in \Lambda$, then $\prod_{\alpha \in \Lambda} X_\alpha$ is simply the space of all functions $f : \Lambda \to X$. It is sometimes written X^Λ.

For each $\beta \in \Lambda$ there is a projection onto X_β given by

$$\pi_\beta : \prod_{\alpha \in \Lambda} X_\alpha \to X_\beta$$
$$f \mapsto f(\beta)$$

If each X_α is endowed with a topology τ_α, we define the *product topology* on $\prod_{\alpha \in \Lambda} X_\alpha$ as the smallest topology that makes each projection continuous. More precisely, since π_α is continuous if and only if $\pi_\alpha^{-1}(V_\alpha)$ is open for every open set V_α of X_α, τ is the smallest topology that contains the family

$$\mathcal{F} = \left\{ \pi_\alpha^{-1}(V_\alpha) : V_\alpha \in \tau_\alpha, \alpha \in \Lambda \right\}. \tag{25}$$

Proposition 177 *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ *be a collection of topological spaces. Then a base for the product topology is given by all sets of the form*

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha),$$

where Λ_0 *is a finite subset of* Λ *and* V_α *is an open set of* X_α*, or, equivalently, by all sets of the form*

$$\prod_{\alpha\in\Lambda} V_\alpha,$$

where V_α *is an open set of* X_α *and* $V_\alpha = X_\alpha$ *for all but finitely many* $\alpha \in \Lambda$*.*

Proof. In view of Proposition 119 and of Example 123(iii), a base for the topology τ is given by finite intersections of elements of $\mathcal{F}$, precisely,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha)$$

where Λ_0 is a finite subset of Λ and V_α is an open set of X_α. Now, for $\beta \in \Lambda_0$,

$$\begin{aligned}\pi_\beta^{-1}(V_\beta) &= \left\{ f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha : \; f(\alpha) \in X_\alpha \text{ for every } a \in \Lambda \setminus \{\beta\} \text{ and } f(\beta) \in V_\beta \right\} \\ &= \prod_{\alpha\in\Lambda,\, \alpha<\beta} X_\alpha \times V_\beta \times \prod_{\alpha\in\Lambda\; \alpha>\beta} X_\alpha.\end{aligned}$$

Hence,

$$\bigcap_{\alpha\in\Lambda_0} \pi_\alpha^{-1}(V_\alpha) = \prod_{\alpha\in\Lambda} V_\alpha,$$

where $V_\alpha := X_\alpha$ for all $a \in \Lambda \setminus \Lambda_0$. ■

Exercise 178 *Given a nonempty set* X*, consider the space* $X^{\mathbb{R}}$ *of all functions* $f : X \to \mathbb{R}$*. What is the relation between the topology introduced in Example 123 and the product topology?*

Why this funny topology? Why not take the smallest topology that contains all sets of the form $\prod_{\alpha\in\Lambda} V_\alpha$, where V_α is an open set of X_α? This topology is called the *box topology*. If Λ is finite, then the product topology and the box topology coincide. However, if Λ is infinite, then the box topology has too many open sets, and so a lot of properties that are true for the product topology fails for the box topology (compactness, connectedness, continuity of the projections, etc.).

Exercise 179 *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ *be a collection of topological spaces. Prove that for each* $\beta \in \Lambda$ *the projection* $\pi_\beta : \prod_{\alpha\in\Lambda} X_\alpha \to X_\beta$ *is continuous and open but not closed.*

Theorem 180 *Let* (X,τ) *and* (X_α,τ_α), $\alpha\in\Lambda$, *be topological spaces and let*

$$f : X \to \prod_{\alpha\in\Lambda} X_\alpha.$$

Then f *is continuous if and only if* $\pi_\beta \circ f : X \to X_\beta$ *is continuous for every* $\beta\in\Lambda$.

Example 181 *Consider the set* $\mathbb{R}^{\mathbb{N}} = \{g : \mathbb{N}\to\mathbb{R}\}$ *and the function*

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^{\mathbb{N}} \\ &x \mapsto (x, x, \ldots) \end{aligned}$$

For every $n\in\mathbb{R}^{\mathbb{N}}$ *we have that* $\pi_n \circ f : \mathbb{R}\to\mathbb{R}$ *is the function* $\pi_n \circ f(x) = x$, *which is continuous. Hence,* f *is continuous with respect to the product topology. However,* f *is not continuous with respect to the box topology. Indeed, consider the open set*

$$\begin{aligned} B &= (-1,1)\times\left(-\frac{1}{2},\frac{1}{2}\right)\times\cdots\times\left(-\frac{1}{n},\frac{1}{n}\right)\cdots \\ &= \prod_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right). \end{aligned}$$

If f *were continuous, then the set* $f^{-1}(B)$ *would be open. Note that* $0\in f^{-1}(B)$, *and so* $f^{-1}(B)$ *should contain an interval* $(-\delta,\delta)$ *for some* $\delta > 0$. *Hence,* $f((-\delta,\delta))\subset B$, *but then taking projections, we get*

$$(\pi_n \circ f)((-\delta,\delta)) = (-\delta,\delta) \subset \left(-\frac{1}{n},\frac{1}{n}\right)$$

for all $n\in\mathbb{N}$, *which is a contradiction.*

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Proof of Theorem 180. Let f be continuous. Since π_β is continuous by Exercise 179, it follows that $\pi_\beta \circ f$ is continuous. Conversely, assume that each $\pi_\beta \circ f$ is continuous.

Since a subbase for the product topology is given by all sets of the form $\pi_\alpha^{-1}(V_\alpha)$, where V_α is an open set of X_α, by Exercise 156, it suffices to show that $f^{-1}\left(\pi_\alpha^{-1}(V_\alpha)\right)$ is open in X. But

$$f^{-1}\left(\pi_\alpha^{-1}(V_\alpha)\right) = (\pi_\alpha \circ f)^{-1}((V_\alpha)),$$

which is open in X, since $\pi_\alpha \circ f$ is continuous, again by Exercise 156. Hence, f is continuous. ■

Exercise 182 *Let (X, d) and (X_α, d_α), $\alpha \in \Lambda$, be metric spaces and let*

$$f : X \to \prod_{\alpha\in\Lambda} X_\alpha.$$

Prove f is continuous if and only if that $\pi_\alpha \circ f$ is continuous for every $\alpha \in \Lambda$ and for every $x \in X$ there exists a ball $B(x, r)$ such that $\pi_\alpha \circ f : B(x, r) \to X_\alpha$ is constant for all but finitely many $\alpha \in \Lambda$.

To prove the next theorem we need transfinite induction. Consider an arbitrary set X and a binary relation $\prec$ on X. We call $(X, \prec)$ a *linear ordering* if

(i) $\prec$ is *transitive* on X:

$$\text{for all } x, y, z \in X, \text{ if } x \prec y \text{ and } y \prec z \text{ then } x \prec z;$$

(ii) *trichotomy* holds:

$$\text{for all } x, y \in X \text{ we have that } x \prec y \text{ or } y \prec x \text{ or } x = y;$$

(iii) $\prec$ is *irreflexive*:

$$\text{for all } x \in X \text{ the property } x \prec x \text{ does not hold.}$$

We say that $x \preccurlyeq y$ if $x \prec y$ or $x = y$.

We say that $(X, \prec)$ is a *well-ordering* if it is a linear ordering and

for every $E \subset X$ with $E \neq \emptyset$ there exists $x \in E$
such that $x \preccurlyeq y$ for all $y \in E$,

The element x is called the $\prec$-*least* element of E. The usual proof by induction may be extended to well-orderings in the following way.

Proposition 183 (Proofs by induction) *Let $(X, \prec)$ be a well-ordering. Let $P(x)$ be a statement about a variable x. Suppose that for all $y \in X$,*

$$\textit{if } P(x) \textit{ holds for all } x \prec y \textit{ then } P(y) \textit{ holds.}$$

If $P(x_0)$ is true for the $\prec$-least element x_0 of X, then $P(y)$ holds for all $y \in X$.

Another fundamental assumption of mathematics is that for every set X, there exists a binary relation $\prec$ on X such that $(X, \prec)$ is a well-ordering. This is the *axiom of choice* (AC) in set theory.

Theorem 184 (Tychonoff's theorem) *Wright. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of compact topological spaces. Then $\prod_{\alpha\in\Lambda} X_\alpha$ is compact.*

Proof. By the axiom of choice, we may assume that Λ is well-ordered; that is, there is an order relation $\leq$ such that every subset of Λ has a smallest element. Assume by contradiction that $\prod_{\alpha\in\Lambda} X_\alpha$ is not compact. Then there is an open cover $\mathcal{W}$ of $\prod_{\alpha\in\Lambda} X_\alpha$ with the property that no finite subfamily of $\mathcal{W}$ covers $\prod_{\alpha\in\Lambda} X_\alpha$.

We claim that for every $\beta \in \Lambda$ there exists $x_\beta \in X_\beta$ with the property that if W is any open set containing

$$\prod_{\alpha\in\Lambda,\,\alpha\leq\beta} \{x_\alpha\} \times \prod_{\alpha\in\Lambda\,\alpha>\beta} X_\alpha,$$

then no finite subfamily of $\mathcal{W}$ covers W. We use Proposition 183. Let $\alpha_0 \in \Lambda$ be the least element of Λ. Reasoning as in Step 1 of the proof of Theorem 175 (with X replaced by X_{α_0} and Y by $\prod_{\alpha\in\Lambda\,\alpha>\alpha_0} X_\alpha$) we can prove that there exists $x_{\alpha_0} \in X_{\alpha_0}$ such that if W is any open set containing

$$\{x_{\alpha_0}\} \times \prod_{\alpha\in\Lambda\,\alpha>\alpha_0} X_\alpha,$$

then no finite subfamily of $\mathcal{W}$ covers W.

Assume next that $x_\alpha \in X_\alpha$ have been chosen for all $\alpha < \beta$. We claim that there exists $x_\beta \in X_\beta$ with the if W is any open set containing

$$\prod_{\alpha\in\Lambda,\,\alpha\leq\beta} \{x_\alpha\} \times \prod_{\alpha\in\Lambda\,\alpha>\beta} X_\alpha,$$

then no finite subfamily of $\mathcal{W}$ covers W. If this is not the case, then for all $x \in X_\beta$ there is an open set W_x containing

$$\prod_{\alpha\in\Lambda,\,\alpha<\beta} \{x_\alpha\} \times \{x\} \times \prod_{\alpha\in\Lambda\,\alpha>\beta} X_\alpha,$$

and a finite subfamily of $\mathcal{W}$ that covers W_x. Without loss of generality, we may assume that W_x has the form

$$W_x = \prod_{\alpha\in\Lambda,\,\alpha<\beta} U_{\alpha,x} \times U_{\beta,x} \times \prod_{\alpha\in\Lambda\,\alpha>\beta} X_\alpha,$$

where $U_{\alpha,x}$ is open in X_a for every $\alpha \in \Lambda$ with $\alpha \leq \beta$ and $U_{\alpha,x} = X_\alpha$ for all but finitely many $\alpha \in \Lambda$ with $\alpha \leq \beta$. Since $\{U_{\beta,x}\}_{x \in X_\beta}$ is an open cover of X_β, by the compactness of X_β there exist $x_1, \ldots, x_m$ such that

$$U_{\beta,x_1} \cup \cdots \cup U_{\beta,x_m} = X_\beta.$$

For each x_i, $i = 1, \ldots, m$, find a finite subfamily $\mathcal{W}_i$ of $\mathcal{W}$ that covers W_{x_i}. Then the subfamily

$$\{W : W \in \mathcal{W}_i \text{ for some } i = 1, \ldots, m\}$$

is finite and covers

$$\prod_{\alpha \in \Lambda, \, \alpha < \beta} (U_{\alpha,x_1} \cap \cdots \cap U_{\alpha,x_m}) \times X_\beta \times \prod_{\alpha \in \Lambda \, \alpha > \beta} X_\alpha,$$

which is an open set covering

$$\prod_{\alpha \in \Lambda, \, \alpha < \beta} \{x_\alpha\} \times \prod_{\alpha \in \Lambda \, \alpha \geq \beta} X_\alpha.$$

This contradicts the way the points x_α have been chosen. Indeed, since $\bigcap_{i=1}^m U_{\alpha,x_i} = X_\alpha$, for all but finitely many $\alpha \in \Lambda$ with $\alpha \leq \beta$, let α_0 be the maximum $\alpha \in \Lambda$ with $\alpha \leq \beta$ for which $\bigcap_{i=1}^m U_{\alpha,x_i} \neq X_\alpha$. Then we have contradicted the choice of x_{α_0}. This proves the claim. Thus we have constructed a family $\{x_\alpha\}_{\alpha \in \Lambda}$. The function

$$f : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha$$
$$\alpha \mapsto x_\alpha$$

Then no element of $\mathcal{W}$ covers f. Indeed, if $W \in \mathcal{W}$ covers f, then it must contains an open set of the form

$$\prod_{\alpha \in \Lambda} U_g$$

where U_α is a neighborhood of x_α for every $\alpha \in \Lambda$ and $U_{\alpha,x} = X_\alpha$ for all but finitely many $\alpha \in \Lambda$, which contradicts the the way the points x_α have been chosen. ■

Exercise 185 *Prove that* $[0,1]^{\mathbb{N}} = \{f : \mathbb{N} \to [0,1]\}$ *with the box topology is not compact.*

Exercise 186 *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ *be a collection of nonempty topological spaces, and let* $E_\alpha \subset X_\alpha$ *be nonempty for every* $\alpha \in \Lambda$. *Fix* $g \in \prod_{\alpha \in \Lambda} E_\alpha$ *and consider the set*

$$E := \left\{ f \in \prod_{\alpha \in \Lambda} E_\alpha : \, f(\alpha) = g(\alpha) \, \text{ for all but finitely many } \alpha \in \Lambda \right\}.$$

Prove that

$$\overline{E} = \overline{\prod_{\alpha \in \Lambda} E_\alpha}.$$

Exercise 187 *Prove that in* $\mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \to \mathbb{R}\}$ *with the box topology, the set*

$$U := \{f : \mathbb{N} \to \mathbb{R} : \ f \text{ is bounded}\}$$

is both open and closed, so that $\mathbb{R}^{\mathbb{N}}$ *is not connected.*

Exercise 188 *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ *be a collection of topological spaces and let* $E_\alpha \subset X_\alpha$ *be nonempty sets for every* $\alpha \in \Lambda$.

(i) Prove that

$$\overline{\prod_{\alpha \in \Lambda} E_\alpha} = \prod_{\alpha \in \Lambda} \overline{E_\alpha}.$$

(ii) Is it true that

$$\left(\prod_{\alpha \in \Lambda} E_\alpha\right)^\circ = \prod_{\alpha \in \Lambda} (E_\alpha)^\circ?$$

(iii) Prove that $\prod_{\alpha \in \Lambda} E_\alpha$ *is closed if and only if* E_α *is closed for every* $\alpha \in \Lambda$.

Exercise 189 *Let* $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$ *be a collection of Hausdorff topological spaces. Prove that* $\prod_{\alpha \in \Lambda} X_\alpha$ *is a Hausdorff space.*

Example 190 $2^{[0,1]}$ *with the product topology is compact but not sequentially compact.*

2.7 Compactification

In studying a noncompact topological space X it is often useful to construct a space that contains X and that is compact. The extended real numbers are such an example.

The simplest type of compactification is given by adding one point to X.

Theorem 191 (Alexandroff) *Let (X,τ) be a topological space. Let ∞ denote a point not in X and consider the set $X^\infty := X \cup \{\infty\}$. Let τ_∞ be the collection of all subsets $U \subset X^\infty$ such that either U is an open set of X or $\infty \in U$ and $X \backslash U$ is a closed compact set of X. Then (X^∞, τ_∞) is a compact topological space. Moreover, (X^∞, τ_∞) is a Hausdorff space if and only if (X,τ) is Hausdorff and locally compact.*

Proof. Step 1: We prove that (X^∞, τ_∞) is a topological space. We begin by observing that if U belongs to τ_∞ if and only if

(i) $U \cap X$ belongs to τ

(ii) if $\infty \in U$, then $X \setminus U$ is compact set of X.

By (i), finite intersections and arbitrary unions of elements of τ_∞ intersect X in open sets. If $U_1, U_2 \in \tau_\infty$ and $\infty \in U_1 \cap U_2$, then by De Morgan's laws

$$X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2),$$

and since both $X \setminus U_1$ and $X \setminus U_2$ are closed and compact, so is their union. This shows that finite intersections of elements of τ_∞ are still in τ_∞. If $\{U_\alpha\}_{\alpha\in\Lambda}$ is an arbitrary family of elements of τ_∞ and if

$$\infty \in \bigcup_{\alpha\in\Lambda} U_\alpha,$$

then $\infty \in U_\beta$ for some $\beta \in \Lambda$. Hence, $X \setminus U_\beta$ is closed and compact. Since $\bigcup_{\alpha\in\Lambda} U_\alpha \cap X$ is open, $X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ is closed and since

$$X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha \subset X \setminus U_\beta,$$

by Proposition 167, $X \setminus \bigcup_{\alpha\in\Lambda} U_\alpha$ is compact. Hence, $\bigcup_{\alpha\in\Lambda} U_\alpha$ belongs to τ_∞. The set X^∞ belongs to τ_∞, since $X^\infty \setminus X^\infty = \emptyset$ is closed and compact, while the empty set belongs to τ_∞ by (i). Thus, (X^∞, τ_∞) is a topological space.

Step 2: We prove that (X^∞, τ_∞) is compact. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a family of elements of τ_∞ such that

$$X^\infty = \bigcup_{\alpha\in\Lambda} U_\alpha.$$

Then $\infty \in U_\beta$ for some $\beta \in \Lambda$. Hence, $X \setminus U_\beta$ is closed and compact. Since

$$X \setminus U_\beta \subset \bigcup_{\alpha \in \Lambda} U_\alpha \cap X,$$

there exist $\alpha_1, \ldots, \alpha_m \in \Lambda$ such that

$$X \setminus U_\beta \subset \bigcup_{i=1}^{m} U_{\alpha_i} \cap X.$$

The finite family $\{U_{\alpha_1}, \ldots, U_{\alpha_m}, U_\beta\}$ covers X^∞.
Step 3: Finally, we show that (X^∞, τ_∞) is a Hausdorff space if and only if (X, τ) is Hausdorff and locally compact. Assume that (X^∞, τ_∞) is a Hausdorff space. Then (X, τ) is a Hausdorff space in view of property (i). To prove that it is locally compact, let $x \in X$. Since (X^∞, τ_∞) is a Hausdorff space, there exist two disjoint neighborhoods U and V of x and ∞, respectively. Then $X \setminus V$ is closed and compact and $U \subset X \setminus V$. Hence, $\overline{U} \subset X \setminus V$, and so $\overline{U}$ is compact by Proposition 167.

Conversely, assume that (X, τ) is Hausdorff and locally compact. Let $x, y \in X^\infty$ be two distinct points. If neither of them is ∞, then $x, y \in X$, and so since (X, τ) is a Hausdorff space, there exist two disjoint neighborhoods $U \subset X$ and $V \subset X$ of x and y, respectively. By (i), U and V belong to τ_∞. If $y = \infty$, since (X, τ) is locally compact, choose a neighborhood U of x such that $\overline{U}$ is compact. Then $X_\infty \setminus \overline{U}$ belongs to τ_∞. Hence, U and $X_\infty \setminus \overline{U}$ are two disjoint neighborhoods of x and ∞. ■

Exercise 192 *Prove that the circle is the one-point compactification of* $(0, 1)$.

Exercise 193 *Describe the one-point compactification of the following sets of* $\mathbb{R}^2$ *with the usual topology.*

(i) $\{(x, y) : x \in (0, 1], y = 0\}$

(ii) $\left\{\left(\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(iii) $\left\{(x, y) : x^2 + y^2 < 1\right\}$

(iv) $\left\{(x, y) : x^2 + y^2 < 1\right\} \cup \{(0, 1)\}$

(v) $\{(x, y) : -1 \le x \le 1\}$.

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Given a topological space (X, τ) and a function $f : X \to \mathbb{R}$, the *support* of f is the closed set

$$\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

Exercise 194 *Let (X, τ) be a topological space and consider the space*

$$C_c(X) := \{f : X \to \mathbb{R} \text{ continuous, } \operatorname{supp} f \text{ is a compact set of } X\}$$

with the metric

$$d(f, g) := \max_{x \in X} |f(x) - g(x)|.$$

The completion of $C_c(X)$ is the space $C_0(X)$ of functions vanishing at infinity. Prove that $f \in C_0(X)$ if and only if for every $\varepsilon > 0$ there exists a closed compact set $K \subset X$ such that

$$|f(x)| < \varepsilon$$

for all $x \in X \setminus K$.

Exercise 195 *Let (X, τ) be a topological space. Prove that $g \in C(X^\infty)$ if and only if*

$$(g - c)|_X = f$$

for some $f \in C_0(X)$ and $c \in \mathbb{R}$. Show also that

$$\|g - c\|_{C(X^\infty)} = \max \|f\|_{C_0(X)}. \tag{26}$$

2.8 Stone–Čech's Compactification

Given a topological space (X, τ), a *compactification* of X is a pair (h, Y), where (Y, τ_Y) is a compact topological space and $h : X \to Y$ is one-to-one, continuous, $h(X)$ is dense in Y and $h^{-1} : h(X) \to X$ is continuous; that is, h is a homeomorphism of X into a dense subset of Y. A compactification (h, Y) of X is Hausdorff if the space Y is Hausdorff.

Note that if (X^∞, τ_∞) is the one-point-compactification of X, then (i, X^∞) is a compactification, where $I : X \to X^\infty$ is the identity function. For noncompact spaces, the one-point-compactification is a way the smallest compactification. We now construct the largest.

Given a topological space (X, τ_X) consider the space $C_b(X)$ of all real-valued continuous bounded functions $f : X \to \mathbb{R}$. For every $f \in C_b(X)$ there exists $t_f > 0$ such that $f(x) \in [-t_f, t_f]$ for all $x \in X$. Consider

$$Y_0 := \prod_{f \in C_b(X)} [-t_f, t_f] = \{g : C_b(X) \to \mathbb{R} : g(f) \in [-t_f, t_f] \text{ for every } f \in C_b(X)\}.$$

By Tychonoff's theorem, this space is compact with the product topology τ_{Y_0}. Consider the evaluation map $e : X \to Y_0$ defined as follows: for every $x \in X$, $e(x) : C_b(X) \to \mathbb{R}$ is the function

$$e(x)(f) := f(x) \in [-t_f, t_f], \quad f \in C_b(X).$$

We claim that the function e is continuous. To see this, note that for every $f \in C_b(X)$ the projection π_f is given by

$$\begin{aligned} \pi_f : Y_0 &\to [-t_f, t_f] \\ g &\mapsto g(f) \end{aligned}$$

and so $\pi_f \circ e : X \to [-t_f, t_f]$ is the function f itself, since for every $x \in X$,

$$(\pi_f \circ e)(x) = \pi_f(e(x)) = e(x)(f) = f(x). \tag{27}$$

Since f is continuous, it follows from Theorem 180 that e is continuous. Define $\beta(X) := \overline{e(X)} \subset Y_0$. Since Y_0 is compact and $\beta(X)$ is closed, we have that $\beta(X)$ is compact (see Proposition 167). Thus we can consider the pair $(e, \beta(X))$. We want to see when this pair is a compactification of X. We are missing two properties. We need e to be one-to-one and we need $e^{-1} : e(X) \to X$ to be continuous.

The evaluation map e is one-to-one if for every $x, y \in X$ with $x \neq y$, we have that $e(x) \neq e(y)$. This is equivalent to say that there exists $f \in C_b(X)$ such $f(x) = e(x)(f) \neq e(y)(f) = f(y)$. This property says that bounded continuous functions *separate* points. Assume that this is the case. Then we can consider the inverse function $e^{-1} : e(X) \to X$.

Note that $e^{-1} : e(X) \to X$ is continuous if and only if $e : X \to e(X)$ is open.

Definition 196 *A topological space (X, τ) is* completely regular *if for every $x_0 \in X$ and every closed set $C \subset X$ that does not contain x_0, there exists a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f = 0$ on C.*

Exercise 197 *Prove that a metric space is completely regular.*

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Assume that X is completely regular. Fix an open set $U \subset X$ and let $x_0 \in U$. Since X is completely regular, there exists $f_0 \in C_b(X)$ such that $f_0(x_0) = 1$ and $f_0 = 0$ on $X \setminus U$. The set $V_{f_0} := f_0^{-1}((0,\infty))$ is open in X, contains x_0 and $V_{f_0} \subset U$. Moreover,

$$\begin{aligned} e(V_{f_0}) &= \{g \in Y_0 : g(f_0) > 0\} \cap e(X) \\ &= \prod_{f \in C_b(X)} A_f \cap e(X). \end{aligned}$$

where $A_f = [-t_f, t_f]$ if $f \neq f_0$ and $A_{f_0} = (0, t_{f_0}]$, which is open in $[-t_{f_0}, t_{f_0}]$. Thus, $e(V_{f_0})$ is an open neighborhood of $e(x_0)$ in $e(X)$.

Thus we have proved the following.

Theorem 198 (Stone–Čech) *Let (X,τ) be a completely regular topological space such that bounded continuous functions separate points. Then $(e,\beta(X))$ is a compactification of X.*

An important property of the Stone–Čech compactification of X is that every bounded continuous function $f_1 : X \to \mathbb{R}$ can be uniquely "extended" to a continuous function $F_1 : \beta(X) \to \mathbb{R}$. More precisely, $f_1 \circ e^{-1} : e(X) \to \mathbb{R}$ can be uniquely "extended" to a continuous function $F_1 : \beta(X) \to \mathbb{R}$. To see this, note that the projection mapping

$$\begin{aligned} \pi_{f_1} : \beta(X) &\to [-t_{f_1}, t_{f_1}] \\ g &\mapsto g(f_1) \end{aligned}$$

is continuous. Moreover, if $g \in e(X)$, then there exists $x \in X$ such that $g = e(x)$ and so

$$\left(f_1 \circ e^{-1}\right)(g) = \left(f_1 \circ e^{-1}\right)(e(x)) = f_1(x).$$

On the other hand, by (27),

$$\pi_{f_1}(g) = \pi_{f_1}(e(x)) = e(x)(f_1) = f_1(x).$$

Hence, $\left(f_1 \circ e^{-1}\right)(g) = \pi_{f_1}(g)$ for all $g \in e(X)$. Thus, if we identify X with $e(X)$, then π_{f_1} can be considered as a continuous extension of f_1.

Note that this extension is actually unique. Indeed, we have the following.

Exercise 199 *Let (Y,τ_Y) and (Z,τ_Z) be topological spaces, with Z Hausdorff and let $f : E \to Z$ be a continuous function, where $E \subset Y$. Then there is at most one extension of f to $\overline{E}$.*

2.9 Normal Spaces

Definition 200 *A topological space (X,τ) is a* normal space *if for every pair of disjoint closed sets $C_1, C_2 \subset X$ there exist two disjoint open set U_1, U_2 such that $U_1 \supset C_1$ and $U_2 \supset C_2$.*

Remark 201 *Note that* $\overline{U_1} \cap C_2 = \emptyset$. *Indeed, assume by contradiction that there exists* $x \in \overline{U_1} \cap C_2$. *Since* $U_2 \supset C_2$, *we have that* $x \in U_2$, *which implies that* U_2 *is a neighborhood of* x. *Using the fact that* $x \in \overline{U_1}$ *and Proposition 128, we conclude that* $U_1 \cap U_2$ *is nonempty, which is a contradiction.*

The next result shows that a metric space (X, d) is a normal space.

Proposition 202 *Let* (X, d) *be a metric space and let* τ *be the topology determined by* d. *Then* (X, τ) *is a normal space.*

Proof. If $C_1, C_2 \subset X$ are disjoint and closed, then the open sets

$$U_1 := \{x \in X : \operatorname{dist}(x, C_1) < \operatorname{dist}(x, C_2)\},$$
$$U_2 := \{x \in X : \operatorname{dist}(x, C_1) > \operatorname{dist}(x, C_2)\}$$

are two disjoint neighborhoods of C_1 and C_2, respectively. ■

Exercise 203 *Prove that every compact Hausdorff space is normal.*

Exercise 204 *Let (X,τ) be a topological space and let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a (possibly uncountable) family of lower semicontinuous (respectively sequentially lower semicontinuous) functions, $f_\alpha : X \to \mathbb{R}$. Assume that the function*

$$f_+ := \sup_{\alpha\in\Lambda} f_\alpha$$

is real-valued.

1. *Prove that f_+ is lower semicontinuous.*
2. *Prove that if Λ is a finite set, then then the function*

   $$f_- := \min_{\alpha} f_\alpha$$

   is still lower semicontinuous.
3. *Prove that if Λ is infinite and f_- is real-valued, then f_- may not be lower semicontinuous.*

Exercise 205 *Let (X,τ) be a topological space and let $E \subset X$.*

(i) Prove that the characteristic function *of E, defined by*

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise}, \end{cases}$$

is lower semicontinuous if and only if E is open.

(ii) Prove that χ_E is sequentially lower semicontinuous if and only if E is sequentially open (that is, $X \setminus E$ is sequentially closed).

(iii) Prove that there exist sequentially lower semicontinuous functions that are not lower semicontinuous.

The next theorem gives an important characterization of normal spaces.

Theorem 206 (Urysohn's lemma) *A topological space (X,τ) is normal if and only if for all disjoint closed sets $C_1, C_2 \subset X$ there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1 and $f \equiv 0$ in C_2.*

Lemma 207 *Let (X,τ) be a normal space and let Let $C \subset U \subset X$, where C is closed and U is open. Then there exists an open set $V \subset X$ such that*

$$C \subset V \subset \overline{V} \subset U.$$

Proof. Since the sets C and $X \setminus U$ are closed and disjoint, by Remark 201, we may find an open set $V \subset X$ such that $V \supset C$ and $\overline{V} \cap (X \setminus U) = \emptyset$. In particular, $\overline{V} \subset U$. ■

We now turn to the proof of Urysohn's lemma.

Proof of Urysohn's lemma. Step 1: Assume that X is normal. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Set $r_0 := 0$ and $r_1 := 1$ and let $\{r_n\}_{n=2}^{\infty}$ be an enumeration of the rationals numbers in $(0, 1)$. By the previous lemma applied to C_1 and $X \setminus C_2$ there exists an open set $V_0 \subset X$ such that

$$C_1 \subset V_0 \subset \overline{V_0} \subset X \setminus C_2.$$

Again by the previous lemma, this time with C_1 and V_0, there exists an open set $V_1 \subset X$ such that

$$C_1 \subset V_1 \subset \overline{V_1} \subset V_0,$$

and so,

$$C_1 \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset X \setminus C_2.$$

Inductively, assume that given $n \in \mathbb{N}$ there exist open sets $V_{r_1}, \ldots, V_{r_n} \subset X$ such that if $r_i < r_j$, then $\overline{V_{r_j}} \subset V_{r_i}$. Consider r_{n+1}. Since $r_0 = 0 < r_{n+1} < r_1 = 1$, one of the numbers $r_1, \ldots, r_n$, say r_i, will be the largest below r_{n+1}, and one, say r_j, will be the smallest greater than r_{n+1}. Then $\overline{V_{r_j}} \subset V_{r_i}$, and so by the previous lemma we may find an open set $V_{r_{n+1}} \subset X$ such that

$$\overline{V_{r_j}} \subset V_{r_{n+1}} \subset \overline{V_{r_{n+1}}} \subset V_{r_i}.$$

Thus, by induction, we can construct a sequence of open sets $\{V_r\}_{r \in [0,1] \cap \mathbb{Q}}$ with the properties that $C_1 \subset V_r$, $\overline{V_r} \subset X \setminus C_2$, and for all $r, s \in [0, 1] \cap \mathbb{Q}$ with $r < s$ we have $\overline{V_s} \subset V_r$. For $r, s \in [0, 1] \cap \mathbb{Q}$ define the functions

$$f_r(x) := \begin{cases} r & \text{if } x \in V_r, \\ 0 & \text{otherwise}, \end{cases} \qquad g_s(x) := \begin{cases} 1 & \text{if } x \in \overline{V_s}, \\ s & \text{otherwise}, \end{cases}$$

and

$$f := \sup_{r \in [0,1] \cap \mathbb{Q}} f_r, \quad g := \inf_{s \in [0,1] \cap \mathbb{Q}} g_s.$$

Then f is lower semicontinuous, g is upper semicontinuous, and $0 \le f \le 1$. If $x \in C_1 \subset V_1$, then, since $V_1 \subset V_r$ for all $r \in [0, 1] \cap \mathbb{Q}$, we have that $x \in V_r$ for all $r \in [0, 1] \cap \mathbb{Q}$, and so $f_r(x) = r$ and

$$f(x) = \sup_{r \in [0,1] \cap \mathbb{Q}} f_r(x) = \sup_{r \in [0,1] \cap \mathbb{Q}} r = 1,$$

while if $x \in C_2$, then, since $\overline{V_0} \subset X \setminus C_2$ and $\overline{V_r} \subset V_0$ for all $r \in [0, 1] \cap \mathbb{Q}$, we have that $x \notin V_r$ for any $r \in [0, 1] \cap \mathbb{Q}$, and so $f_r(x) = 0$ and

$$f(x) = \sup_{r \in [0,1] \cap \mathbb{Q}} f_r(x) = \sup_{r \in [0,1] \cap \mathbb{Q}} 0 = 0.$$

To conclude the first part of the proof, it remains to prove that f is continuous. It is enough to show that $f = g$. Given $x \in X$, if $f_r(x) > g_s(x)$ for some $r, s \in [0,1] \cap \mathbb{Q}$, then, necessarily, $r > s$, $x \in V_r$ and $x \notin \overline{V_s}$. But $s < r$ implies $\overline{V_r} \subset V_s$, which is a contradiction. Hence, $f_r \leq g_s$ for all $r, s \in [0,1] \cap \mathbb{Q}$, and so, taking first the supremum over all r and then the infimum over all s, we conclude that $f \leq g$. Now assume by contradiction that $f(x) < g(x)$ for some $x \in X$. Then by the density of the rationals numbers we may find $r, s \in [0,1] \cap \mathbb{Q}$ such that

$$f(x) < r < s < g(x).$$

Since $f(x) < r$, it follows that $x \notin V_r$. On the other hand, since $s < g(x)$, we have that $x \in \overline{V_s}$. This contradicts the fact that $\overline{V_s} \subset V_r$ and completes the first part of the proof.

Step 2: Assume that for all disjoint closed sets $C_1, C_2 \subset X$ there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1 and $f \equiv 0$ in C_2. We claim that X is normal. Indeed, let $C_1, C_2 \subset X$ be disjoint closed sets and let $f : X \to [0,1]$ be as above. Then the sets $f^{-1}\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ and $f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ are open, disjoint, and contain C_1 and C_2, respectively. This concludes the proof. ∎

Exercise 208 *Let (X, τ) be a normal space.*

(i) Prove that given a proper set closed set $C \subset X$, there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 0$ in C and $f > 0$ in $X \setminus C$ if and only if C is a G_δ set.

(ii) Let $C_1, C_2 \subset X$ be two disjoint closed sets. Prove that there exists a continuous function $f : X \to [0,1]$ such that $f \equiv 1$ in C_1, $f \equiv 0$ in C_2, and $0 < f < 1$ in $X \setminus (C_1 \cup C_2)$ if and only if C_1, C_2 are G_δ sets.

Friday, November 18, 2009

To prove the next theorem, we need a few facts about the induced topology. Next we introduce the notion of induced topology.

Definition 209 *Given a topological space (X,τ) consider a subset $E\subset X$. Consider the family of sets*

$$\tau_E := \{U\cap E : U\in\tau\}.$$

Then (E,τ_E) is a topological space. The topology τ_E is called the relative*, or* induced*,* topology *on E. An element of τ_E is called a* relatively open set*.*

Exercise 210 *Consider a topological space (X,τ) and a subset $E\subset X$. For every $F\subset E$, let $\overline{F}^{\tau_E}$ denote the closure of F with respect to the topology τ_E. Prove that*

$$\overline{F}^{\tau_E} = \overline{F}\cap E.$$

Remark 211 *It follows from the previous exercise that if E is closed (with respect to τ), then $\overline{F}^{\tau_E}$ is closed (with respect to τ), since it is given by the intersection of two closed sets (with respect to τ).*

Remark 212 *Let (X,τ_X) and (Y,τ_Y) be two topological spaces, and let $f: E\to Y$, where $E\subset Y$. We recall that given $x_0\in E$, the function f is said to be* continuous at x_0 *if for every neighborhood $V\subset Y$ of $f(x_0)$, there exists a neighborhood $U\subset X$ of x_0 such that*

$$f(x)\in V$$

for all $x\in U\cap E$. Since $U\cap E\in\tau_E$, we have that f is continuous at x_0 if and only if $f:(E,\tau_E)\to Y$ is continuous at x_0.

Theorem 213 (Tietze's extension theorem) *A topological space (X,τ) is normal if and only if for every closed set $C\subset X$ and every continuous function $f: C\to\mathbb{R}$ there exists a continuous function $F: X\to\mathbb{R}$ such that $F(x)=f(x)$ for all $x\in C$. Moreover, if $f(C)\subset[a,b]$, then F may be constructed so that*

$$F(C)\subset[a,b].$$

Proof. Step 1: Assume that X is normal, let $C\subset X$ be a closed set and let $f: C\to[-1,1]$ be a continuous function. Then the sets $f^{-1}\left(\left[\frac{1}{3},\infty\right)\right)$ and $f^{-1}\left(\left(-\infty,-\frac{1}{3}\right]\right)$ are disjoint closed subsets of C with respect to the induced topology τ_C but since C is closed, it follows by the previous exercise that they are actually closed with respect to τ. Hence, we may apply Urysohn's lemma to find a continuous function $f_1: X\to\left[-\frac{1}{3},\frac{1}{3}\right]$ such that $f_1\equiv\frac{1}{3}$ in $f^{-1}\left(\left[\frac{1}{3},\infty\right)\right)$ and $f_1\equiv-\frac{1}{3}$ in $f^{-1}\left(\left(-\infty,-\frac{1}{3}\right]\right)$. We claim that

$$|f-f_1|\le\frac{2}{3}\quad\text{on } C.$$

Indeed, if $f(x) \in \left[-1, -\frac{1}{3}\right]$, then $f_1(x) = -\frac{1}{3}$; if $f(x) \in \left[\frac{1}{3}, 1\right]$, then $f_1(x) = \frac{1}{3}$; while if $f(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$, then so $f_1(x) \in \left[-\frac{1}{3}, \frac{1}{3}\right]$.

Repeat this construction with $f - f_1$ in place of f and $(f - f_1)^{-1}\left(\left[\frac{2}{9}, \infty\right)\right)$ and $(f - f_1)^{-1}\left(\left(-\infty, -\frac{2}{9}\right]\right)$ in place of $f^{-1}\left(\left[\frac{1}{3}, \infty\right)\right)$ and $f^{-1}\left(\left(-\infty, -\frac{1}{3}\right]\right)$, respectively, to find a continuous function $f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that $f_2 \equiv \frac{2}{9}$ in $(f - f_1)^{-1}\left(\left[\frac{2}{9}, \infty\right)\right)$ and $f_2 \equiv -\frac{2}{9}$ in $(f - f_1)^{-1}\left(\left(-\infty, -\frac{2}{9}\right]\right)$. As before, we can prove that

$$|(f - f_1) - f_2| \leq \left(\frac{2}{3}\right)^2 \quad \text{on } C.$$

Inductively for every $n \in \mathbb{N}$ we can construct a continuous function

$$f_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^{n-1}, \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}\right] \tag{28}$$

such that

$$|f - f_1 - \cdots - f_n| \leq \left(\frac{2}{3}\right)^n \quad \text{on } C. \tag{29}$$

Define

$$F(x) := \sum_{n=1}^{\infty} f_n(x), \quad x \in X.$$

Note that by (28),

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 1, \tag{30}$$

and so F is well-defined and takes values in $[-1, 1]$. In turn, since the series converges, it follows from (29) that for every $x \in C$,

$$\begin{aligned} |f(x) - F(x)| &= \left| f(x) - \lim_{m \to \infty} \sum_{n=1}^{m} f_n(x) \right| \\ &\leq \lim_{m \to \infty} \left| f(x) - \sum_{n=1}^{m} f_n(x) \right| \leq \lim_{m \to \infty} \left(\frac{2}{3}\right)^m = 0, \end{aligned}$$

and so $F = f$ on C. It remains to show that F is continuous. Fix $x \in X$ and $\varepsilon > 0$ and find n_ε so large that

$$\sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \leq \frac{\varepsilon}{2}. \tag{31}$$

Since $f_1, \ldots, f_{n_\varepsilon}$ are continuous at x, for every $n = 1, \ldots, n_\varepsilon$ there exists a neighborhood U_n of x such that if $y \in U_n$, then

$$|f_n(y) - f_n(x)| \leq \frac{\varepsilon}{2n_\varepsilon}. \tag{32}$$

Take $U := \bigcap_{n=1}^{n_\varepsilon} U_n$. Then by (30), (31), and (32), for every $y \in U$,

$$
\begin{aligned}
|F(y) - F(x)| &= \left| \sum_{n=1}^{n_\varepsilon} f_n(y) - \sum_{n=1}^{n_\varepsilon} f_n(x) \right| + \sum_{n=n_\varepsilon+1}^{\infty} |f_n(x)| + \sum_{n=n_\varepsilon+1}^{\infty} |f_n(y)| \\
&\le \sum_{n=1}^{n_\varepsilon} |f_n(y) - f_n(x)| + \frac{2}{3} \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{2}{3}\right)^{n-1} \le n_\varepsilon \frac{\varepsilon}{2n_\varepsilon} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which shows that F is continuous at x. Note that the proof continues to work if in place of $f : C \to [-1,1]$ we have $f : C \to [a,b]$, with the only change that in this case $F : X \to [a,b]$.

Step 2: Assume that X is normal, let $C \subset X$ be a closed set and let $f : C \to \mathbb{R}$ be a continuous function. Since $(-1,1)$ is homeomorphic to $\mathbb{R}$, we can construct an homeomorphism $g : \mathbb{R} \to (-1,1)$. Consider the function $h := g \circ f : C \to [-1,1]$. Since h is continuous, by Step 1 there exists a continuous function $H : X \to [-1,1]$ such that $H = h$ on C. The problem is that H can take values -1 and 1. To avoid this, let $C_1 := H^{-1}(\{-1,1\})$. Then C_1 and C are closed and disjoint, and so by Urysohn's lemma we may find a continuous function $h_1 : X \to [0,1]$ such that $h_1 \equiv 0$ in C_1 and $h_1 \equiv 1$ in C. Then $H_1 := H h_1$ is continuous, $H_1 = H = h$ on C and $H_1 : X \to (-1,1)$. Since g^{-1} is continuous, the function $F := g^{-1} \circ H_1$ is continuous and real-valued and for $x \in C$,

$$
F(x) = g^{-1}(H_1(x)) = g^{-1}(h(x)) = g^{-1}(g \circ f(x)) = f(x).
$$

Step 3: Assume that or every closed set $C \subset X$ and every continuous function $f : C \to \mathbb{R}$ there exists a continuous function $F : X \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in C$. Assume also that if $f(C) \subset [a,b]$, then $F(C) \subset [a,b]$. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Then $C_1 \cup C_2$ is closed. Define $f := 1$ in C_1 and $f := 0$ in C_2. Then $f : C_1 \cup C_2 \to [0,1]$ is continuous, and so by hypothesis there exists a continuous function $F : X \to [0,1]$ such that $F(x) = f(x)$ for all $x \in C_1 \cup C_2$. Thus, we are in a position to apply Urysohn's lemma (or repeat Step 2 of its proof) to conclude that X is normal. ■

Monday, November 23, 2009

Next we study partitions of unity for normal spaces.

Definition 214 *Let X be a topological space and let $\mathcal{F}$ be a collection of subsets of X. Then*

(i) $\mathcal{F}$ *is* point finite *if every $x \in X$ belongs to only finitely many $U \in \mathcal{F}$,*

(ii) $\mathcal{F}$ *is* locally finite *if every $x \in X$ has a neighborhood meeting only finitely many $U \in \mathcal{F}$,*

(iii) $\mathcal{G} \subset \mathcal{P}(X)$ *is a* refinement *of $\mathcal{F}$ if*

$$\bigcup_{G \in \mathcal{G}} G = \bigcup_{F \in \mathcal{F}} F$$

and every element of $\mathcal{G}$ is contained in some element of $\mathcal{F}$.

Theorem 215 *A topological space (X, τ) is normal if and only if for every point finite open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X there exists another open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X with the property that $\overline{V_\alpha} \subset U_\alpha$.*

Proof. Assume that (X, τ) is normal and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a point finite open cover of X. By the axiom of choice, we may assume that Λ is well-ordered; that is, there is an order relation $\leq$ such that every subset of Λ has a smallest element. Let $\alpha_0 \in \Lambda$ be the least element of Λ. We claim that for every $\beta \in \Lambda$ there is an open set V_β with the properties that

$$C_\beta \subset V_\beta \subset \overline{V_\beta} \subset U_\beta,$$

where

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha < \beta} V_\alpha \right) \cup \left(\bigcup_{\alpha > \beta} U_\alpha \right) \right).$$

To construct V_β we use transfinite induction on Λ. Define

$$C_{\alpha_0} := X \setminus \bigcup_{\alpha > \alpha_0} U_\alpha.$$

Then $C_{\alpha_0} \subset U_{\alpha_0}$ and is closed. By Lemma 207 there exists an open set V_{α_0} such that

$$C_{\alpha_0} \subset V_{\alpha_0} \subset \overline{V_{\alpha_0}} \subset U_{\alpha_0}.$$

Suppose that V_α has been chosen for every $\alpha \in \Lambda$ with $\alpha < \beta$ and define

$$C_\beta := X \setminus \left(\left(\bigcup_{\alpha < \beta} V_\alpha \right) \cup \left(\bigcup_{\alpha > \beta} U_\alpha \right) \right).$$

To prove that $C_\beta \subset U_\beta$, fix $x \in C_\beta$. Then

$$x \notin V_\alpha \text{ for any } \alpha < \beta \text{ and } x \notin U_\alpha \text{ for any } \alpha > \beta. \tag{33}$$

Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is a point finite cover, x belongs to only finitely many U_α, say, $U_{\alpha_1}, \ldots, U_{\alpha_m}$. Without loss of generality, we may assume that $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. Hence, by (33), we have that $\alpha_m \leq \beta$. We claim that $\alpha_m = \beta$. Indeed, if $\alpha_m < \beta$, then by (33), $x \notin V_{\alpha_m}$ but

$$x \in C_{\alpha_m} = X \setminus \left(\left(\bigcup_{\alpha<\alpha_m} V_\alpha \right) \cup \left(\bigcup_{\alpha>\alpha_m} U_\alpha \right) \right),$$

which contradicts the fact that $C_{\alpha_m} \subset V_{\alpha_m}$. This shows that $\alpha_m = \beta$, so that $x \in U_\beta$.

By Lemma 207 there exists an open set V_β such that

$$C_\beta \subset V_\beta \subset \overline{V_\beta} \subset U_\beta.$$

Thus, by Proposition 183 we have constructed a family of open sets $\{V_\alpha\}_{\alpha\in\Lambda}$.

It remains to show that it covers X. Fix $x \in X$. Since $\{U_\alpha\}_{\alpha\in\Lambda}$ is a point finite cover, x belongs to only finitely many U_α, say, $U_{\alpha_1}, \ldots, U_{\alpha_m}$. As before assume that $\alpha_m = \max\{\alpha_1, \ldots, \alpha_m\}$. Then $x \notin U_\alpha$ for $\alpha > \alpha_m$. There are now two cases. If $x \in \bigcup_{\alpha<\alpha_m} V_\alpha$, then there is nothing to prove. If $x \notin \bigcup_{\alpha<\alpha_m} V_{\alpha_m}$, then $x \in C_{\alpha_m}$ and so $x \in V_{\alpha_m}$. This shows that $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X.

Conversely, assume that every point finite open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there exists another open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{V_\alpha} \subset U_\alpha$. We claim that (X, τ) is normal. Let $C_1, C_2 \subset X$ be two disjoint closed sets. Then $\{X \setminus C_1, X \setminus C_2\}$ is a point finite open cover of X. Hence, there exist two open sets V_1 and V_2 such that

$$X = V_1 \cup V_2, \quad \overline{V_1} \subset X \setminus C_1, \quad \overline{V_2} \subset X \setminus C_2.$$

The sets $X \setminus \overline{V_1}$ and $X \setminus \overline{V_2}$ are open, disjoint, and contain C_1 and C_2, respectively. ■

As a corollary of the previous theorem we can show that in a normal space every locally finite open cover admits a partition of unity subordinated to it.

Definition 216 *If (X, τ) is a topological space, a* partition of unity *on X is a family $\{\varphi_i\}_{i\in\Lambda}$ of continuous functions $\varphi_i : X \to [0, 1]$ such that*

$$\sum_{i\in\Lambda} \varphi_i(x) = 1$$

for all $x \in X$. A partition of unity is locally finite *if for every $x \in X$ there exists a neighborhood U of x such that the set $\{i \in \Lambda : U \cap \operatorname{supp} \varphi_i \neq \emptyset\}$ is finite. If $\{U_j\}_{j\in\Xi}$ is an open cover of X, a partition of unity* subordinated to the cover *$\{U_j\}_{j\in\Xi}$ is a partition of unity $\{\varphi_i\}_{i\in\Lambda}$ such that for every $i \in \Lambda$, $\operatorname{supp} \varphi_i \subset U_j$ for some $j \in \Xi$.*

Monday, November 30, 2009

Theorem 217 *Let (X,τ) be a normal space and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite open cover of X. Then there exists a partition of unity subordinated to it.*

Proof. Since a locally finite cover is point finite, by the previous theorem there exists another open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{V_\alpha}\subset U_\alpha$. Note that $\{\overline{V_\alpha}\}_{\alpha\in\Lambda}$ remains locally finite.

By Lemma 207, there exists an open set W_α such that

$$\overline{V_\alpha}\subset W_\alpha\subset\overline{W_\alpha}\subset U_\alpha.$$

Since the closed sets sets $\overline{V_\alpha}$ and $X\setminus W_\alpha$ are disjoint, there exists a continuous function $f_\alpha : X\to[0,1]$ such that $f_\alpha\equiv 1$ in $\overline{V_\alpha}$ and $f_\alpha\equiv 0$ in $X\setminus W_\alpha$. Hence,

$$\{x\in X:\ f_\alpha(x)>0\}\subset W_\alpha$$

and so,

$$\operatorname{supp} f_\alpha\subset\overline{W_\alpha}\subset U_\alpha.$$

Define

$$f(x):=\sum_{\alpha\in\Lambda}f_\alpha(x),\quad x\in X.$$

Since $\{V_\alpha\}_{\alpha\in\Lambda}$ covers X, for every $x\in X$ there exists $\alpha\in\Lambda$ such that $x\in V_\alpha$, and so $f_\alpha(x)=1$. Thus, $f>0$. Moreover, since $\{\overline{V_\alpha}\}_{\alpha\in\Lambda}$ is locally finite for every $x\in X$ there exists a neighborhood U of x that intersects only finitely many $\overline{V_\alpha}$. Thus, f reduces to a finite sum in U. In particular, $f<\infty$ in U and f is continuous in x. By the arbitrariness of x, we have that f is continuous. Hence the function

$$\varphi_\alpha(x):=\frac{f_\alpha(x)}{f(x)},\quad x\in X,$$

is well-defined and continuous. Since $\operatorname{supp}\varphi_\alpha=\operatorname{supp}f_\alpha\subset U_\alpha$, the family $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a locally finite partition of unity subordinated to $\{U_\alpha\}_{\alpha\in\Lambda}$. ■

2.10 Metrization

A topological space (X,τ) is *metrizable* if its topology can be determined by a metric. The metrizability and the normability of a given topology depend on the properties of a base (see Theorems 218 and 224 below).

In view of and Theorem 219, in order for topological space (X,τ) to be metrizable it is necessary that (X,τ) be Hausdorff and normal. Thus in the next two theorems, without loss of generality, we will assume that these two properties are satisfied.

Theorem 218 (Urysohn's metrization theorem) *A topological space (X,τ) is metrizable and separable if and only if X is Hausdorff, normal, and it has a countable base.*

Proof. If (X, d) is a metric space, then by Propositions 132 and 202 it is Hausdorff and normal. Moreover, if (X, d) is separable, then there exists a sequence $\{x_n\} \subset X$ which is dense in X. The countable family of balls $\left\{B\left(x_n, \frac{1}{k}\right)\right\}_{k,n\in\mathbb{N}}$ is a base for the topology τ determined by d.

Conversely, assume that (X, τ) is a Hausdorff normal space with a countable base $\mathcal{B} = \{B_n\}_n$. We will show that X is homeomorphic (in the topological sense) to a subset of ℓ^2, which is separable.

Step 1: We claim that every closed set is a G_δ, or, equivalently, that every open set is an F_σ set. Fix an open set $U \subset X$. Fix $x \in U$. Since X is Hausdorff, the singleton $\{x\}$ is closed, and $\{x\} \subset U$. By Lemma 207 there exists an open set $V \subset X$ such that

$$\{x\} \subset V \subset \overline{V} \subset U.$$

Since $\mathcal{B}$ is a base, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset V$, and so

$$\{x\} \subset B_x \subset \overline{B_x} \subset \overline{V} \subset U.$$

This shows that

$$U = \bigcup_{\overline{B_n} \subset U} \overline{B_n};$$

that is, that U is an F_σ set.

Step 2: By the previous step and Exercise 208 for each element B_n in the base $\mathcal{B}$ there exists a continuous function $\varphi_n : X \to [0, 1]$ with the property that

$$\varphi_n(x) = 0 \text{ for } x \in X \setminus B_n, \quad \varphi_n(x) > 0 \text{ for } x \in B_n. \tag{34}$$

Define

$$\psi_n(x) := \frac{1}{n} \frac{\varphi_n(x)}{\sqrt{1 + (\varphi_n(x))^2}}, \quad x \in X.$$

Then ψ_n is continuous. Moreover,

$$\sum_n (\psi_n(x))^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{(\varphi_n(x))^2}{1 + (\varphi_n(x))^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

which shows that for every fixed $x \in X$, $\{\psi_n(x)\}_n$ belongs ℓ^2. Hence, the map

$$\begin{aligned} f : X &\to \ell^2 \\ x &\mapsto \{\psi_n(x)\}_n \end{aligned}$$

is well-defined.

We claim that f is one-to-one. To see this, let $x, y \in X$ with $x \neq y$. Since X is Hausdorff, there exists B_n such that $x \in B_n$ and $y \in X \setminus B_n$. It follows by (34) that $\psi_n(x) > 0$, while $\psi_n(y) = 0$. Hence, $f(x) \neq f(y)$ and the claim is proved. ■

Proof. Next we claim that f is continuos. Fix $x_0 \in X$ and $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n^2} \leq \frac{\varepsilon^2}{8}.$$

Since each ψ_n, $n = 1, \ldots, n_\varepsilon$, is continuous at x_0 and we have a finite number of them, there exists a neighborhood $V \subset U$ of x_0 such that

$$|\psi_n(x) - \psi_n(x_0)| \leq \frac{\varepsilon}{\sqrt{2n_\varepsilon}}$$

for all $x \in V$ and $n = 1, \ldots, n_\varepsilon$. Then for $x \in V$,

$$\sum_{n=1}^{n_\varepsilon} (\psi_n(x) - \psi_n(x_0))^2 \leq n_\varepsilon \frac{\varepsilon^2}{2n_\varepsilon} = \frac{\varepsilon^2}{2},$$

while

$$\begin{aligned}\sum_{n=n_\varepsilon+1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2 &\leq 2 \sum_{n=n_\varepsilon+1}^{\infty} \left[(\psi_n(x))^2 + (\psi_n(x_0))^2\right] \\ &\leq 2 \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^2}\right) \leq \frac{4\varepsilon^2}{8} = \frac{\varepsilon^2}{2}.\end{aligned}$$

Hence,

$$d_2(f(x), f(x_0)) \leq \varepsilon$$

for all $x \in V$, which shows continuity at x_0.

Finally, we prove that $f^{-1} : f(X) \to X$ is continuous. Let $y_0 \in f(X)$, then there exists $x_0 \in X$ such that $f(x_0) = y_0$. Consider a neighborhood U of x_0. Since $\mathcal{B}$ is a base, there exists B_n in $\mathcal{B}$ such that $B_n \subset U$ so that $\psi_n(x_0) > 0$. Let $\delta := \psi_n(x_0)$. If $d_2(f(x), f(x_0)) < \delta$, then

$$\begin{aligned}|\psi_n(x) - \psi_n(x_0)| &\leq \left(\sum_{n=1}^{\infty} (\psi_n(x) - \psi_n(x_0))^2\right)^{\frac{1}{2}} \\ &= d_2(f(x), f(x_0)) < \delta = \psi_n(x_0),\end{aligned}$$

which implies that $\psi_n(x) > 0$, and so, by (34), that $x \in B_n \subset U$. This shows that f^{-1} is continuous at $y_0 = f(x_0)$. ■

Next we drop the separability. The next theorem was first proved by Stone in 1948. The present proof is due to Ornstein (see also a proof of M.E. Rudin).

Theorem 219 *Let (X, d) be a metric space. Then every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X admits a point finite refinement.*

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By the axiom of choice we may assume that Λ is well-ordered. Fix $\alpha \in \Lambda$. A *chosen* ball (with respect to α) is a ball $B\left(x, \frac{1}{2^{n_x+1}}\right)$ such that

(i) $B\left(x, \frac{1}{2^{n_x}}\right) \subset U_\alpha$,

(ii) n_x is the smallest integer for which (i) holds,

(iii) $B\left(x, \frac{1}{2^{n_x}}\right) \subset U_\beta$ for some $\beta < \alpha$.

Step 1: Let $\mathcal{B}_\alpha := \left\{B\left(x, \frac{1}{2^{n_x+1}}\right) : B\left(x, \frac{1}{2^{n_x+1}}\right) \text{ is a chosen ball}\right\}$ and define the open set

$$V_\alpha := U_\alpha \setminus \overline{\bigcup_{B \in \mathcal{B}_\alpha} B}.$$

We claim that $\{V_\alpha\}_{\alpha \in \Lambda}$ is still an open cover of X. Indeed, assume by contradiction that there exists $x \in X$ that is not covered by $\{V_\alpha\}_{\alpha \in \Lambda}$. Let U_α be the first element that contains x (recall that the set $\{\beta \in \Lambda : x \in U_\beta\}$ has a minimum). Then $B(x, r) \subset U_\alpha$ for some $r > 0$. Since $x \notin V_\alpha$ and x is not in any chosen ball (with respect to α) in view of (iii), it follows that x must be an accumulation point of chosen balls, namely, there exist two sequences $\left\{B\left(x_k, \frac{1}{2^{n_{x_k}+1}}\right)\right\} \subset \mathcal{B}_\alpha$ and $y_k \in B\left(x_k, \frac{1}{2^{n_{x_k}+1}}\right)$ such that $y_k \to x$ as $k \to \infty$ (the balls could be repeated). Note that n_{x_k} cannot approach infinity along a subsequence, not relabeled, since this would imply that

$$B\left(x_k, \frac{1}{2^{n_{x_k}-1}}\right) \subset B(x, r) \subset U_\alpha$$

for infinitely many k and this would contradict (ii). Hence,

$$\min_k \frac{1}{2^{n_{x_k}+1}} = \frac{1}{2^{n_0+1}}$$

for some $n_0 \in \mathbb{N}$. Let $k \in \mathbb{N}$ be so large that $d(x, y_k) < \frac{1}{2^{n_0+1}}$, then

$$d(x, x_k) \leq d(x, y_k) + d(y, x_k) < \frac{1}{2^{n_0+1}} + \frac{1}{2^{n_{x_k}+1}} \leq \frac{2}{2^{n_{x_k}+1}} = \frac{1}{2^{n_{x_k}}};$$

that is, $x \in B\left(x_k, \frac{1}{2^{n_{x_k}}}\right)$. But then by (iii), it follows that x must belong to U_β for some $\beta < \alpha$, which contradicts the choice of α and proves the claim. ■

Step 2: Next we prove that for every $x \in X$ there exists a finite number of V_α that contain x. By the previous step there exists V_β such that $x \in V_\beta$. By construction, this means that U_β is the first element to contain some ball $B\left(x, \frac{1}{2^m}\right)$. Note that if U_β is the first element to contain some ball $B\left(x, \frac{1}{2^m}\right)$ and U_γ is the first element to contain some ball $B\left(x, \frac{1}{2^n}\right)$, and if, say, $n > m$, then $\gamma \leq \beta$. Hence, the family α such that V_α contain x form a descending sequence. Since Λ is well-ordered, only a finite number are distinct.

Theorem 220 *Let (X, d) be a metric space. Then every point finite open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X admits a locally finite refinement.*

Proof. We construct a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$ that is locally finite. For every $x \in X$ let

$$r_x := \frac{1}{2} \sup \{r > 0 : \ B(x, r) \subset V_\alpha \text{ for some } \alpha \in \Lambda\}.$$

If $r_x = \infty$ for some x, then one could construct a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$ given by sequence of balls $\{B(x, n)\}_{n\in\mathbb{N}}$, which is a locally finite. Thus, assume that $r_x < \infty$ for all $x \in X$. For every $\beta \in \Lambda$, let W_β be the union of all balls $B\left(x, \frac{r_x}{2}\right)$ such that V_β is the first open set in the cover $\{V_\alpha\}_{\alpha\in\Lambda}$ to contain $B(x, r_x)$. By construction, $\{W_\alpha\}_{\alpha\in\Lambda}$ is still an open cover of X. Moreover, $W_\alpha \subset V_\alpha \subset U_\alpha$ for every $\alpha \in \Lambda$, and so $\{W_\alpha\}_{\alpha\in\Lambda}$ is a refinement of $\{V_\alpha\}_{\alpha\in\Lambda}$. It remains to show that $\{W_\alpha\}_{\alpha\in\Lambda}$ is locally finite.

We claim that if $W_\alpha \cap B\left(x, \frac{r_x}{8}\right) \neq \emptyset$, then $x \in V_\alpha$. Indeed, assume the contrary for some $x \in X$. Then there exists $y \in W_\alpha$ such that $B\left(y, \frac{r_y}{2}\right) \cap B\left(x, \frac{r_x}{8}\right) \neq \emptyset$ such that $x \notin V_\alpha$. Since $y \in W_\alpha$, we have that $B(y, r_y) \subset V_\alpha$. In particular, $x \notin B(y, r_y)$, and so

$$r_y < d(x, y) < \frac{r_y}{2} + \frac{r_x}{8},$$

which implies that $\frac{r_y}{2} < \frac{r_x}{8}$. Hence,

$$d(x, y) < \frac{r_y}{2} + \frac{r_x}{8} < \frac{r_x}{8} + \frac{r_x}{8} = \frac{r_x}{4};$$

that is, $y \in B\left(x, \frac{r_x}{4}\right)$. In turn, $B\left(y, \frac{5r_y}{2}\right) \subset B(x, r_x)$. By the definition of r_x, this implies that $B(y, 5r_y)$ belongs to some V_β, which contradicts the definition of r_y. Hence, the claim holds. Thus, for every $x \in X$, the only open sets W_α that intersect $B\left(x, \frac{r_x}{8}\right)$ are those for which V_α contains x. By the previous step these V_α are finite and the proof is complete. ■

Definition 221 *Let X be a topological space and let $\mathcal{F}$ be a collection of subsets of X. Then $\mathcal{F}$ is* σ-locally finite *if*

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n,$$

where each $\mathcal{F}_n$ is a locally finite collection in X.

Corollary 222 *Let (X, d) be a metric space. Then every open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of X admits a locally finite partition of unity subordinated to it.*

Corollary 223 *Let (X, d) be a metric space. Then X admits a σ-locally finite base.*

Proof. For every $x \in X$ and $n \in \mathbb{N}$ consider the ball $B\left(x, \frac{1}{n}\right)$. Then $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X, n\in\mathbb{N}}$ is a base. Fix $n \in \mathbb{N}$ and consider the open cover of X, $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X}$. By the previous theorems, there exists a locally finite open refinement $\mathcal{V}_n$ of $\left\{B\left(x, \frac{1}{n}\right)\right\}_{x\in X}$. ■

It turn out that the previous condition is also sufficient for metrizability.

Theorem 224 (Nagata–Smirnov's metrization theorem) *A topological space (X, τ) is metrizable if and only if it is Hausdorff, normal, and has a σ-locally finite base.*

END OF THE COURSE

Lemma 225 *Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a locally finite family of sets. Then*

$$\overline{\bigcup_{\alpha\in\Lambda} E_\alpha} = \bigcup_{\alpha\in\Lambda} \overline{E_\alpha}.$$

In particular, the union of a locally finite family of closed sets is closed.

Proof. Define

$$E := \bigcup_{\alpha\in\Lambda} E_\alpha.$$

Indeed, by Proposition 129, we have that

$$\overline{E} \supset \bigcup_{\alpha\in\Lambda} \overline{E_\alpha}.$$

To prove the opposite inclusion, let $x \in \overline{E}$. Since $\{E_\alpha\}_{\alpha\in\Lambda}$ is locally finite, there exists a neighborhood U of x that intersects only finitely many E_α, say, $E_{\alpha_1}, \ldots, E_{\alpha_m}$. Let V be a neighborhood of x. Since $x \in \overline{E}$, by Proposition 128 we have that $(V \cap U) \cap E$ is nonempty. By the choice of U, this implies that

$$(V \cap U) \cap \bigcup_{i=1}^{m} E_{\alpha_i} \neq \emptyset.$$

Hence,

$$y \in \overline{\bigcup_{i=1}^{m} E_{\alpha_i}} = \bigcup_{i=1}^{m} \overline{E_{\alpha_i}},$$

where we have used again Proposition 129. This concludes the proof. ■

We now turn to the proof of Nagata–Smirnov's metrization theorem.

Proof. If (X, d) is a metric space, then by Propositions 132 and 202 it is Hausdorff and normal. In Theorem 219, we will see that X has a σ-locally finite base.

Conversely, assume that X is Hausdorff, normal, and has a σ-locally finite base, that is, a base of the form

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n,$$

where each $\mathcal{B}_n = \{B_{n,\alpha}\}_{\alpha\in\Lambda_n}$ is locally finite, then X is metrizable. We will show that X is homeomorphic (in the topological sense) to a subset of an ℓ^2 metric space.

Step 1: We claim that every closed set is a G_δ, or, equivalently, that every open set is an F_σ set. Fix an open set $U \subset X$. As in Step 1 of the proof of

Urysohn's metrization theorem, for every $x \in U$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset \overline{B_x} \subset U$. For every $n \in \mathbb{N}$, let

$$C_n := \bigcup_{B_x \in \mathcal{B}_n} \overline{B_x}.$$

By Lemma 225, C_n is closed. Moreover, $C_n \subset U$. Since

$$U = \bigcup_{n=1}^{\infty} C_n,$$

it follows that U is an F_σ set.
Step 2: By the previous step and Exercise 208 for each element $B_{n,\alpha}$ in the base $\mathcal{B}$ there exists a continuous function $\varphi_{\alpha,n} : X \to [0,1]$ with the property that

$$\varphi_{\alpha,n}(x) = 0 \text{ for } x \in X \setminus B_{\alpha,n}, \quad \varphi_{\alpha,n}(x) > 0 \text{ for } x \in B_{\alpha,n}. \tag{35}$$

Define

$$\psi_{\alpha,n}(x) := \frac{1}{n} \frac{\varphi_{\alpha,n}(x)}{\sqrt{1 + \sum_\beta (\varphi_{\beta,n}(x))^2}}, \quad x \in X.$$

Note that since each $\mathcal{B}_n$ is locally finite, for every $x \in X$ there exists a neighborhood U of x such that $\varphi_{\beta,n} = 0$ in U for all β except at most finitely many. Thus, the infinite sum $\sum_\beta (\varphi_{\beta,n})^2$ reduces to a finite sum in U. This shows that $\psi_{\alpha,n}$ is continuous. Moreover,

$$\sum_{\alpha,n} (\psi_{\alpha,n}(x))^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_\alpha \frac{(\varphi_{\alpha,n}(x))^2}{1 + \sum_\beta (\varphi_{\beta,n}(x))^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

which shows that for every fixed $x \in X$, $\{\psi_{\alpha,n}(x)\}_{\alpha,n}$ belongs to the ℓ^2 space

$$\ell^2 := \left\{ \{a_{\alpha,n}\}_{\alpha,n} : \sum_{\alpha,n} a_{\alpha,n}^2 < \infty \right\}$$

with metric

$$d_2(a,b) := \left(\sum_{\alpha,n} (a_{\alpha,n} - b_{\alpha,n})^2 \right)^{\frac{1}{n}},$$

where $a = \{a_{\alpha,n}\}_{\alpha,n}$ and $b = \{a_{\alpha,n}\}_{\alpha,n}$. Hence, the map

$$\begin{aligned} f : X &\to \ell^2 \\ x &\mapsto \{\psi_{\alpha,n}(x)\}_{\alpha,n} \end{aligned}$$

is well-defined.

We claim that f is one-to-one. To see this, let $x, y \in X$ with $x \neq y$. Then there exist $B_{\alpha,n}$ such that $x \in B_{\alpha,n}$ and $y \in X \setminus B_{\alpha,n}$. It follows by (35) that $\psi_{\alpha,n}(x) > 0$, while $\psi_{\alpha,n}(y) = 0$. Hence, $f(x) \neq f(y)$ and the claim is proved.

Next we claim that f is continuos. Fix $x_0 \in X$ and $\varepsilon > 0$ and find $n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty} \frac{1}{n^2} \leq \frac{\varepsilon^2}{8}.$$

By local finiteness, for every $n = 1, \ldots, n_\varepsilon$ we may find a neighborhood of x_0 that intersects finitely many $B_{\alpha,n}$ in $\mathcal{B}_n$. By intersecting this *finite* number of neighborhoods, we obtain a neighborhood U of x that intersects finitely many $B_{\alpha,n}$ for $n = 1, \ldots, n_\varepsilon$, say, say, $B_{\alpha_1,n_1}, \ldots, B_{\alpha_m,n_m}$. Since each ψ_{α_i,n_i} is continuous at x_0 and we have a finite number of them, there exists a neighborhood $V \subset U$ of x_0 such that

$$|\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0)| \leq \frac{\varepsilon}{\sqrt{2m}}$$

for all $x \in V$ and $i = 1, \ldots, m$. Then for $x \in V$,

$$\sum_{n=1}^{n_\varepsilon} \sum_{\alpha} (\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2 = \sum_{i=1}^{m} (\psi_{\alpha_i,n_i}(x) - \psi_{\alpha_i,n_i}(x_0))^2 \leq m\frac{\varepsilon^2}{2m} = \frac{\varepsilon^2}{2},$$

while

$$\begin{aligned}\sum_{n=n_\varepsilon+1}^{\infty} \sum_{\alpha} (\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2 &\leq 2 \sum_{n=n_\varepsilon+1}^{\infty} \sum_{\alpha} \left[(\psi_{\alpha,n}(x))^2 + (\psi_{\alpha,n}(x_0))^2\right] \\ &\leq 2 \sum_{n=n_\varepsilon+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^2}\right) \leq \frac{4\varepsilon^2}{8} = \frac{\varepsilon^2}{2}.\end{aligned}$$

Hence,

$$d_2(f(x), f(x_0)) \leq \varepsilon$$

for all $x \in V$, which shows continuity at x_0.

Finally, we prove that $f^{-1} : f(X) \to X$ is continuous. Let $y_0 \in f(X)$, then there exists $x_0 \in X$ such that $f(x_0) = y_0$. Consider a neighborhood U of x_0. Since $\mathcal{B}$ is a base, there exists $B_{\alpha,n}$ in $\mathcal{B}$ such that $B_{\alpha,n} \subset U$ so that $\psi_{\alpha,n}(x_0) > 0$. Let $\delta := \psi_{\alpha,n}(x_0)$. If $d_2(f(x), f(x_0)) < \delta$, then

$$\begin{aligned}|\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0)| &\leq \left(\sum_{n=1}^{\infty} \sum_{\alpha} (\psi_{\alpha,n}(x) - \psi_{\alpha,n}(x_0))^2\right)^{\frac{1}{2}} \\ &= d_2(f(x), f(x_0)) < \delta = \psi_{\alpha,n}(x_0),\end{aligned}$$

which implies that $\psi_{\alpha,n}(x) > 0$, and so, by (35), that $x \in B_{\alpha,n} \subset U$. This shows that f^{-1} is continuous at $y_0 = f(x_0)$. ■

2.11 Paracompact Spaces and Partitions of Unity

Theorem 217 leaves open the question of the existence of a partition of unity subordinated to an arbitrary open cover of X. This problem brings us to the definition of paracompact spaces.

Definition 226 *A topological space* (X, τ) *is* paracompact *if it is Hausdorff and if for every an open cover* $\{U_\alpha\}_{\alpha\in\Lambda}$ *of* X*, there exists an open cover of* X *that is a locally finite refinement of* $\{U_\alpha\}_{\alpha\in\Lambda}$*.*

An important class of paracompact spaces is given by metric spaces. This follows from Theorems 219 and 220.

Next we show that paracompact spaces are normal spaces.

Proposition 227 *Let* (X, τ) *be a paracompact space. Then* (X, τ) *is a normal space.*

Proof. Step 1: Let $C \subset X$ be a closed set and let $x \in X \setminus C$. We claim that there exist two disjoint neighborhoods of x and C. Since X is a Hausdorff space, for every $y \in C$, there exist two disjoint neighborhoods V_x and V_y of x and y. Note that since $V_x \cap V_y = \emptyset$, by Proposition 128, we have that $x \notin \overline{V_y}$. Then the sets V_y, $y \in C$, and $X \setminus C$ form an open cover of X. Since X is paracompact, there exists a locally finite refinement $\{U_\alpha\}_{\alpha\in\Lambda}$. Define

$$U := \bigcup_{\alpha\in\Lambda:\, U_\alpha\cap C\neq\emptyset} U_\alpha.$$

Then U is open and contains C. Moreover, by the previous lemma,

$$\overline{U} = \bigcup_{\alpha\in\Lambda:\, U_\alpha\cap C\neq\emptyset} \overline{U_\alpha}.$$

Since by construction each U_α such that $U_\alpha \cap C \neq \emptyset$ is contained in some V_y, it follows that $\overline{U_\alpha} \subset \overline{V_y}$, and so $x \notin \overline{U_\alpha}$. In turn, $x \notin \overline{U}$. The sets $X \setminus \overline{U}$ and U are open, disjoint, and contain x and C, respectively.

Step 2: Let $C_1, C_2 \subset X$ be two closed disjoint sets. By the previous step, for each $y \in C_2$ we may find an open set V_y such that $V_y \cap C_1 = \emptyset$. Define U as in the previous step with C replaced with C_2. Then exactly as before we can show that $C_1 \cap \overline{U} = \emptyset$, and so the sets $X \setminus \overline{U}$ and U are open, disjoint, and contain C_1 and C_2, respectively. This shows that X is normal. ■

Next we present several characterizations of paracompact spaces.

Theorem 228 (Michael) *Let* (X, τ) *be a normal space. Then the following are equivalent.*

(i) (X, τ) *is paracompact.*

(ii) For every open cover of X *there exists a locally finite refinement (not necessarily open).*

(iii) For every open cover of X *there exists a closed, locally finite refinement.*

(iv) For every open cover of X *there exists a* σ*-locally finite open refinement.*

Proof. (i)$\Longrightarrow$(ii) There is nothing to prove.

(ii)$\Longrightarrow$(iii) Assume (ii) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By Theorem 215 there exists an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of X with the property that $\overline{V_\alpha}\subset U_\alpha$ for every $\alpha\in\Lambda$. Now apply (ii) to $\{V_\alpha\}_{\alpha\in\Lambda}$ to find a locally finite refinement $\mathcal{E}$ of $\{V_\alpha\}_{\alpha\in\Lambda}$. Let $\mathcal{C}:=\left\{\overline{E}:\,E\in\mathcal{E}\right\}$. Then $\mathcal{C}$ is still a refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. It remains to show that $\mathcal{C}$ is still locally finite. Let $x\in X$. Since $\mathcal{E}$ is locally finite, there exists a neighborhood U of x that intersects only finitely many elements of $\mathcal{E}$, say, $E_1,\dots,E_m$. If $\overline{E}\cap U\neq\emptyset$, then by Proposition 128, $E\cap U\neq\emptyset$, and so E is one of the $E_1,\dots,E_m$. Thus, $\mathcal{C}:=\left\{\overline{E}:\,E\in\mathcal{E}\right\}$ is locally finite.

(iii)$\Longrightarrow$(i) Assume (iii) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By (iii) there exists a closed locally finite refinement $\mathcal{C}$ of $\{U_\alpha\}_{\alpha\in\Lambda}$. Hence, for every $x\in X$ there exists a neighborhood V_x of x that intersects only finitely many elements of $\mathcal{C}$. Since $\{V_x\}_{x\in X}$ is an open cover of X, we can apply (iii) once more to find a closed locally finite refinement $\mathcal{K}$ of $\{V_x\}_{x\in X}$. For every $C\in\mathcal{C}$ define $\mathcal{K}_C:=\{K\in\mathcal{K}:\,K\cap C=\emptyset\}$ and set

$$D_C:=X\setminus\bigcup_{K\in\mathcal{K}_C}K.$$

Since $\mathcal{K}_C$ is locally finite, by Lemma 225 the set $\bigcup_{K\in\mathcal{K}_C}K$ is closed, and thus D_C is open. Moreover, since $\mathcal{K}$ is a cover of X, it follows from the definition of $\mathcal{K}_C$ that D_C contains C. Thus, the family $\{D_C\}_{C\in\mathcal{C}}$ is an open cover of X. Moreover, if $K\in\mathcal{K}$, then K intersects D_C if and only if K intersects C. We claim that $\{D_C\}_{C\in\mathcal{C}}$ is locally finite. To see this, fix $x\in X$. Since $\mathcal{K}$ is locally finite there exist a neighborhood U of x that intersects only finitely many elements of $\mathcal{K}$, say, $K_1,\dots,K_m$. On the other hand, if $U\cap D_C\neq\emptyset$ for some $C\in\mathcal{C}$, then there is $y\in U$ such that $y\notin\bigcup_{K\in\mathcal{K}_C}K$. Since $\mathcal{K}$ is a cover, y belongs to some $K\notin\mathcal{K}_C$. Thus, $U\cap K\neq\emptyset$ for some $K\notin\mathcal{K}_C$, which implies that one of the $K_1,\dots,K_m$ intersects $C\cap U$. This means that one of the $K_1,\dots,K_m$ intersects D_C. Since each K_i is contained in some V_z and V_z intersects only finitely many C in $\mathcal{C}$, we have that $U\cap D_C\neq\emptyset$ for finitely many $C\in\mathcal{C}$, which shows that $\{D_C\}_{C\in\mathcal{C}}$ is locally finite.

Since $\mathcal{C}$ is refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$, for each $C\in\mathcal{C}$ we may find $\alpha_C\in\Lambda$ such that $C\subset U_{\alpha_C}$. Finally, define

$$\mathcal{V}:=\{D_C\cap U_{\alpha_C}:\,C\in\mathcal{C}\}.$$

We claim that is locally finite open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Since the open set $D_C\cap U_{\alpha_C}$ contains C and $\mathcal{C}$ is a cover of X, the family $\mathcal{V}$ is an open refinement of $\{U_\alpha\}_{\alpha\in\Lambda}$. Since $\{D_C\}_{C\in\mathcal{C}}$ is locally finite, then so is $\mathcal{V}$.

It remains to show that (iv)$\Longrightarrow$(ii). Assume (iv) and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of X. By (iv) there exists an open refinement of the form

$$\mathcal{V}=\bigcup_{n=1}^{\infty}\mathcal{V}_n,$$

where each $\mathcal{V}_n$ is locally finite. For every $n\in\mathbb{N}$ and for every $V\in\mathcal{V}_n$ define

$$\mathcal{V}_V:=\{U\in\mathcal{V}:\,U\in\mathcal{V}_k\text{ for some }k<n\}$$

and set

$$E_V := V \setminus \bigcup_{U \in \mathcal{V}_V} U.$$

Then $E_V \subset V$ and $\{E_V\}_{V \in \mathcal{V}}$ is a cover of X. Note that the family $\{E_V\}_{V \in \mathcal{V}}$ is locally finite. Indeed, given $x \in X$, let $n \in \mathbb{N}$ be the first integer such that x belongs to some V in the family $\mathcal{V}_n$. Then V is a neighborhood of x that does not intersect any E_U for $U \in \mathcal{V}_k$ with $k > n$ (since V has been removed from each such E_U). Thus, V can only intersect sets E_U such that $U \in \mathcal{V}_k$ with $k \leq n$. Since each $\mathcal{V}_k$ is locally finite for $k \leq n$, we may find a neighborhood W_k of x that intersects only finitely many U in $\mathcal{V}_k$., and, in turn, only finitely many of the corresponding E_U. The neighborhood

$$V \cap \bigcap_{k=1}^{n} W_k$$

of x intersects finitely many E_U. This shows that $\{E_V\}_{V \in \mathcal{V}}$ is locally finite and concludes the proof. ■

The importance of paracompact spaces comes from the following theorem.

Theorem 229 (Michael) *Let (X, τ) be a normal space. Then the following are equivalent.*

(i) (X, τ) *is paracompact.*

(ii) For every open cover of X there exists a locally finite partition of unity subordinated to it.

(iii) For every open cover of X there exists a partition of unity subordinated to it.

Proof. Step 1: Assume that (X, τ) is paracompact and let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X. Since (X, τ) is paracompact there exists a locally finite refinement of $\{U_\alpha\}_{\alpha \in \Lambda}$ and (X, τ) is a normal space, by the previous proposition. We are now in a position to apply Theorem 217 to find locally finite partition of unity subordinated to the refinement, and in particular, to $\{U_\alpha\}_{\alpha \in \Lambda}$.
Step 2: Assume that every open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X admits a locally finite partition of unity $\{\varphi_i\}_{i \in I}$ subordinated to it. Fix any two such $\{U_\alpha\}_{\alpha \in \Lambda}$ and $\{\varphi_i\}_{i \in I}$. For every $n \in \mathbb{N}$ and every $i \in I$ consider the set

$$V_{i,n} := \left\{ x \in X : \varphi_i(x) > \frac{1}{n} \right\}.$$

Then $V_{n,i}$ is open. We claim that $\{V_{i,n}\}_{i \in I, n \in \mathbb{N}}$ is an open cover of X that is a locally finite refinement of $\{U_\alpha\}_{\alpha \in \Lambda}$. Indeed, if $x \in X$, then

$$\sum_{i \in I} \varphi_i(x) = 1,$$

and so there exists $i \in I$ such that $\varphi_i(x) \in (0,1]$. It follows that $\varphi_i(x) > \frac{1}{n}$ for all $n \in \mathbb{N}$ sufficiently large, and so $x \in V_{i,n}$ for all $n \in \mathbb{N}$ sufficiently large. This shows that $\{V_{i,n}\}_{i \in I, n \in \mathbb{N}}$ is an open cover of X. Moreover, since $V_{i,n} \subset \operatorname{supp} \varphi_i$ and $\{\varphi_i\}_{i \in I}$ is subordinated to $\{U_\alpha\}_{\alpha \in \Lambda}$, each $\operatorname{supp} \varphi_i$ (and in turn each $V_{i,n}$) is contained in some U_α. Thus, $\{V_{i,n}\}_{i \in I, n \in \mathbb{N}}$ is a refinement of $\{U_\alpha\}_{\alpha \in \Lambda}$. We claim that $\{V_{i,n}\}_{i \in I, n \in \mathbb{N}}$ is σ-locally finite. Fix $x_0 \in X$ and $n \in \mathbb{N}$. Write

$$1 = \sum_{i \in I} \varphi_i(x_0) = \sum_{i \in I_0} \varphi_i(x_0),$$

where $I_0 := \{i \in I : \varphi_i(x_0) > 0\}$ is countable. Since the series $\sum_{i \in I_0} \varphi_i(x_0)$ is convergent, there exists a finite subset $I_1 \subset I_0$ such that

$$\sum_{i \in I_1} \varphi_i(x_0) > 1 - \frac{1}{2n}.$$

By continuity and the fact that I_1 is finite, we may find an open neighborhood U of x_0 such that

$$\sum_{i \in I_1} \varphi_i(x) > 1 - \frac{1}{n}$$

for all $x \in U$. Note that if i does not belong I_1, then U cannot intersect $V_{i,n}$ (the sum would be greater than one). Hence, U intersect only finitely many $V_{i,n}$ (recall that n is fixed). The result now follows from the previous theorem. ■